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A New Representation for Srivastava's λ-Generalized Hurwitz-Lerch Zeta Functions

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Abstract: Taking inspiration principally from some of the latest research, we develop a new series representation for the λ-generalized Hurwitz-Lerch zeta functions. This representation led to important new results. The Fourier transform played a foundational role in this work. The duality property of the Fourier transform became significant for checking the consistency of the results. Some known data has been verified as special cases of the results obtained in this investigation.

Keywords: Hurwitz-Lerch zeta function; generalized functions; analytic number theory; λ-generalized Hurwitz-Lerch zeta functions; derivative properties; series representation

1. Introduction

The Hurwitz-Lerch zeta function has always been remained a focal point for numerous investigators because of its influence on analytic number theory and further practical disciplines. Recently, Srivastava [1] offered a substantially innovative class of Hurwitz-Lerch zeta functions, namely, λ-generalized Hurwitz-Lerch zeta functions. The exploration of its diverse forms has garnered notable concern, and numerous papers have consequently been presented on this subject. Jankov et al. [2] and Srivastava et al. [3] have offered inequalities by considering diverse cases of these functions. Srivastava et al. [4], have presented a nonlinear operator connected to λ-generalized Hurwitz-Lerch zeta functions, in order to investigate the inclusion properties of the definite subclass of a special type of meromorphic functions. Srivastava and Gaboury [5] have considered new expansion formulas for such functions (see, for related data, [6,7]; see also more systematically supplementary revisions cited in these publications). Luo and Raina [8] have discussed an interesting series representation. They also acquired some new inequalities comprising Srivastava's λ-generalized Hurwitz-Lerch zeta functions.

By taking inspiration from all these outcomes, in our current investigation, we consistently present all the special cases of this newly concentrated family of Srivastava's λ-generalized Hurwitz-Lerch zeta functions in the form of a table. On the one hand, we take account of extended Fermi-Dirac and Bose-Einstein functions defined by Srivastava et al. [9], and on the other, we focus on the close relationship of these functions with the family of zeta and related functions. The purpose of this analysis is to discover some fascinating innovative outcomes for Srivastava's λ-generalized Hurwitz-Lerch zeta functions and their different cases by succeeding the methodology of Chaudhry & Qadir [10], Tassaddiq & Qadir [11,12], Tassaddiq [13], Lail & Qadir [14], and Tassaddiq [15]. In these articles [10–15], the authors have investigated new representations for gamma, generalized gamma, extended Fermi-Dirac and Bose-Einstein functions, and Hypergeometric functions, respectively, in terms of complex delta functions. More recently, Tassaddiq [16] has obtained some new results for Srivastava's λ-generalized Hurwitz-Lerch zeta functions by using its Mellin transform representation.

In the present work, we acquire a different representation for the recently introduced family of the λ-generalized Hurwitz-Lerch zeta functions in terms of complex delta functions. We validate this over the space of entire test functions denoted by $\mathbf{Z}$. In the usual sense, we can think of a function being defined in the form of an integral or a series of some variables, or in terms of elementary functions. Nevertheless, it requires consideration as an object in itself, characterized by an integral or a series. This is the only possibility to study the function further than its original domain of description. This is necessary for diverse applications of any function. This concern comes to be principally significant while talking about the concept of higher transcendental functions. Such functions have different series, asymptotic, and integral representations to express functions in diverse domains and to give more simple proofs of its properties when compared to others. Therefore, our new representation is a powerful modeling tool that generalizes the domain of the λ-generalized Hurwitz-Lerch zeta functions from complex numbers to complex functions. It applies to functionals that depend on functions, rather than functions that depend on numbers. Since the methodology used is new, therefore each general result in this paper has the capacity to obtain similar new results for well-studied functions. It provides a computational technique to evaluate integrals of the products of these functions. The stability of the results is confirmed by means of classical methods. In any case, this investigation evidence is meaningful for delivering substantial and innovative results. The approach used is simple and interesting.

Next, we will present the basic definitions and preliminaries by dividing this section into two sections, namely (2.1) and (2.2). In Section 2.1, we discuss preliminaries related to Srivastava's λ-generalized Hurwitz-Lerch zeta functions, while in Section 2.2, we discuss basic preliminaries relevant with distributions (generalized functions) that are necessary to understand the results presented in this paper. The organization of the ensuing sections of this paper is as follows: We present a new representation of the λ-generalized Hurwitz-Lerch zeta functions in Section 3. We achieve analogous outcomes for new associated functions. We discuss the convergence and consequences of new representation in Section 4. We present the Fourier transform representation in Section 5. We check the validity of the results achieved by new representation in Section 5. We summarize our present analysis in the last Section 6. Some interesting new formulae created by giving variations to different parameters are presented in Appendix A.

2. Materials and Methods

2.1. Srivastava's λ-Generalized Hurwitz-Lerch Zeta Functions

Consider the ordinary symbolizations

$$\mathbf{N} := \{\mathbf{1},\mathbf{2},\cdots\};\ \mathbf{N_0} := \mathbf{N} \cup \{\mathbf{0}\};\ \mathbf{Z}^- := \{-\mathbf{1},-\mathbf{2},\cdots\};\ \mathbf{Z}_0^- := \mathbf{Z}^- \cup \{\mathbf{0}\} \tag{1}$$

where $\mathbf{Z}^-$ is the set of negative integers. The symbols $\mathbf{R}, \mathbf{R}^+$, and $\mathbf{C}$ symbolize the sets of real, positive real, and complex numbers, individually throughout the paper.

The standard Fox-Wright function is an extension for the generalized hypergeometric function that is defined by ([8] (p. 2219) Equation (1)) (see also [3], (p. 516), Equation (1)) and [17] p. (493), Equation (2))

$$\mathbf{p}^{\Psi^*}\mathbf{q}\left[\begin{matrix}(\boldsymbol{\lambda}_1,\boldsymbol{\rho}_1),\ldots, & (\boldsymbol{\lambda}_p,\boldsymbol{\rho}_p)\\ (\boldsymbol{\mu}_1,\boldsymbol{\sigma}_1),\ldots, & (\boldsymbol{\mu}_q,\boldsymbol{\sigma}_q)\end{matrix};\mathbf{z}\right] = \sum_{\chi=0}^{\infty}\frac{([\boldsymbol{\lambda}_p])_{\rho_p\chi}}{([\boldsymbol{\mu}_q])_{\sigma_q\chi}}\frac{\mathbf{z}^{\chi}}{\chi!} \tag{2}$$

$$\left(\boldsymbol{\lambda}\mathbf{j}\,,\mu_k\ \in\ \mathbf{C}\ \textbf{and}\ \boldsymbol{\rho}_j\,,\boldsymbol{\sigma}_k \in\ \mathbf{R}_+(\,\mathbf{j}\ =\ \mathbf{1},\cdots,\mathbf{p};\ \mathbf{k}\ =\ \mathbf{1},\cdots,\mathbf{q})\right).$$

Pochammar symbols $([\boldsymbol{\lambda}_p])_{\rho_p\chi} := [\boldsymbol{\lambda}_1]_{\rho_p}\chi\ \cdots\ [\boldsymbol{\lambda}_p]_{\rho_p\chi}$ are the shifted factorial, defined in terms of the basic gamma function as follows:

$$(\boldsymbol{\lambda})_{\rho} = \frac{\boldsymbol{\Gamma}(\boldsymbol{\lambda}+\boldsymbol{\rho})}{\boldsymbol{\Gamma}(\boldsymbol{\lambda})} = \begin{cases} \mathbf{1}\ (\boldsymbol{\rho}=\mathbf{0},\boldsymbol{\rho}\in\ \mathbb{C}\setminus\{\mathbf{0}\}) \\ \boldsymbol{\lambda}(\boldsymbol{\lambda}+\mathbf{1})\ldots(\boldsymbol{\lambda}+\boldsymbol{\chi}-\mathbf{1})\ (\boldsymbol{\rho}=\boldsymbol{\chi}\in\ \mathbb{N};\boldsymbol{\lambda}\in\ \mathbb{C}),\end{cases} \tag{3}$$

$$\Delta := \sum_{j=1}^{q} \sigma_j - \sum_{j=1}^{p} \rho_j \text{ and } \nabla :=\left(\prod_{j=1}^{p} \rho_j^{-\rho_j}\right) \cdot \left(\prod_{j=1}^{q} \sigma_j^{\sigma_j}\right).$$

The series given by (2) converges in the complete complex z-plane for $\Delta > -1$; and if $\Delta = 0$, the series (2) converges for specific values of $|z| < \nabla$. For more a comprehensive exchange of such functions, we refer the interested reader to see the references [18–23].

Srivastava's λ-generalized Hurwitz-Lerch zeta functions as presented by ([1], p. 1487, Equation (4))

$$\Phi_{\lambda_1,\dots,\lambda_p,\mu_1,\dots,\mu_q}^{(\rho_1,\dots,\rho_p,\sigma_1,\dots,\sigma_q)}(z,s,a;b,\lambda) := \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \exp\left(-at - \frac{b}{t^\lambda}\right) p^{\Psi^*} q \left[\begin{matrix} (\lambda_1,\rho_1),\dots, & (\lambda_p,\rho_p) \\ (\mu_1,\sigma_1),\dots, & (\mu_q,\sigma_q) \end{matrix}; ze^{-t}\right] dt \tag{4}$$

$$(\min[\Re(a),\Re(s)] > 0; \Re(b) \geqq 0; \lambda \geqq 0)$$

is central for this research paper. Luo and Raina obtained the following series representation ([8], p. 2221, Equation (6))

$$\Phi_{\lambda_1,\dots,\lambda_p,\mu_1,\dots,\mu_q}^{(\rho_1,\dots,\rho_p,\sigma_1,\dots,\sigma_q)}(z,s,a;b,\lambda) = \frac{1}{\lambda\Gamma(s)} \sum_{\chi=0}^{\infty} \frac{([\lambda_p])_{\rho_p\chi}}{([\mu_q])_{\sigma_q\chi}} \frac{z^\chi}{\chi!} Z_{\frac{1}{\lambda}}^{\frac{s}{\lambda}}(a+\chi)^\lambda \, b \frac{z^\chi}{(\chi+a)^s} \tag{5}$$

$$(\lambda j \in R (j = 1,..,p) \text{and } \mu_j \in R\backslash Z - 0 \ (j = 1,\dots,q); \rho j > 0 (j,\dots,p); \sigma j > 0 (j = 1,\dots,q); 1 + \Delta \geq 0)$$

so that, obviously, one can get the following association with extended Hurwitz-Lerch zeta functions ([17], p. 503, Equation (6.2)) (see also [3,24])

$$\Phi_{\lambda_1,\dots,\lambda_p,\mu_1,\dots,\mu_q}^{(\rho_1,\dots,\rho_p,\sigma_1,\dots,\sigma_q)}(z,s,a;0,\lambda) = \Phi_{\lambda_1,\dots,\lambda_p,\mu_1,\dots,\mu_q}^{(\rho_1,\dots,\rho_p,\sigma_1,\dots,\sigma_q)}(z,s,a) = e^b \Phi_{\lambda_1,\dots,\lambda_p,\mu_1,\dots,\mu_q}^{(\rho_1,\dots,\rho_p,\sigma_1,\dots,\sigma_q)}(z,s,a;b,0). \tag{6}$$

By making use of Equations (4)-(6), we list all the items in the subsequent table that are straightforward to achieve in view of different values of the parameters as specified column and row wise on the next page.

Now if we go through the previous research, we notice that the different cases of λ-generalized Hurwitz-Lerch zeta functions specified in the third column and second row, explicitly $\Theta_\mu^\lambda(\mp z,s,a;b)$, have been defined and explored by [25], (p. 90), Equation (1.6), and [26]. Some of its most interesting versions were studied and considered by [27]. The original class of zeta functions specifically and explicitly is: Hurwitz-Lerch zeta function $\Phi(\pm z,s,a)$, [28], (p. 27), Equation (1.11), extended Fermi-Dirac $\Theta_a(x;s)$, [9], (p. 9), Equation (3.14), extended Bose- Einstein $\Psi_a(x;s)$, [9], (p. 115), Equation (4.4), Fermi-Dirac $F_s(x)$, [9], p. 109, Equation (1.12)], Bose-Einstein $B_s(x)$, [9], (p. 109), Equation (1.12), Polylogarithm $\phi(z,s)$, [28], (Chapter 1], Hurwitz zeta $\zeta(s,a)$ [28], (Chapter 1), and Riemann zeta functions $\zeta(s)$, [28] (Chapter 1), respectively are listed in the last column of Table 1. Two of the items in the first row specifically $\Phi_{\lambda_1,\dots,\lambda_p,\mu_1,\dots,\mu_q}^{(\rho_1,\dots,\rho_p,\sigma_1,\dots,\sigma_q)}(\pm z,s,a)$ are defined by [1], (p. 1486), Equation 1.11 (see also [17]) and $\Phi_\mu^*(\pm z,s,a)$ defined by [29], p. 100, Equation (1.5). The extended Riemann zeta $\zeta_b(s)$ [30], (p. 308) and Hurwitz zeta functions $\zeta_b(s,a)$ [30], (p. 308) are noticeable in the last two rows. For additional comprehensive study of zeta and related functions, we refer the reader to [1–32] and related discussions therein.

Table 1. Different Special Cases of λ-Generalized Hurwitz-Lerch Zeta Functions.

$min[\Re(a),\Re(s)] > 0; \Re(b) \geqq 0; \lambda \geqq 0;$ $\boldsymbol{\rho = \rho_{1,\dots,}\rho_{p;,}\sigma = \sigma_1,\dots,\sigma_q; \lambda^* = \lambda_1,\dots,\lambda_p; \mu = \mu_1,\dots,,\mu_q}$		(p−1 = q = 0; λ_1 = μ; ρ_1 =1)					(p−1 = q = 0; λ_1 = μ; ρ_1 =1)	
			$\lambda = 1$	$\mu = 1$	$\lambda = \mu = 1$	$b = 0$	$b = 0$	$\mu = 1;\ b = 0$
λ-Generalized Hurwitz-Lerch Zeta Functions	$\Phi^{(\rho;\sigma)}_{\lambda^*,\mu}(\pm z, s, a; b,$ [1], (p. 1487), Equation (1.14)	$\Theta^{\lambda}_{\mu}(\mp z, s, a; b)$ [25], (p. 90), Equation (6) and [26]	$\Phi^*_{\mu}(\pm z, s, a, b)$	$\Phi_b(\pm z, s, a,$	$\Phi_b(\pm z, s, a)$	$\Phi^{(\rho;\sigma)}_{\lambda^*,\mu}(\pm z, s$ ([1], p. 1486, Equation (1.11)) & [17]	$\Phi^*_{\mu}(\pm z, s, a)$ ([29], p. 100, Equation (1.5))	$\Phi(\pm z, s, a)$ ([28], p. 27, Equation (1.11))
$\Phi^{(\rho;\sigma)}_{\lambda^*,\mu}(\pm e^{-x}, s, a; b, \lambda)$ λ-Generalized Extended Fermi-Dirac and Extended Bose-Einstein Functions	$\Theta^{(\rho;\sigma)}_{\lambda^*,\mu}(x, s, a; b, \lambda$	$\Theta^{\lambda}_{\mu}(x, s, a; b)$	$\Theta^*_{\mu}(x, s, a, b)$	$\Theta_b(x, s, a; \lambda$	$\Theta_b(x, s, a)$	$\Theta^{(\rho;\sigma)}_{\lambda^*,\mu}(x, s,$	$\Theta^*_{\mu}(x, s, a)$ ([27], p. 12, Equation (45))	$\Theta_a(x; s)$ ([9], p. 9, Equation (3.14))
	$\Psi^{(\rho;\sigma)}_{\lambda^*,\mu}(x, s, a; b, \lambda$	$\Psi^{\lambda}_{\mu}(x, s, a; b)$	$\Psi^*_{\mu}(x, s, a, b)$	$\Psi_b(x, s, a, \lambda$	$\Psi_b(x, s, a)$	$\Psi^{(\rho;\sigma)}_{\lambda^*,\mu}(x, s,$	$\Psi^*_{\mu}(x, s, a)$ ([27], p. 12, Equation (45))	$\Psi_a(x; s)$ [9], p. 115, Equation (4.4)
$\Phi^{(\rho;\sigma)}_{\lambda^*,\mu}(\pm z, s, 1; b, \lambda)$ λ-Generalized Polylogarithm Functions	$\phi^{(\rho;\sigma)}_{\lambda^*,\mu}(\pm z, s; b, \lambda)$	$\phi^{\lambda}_{\mu}(\mp z, s, a; b)$	$\phi^*_{\mu}(z, s, b)$	$\phi_b(z, s, \lambda)$	$\phi_b(z, s)$	$\phi^{(\rho;\sigma)}_{\lambda^*,\mu}(z, s)$	$\phi^*_{\mu}(z, s)$ ([27], p. 12, Equation (42))	$\phi(z, s)$ [28], (Chapter 1)
$\Phi^{(\rho;\sigma)}_{\lambda^*,\mu}(\pm e^{-x}, s+1, 1; b, \lambda)$ λ-Generalized Fermi-Dirac and Bose Einstein Functions	$F^{(\rho;\sigma)}_{\lambda^*,\mu}(x, s; b, \lambda)$	$F^{\lambda}_{\mu}(x, s, a; b)$	$F^*_{\mu}(x, s, b)$	$F_b(x, s, \lambda)$	$F_b(x, s)$	$F^{(\rho;\sigma)}_{\lambda^*,\mu}(x, s)$	$F^*_{\mu}(x, s)$ ([27], p. 12, Equation (45))	$F_s(x)$ ([9], p. 109, Equation (1.12)
	$B^{(\rho;\sigma)}_{\lambda^*,\mu}(x, s; b, \lambda)$	$B^{\lambda}_{\mu}(x, s, a; b)$	$B^*_{\mu}(x, s, b)$	$B_b(x, s, \lambda)$	$B_b(x, s)$	$B^{(\rho;\sigma)}_{\lambda^*,\mu}(x, s)$	$B^*_{\mu}(x, s)$ ([27], p. 12, Equation (45))	$B_s(x)$ ([9], p. 109, Equation (1.12))
$\Phi^{(\rho;\sigma)}_{\lambda^*,\mu}(\pm 1, s, a; b, \lambda)$ λ-Generalized Hurwitz zeta Functions	$\zeta^{(\rho;\sigma)}_{\lambda^*,\mu}(s, a; b, \lambda)$	$\zeta^{\lambda}_{\mu}(s, a; b)$	$\zeta^*_{\mu}(s, a, b)$	$\zeta_b(s, a, \lambda)$	$\zeta_b(s, a)$ [30], p. 308	$\zeta^{(\rho;\sigma)}_{\lambda^*,\mu}(s, a)$	$\zeta^*_{\mu}(s, a)$ [27]	$\zeta(s, a)$ [28], (Chapter 1)
$\Phi^{(\rho;\sigma)}_{\lambda^*,\mu}(\pm 1, s, 1; b, \lambda)$ λ-Generalized Riemann Zeta Functions	$\zeta^{(\rho;\sigma)}_{\lambda^*,\mu}(s)$	$\zeta^{\lambda}_{\mu}s; b)$	$\zeta^*_{\mu}(s, b)$	$\zeta_b(s, \lambda)$	$\zeta_b(s)$ [30], p. 308	$\zeta^{(\rho;\sigma)}_{\lambda^*,\mu}(s)$	$\zeta^*_{\mu}(s)$ [27]	$\zeta(s)$ [28], (Chapter 1)

2.2. Distributions and Test Functions

Continuous linear functionals that act on some space of test functions are commonly known as generalized functions (or distributions). These are the elements of the corresponding dual space of test functions. A review of such elements is significant, because they not only have locally integrable functions, but also consist of additional objects that are not regular distributions. Consequently, several actions such as integration, differentiation, and limits that are defined for functions can be applied to functionals. A delta functional commonly used in singular distribution is defined by

$$\langle \boldsymbol{\delta}(\mathbf{u} - \mathbf{a}), \boldsymbol{\varphi}(\mathbf{t})\rangle = \boldsymbol{\varphi}(\mathbf{a})\ (\forall \boldsymbol{\varphi} \in \mathbf{D}, \mathbf{a} \in \mathbf{R}), \tag{7}$$

where for a non-zero $\boldsymbol{a}$, $\boldsymbol{\delta}(-\mathbf{u}) = \boldsymbol{\delta}(\mathbf{u})$; $\boldsymbol{\delta}(\mathbf{au}) = \frac{\boldsymbol{\delta}(\mathbf{t})}{|\mathbf{a}|}$.

A multi-volume presentation [33] (Vol. I-V) by Gelfand and Shilov is a great treatise on such functions. The commonly used spaces of test functions are the spaces of compact support functions denoted by $\mathbf{D}$, and the space of rapidly decaying functions denoted by $\mathbf{S}$, that also have derivatives of all orders. The spaces $\mathbf{D}'$ and $\mathbf{S}'$ are the dual spaces of $\mathbf{D}$ and $\boldsymbol{S}$. Spaces $\mathbf{S}$ and $\mathbf{S}'$ are closed under the Fourier transform, but $\boldsymbol{D}$ and $\boldsymbol{D}'$ are not. The Fourier transform of the elements of $\boldsymbol{D}'$ are continuous linear functionals acting on the elements of $\boldsymbol{Z}$ that comprises of entire functions such that their Fourier transforms are in $\boldsymbol{D}$ [34]. The entire function $\boldsymbol{\varphi \epsilon Z}$ does not vanish on some interval $\boldsymbol{a} < \boldsymbol{u} < \boldsymbol{b}$, but vanishes universally. Accordingly

$$\mathbf{Z}' \supset \mathbf{S}' \supset \mathbf{S} \supset \mathbf{Z};\ \mathbf{D} \cap \mathbf{Z} \equiv \mathbf{0};\ \mathbf{D}' \supset \mathbf{S}' \supset \mathbf{S} \supset \mathbf{D}. \tag{8}$$

The elements of Z consist of entire analytic functions satisfying the following set of inequalities

$$|\mathbf{s}^{\mathbf{q}}\ \boldsymbol{\varphi}(\mathbf{s})| \le \mathbf{C}_{\mathbf{q}} \mathbf{e}^{\mathbf{a}|\boldsymbol{\tau}|};\ (\mathbf{q} = \mathbf{0}, \mathbf{1}, \mathbf{2}, \dots.) \tag{9}$$

where the constants $\boldsymbol{a}$ and $\boldsymbol{C_q}$ may depend on $\boldsymbol{\varphi}$. By ([33], Vol 1, p. 169, Equation (8)), we take the Fourier transform of exponential function

$$\mathbf{F}[\mathbf{e}^{\boldsymbol{\alpha}\mathbf{t}};\ \boldsymbol{\omega}] = \mathbf{2}\boldsymbol{\pi}\boldsymbol{\delta}(\boldsymbol{\omega} - \mathbf{i}\boldsymbol{\alpha}) \tag{10}$$

as an example of distribution that is an element of $\boldsymbol{Z}'$ and for $\forall \boldsymbol{g} \in \boldsymbol{Z}'$ ([33], (p. 159), Equation (4)), see also ([34], p. 201, Equation (9))

$$\boldsymbol{g}(\mathbf{s} + \mathbf{b}) = \sum_{\mathbf{r}=\mathbf{0}}^{\infty} \boldsymbol{g}^{(\mathbf{r})}(\mathbf{s}) \frac{\mathbf{b}^{\mathbf{r}}}{\mathbf{r}!}. \tag{11}$$

So that we have the following basic identity

$$\boldsymbol{\delta}(\mathbf{s} + \mathbf{b}) = \sum_{\mathbf{r}=\mathbf{0}}^{\infty} \boldsymbol{\delta}^{(\mathbf{r})}(\mathbf{s}) \frac{\mathbf{b}^{\mathbf{r}}}{\mathbf{r}!};\ \langle \boldsymbol{\delta}^{(\mathbf{r})}(\mathbf{s}), \boldsymbol{\varphi}(\mathbf{s})\rangle = (-\mathbf{1})^{\mathbf{r}} \boldsymbol{\varphi}^{(\mathbf{r})}(\mathbf{0}). \tag{12}$$

For an additional extensive study of these spaces, we refer the reader to [33] (Vol. I-V), [34,35] and the related bibliography therein.

Throughout this investigation, conditions on the parameters will be considered standard as given in (1)-(6) unless otherwise stated.

3. Results

New Series Representation of the λ-Generalized Hurwitz-Lerch Zeta Functions

Theorem 1. *λ-generalized Hurwitz-Lerch zeta functions have the following representation*

$$\begin{aligned}\boldsymbol{\Gamma}(\mathbf{s})\boldsymbol{\Phi}_{\boldsymbol{\lambda}_1,\dots,\boldsymbol{\lambda}_p,\boldsymbol{\mu}_1,\dots,\boldsymbol{\mu}_q}^{(\boldsymbol{\rho}_1,\dots,\boldsymbol{\rho}_p,\boldsymbol{\sigma}_1,\dots,\boldsymbol{\sigma}_q)}&(\mathbf{z}, \mathbf{s}, \mathbf{a}; \mathbf{b}, \boldsymbol{\lambda}) \\ &= \mathbf{2}\boldsymbol{\pi} \sum_{\boldsymbol{\chi},\boldsymbol{\xi},\boldsymbol{\psi}=\mathbf{0}}^{\infty} \frac{([\boldsymbol{\lambda}_{\mathbf{p}}])_{\boldsymbol{\rho}_{\mathbf{p}}\boldsymbol{\chi}}}{([\boldsymbol{\mu}_{\mathbf{q}}])_{\boldsymbol{\sigma}_{\mathbf{q}}\boldsymbol{\chi}}} \frac{(\mathbf{z})^{\boldsymbol{\chi}}}{\boldsymbol{\chi}!} \frac{(-(\boldsymbol{\chi} + \mathbf{a}))^{\boldsymbol{\xi}}}{\boldsymbol{\xi}!} \frac{(-\mathbf{b})^{\boldsymbol{\psi}}}{\boldsymbol{\psi}!} \boldsymbol{\delta}(\mathbf{s} + \boldsymbol{\xi} - \boldsymbol{\lambda}\boldsymbol{\psi}).\end{aligned} \tag{13}$$

Proof: Let us first replace $t = e^y$ and $s = \sigma + i\tau$ in Equation (4), then we get

$$\Phi_{\lambda_1,\ldots,\lambda_p,\mu_1,\ldots,\mu_q}^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}(z,s,a) = \tag{14}$$

$$= \frac{1}{\Gamma(s)}\int_{-\infty}^{\infty} e^{y(\sigma+i\tau)}\exp\left(-ae^y - \frac{b}{e^{\lambda y}}\right) p^{\Psi^*}q\left[\begin{matrix}(\lambda_1,\rho_1),\ldots, & (\lambda_p,\rho_p) \\ (\mu_1,\sigma_1),\ldots, & (\mu_q,\sigma_q)\end{matrix}; z\exp(-e^y)\right]dt,$$

$$(\min[\Re(a),\Re(s)] > 0).$$

Now, writing the series form of the Fox-Wright function

$$p^{\Psi^*}q\left[\begin{matrix}(\lambda_1,\rho_1),\ldots, & (\lambda_p,\rho_p) \\ (\mu_1,\sigma_1),\ldots, & (\mu_p,\sigma_p)\end{matrix}; z\exp(-e^y)\right] = \sum_{\chi=0}^{\infty}\frac{([\lambda_p])_{\rho_p\chi}}{([\mu_q])_{\sigma_q\chi}}\frac{z^\chi}{\chi!}\exp(-\chi e^y) \tag{15}$$

and then collecting and expanding the exponential terms

$$e^{\sigma y}\exp\left(-(a+\chi)e^y - \frac{b}{e^{\lambda y}}\right) = \sum_{\xi,\psi=0}^{\infty}\frac{([\lambda_p])_{\rho_p\chi}}{([\mu_q])_{\sigma_q\chi}}\frac{(z)^\chi}{\chi!}\frac{(-(\chi+a))^\xi}{\xi!}\frac{(-b)^\psi}{\psi!}, \tag{16}$$

we get

$$\Phi_{\lambda_1,\ldots,\lambda_p,\mu_1,\ldots,\mu_q}^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}(z,s,a;b,\lambda) \tag{17}$$

$$= \sum_{\chi,\xi,\psi=0}^{\infty}\frac{([\lambda_p])_{\rho_p\chi}}{([\mu_q])_{\sigma_q\chi}}\frac{(z)^\chi}{\chi!}\frac{(-(\chi+a))^\xi}{\xi!}\frac{(-b)^\psi}{\psi!}\int_{-\infty}^{\infty} e^{i\tau y}\, e^{(\sigma+\xi-\psi\lambda)y}dy.$$

The order of summation and integration is interchangeable due to uniform convergence of the integral. By using Equation (10), we get

$$\int_{-\infty}^{\infty} e^{i\tau y}\, e^{(\sigma+\xi-\psi\lambda)y}dy = F\left[e^{(\sigma+\xi-\psi\lambda)y};\tau\right] = 2\pi\delta(\tau - i(\sigma+\xi-\psi\lambda)) \tag{18}$$

$$= 2\pi\,\delta[\frac{1}{i}\,(i\tau - (\sigma+\xi-\psi\lambda))] = 2\pi|\,i\,|\delta(\sigma + i\tau + \xi - \psi\lambda)$$

$$= 2\pi\delta(s+\xi-\psi\lambda).$$

The above Equations (17) and (18) lead to the required result.

Remark 1. *We can get analogous outcomes for further associated functions as enumerated row-wise in Table 1, in view of altered parameter values in the form of following corollaries.*

Corollary 1. *λ-Generalized Extended Fermi-Dirac functions have the following representation*

$$\Gamma(s)\Theta_{\lambda_1,\ldots,\lambda_p,\mu_1,\ldots,\mu_q}^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}(x,s,a;b,\lambda)$$

$$= 2\pi\sum_{\chi,\xi,\psi=0}^{\infty}\frac{([\lambda_p])_{\rho_p\chi}}{([\mu_q])_{\sigma_q\chi}}\frac{(-e^{-x})^\chi}{\chi!}\frac{(-(\chi+a))^\xi}{\xi!}\frac{(-b)^\psi}{\psi!}\delta(s+\xi-\lambda\psi) \tag{19}$$

and λ-Generalized Extended Bose-Einstein functions have the following representation

$$\Gamma(s)\Psi_{\lambda_1,\ldots,\lambda_p,\mu_1,\ldots,\mu_q}^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}(x,s,a;b,\lambda)$$

$$= 2\pi\sum_{\chi,\xi,\psi=0}^{\infty}\frac{([\lambda_p])_{\rho_p\chi}}{([\mu_q])_{\sigma_q\chi}}\frac{(e^{-x})^\chi}{\chi!}\frac{(-(\chi+a))^\xi}{\xi!}\frac{(-b)^\psi}{\psi!}\delta(s+\xi-\lambda\psi). \tag{20}$$

Proof. This holds by simply replacing $z \longrightarrow \pm e^{-x}$ on both sides of (13) and by means of the corresponding item specified in column 2 and row 2 of Table 1.

Corollary 2. *λ-Generalized Fermi-Dirac functions have the following representation*

$$\Gamma(s)F^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}_{\lambda_1,\ldots,\lambda_p,\mu_1,\ldots,\mu_q}(x,s;b,\lambda) = 2\pi \sum_{\chi,\xi,\psi=0}^{\infty} \frac{([\lambda_p])_{\rho_p\chi}}{([\mu_q])_{\sigma_q\chi}} \frac{(-e^{-x})^{\chi}}{\chi!} \frac{(-(\chi+1))^{\xi}}{\xi!} \frac{(-b)^{\psi}}{\psi!} \delta(s+\xi-\lambda\psi) \tag{21}$$

and λ-Generalized Extended Bose-Einstein functions have the following representation

$$\Gamma(s)B^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}_{\lambda_1,\ldots,\lambda_p,\mu_1,\ldots,\mu_q}(x,s;b,\lambda) = 2\pi \sum_{\chi,\xi,\psi=0}^{\infty} \frac{([\lambda_p])_{\rho_p\chi}}{([\mu_q])_{\sigma_q\chi}} \frac{(e^{-x})^{\chi}}{\chi!} \frac{(-(\chi+a))^{\xi}}{\xi!} \frac{(-b)^{\psi}}{\psi!} \delta(s+\xi-\lambda\psi). \tag{22}$$

Proof. Both results hold by simply replacing $z \longrightarrow \pm e^{-x}$; $a \longrightarrow 1$ on both sides of (13) and in view of defined item from Table 1 reliable on these parameter values.

Corollary 3. *λ-Generalized Polylogarithm functions has the following representation*

$$\Gamma(s)\phi^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}_{\lambda_1,\ldots,\lambda_p,\mu_1,\ldots,\mu_q}(z,s;b,\lambda) = 2\pi z \sum_{\chi,\xi,\psi=0}^{\infty} \frac{([\lambda_p])_{\rho_p\chi}}{([\mu_q])_{\sigma_q\chi}} \frac{(z)^{\chi}}{\chi!} \frac{(-(\chi+1))^{\xi}}{\xi!} \frac{(-b)^{\psi}}{\psi!} \delta(s+\xi-\lambda\psi). \tag{23}$$

Proof. This holds by simply replacing $a \longrightarrow 1$ on both sides of (13) and using the precise element from Table 1 equivalent to these constraint values.

Corollary 4. *λ-Generalized Hurwitz zeta functions has the following representation*

$$\Gamma(s)\zeta^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}_{\lambda_1,\ldots,\lambda_p,\mu_1,\ldots,\mu_q}(s,a;b,\lambda) = 2\pi \sum_{\chi,\xi,\psi=0}^{\infty} \frac{([\lambda_p])_{\rho_p\chi}}{([\mu_q])_{\sigma_q\chi}} \frac{1}{\chi!} \frac{(-(\chi+a))^{\xi}}{\xi!} \frac{(-b)^{\psi}}{\psi!} \delta(s+\xi-\lambda\psi). \tag{24}$$

Proof. This holds by simply replacing $z \longrightarrow 1$ on both sides of (13) and, in view of particular items from Table 1, stable with these parameter values.

Corollary 5. *λ-Generalized Riemann zeta functions has the following representation*

$$\Gamma(s)\zeta^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}_{\lambda_1,\ldots,\lambda_p,\mu_1,\ldots,\mu_q}(s;b,\lambda) = 2\pi \sum_{\chi,\xi,\psi=0}^{\infty} \frac{([\lambda_p])_{\rho_p\chi}}{([\mu_q])_{\sigma_q\chi}} \frac{1}{\chi!} \frac{(-(\chi+1))^{\xi}}{\xi!} \frac{(-b)^{\psi}}{\psi!} \delta(s+\xi-\lambda\psi). \tag{25}$$

Proof. This holds by simply replacing $z \longrightarrow 1; a \longrightarrow 1$ on both sides of (13) and, in view of certain components from Table 1, is firm with these considered values.

Remark 1. *We can get similar representations for other special cases of these functions by considering different parameter variations in view of Table 1 column-wise.*

By putting $\boldsymbol{\lambda} = \mathbf{0}$ in the above results (13), and in view of the relation (6), we get the following new results:

$$\Gamma(s)\Phi_{\lambda_1,\ldots,\lambda_p,\mu_1,\ldots,\mu_q}^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}(z,s,a) = 2\pi e^{b} \sum_{\chi,\xi=0}^{\infty} \frac{([\lambda_p])_{\rho_p\chi}}{([\mu_q])_{\sigma_q\chi}} \frac{(z)^{\chi}}{\chi!} \frac{(-(\chi+a))^{\xi}}{\xi!} \delta(s+\xi). \tag{26}$$

Next by considering $\mathbf{b} = \mathbf{0}$ in (26), we get the following results:

$$\Gamma(s)\Phi_{\lambda_1,\ldots,\lambda_p,\mu_1,\ldots,\mu_q}^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}(z,s,a) = 2\pi \sum_{\chi,\xi=0}^{\infty} \frac{([\lambda_p])_{\rho_p\chi}}{([\mu_q])_{\sigma_q\chi}} \frac{(z)^{\chi}}{\chi!} \frac{(-(\chi+a))^{\xi}}{\xi!} \delta(s+\xi). \tag{27}$$

Considering, $p-1 = q = 0$; ($\boldsymbol{\lambda_1} = \boldsymbol{\mu}$; $\boldsymbol{\rho_1} = 1$), the above Equation (13) would reduce immediately to the following form

$$\Gamma(s)\Theta_{\mu}^{\lambda}(z,s,a;b) = 2\pi \sum_{\chi,\xi,\psi=0}^{\infty} (\mu)_{\chi} \frac{(z)^{\chi}}{\chi!} \frac{(-(\chi+a))^{\xi}}{\xi!} \frac{(-b)^{\psi}}{\psi!} \delta(s+\xi-\lambda\psi). \tag{28}$$

Next, specifying $\boldsymbol{\mu} = 1$ in (28), one can get the following new result as special case

$$\Gamma(s)\Theta(z,s,a;b,\lambda) = 2\pi \sum_{\chi,\xi,\psi=0}^{\infty} (z)^{\chi} \frac{(-(\chi+a))^{\xi}}{\xi!} \frac{(-b)^{\psi}}{\psi!} \delta(s+\xi-\lambda\psi). \tag{29}$$

Next, again by giving variations to different parameters, we can get similar representations for other special cases of these functions.

By putting $\boldsymbol{\lambda} = \mathbf{0}$ in the above result (29), we get the following new result

$$\Gamma(s)\Phi(z,s,a;b) = 2\pi e^{b} \sum_{\chi,\xi=0}^{\infty} (z)^{\chi} \frac{(-(\chi+a))^{\xi}}{\xi!} \delta(s+\xi). \tag{30}$$

By putting $\mathbf{b} = \mathbf{0}$ in (30), we get the following results for the original family of Hurwitz-Lerch zeta function and its special cases ([13], Chapter 4):

$$\Gamma(s)\Phi(z,s,a) = 2\pi \sum_{\chi,\xi=0}^{\infty} (z)^{\chi} \frac{(-(\chi+a))^{\xi}}{\xi!} \delta(s+\xi), \tag{31}$$

$$\Gamma(s)\phi(z,s) = 2\pi z \sum_{\chi,\xi=0}^{\infty} (z)^{\chi} \frac{(-(\chi+1))^{\xi}}{\xi!} \delta(s+\xi), \tag{32}$$

$$\Gamma(s)\zeta(s,a) = 2\pi \sum_{\chi,\xi=0}^{\infty} \frac{(-(\chi+a))^{\xi}}{\xi!} \delta(s+\xi), \tag{33}$$

$$\Gamma(s)\zeta(s) = 2\pi \sum_{\chi,\xi=0}^{\infty} \frac{(-(\chi+1))^{\xi}}{\xi!} \delta(s+\xi). \tag{34}$$

Remark 1. *Note that we have obtained a demonstration given in the form of complex delta functions that is only meaningful in the sense of distributions once defined as an inner product with some suitable function. For example, divide both sides of (34) in the usual sense*

$$1 = \frac{\sum_{\chi,\xi=0}^{\infty} \frac{(-(\chi+1))^{\xi}}{\xi!} \delta(s+\xi)}{\Gamma(s)\zeta(s)}. \tag{35}$$

In addition, we get

$$\mathbf{1} = \sum_{\chi,\xi=0}^{\infty} \frac{(-(\chi+\mathbf{1}))^{\xi}}{\xi!} \frac{\mathbf{1}}{\Gamma(-\xi)\zeta(-\xi)}, \tag{36}$$

where the product $\Gamma(-\xi)\zeta(-\xi)$ contributes only for even values of ξ, because zeros of zeta cancel the poles of gamma functions while for other values of ξ, the right-hand side sum will vanish due to $\Gamma(-\xi)$ in the reciprocal. Therefore, we get

$$\mathbf{1} = \sum_{\chi,\xi=1}^{\infty} \frac{(\chi)^{2\xi}}{(2\xi)!} + \mathbf{0} \Rightarrow \mathbf{1} = \sum_{\chi=0}^{\infty} \cosh(\chi) \Rightarrow \mathbf{1} = \infty \tag{37}$$

that leads to an obvious contradiction.

Meanwhile, if we consider the inner product

$$\langle \Gamma(s)\zeta(s), \frac{\mathbf{1}}{\Gamma(s)\zeta(s)} \rangle = \sum_{\chi,\xi=0}^{\infty} \frac{(-(\chi+\mathbf{1}))^{\xi}}{\xi!} \langle \delta(s+\xi), \frac{\mathbf{1}}{\Gamma(s)\zeta(s)} \rangle \tag{38}$$

then we get

$$\int_{s\in\mathbb{C}} \mathbf{1}\ ds = \sum_{\chi,\xi=0}^{\infty} \frac{(-(\chi+\mathbf{1}))^{\xi}}{\xi!} \frac{\mathbf{1}}{\Gamma(-\xi)\zeta(-\xi)}. \tag{39}$$

Due to the reason as stated above we get

$$\int_{s\in\mathbb{C}} \mathbf{1}\ ds = \sum_{\chi,\xi=1}^{\infty} \frac{(\chi)^{2\xi}}{(2\xi)!} + \mathbf{0}, \tag{40}$$

$$\int_{s\in\mathbb{C}} \mathbf{1}\ ds = \int_{-\infty}^{+\infty} \mathbf{1}\, ds = \sum_{\chi=0}^{\infty} \cos h(\chi)\,, \tag{41}$$

and both sides diverge. Therefore, we need to be very rigorous in selecting a class of functions for which this representation is meaningful or convergent.

4. Convergence and Applications of New Series Representation

The representation of the λ -generalized Hurwitz-Lerch zeta function

$$\Phi_{\lambda_1,\dots,\lambda_p,\mu_1,\dots,\mu_q}^{(\rho_1,\dots,\rho_p,\sigma_1,\dots,\sigma_q)}(z,s,a;b,\lambda)$$

and related functions is attained in the form of the series of delta function that is defined simply if converges as distributions or generalized functions. Therefore, these new representations are well defined for the functions for which these infinite series converge. Meanwhile, the complex delta function acts as a continuous linear functional on the space Z. Hence, it is straightforward that the series of delta functions are obviously the continuous linear functionals acting on the space Z. (The results may also be true for some larger spaces ,but here in our present investigation, we are just restricting to Z). Therefore, $\forall \Lambda(s) \epsilon Z$, we get from (13)

$$\begin{aligned} &\langle \Gamma(s)\Phi_{\lambda_1,\dots,\lambda_p,\mu_1,\dots,\mu_q}^{(\rho_1,\dots,\rho_p,\sigma_1,\dots,\sigma_q)}(z,s,a;b,\lambda), \Lambda(s)\rangle = \\ &2\pi \sum_{\chi,\xi,\psi=0}^{\infty} \frac{([\lambda_p])_{\rho_p\chi}}{([\mu_q])_{\sigma_q\chi}} \frac{(z)^{\chi}}{\chi!} \frac{(-(\chi+a))^{\xi}}{\xi!} \frac{(-b)^{\psi}}{\psi!} \langle \delta(s+\xi-\lambda\psi), \Lambda(s)\rangle;\ (\forall \Lambda(s) \epsilon Z) \\ &= \sum_{\chi,\xi,\psi=0}^{\infty} \frac{([\lambda_p])_{\rho_p\chi}}{([\mu_q])_{\sigma_q\chi}} \frac{(z)^{\chi}}{\chi!} \frac{(-(\chi+a))^{\xi}}{\xi!} \frac{(-b)^{\psi}}{\psi!} \Lambda(\lambda\psi-\xi). \end{aligned} \tag{42}$$

Here, in the above equation, we have used the shifting property of delta functions as follows

$$\langle \boldsymbol{\delta(s+\xi-\lambda\psi),\Lambda(s)}\rangle = \boldsymbol{\Lambda(\lambda\psi-\xi)}, \tag{43}$$

which being the elements of space $\boldsymbol{Z}$ are slowly increasing (bounded by a polynomial) test functions and note that sum over the coefficients is

$$\begin{aligned}\textbf{sum over the coefficients} &= \sum_{\chi,\xi,\psi=0}^{\infty} \frac{([\lambda_p])_{\rho_p\chi}}{([\mu_q])_{\sigma_q\chi}} \frac{(z)^{\chi}}{\chi!} \frac{(-(\chi+a))^{\xi}}{\xi!} \frac{(-b)^{\psi}}{\psi!} \\ &= \exp(-a-b) p^{\Psi^*} q\left[\begin{matrix}(\lambda_1,\rho_1),\ldots, & (\lambda_p,\rho_p) \\ (\mu_1,\sigma_1),\ldots, & (\mu_q,\sigma_q)\end{matrix}; \frac{z}{e}\right],\end{aligned} \tag{44}$$

which is finite and well defined. Therefore, by using the famous Abel convergent test or by ([35], Proposition 1, p. 46), it is obvious that new series given by (13) converges for $\forall \boldsymbol{\Lambda(s)\epsilon Z}$, which leads to a similar fact for its special and other related cases given in Section 3 and Appendix A.

As already mentioned, in our present investigation, we proved the convergence for slowly increasing functions, but it can now be observed that the series converges for a larger space of functions. Therefore, the condition is necessary and not sufficient, that means for $\forall \boldsymbol{\Lambda(s)\epsilon Z}$, the series is convergent but if the series is convergent, then $\boldsymbol{\Lambda(s)}$ may belong to some other large space for which delta function is meaningful.

Next, by using the new representation of $\boldsymbol{\Phi}_{\lambda_1,\ldots,\lambda_p,\mu_1,\ldots,\mu_q}^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}(\boldsymbol{z,s,a;b,\lambda})$, we can find some new integral formulae and verify them by using classical Fourier transform. First, we consider a simple example of a specific set of functions

$$\boldsymbol{\Lambda(s)} = \boldsymbol{\omega^{s\beta}}\ (\boldsymbol{\omega \neq 0};\ \boldsymbol{s \in C};\ \boldsymbol{\beta \epsilon \mathbb{R}}). \tag{45}$$

By taking the inner product of these functions with (13) and using the basic (shift) property of delta functions, we get

$$\begin{aligned}&\int_{s\epsilon\mathbb{C}} \omega^{s\beta}\, \Gamma(s) \Phi_{\lambda_1,\ldots,\lambda_p,\mu_1,\ldots,\mu_q}^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}(z,s,a;b,\lambda) ds = \\ &2\pi \sum_{\chi,\xi,\psi=0}^{\infty} \frac{([\lambda_p])_{\rho_p\chi}}{([\mu_q])_{\sigma_q\chi}} \frac{(z)^{\chi}}{\chi!} \frac{(-(\chi+a))^{\xi}}{\xi!} \frac{(-b)^{\psi}}{\psi!}\, \omega^{(\lambda\psi-\xi)\beta} \\ &= 2\pi \exp\left(-a\omega^{-\beta} - \frac{b}{\omega^{-\lambda\beta}}\right) p^{\Psi^*} q\left[\begin{matrix}(\lambda_1,\rho_1),\ldots, & (\lambda_p,\rho_p) \\ (\mu_1,\sigma_1),\ldots, & (\mu_q,\sigma_q)\end{matrix}; z.\exp(-\omega^{-\beta})\right].\end{aligned} \tag{46}$$

Similarly, by considering the action of $\boldsymbol{\Lambda(s)}$ for representations (19)–(34), we can get the following new results:

$$\begin{aligned}&\int_{s\epsilon\mathbb{C}} \omega^{s\beta}\, \Gamma(s) \Theta_{\lambda_1,\ldots,\lambda_p,\mu_1,\ldots,\mu_q}^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}(x,s,a;b,\lambda) ds \\ &= 2\pi \exp\left(-a\omega^{-\beta} - \frac{b}{\omega^{-\lambda\beta}}\right) p^{\Psi^*} q\left[\begin{matrix}(\lambda_1,\rho_1),\ldots, & (\lambda_p,\rho_p) \\ (\mu_1,\sigma_1),\ldots, & (\mu_q,\sigma_q)\end{matrix}; -\exp(-x-\omega^{-\beta})\right];\end{aligned} \tag{47}$$

$$\begin{aligned}&\int_{s\epsilon\mathbb{C}} \omega^{s\beta}\, \Gamma(s) \Psi_{\lambda_1,\ldots,\lambda_p,\mu_1,\ldots,\mu_q}^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}(x,s,a;b,\lambda) ds \\ &= 2\pi \exp\left(-a\omega^{-\beta} - \frac{b}{\omega^{-\lambda\beta}}\right) p^{\Psi^*} q\left[\begin{matrix}(\lambda_1,\rho_1),\ldots, & (\lambda_p,\rho_p) \\ (\mu_1,\sigma_1),\ldots, & (\mu_q,\sigma_q)\end{matrix}; \exp(-x-\omega^{-\beta})\right];\end{aligned} \tag{48}$$

$$\int_{s\epsilon\mathbb{C}} \omega^{s\beta}\, \Gamma(s) F_{\lambda_1,\ldots,\lambda_p,\mu_1,\ldots,\mu_q}^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}(x,s;b,\lambda) ds \tag{49}$$

$$= 2\pi\exp\left(-\omega^{-\beta} - \frac{b}{\omega^{-\lambda\beta}}\right) p^{\Psi^*} q\left[\begin{matrix}(\lambda_1,\rho_1),\dots, & (\lambda_p,\rho_p)\\ (\mu_1,\sigma_1),\dots, & (\mu_q,\sigma_q)\end{matrix}; -\exp(-x-\omega^{-\beta})\right];$$

$$\int_{s\in\mathbb{C}} \omega^{s\beta}\,\Gamma(s) B_{\lambda_1,\dots,\lambda_p,\mu_1,\dots,\mu_q}^{(\rho_1,\dots,\rho_p,\sigma_1,\dots,\sigma_q)}(x,s;b,\lambda)ds \tag{50}$$

$$= 2\pi\exp\left(-\omega^{-\beta} - \frac{b}{\omega^{-\lambda\beta}}\right) p^{\Psi^*} q\left[\begin{matrix}(\lambda_1,\rho_1),\dots, & (\lambda_p,\rho_p)\\ (\mu_1,\sigma_1),\dots, & (\mu_q,\sigma_q)\end{matrix}; \exp(-x-\omega^{-\beta})\right];$$

$$\int_{s\in\mathbb{C}} \omega^{s\beta}\,\Gamma(s) \Phi_{\lambda_1,\dots,\lambda_p,\mu_1,\dots,\mu_q}^{(\rho_1,\dots,\rho_p,\sigma_1,\dots,\sigma_q)}(z,s;b,\lambda)ds \tag{51}$$

$$= 2\pi z.\exp\left(-\omega^{-\beta} - \frac{b}{\omega^{-\lambda\beta}}\right) p^{\Psi^*} q\left[\begin{matrix}(\lambda_1,\rho_1),\dots, & (\lambda_p,\rho_p)\\ (\mu_1,\sigma_1),\dots, & (\mu_q,\sigma_q)\end{matrix}; z\exp(-\omega^{-\beta})\right];$$

$$\int_{s\in\mathbb{C}} \omega^{s\beta}\,\Gamma(s) \zeta_{\lambda_1,\dots,\lambda_p,\mu_1,\dots,\mu_q}^{(\rho_1,\dots,\rho_p,\sigma_1,\dots,\sigma_q)}(s,a;b;\lambda)ds \tag{52}$$

$$= 2\pi\exp\left(-a\omega^{-\beta} - \frac{b}{\omega^{-\lambda\beta}}\right) p^{\Psi^*} q\left[\begin{matrix}(\lambda_1,\rho_1),\dots, & (\lambda_p,\rho_p)\\ (\mu_1,\sigma_1),\dots, & (\mu_q,\sigma_q)\end{matrix}; \exp(-\omega^{-\beta})\right];$$

$$\int_{s\in\mathbb{C}} \omega^{s\beta}\,\Gamma(s) \zeta_{\lambda_1,\dots,\lambda_p,\mu_1,\dots,\mu_q}^{(\rho_1,\dots,\rho_p,\sigma_1,\dots,\sigma_q)}(s;b,\lambda)ds \tag{53}$$

$$= 2\pi\exp\left(-\omega^{-\beta} - \frac{b}{\omega^{-\lambda\beta}}\right) p^{\Psi^*} q\left[\begin{matrix}(\lambda_1,\rho_1),\dots, & (\lambda_p,\rho_p)\\ (\mu_1,\sigma_1),\dots, & (\mu_q,\sigma_q)\end{matrix}; \exp(-\omega^{-\beta})\right].$$

By putting $b = 0$, in (46), we get the following new results: (and if we put $\lambda = 0$, we get e^b times the following results (54)):

$$\int_{s\in\mathbb{C}} \omega^{s\beta}\,\Gamma(s) \Phi_{\lambda_1,\dots,\lambda_p,\mu_1,\dots,\mu_q}^{(\rho_1,\dots,\rho_p,\sigma_1,\dots,\sigma_q)}(z,s,a)ds= \tag{54}$$

$$\sum_{\chi,\xi=0}^{\infty} \frac{(z)^\chi}{\chi!}\,\frac{(-(\chi+a))^\xi}{\xi!}\,\omega^{(\lambda\psi-\xi)\beta}$$

$$= 2\pi\exp(-a\omega^{-\beta})\, p^{\Psi^*} q\left[\begin{matrix}(\lambda_1,\rho_1),\dots, & (\lambda_p,\rho_p)\\ (\mu_1,\sigma_1),\dots, & (\mu_q,\sigma_q)\end{matrix}; z.\exp(-\omega^{-\beta})\right].$$

By considering $p\mathbf{-1} = q = 0$ ($\lambda_1 = \mu$; $\rho_1 = 1$), $b \neq 0$ in equation (46) we can get the following

$$\int_{s\in\mathbb{C}} \omega^{s\beta}\,\Gamma(s)\Theta_\mu^\lambda(z,s,a;b)ds = \frac{2\pi\exp\left(-(a-1)\omega^{-\beta} - \frac{b}{\omega^{-\lambda\beta}}\right)}{(\exp(\omega^{-\beta})-z)^\mu}. \tag{55}$$

Taking $b = 0$ in the above results (55) leads to the following new result:

$$\int_{s\in\mathbb{C}} \omega^{s\beta}\,\Gamma(s)\Phi_\mu^*(z,s,a)ds = \frac{2\pi\exp(-(a-1)\omega^{-\beta})}{(\exp(\omega^{-\beta})-z)^\mu}. \tag{56}$$

By considering other parametric values as, $p\mathbf{-1} = q = 0$; $\lambda_1 = \mu$; $\rho_1 = 1$; $b \neq 0$; $\lambda = \mu = 1$ the above result (54) shrinks instantly to the subsequent result:

$$\int_{s\in\mathbb{C}} \omega^{s\beta}\Gamma(s)\Phi_b(z,s,a)ds = \frac{2\pi\exp\left(-(a-1)\omega^{-\beta} - \frac{b}{\omega^{-\lambda\beta}}\right)}{(\exp(\omega^{-\beta})-z)}. \tag{57}$$

Next, by putting $b = 0$ in the above Equations (57), we get the following [13], (Chapter 4):

$$\int_{s\in\mathbb{C}} \omega^{s\beta}\,\Gamma(s)\Phi(z,s,a)ds = \frac{\exp\left(-(a-1)\omega^{-\beta}\right)}{\left(\exp(\omega^{-\beta})-z\right)}. \tag{58}$$

Remark 2. *Results obtained in this section give insights for further new results. For example, consider* $\omega = \frac{1}{e}$, *then we get the Laplace transform of the* λ*-generalized Hurwitz-Lerch zeta functions and the related family of functions. Before going on further with this new representation, we consider the consistency of the new results in the subsequent section.*

5. Fourier Transform Representation

The main purpose of this section is to verify the consistency of the results obtained by the new series representation with the classical Fourier transform representation. Different transform representations have always been of interest for such functions.

By replacing $t = e^y$ and $s = \sigma + i\tau$ in Equation (4), the Fourier transform representation of λ-generalized Hurwitz-Lerch zeta functions is

$$\Gamma(s)\Phi_{\lambda_1,\dots,\lambda_p,\mu_1,\dots,\mu_q}^{(\rho_1,\dots,\rho_p,\sigma_1,\dots,\sigma_q)}(z,s,a;b,\lambda) \tag{59}$$

$$= \sqrt{2\pi}\mathcal{F}\left[e^{\sigma y}\exp(-ae^y - \tfrac{b}{e^{\lambda y}})\,{}_p\Psi^*_q\left[\begin{matrix}(\lambda_1,\rho_1),\dots, & (\lambda_p,\rho_p)\\ (\mu_1,\sigma_1),\dots, & (\mu_q,\sigma_q)\end{matrix}; z\exp(-e^y)\right];\tau\right]$$

$$(\min[\Re(a),\Re(s)] > 0; \Re(b) \geqq 0; \lambda \geqq 0).$$

Similarly for the λ-generalized extended Fermi-Dirac functions

$$\Gamma(s)\Theta_{\lambda_1,\dots,\lambda_p,\mu_1,\dots,\mu_q}^{(\rho_1,\dots,\rho_p,\sigma_1,\dots,\sigma_q)}(x,s,a;b,\lambda) \tag{60}$$

$$= \sqrt{2\pi}\mathcal{F}\left[e^{\sigma y}\exp(-ae^y - \tfrac{b}{e^{\lambda y}})\,{}_p\Psi^*_q\left[\begin{matrix}(\lambda_1,\rho_1),\dots, & (\lambda_p,\rho_p)\\ (\mu_1,\sigma_1),\dots, & (\mu_q,\sigma_q)\end{matrix}; -e^{-x}\exp(-e^y)\right];\tau\right]$$

$$(\min[\Re(a),\Re(s)] > 0; \Re(b) \geqq 0; \lambda \geqq 0)$$

and Extended Bose-Einstein Functions

$$\Gamma(s)\Psi_{\lambda_1,\dots,\lambda_p,\mu_1,\dots,\mu_q}^{(\rho_1,\dots,\rho_p,\sigma_1,\dots,\sigma_q)}(x,s,a;b,\lambda) \tag{61}$$

$$=\sqrt{2\pi}\ \mathcal{F}\left[e^{\sigma y}\exp(-ae^y - \tfrac{b}{e^{\lambda y}})\,{}_p\Psi^*_q\left[\begin{matrix}(\lambda_1,\rho_1),\dots, & (\lambda_p,\rho_p)\\ (\mu_1,\sigma_1),\dots, & (\mu_q,\sigma_q)\end{matrix}; e^{-x}\exp(-e^y)\right];\tau\right]$$

$$(\min[\Re(a),\Re(s)] > 0; \Re(b) \geqq 0; \lambda \geqq 0).$$

For λ-generalized Fermi-Dirac functions

$$\Gamma(s)F_{\lambda_1,\dots,\lambda_p,\mu_1,\dots,\mu_q}^{(\rho_1,\dots,\rho_p,\sigma_1,\dots,\sigma_q)}(x,s;b,\lambda) \tag{62}$$

$$= \sqrt{2\pi}\mathcal{F}\left[e^{\sigma y}\exp(-e^y - \tfrac{b}{e^{\lambda y}})\,{}_p\Psi^*_q\left[\begin{matrix}(\lambda_1,\rho_1),\dots, & (\lambda_p,\rho_p)\\ (\mu_1,\sigma_1),\dots, & (\mu_q,\sigma_q)\end{matrix}; -e^{-x}\exp(-e^y)\right];\tau\right]$$

$$(\min[\Re(a),\Re(s)] > 0; \Re(b) \geqq 0; \lambda \geqq 0)$$

and Bose-Einstein functions

$$\Gamma(s)B_{\lambda_1,\dots,\lambda_p,\mu_1,\dots,\mu_q}^{(\rho_1,\dots,\rho_p,\sigma_1,\dots,\sigma_q)}(x,s;b,\lambda) \tag{63}$$

$$= \sqrt{2\pi}\mathcal{F}\left[e^{\sigma y}\exp(-e^{y} - \frac{b}{e^{\lambda y}})p^{\Psi^*}q\left[\begin{matrix}(\lambda_1,\rho_1),\dots, & (\lambda_p,\rho_p)\\ (\mu_1,\sigma_1),\dots, & (\mu_q,\sigma_q)\end{matrix}; e^{-x}\exp(-e^{y})\right];\tau\right]$$

$$(\min[\Re(a),\Re(s)] > 0; \Re(b) \geqq 0; \lambda \geqq 0).$$

For λ-generalized Polylogarithm functions

$$\Gamma(s)\phi_{\lambda_1,\dots,\lambda_p,\mu_1,\dots,\mu_q}^{(\rho_1,\dots,\rho_p,\sigma_1,\dots,\sigma_q)}(z,s;b,\lambda) \tag{64}$$

$$= \sqrt{2\pi}\mathcal{F}\left[ze^{\sigma y}\exp(-e^{y} - \frac{b}{e^{\lambda y}})p^{\Psi^*}q\left[\begin{matrix}(\lambda_1,\rho_1),\dots, & (\lambda_p,\rho_p)\\ (\mu_1,\sigma_1),\dots, & (\mu_q,\sigma_q)\end{matrix}; z\exp(-e^{y})\right];\tau\right]$$

$$(\min[\Re(a),\Re(s)] > 0; \Re(b) \geqq 0; \lambda \geqq 0)$$

For λ-generalized Hurwitz zeta functions

$$\Gamma(s)\zeta_{\lambda_1,\dots,\lambda_p,\mu_1,\dots,\mu_q}^{(\rho_1,\dots,\rho_p,\sigma_1,\dots,\sigma_q)}(s,a;b,\lambda) \tag{65}$$

$$= \sqrt{2\pi}\mathcal{F}\left[e^{\sigma y}\exp(-ae^{y} - \frac{b}{e^{\lambda y}})p^{\Psi^*}q\left[\begin{matrix}(\lambda_1,\rho_1),\dots, & (\lambda_p,\rho_p)\\ (\mu_1,\sigma_1),\dots, & (\mu_q,\sigma_q)\end{matrix}; \exp(-e^{y})\right];\tau\right]$$

$$(\min[\Re(a),\Re(s)] > 0; \Re(b) \geqq 0; \lambda \geqq 0).$$

For λ-generalized Riemann zeta functions

$$\Gamma(s)\zeta_{\lambda_1,\dots,\lambda_p,\mu_1,\dots,\mu_q}^{(\rho_1,\dots,\rho_p,\sigma_1,\dots,\sigma_q)}(s;b,\lambda) \tag{66}$$

$$= \sqrt{2\pi}\mathcal{F}\left[e^{\sigma y}\exp(-e^{y} - \frac{b}{e^{\lambda y}})p^{\Psi^*}q\left[\begin{matrix}(\lambda_1,\rho_1),\dots, & (\lambda_p,\rho_p)\\ (\mu_1,\sigma_1),\dots, & (\mu_q,\sigma_q)\end{matrix}; \exp(-e^{y})\right];\tau\right]$$

$$(\min[\Re(a),\Re(s)] > 0; \Re(b) \geqq 0; \lambda \geqq 0).$$

Similarly, by giving variations to different parameters, we can get similar representations for other special cases of these functions in consideration of Table 1.

6. Verification of the Results Obtained by New Representation

For the Fourier transform of any function $f(t)$, duality property holds as

$$\mathcal{F}\left[\sqrt{2\pi}\mathcal{F}[f(t);\tau];\beta\right] = 2\pi f(-\beta). \tag{67}$$

Hence, from (59)–(66), by applying (67), we obtain the following

$$\mathcal{F}\left\{\Gamma(\sigma+i\tau)\Phi_{\lambda_1,\dots,\lambda_p,\mu_1,\dots,\mu_q}^{(\rho_1,\dots,\rho_p,\sigma_1,\dots,\sigma_q)}(z,\sigma+i\tau,a;b,\lambda);\beta\right\}= \tag{68}$$

$$\mathcal{F}\left\{\sqrt{2\pi}\mathcal{F}\left\{e^{\sigma y}e^{-ae^{y}-\frac{b}{e^{\lambda y}}}p^{\Psi^*}q\left[\begin{matrix}(\lambda_1,\rho_1),\dots, & (\lambda_p,\rho_p)\\ (\mu_1,\sigma_1),\dots, & (\mu_q,\sigma_q)\end{matrix}; ze^{-e^{y}}\right];\tau\right\};\beta\right\} = f(-\beta)$$

$$= 2\pi e^{-\sigma\beta}\exp(-ae^{-\beta} - \frac{b}{e^{-\lambda\beta}})p^{\Psi^*}q\left[\begin{matrix}(\lambda_1,\rho_1),\dots, & (\lambda_p,\rho_p)\\ (\mu_1,\sigma_1),\dots, & (\mu_q,\sigma_q)\end{matrix}; z\exp(-e^{-\beta})\right]$$

$$(\min[\Re(a),\Re(s)] > 0; \Re(b) \geqq 0; \lambda \geqq 0),$$

$$\text{Or } \int_{-\infty}^{+\infty} e^{i\tau\beta}\,\Gamma(\sigma+i\tau)\Phi_{\lambda_1,\dots,\lambda_p,\mu_1,\dots,\mu_q}^{(\rho_1,\dots,\rho_p,\sigma_1,\dots,\sigma_q)}(z,\sigma+i\tau,a;b,\lambda)d\tau \tag{69}$$

$$= 2\pi e^{-\sigma\beta}\exp\left(-ae^{-\beta} - \frac{b}{e^{-\lambda\beta}}\right)p^{\Psi^*}q\left[\begin{matrix}(\lambda_1,\rho_1),\dots, & (\lambda_p,\rho_p)\\ (\mu_1,\sigma_1),\dots, & (\mu_q,\sigma_q)\end{matrix}; z\exp\left(-e^{-\beta}\right)\right],$$

which is the special case of our main result (46) for $\boldsymbol{w = e; s = \sigma + i\tau}$ and verifies that results obtained by the new representation are consistent with the classical results.

If we put $\boldsymbol{\beta = 0}$ in the above equation (69), we get the following integral:

$$\int_{-\infty}^{+\infty} \Gamma(\sigma + i\tau)\Phi_{\lambda_1,\dots,\lambda_p,\mu_1,\dots,\mu_q}^{(\rho_1,\dots,\rho_p,\sigma_1,\dots,\sigma_q)}(z, \sigma + i\tau, a; b, \lambda)d\tau = 2\pi e^{-a-b}{}_p\Psi^*_q\left[\begin{matrix}(\lambda_1,\rho_1),\dots, & (\lambda_p,\rho_p) \\ (\mu_1,\sigma_1),\dots, & (\mu_q,\sigma_q)\end{matrix}; \frac{z}{e}\right], \tag{70}$$

which is also a specific case of our main result (46). It shows that our new representation produces new results that cannot be found by other methods, but special cases of our obtained results are consistent with the classical results.

Similarly, by considering different parametric values in the above equations and as given in Table 1 in Section 2, we can get the following list of integrals: (one can also note that results obtained by new representation are not only more general than the results obtained by Fourier transform representation but also consistent with the special cases of these results)

$$\int_{-\infty}^{+\infty} e^{i\tau\beta}\,\Gamma(\sigma + i\tau)\Psi_{\lambda_1,\dots,\lambda_p,\mu_1,\dots,\mu_q}^{(\rho_1,\dots,\rho_p,\sigma_1,\dots,\sigma_q)}(x, \sigma + i\tau, a; b, \lambda)d\tau = 2\pi e^{-\sigma\beta}\exp\left(-ae^{-\beta} - \frac{b}{e^{-\lambda\beta}}\right){}_p\Psi^*_q\left[\begin{matrix}(\lambda_1,\rho_1),\dots, & (\lambda_p,\rho_p) \\ (\mu_1,\sigma_1),\dots, & (\mu_q,\sigma_q)\end{matrix}; e^{-x}\exp\left(-e^{-\beta}\right)\right]; \tag{71}$$

$$\int_{-\infty}^{+\infty} e^{i\tau\beta}\,\Gamma(\sigma + i\tau)\Theta_{\lambda_1,\dots,\lambda_p,\mu_1,\dots,\mu_q}^{(\rho_1,\dots,\rho_p,\sigma_1,\dots,\sigma_q)}(x, \sigma + i\tau, a; b, \lambda)d\tau = 2\pi e^{-\sigma\beta}\exp\left(-ae^{-\beta} - \frac{b}{e^{-\lambda\beta}}\right){}_p\Psi^*_q\left[\begin{matrix}(\lambda_1,\rho_1),\dots, & (\lambda_p,\rho_p) \\ (\mu_1,\sigma_1),\dots, & (\mu_q,\sigma_q)\end{matrix}; -e^{-x}\exp\left(-e^{-\beta}\right)\right]; \tag{72}$$

$$\int_{-\infty}^{+\infty} e^{i\tau\beta}\,\Gamma(\sigma + i\tau)B_{\lambda_1,\dots,\lambda_p,\mu_1,\dots,\mu_q}^{(\rho_1,\dots,\rho_p,\sigma_1,\dots,\sigma_q)}(x, \sigma + i\tau; b, \lambda)d\tau = 2\pi e^{-\sigma\beta}\exp\left(-e^{-\beta} - \frac{b}{e^{-\lambda\beta}}\right){}_p\Psi^*_q\left[\begin{matrix}(\lambda_1,\rho_1),\dots, & (\lambda_p,\rho_p) \\ (\mu_1,\sigma_1),\dots, & (\mu_q,\sigma_q)\end{matrix}; e^{-x}\exp\left(-e^{-\beta}\right)\right]; \tag{73}$$

$$\int_{-\infty}^{+\infty} e^{i\tau\beta}\,\Gamma(\sigma + i\tau)F_{\lambda_1,\dots,\lambda_p,\mu_1,\dots,\mu_q}^{(\rho_1,\dots,\rho_p,\sigma_1,\dots,\sigma_q)}(x, \sigma + i\tau; b, \lambda)d\tau = 2\pi e^{-\sigma\beta}\exp\left(-e^{-\beta} - \frac{b}{e^{-\lambda\beta}}\right){}_p\Psi^*_q\left[\begin{matrix}(\lambda_1,\rho_1),\dots, & (\lambda_p,\rho_p) \\ (\mu_1,\sigma_1),\dots, & (\mu_q,\sigma_q)\end{matrix}; -e^{-x}\exp\left(-e^{-\beta}\right)\right]; \tag{74}$$

$$\int_{-\infty}^{+\infty} e^{i\tau\beta}\,\Gamma(\sigma + i\tau)\phi_{\lambda_1,\dots,\lambda_p,\mu_1,\dots,\mu_q}^{(\rho_1,\dots,\rho_p,\sigma_1,\dots,\sigma_q)}(z, \sigma + i\tau; b, \lambda)d\tau = 2\pi z e^{-\sigma\beta}\exp\left(-e^{-\beta} - \frac{b}{e^{-\lambda\beta}}\right){}_p\Psi^*_q\left[\begin{matrix}(\lambda_1,\rho_1),\dots, & (\lambda_p,\rho_p) \\ (\mu_1,\sigma_1),\dots, & (\mu_q,\sigma_q)\end{matrix}; z\exp\left(-e^{-\beta}\right)\right]; \tag{75}$$

$$\int_{-\infty}^{+\infty} e^{i\tau\beta}\,\Gamma(\sigma + i\tau)\zeta_{\lambda_1,\dots,\lambda_p,\mu_1,\dots,\mu_q}^{(\rho_1,\dots,\rho_p,\sigma_1,\dots,\sigma_q)}(\sigma + i\tau, a; b, \lambda)d\tau = 2\pi e^{-\sigma\beta}\exp\left(-ae^{-\beta} - \frac{b}{e^{-\lambda\beta}}\right){}_p\Psi^*_q\left[\begin{matrix}(\lambda_1,\rho_1),\dots, & (\lambda_p,\rho_p) \\ (\mu_1,\sigma_1),\dots, & (\mu_q,\sigma_q)\end{matrix}; \exp\left(-e^{-\beta}\right)\right]; \tag{76}$$

$$\int_{-\infty}^{+\infty} \mathbf{e}^{\mathbf{i\tau\beta}}\, \boldsymbol{\Gamma}(\boldsymbol{\sigma}+\mathbf{i\tau})\boldsymbol{\zeta}_{\lambda_1,\dots,\lambda_p,\mu_1,\dots,\mu_q}^{(\rho_1,\dots,\rho_p,\sigma_1,\dots,\sigma_q)}(\boldsymbol{\sigma}+\mathbf{i\tau};\mathbf{b},\boldsymbol{\lambda})\mathbf{d\tau} \tag{77}$$

$$= \mathbf{2\pi e^{-\sigma\beta} exp}\left(-\mathbf{ae^{-\beta}} - \frac{\mathbf{b}}{\mathbf{e^{-\lambda\beta}}}\right)\mathbf{p}^{\Psi^*}\mathbf{q}\left[\begin{matrix}(\lambda_1,\rho_1),\dots, & (\lambda_p,\rho_p) \\ (\mu_1,\sigma_1),\dots, & (\mu_q,\sigma_q)\end{matrix};\mathbf{exp}\left(-\mathbf{e^{-\beta}}\right)\right];$$

$$\int_{-\infty}^{+\infty} \mathbf{e}^{\mathbf{i\tau\beta}}\, \boldsymbol{\Gamma}(\boldsymbol{\sigma}+\mathbf{i\tau})\boldsymbol{\Phi}_{\lambda_1,\dots,\lambda_p,\mu_1,\dots,\mu_q}^{(\rho_1,\dots,\rho_p,\sigma_1,\dots,\sigma_q)}(\mathbf{z},\boldsymbol{\sigma}+\mathbf{i\tau},\mathbf{a})\mathbf{d\tau} \tag{78}$$

$$= \mathbf{2\pi e^{-\sigma\beta} exp}\left(-\mathbf{ae^{-\beta}} - \mathbf{b}\right)\mathbf{p}^{\Psi^*}\mathbf{q}\left[\begin{matrix}(\lambda_1,\rho_1),\dots, & (\lambda_p,\rho_p) \\ (\mu_1,\sigma_1),\dots, & (\mu_q,\sigma_q)\end{matrix};\mathbf{zexp}\left(-\mathbf{e^{-\beta}}\right)\right].$$

For $\boldsymbol{\beta} = \mathbf{0}$, the above results (68)–(78) yield some interesting and simple integral formulae. To confirm the consistency of the results obtained by new representation, it can be noted that the results obtained in this Section (68)–(78) can be generated as special cases of (46)–(48) for $\boldsymbol{\omega} = \mathbf{e}, \mathbf{s} = \boldsymbol{\sigma} + \mathbf{i\tau}$ and vice versa. These are straightforward to obtain by using a basic fact of the Fourier transform and therefore to test the consistency of new representations as they become more important.

7. Discussion and Future Directions

The confluence of distributions (generalized functions) with classical integral transformations has become a remarkably influential tool in the theory of partial differential equations. It has solved various physical and engineering problems that cannot be solved by using classical methods. In this paper, we obtained a new representation for the newly defined family of the $\boldsymbol{\lambda}$-generalized Hurwitz-Lerch zeta functions in terms of complex delta functions such that the definition of these functions is formalized over the space of entire test functions denoted by $\boldsymbol{Z}$. This is significant for advancing the foundations of distributional (generalized function) concepts for such higher transcendental functions and enhancing their applications to solve real-world problems. The Riemann hypothesis is a famous and unsolved problem at present in analytic number theory [31]. It states that "all the non-trivial zeros of the zeta function lie on the real line $\boldsymbol{s} = \mathbf{1/2}$ ". These zeros appear symmetrically as complex conjugates on this line. The integrals of the zeta function and its generalizations are essential in the investigation of Riemann hypothesis and for the study of zeta functions. Such integrals are also important for the study of distributions in statistical inference and reliability theory [1,26,32]. By using this new definition of the $\boldsymbol{\lambda}$-generalized Hurwitz-Lerch zeta functions, one can find such integrals in a simple and uniform way.

$\boldsymbol{\lambda}$-generalized Hurwitz-Lerch zeta functions systematically oversimplify the functions of the zeta family and provide understanding for some other potential new members of this family that are not found in the literature. This element is very useful for achieving new results from one main result. Our main result generates at once significant new results for a class of well-studied functions by applying the methodology of this paper. The Fermi-Dirac and Bose-Einstein functions arose in the distribution functions for quantum statistics that deals with two particular kinds of spin symmetry, namely, bosons and fermions. Their close connection considered in this investigation with the $\boldsymbol{\lambda}$-generalized Hurwitz-Lerch zeta functions have provided some significant new results for these functions that directly develop the future applications of these representations in quantum physics and related fields. The technique to obtain the results by using new representation explores a required simplicity that is always desirous. These are some straightforward examples. It is expected that the approach developed in this investigation will be doubtlessly significant for further exploration of these higher transcendental functions in future research.

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Appendix A. New Results by Considering Special Cases for Section 2 in View of Table 1

$$\Gamma(s)\Theta_{\lambda_1,\ldots,\lambda_p,\mu_1,\ldots,\mu_q}^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}(x,s,a) = 2\pi e^{b}\sum_{\chi,\xi=0}^{\infty}\frac{([\lambda_p])_{\rho_p\chi}}{([\mu_q])_{\sigma_q\chi}}\frac{(-e^{-x})^{\chi}}{\chi!}\frac{(-(\chi+a))^{\xi}}{\xi!}\delta(s+\xi); \tag{A1}$$

$$\Gamma(s)\Psi_{\lambda_1,\ldots,\lambda_p,\mu_1,\ldots,\mu_q}^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}(x,s,a) = 2\pi e^{b}\sum_{\chi,\xi=0}^{\infty}\frac{([\lambda_p])_{\rho_p\chi}}{([\mu_q])_{\sigma_q\chi}}\frac{(e^{-x})^{\chi}}{\chi!}\frac{(-(\chi+a))^{\xi}}{\xi!}\delta(s+\xi); \tag{A2}$$

$$\Gamma(s)B_{\lambda_1,\ldots,\lambda_p,\mu_1,\ldots,\mu_q}^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}(x,s) = 2\pi e^{b}\sum_{\chi,\xi=0}^{\infty}\frac{([\lambda_p])_{\rho_p\chi}}{([\mu_q])_{\sigma_q\chi}}\frac{(e^{-x})^{\chi}}{\chi!}\frac{(-(\chi+1))^{\xi}}{\xi!}\delta(s+\xi); \tag{A3}$$

$$\Gamma(s)F_{\lambda_1,\ldots,\lambda_p,\mu_1,\ldots,\mu_q}^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}(x,s) = 2\pi e^{b}\sum_{\chi,\xi=0}^{\infty}\frac{([\lambda_p])_{\rho_p\chi}}{([\mu_q])_{\sigma_q\chi}}\frac{(-e^{-x})^{\chi}}{\chi!}\frac{(-(\chi+1))^{\xi}}{\xi!}\delta(s+\xi); \tag{A4}$$

$$\Gamma(s)\phi_{\lambda_1,\ldots,\lambda_p,\mu_1,\ldots,\mu_q}^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}(z,s) = 2\pi z e^{b}\sum_{\chi,\xi=0}^{\infty}\frac{([\lambda_p])_{\rho_p\chi}}{([\mu_q])_{\sigma_q\chi}}\frac{(z)^{\chi}}{\chi!}\frac{(-(\chi+1))^{\xi}}{\xi!}\delta(s+\xi); \tag{A5}$$

$$\Gamma(s)\zeta_{\lambda_1,\ldots,\lambda_p,\mu_1,\ldots,\mu_q}^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}(s,a) = 2\pi e^{b}\sum_{\chi,\xi=0}^{\infty}\frac{([\lambda_p])_{\rho_p\chi}}{([\mu_q])_{\sigma_q\chi}}\frac{1}{\chi!}\frac{(-(\chi+a))^{\xi}}{\xi!}\delta(s+\xi); \tag{A6}$$

$$\Gamma(s)\zeta_{\lambda_1,\ldots,\lambda_p,\mu_1,\ldots,\mu_q}^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}(s) = 2\pi e^{b}\sum_{\chi,\xi=0}^{\infty}\frac{([\lambda_p])_{\rho_p\chi}}{([\mu_q])_{\sigma_q\chi}}\frac{1}{\chi!}\frac{(-(\chi+1))^{\xi}}{\xi!}\delta(s+\xi); \tag{A7}$$

$$\Gamma(s)\Theta_{\lambda_1,\ldots,\lambda_p,\mu_1,\ldots,\mu_q}^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}(x,s,a) = 2\pi\sum_{\chi,\xi=0}^{\infty}\frac{([\lambda_p])_{\rho_p\chi}}{([\mu_q])_{\sigma_q\chi}}\frac{(-e^{-x})^{\chi}}{\chi!}\frac{(-(\chi+a))^{\xi}}{\xi!}\delta(s+\xi) \tag{A8}$$

$$\Gamma(s)\Psi_{\lambda_1,\ldots,\lambda_p,\mu_1,\ldots,\mu_q}^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}(x,s,a) = 2\pi\sum_{\chi,\xi=0}^{\infty}\frac{([\lambda_p])_{\rho_p\chi}}{([\mu_q])_{\sigma_q\chi}}\frac{(e^{-x})^{\chi}}{\chi!}\frac{(-(\chi+a))^{\xi}}{\xi!}\delta(s+\xi) \tag{A9}$$

$$\Gamma(s)B_{\lambda_1,\ldots,\lambda_p,\mu_1,\ldots,\mu_q}^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}(x,s) = 2\pi\sum_{\chi,\xi=0}^{\infty}\frac{([\lambda_p])_{\rho_p\chi}}{([\mu_q])_{\sigma_q\chi}}\frac{(e^{-x})^{\chi}}{\chi!}\frac{(-(\chi+1))^{\xi}}{\xi!}\delta(s+\xi) \tag{A10}$$

$$\Gamma(s)F_{\lambda_1,\ldots,\lambda_p,\mu_1,\ldots,\mu_q}^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}(x,s) = 2\pi\sum_{\chi,\xi=0}^{\infty}\frac{([\lambda_p])_{\rho_p\chi}}{([\mu_q])_{\sigma_q\chi}}\frac{(-e^{-x})^{\chi}}{\chi!}\frac{(-(\chi+1))^{\xi}}{\xi!}\delta(s+\xi) \tag{A11}$$

$$\Gamma(s)\phi_{\lambda_1,\ldots,\lambda_p,\mu_1,\ldots,\mu_q}^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}(z,s) = 2\pi z\sum_{\chi,\xi=0}^{\infty}\frac{([\lambda_p])_{\rho_p\chi}}{([\mu_q])_{\sigma_q\chi}}\frac{(z)^{\chi}}{\chi!}\frac{(-(\chi+1))^{\xi}}{\xi!}\delta(s+\xi) \tag{A12}$$

$$\Gamma(s)\zeta_{\lambda_1,\ldots,\lambda_p,\mu_1,\ldots,\mu_q}^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}(s,a) = 2\pi\sum_{\chi,\xi=0}^{\infty}\frac{([\lambda_p])_{\rho_p\chi}}{([\mu_q])_{\sigma_q\chi}}\frac{1}{\chi!}\frac{(-(\chi+a))^{\xi}}{\xi!}\delta(s+\xi) \tag{A13}$$

$$\Gamma(s)\zeta_{\lambda_1,\dots,\lambda_p,\mu_1,\dots,\mu_q}^{(\rho_1,\dots,\rho_p,\sigma_1,\dots,\sigma_q)}(s)=2\pi\sum_{\chi,\xi=0}^{\infty}\frac{([\lambda_p])_{\rho_p\chi}}{([\mu_q])_{\sigma_q\chi}}\frac{1}{\chi!}\frac{(-(\chi+1))^{\xi}}{\xi!}\delta(s+\xi) \tag{A14}$$

$$\Gamma(s)\Theta_{\mu}^{\lambda}(x,s,a;b)=2\pi\sum_{\chi,\xi,\psi=0}^{\infty}(\mu)_\chi\frac{(-e^{-x})^{\chi}}{\chi!}\frac{(-(\chi+a))^{\xi}}{\xi!}\frac{(-b)^{\psi}}{\psi!}\delta(s+\xi-\lambda\psi) \tag{A15}$$

$$\Gamma(s)\Psi_{\mu}^{\lambda}(x,s,a;b)=2\pi\sum_{\chi,\xi,\psi=0}^{\infty}(\mu)_\chi\frac{(e^{-x})^{\chi}}{\chi!}\frac{(-(\chi+a))^{\xi}}{\xi!}\frac{(-b)^{\psi}}{\psi!}\delta(s+\xi-\lambda\psi) \tag{A16}$$

$$\Gamma(s)B_{\mu}^{\lambda}(x,s;b)=2\pi\sum_{\chi,\xi,\psi=0}^{\infty}(\mu)_\chi\frac{(e^{-x})^{\chi}}{\chi!}\frac{(-(\chi+1))^{\xi}}{\xi!}\frac{(-b)^{\psi}}{\psi!}\delta(s+\xi-\lambda\psi) \tag{A17}$$

$$\Gamma(s)F_{\mu}^{\lambda}(x,s;b)=2\pi\sum_{\chi,\xi,\psi=0}^{\infty}(\mu)_\chi\frac{(-e^{-x})^{\chi}}{\chi!}\frac{(-(\chi+1))^{\xi}}{\xi!}\frac{(-b)^{\psi}}{\psi!}\delta(s+\xi-\lambda\psi) \tag{A18}$$

$$\Gamma(s)\phi_{\mu}^{\lambda}(z,s;b)=2\pi z\sum_{\chi,\xi,\psi=0}^{\infty}(\mu)_\chi\frac{(z)^{\chi}}{\chi!}\frac{(-(\chi+1))^{\xi}}{\xi!}\frac{(-b)^{\psi}}{\psi!}\delta(s+\xi-\lambda\psi) \tag{A19}$$

$$\Gamma(s)\zeta_{\mu}^{\lambda}(s,a;b,a)=2\pi\sum_{\chi,\xi,\psi=0}^{\infty}(\mu)_\chi\frac{(e^{-x})^{\chi}}{\chi!}\frac{(-(\chi+a))^{\xi}}{\xi!}\frac{(-b)^{\psi}}{\psi!}\delta(s+\xi-\lambda\psi) \tag{A20}$$

$$\Gamma(s)\zeta_{\mu}^{\lambda}(s;b)=2\pi\sum_{\chi,\xi,\psi=0}^{\infty}(\mu)_\chi\frac{(e^{-x})^{\chi}}{\chi!}\frac{(-(\chi+1))^{\xi}}{\xi!}\frac{(-b)^{\psi}}{\psi!}\delta(s+\xi-\lambda\psi) \tag{A21}$$

$$\Gamma(s)\Phi_{\mu}^{*}(z,s,a,b)=2\pi e^{b}\sum_{\chi,\xi=0}^{\infty}(\mu)_\chi\frac{(z)^{\chi}}{\chi!}\frac{(-(\chi+a))^{\xi}}{\xi!}\delta(s+\xi) \tag{A22}$$

$$\Gamma(s)\Theta_{\mu}^{*}(x,s,a,b)=2\pi e^{b}\sum_{\chi,\xi=0}^{\infty}(\mu)_\chi\frac{(-e^{-x})^{\chi}}{\chi!}\frac{(-(\chi+a))^{\xi}}{\xi!}\delta(s+\xi) \tag{A23}$$

$$\Gamma(s)\Psi_{\mu}^{*}(x,s,a,b)=2\pi e^{b}\sum_{\chi,\xi=0}^{\infty}(\mu)_\chi\frac{(e^{-x})^{\chi}}{\chi!}\frac{(-(\chi+a))^{\xi}}{\xi!}\delta(s+\xi) \tag{A24}$$

$$\Gamma(s)B_{\mu}^{*}(x,s,b)=2\pi e^{b}\sum_{\chi,\xi=0}^{\infty}(\mu)_\chi\frac{(e^{-x})^{\chi}}{\chi!}\frac{(-(\chi+1))^{\xi}}{\xi!}\delta(s+\xi) \tag{A25}$$

$$\Gamma(s)F_{\mu}^{*}(x,s,b)=2\pi e^{b}\sum_{\chi,\xi=0}^{\infty}(\mu)_\chi\frac{(e^{-x})^{\chi}}{\chi!}\frac{(-(\chi+1))^{\xi}}{\xi!}\delta(s+\xi) \tag{A26}$$

$$\Gamma(s)\phi_{\mu}^{*}(z,s;b,\lambda)=2\pi e^{b}z\sum_{\chi,\xi=0}^{\infty}(\mu)_\chi\frac{(z)^{\chi}}{\chi!}\frac{(-(\chi+1))^{\xi}}{\xi!}\delta(s+\xi) \tag{A27}$$

$$\Gamma(s)\zeta_{\mu}^{*}(s,a;b)=2\pi e^{b}\sum_{\chi,\xi=0}^{\infty}(\mu)_\chi\frac{1}{\chi!}\frac{(-(\chi+a))^{\xi}}{\xi!}\delta(s+\xi) \tag{A28}$$

$$\Gamma(s)\zeta^*_\mu(s;b) = 2\pi e^b \sum_{\chi,\xi=0}^{\infty} (\mu)_\chi \frac{1}{\chi!} \frac{(-(\chi+1))^\xi}{\xi!} \delta(s+\xi) \tag{A29}$$

$$\Gamma(s)\Phi^*_\mu(z,s,a) = 2\pi \sum_{\chi,\xi=0}^{\infty} (\mu)_\chi \frac{(z)^\chi}{\chi!} \frac{(-(\chi+a))^\xi}{\xi!} \delta(s+\xi) \tag{A30}$$

$$\Gamma(s)\Theta^*_\mu(x,s,a) = 2\pi \sum_{\chi,\xi=0}^{\infty} (\mu)_\chi \frac{(-e^{-x})^\chi}{\chi!} \frac{(-(\chi+a))^\xi}{\xi!} \delta(s+\xi) \tag{A31}$$

$$\Gamma(s)\Psi^*_\mu(x,s,a) = 2\pi \sum_{\chi,\xi=0}^{\infty} (\mu)_\chi \frac{(e^{-x})^\chi}{\chi!} \frac{(-(\chi+a))^\xi}{\xi!} \delta(s+\xi) \tag{A32}$$

$$\Gamma(s)B^*_\mu(x,s) = 2\pi \sum_{\chi,\xi=0}^{\infty} (\mu)_\chi \frac{(e^{-x})^\chi}{\chi!} \frac{(-(\chi+a))^\xi}{\xi!} \delta(s+\xi) \tag{A33}$$

$$\Gamma(s)F^*_\mu(x,s) = 2\pi \sum_{\chi,\xi=0}^{\infty} (\mu)_\chi \frac{(-e^{-x})^\chi}{\chi!} \frac{(-(\chi+a))^\xi}{\xi!} \delta(s+\xi) \tag{A34}$$

$$\Gamma(s)\phi^*_\mu(z,s;b,\lambda) = 2\pi z \sum_{\chi,\xi=0}^{\infty} (\mu)_\chi \frac{(z)^\chi}{\chi!} \frac{(-(\chi+a))^\xi}{\xi!} \delta(s+\xi) \tag{A35}$$

$$\Gamma(s)\zeta^*_\mu(s,a) = 2\pi \sum_{\chi,\xi=0}^{\infty} (\mu)_\chi \frac{1}{\chi!} \frac{(-(\chi+a))^\xi}{\xi!} \delta(s+\xi) \tag{A36}$$

$$\Gamma(s)\zeta^*_\mu(s) = 2\pi \sum_{\chi,\xi=0}^{\infty} (\mu)_\chi \frac{1}{\chi!} \frac{(-(\chi+1))^\xi}{\xi!} \delta(s+\xi) \tag{A37}$$

$$\Gamma(s)\Theta(x,s,a;b,\lambda) = 2\pi \sum_{\chi,\xi,\psi=0}^{\infty} (-e^{-x})^\chi \frac{(-(\chi+a))^\xi}{\xi!} \frac{(-b)^\psi}{\psi!} \delta(s+\xi-\lambda\psi) \tag{A38}$$

$$\Gamma(s)\Psi(x,s,a;b,\lambda) = 2\pi \sum_{\chi,\xi,\psi=0}^{\infty} (e^{-x})^\chi \frac{(-(\chi+a))^\xi}{\xi!} \frac{(-b)^\psi}{\psi!} \delta(s+\xi-\lambda\psi) \tag{A39}$$

$$\Gamma(s)B(x,s;b,\lambda) = 2\pi \sum_{\chi,\xi,\psi=0}^{\infty} (e^{-x})^\chi \frac{(-(\chi+1))^\xi}{\xi!} \frac{(-b)^\psi}{\psi!} \delta(s+\xi-\lambda\psi) \tag{A40}$$

$$\Gamma(s)F(x,s;b,\lambda) = 2\pi \sum_{\chi,\xi,\psi=0}^{\infty} (-e^{-x})^\chi \frac{(-(\chi+1))^\xi}{\xi!} \frac{(-b)^\psi}{\psi!} \delta(s+\xi-\lambda\psi) \tag{A41}$$

$$\Gamma(s)\phi(z,s;b,\lambda) = 2\pi z \sum_{\chi,\xi,\psi=0}^{\infty} (z)^\chi \frac{(-(\chi+1))^\xi}{\xi!} \frac{(-b)^\psi}{\psi!} \delta(s+\xi-\lambda\psi) \tag{A42}$$

$$\Gamma(s)\zeta(s,a;b,\lambda) = 2\pi \sum_{\chi,\xi,\psi=0}^{\infty} \frac{(-(\chi+a))^\xi}{\xi!} \frac{(-b)^\psi}{\psi!} \delta(s+\xi-\lambda\psi) \tag{A43}$$

$$\boldsymbol{\Gamma(s)\zeta(s;b,\lambda) = 2\pi \sum_{\chi,\xi,\psi=0}^{\infty} \frac{(-(\chi+1))^{\xi}}{\xi!}\frac{(-b)^{\psi}}{\psi!}\delta(s+\xi-\lambda\psi)} \tag{A44}$$

$$\boldsymbol{\Gamma(s)\Phi(z,s,a;b) = 2\pi e^{b} \sum_{\chi,\xi=0}^{\infty} (z)^{\chi}\, \frac{(-(\chi+a))^{\xi}}{\xi!}\delta(s+\xi)} \tag{A45}$$

$$\boldsymbol{\Gamma(s)\Theta(x,s,a;b) = 2\pi e^{b} \sum_{\chi,\xi=0}^{\infty} (-e^{-x})^{\chi}\, \frac{(-(\chi+a))^{\xi}}{\xi!}\delta(s+\xi)} \tag{A46}$$

$$\boldsymbol{\Gamma(s)\Psi(x,s,a;b) = 2\pi e^{b} \sum_{\chi,\xi=0}^{\infty} (e^{-x})^{\chi}\, \frac{(-(\chi+a))^{\xi}}{\xi!}\delta(s+\xi)} \tag{A47}$$

$$\boldsymbol{\Gamma(s)F(x,s;b) = 2\pi e^{b} \sum_{\chi,\xi=0}^{\infty} (-e^{-x})^{\chi}\, \frac{(-(\chi+1))^{\xi}}{\xi!}\delta(s+\xi)} \tag{A48}$$

$$\boldsymbol{\Gamma(s)B(x,s;b) = 2\pi e^{b} \sum_{\chi,\xi=0}^{\infty} (e^{-x})^{\chi}\, \frac{(-(\chi+1))^{\xi}}{\xi!}\delta(s+\xi)} \tag{A49}$$

$$\boldsymbol{\phi(z,s;b) = 2\pi z e^{b} \sum_{\chi,\xi=0}^{\infty} (z)^{\chi}\, \frac{(-(\chi+1))^{\xi}}{\xi!}\delta(s+\xi)} \tag{A50}$$

$$\boldsymbol{\Gamma(s)\zeta(s,a;b) = 2\pi e^{b} \sum_{\chi,\xi=0}^{\infty} \frac{(-(\chi+a))^{\xi}}{\xi!}\delta(s+\xi)} \tag{A51}$$

$$\boldsymbol{\Gamma(s)\zeta(s;b) = 2\pi e^{b} \sum_{\chi,\xi=0}^{\infty} \frac{(-(\chi+1))^{\xi}}{\xi!}\delta(s+\xi)} \tag{A52}$$

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