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Coupled Systems of Sequential Caputo and Hadamard Fractional Differential Equations with Coupled Separated Boundary Conditions

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Abstract: This paper studies the existence and uniqueness of solutions for a new coupled system of nonlinear sequential Caputo and Hadamard fractional differential equations with coupled separated boundary conditions, which include as special cases the well-known symmetric boundary conditions. Banach’s contraction principle, Leray–Schauder’s alternative, and Krasnoselskii’s fixed-point theorem were used to derive the desired results, which are well-illustrated with examples.

Keywords: Caputo fractional derivative; Hadamard fractional derivative; coupled system; separated boundary conditions; existence

MSC: 34A08; 34B10

1. Introduction

Fractional differential equations appear in the mathematical modeling of many real-world phenomena occurring in engineering and scientific disciplines, for instance, see References [1–6]. Mathematical models based on fractional-order integral and differential operators yield more insight into the characteristics of the associated phenomena, as such operators are nonlocal in nature, in contrast to classical ones. In particular, coupled systems of fractional-order differential equations have received great attention in view of their great utility in handling and comprehending practical issues, such as the synchronization of chaotic systems [7,8], anomalous diffusion [9], and ecological effects [10]. For recent theoretical results on the topic, we refer the reader to a series of papers [11–18] and the references cited therein.

Recently, in Reference [19], the authors discussed existence and the uniqueness of solutions for sequential Caputo and Hadamard fractional differential equations subject to separated boundary conditions as

\[
\begin{align*}
\mathcal{C}D^p (H D^q x)(t) &= f(t, x(t)), & t \in [a, b], \\
\alpha_1 x(a) + \alpha_2 (H D^q x)(a) &= 0, & \beta_1 x(b) + \beta_2 (H D^q x)(b) &= 0,
\end{align*}
\]

where \( \mathcal{C}D^p \) and \( H D^q \) are the Caputo and Hadamard fractional derivatives of orders \( p \) and \( q \), respectively, \( 0 < p, q \leq 1 \), starting at a point \( a > 0 \), \( f : [a, b] \times \mathbb{R} \to \mathbb{R} \) is a continuous function and given constants \( \alpha_i, \beta_i \in \mathbb{R}, i = 1, 2 \).
In this paper, we established the existence criteria for a coupled system of sequential Caputo and Hadamard fractional differential equations with coupled separated boundary conditions as:

\[
\begin{align*}
\mathcal{C}D^{p_1}H^i D^{q_1} x(t) &= f(t, x(t), y(t)), \quad t \in [a, b], \\
H^j D^{q_2} \mathcal{C} D^{p_2} y(t) &= g(t, x(t), y(t)), \quad t \in [a, b], \\
\alpha_1 x(a) + \alpha_2 C^D p_2 y(a) &= 0, \quad \beta_1 x(b) + \beta_2 C^D p_2 y(b) = 0, \\
\alpha_3 y(a) + \alpha_4 H^j D^{q_1} x(a) &= 0, \quad \beta_3 y(b) + \beta_4 H^j D^{q_1} x(b) = 0,
\end{align*}
\]

where \(\mathcal{C}D^{p_i}\) and \(H^j D^{q_j}\) are notations of the Caputo and Hadamard fractional derivatives of orders \(p_i\) and \(q_j\), respectively, \(0 < p_i, q_j \leq 1, i = 1, 2\), the nonlinear continuous functions \(f, g : [a, b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}\), \(a > 0, a_i \in \mathbb{R} \setminus \{0\}, \beta_i \in \mathbb{R}, i = 1, \ldots, 4\). Meanwhile, the different definitions of Caputo and Hadamard fractional derivatives that appeared in System (2) are proposed to study the existence theory of solutions of a fractional differential system using a variety of fixed-point theorems. A special case, when \(p_i = q_i = 1, i = 1, 2\), in differential Equation (2) can be presented as:

\[
\begin{align*}
t x' + x' &= f(t, x, y), \quad t y'' = g(t, x, y), \quad t \in [a, b], \\
\alpha_1 x(a) + \alpha_2 y'(a) &= 0, \quad \beta_1 x(b) + \beta_2 y'(b) = 0, \\
\alpha_3 y(a) + \alpha_4 (tx')(a) &= 0, \quad \beta_3 y(b) + \beta_4 (tx')(b) = 0,
\end{align*}
\]

which is mixed type of ordinary differential equations and boundary conditions.

The rest of this paper is organized as follows: Section 2 aims to recall basic definitions and lemmas used in this paper. Section is devoted to the main results concerning the existence and uniqueness of solutions for System (2). The Leray–Schauder alternative and Krasnoselskii’s fixed-point theorem were applied to prove existence, while the uniqueness result was obtained via the Banach contraction mapping principle. Some illustrative examples are presented in Section 4.

2. Preliminaries

To ensure that readers can easily understand the results, we recall some notations and definitions of fractional calculus [3,20].

**Definition 1.** The Caputo fractional derivative of order \(q\) for an at least \(n\)-times differentiable function \(g : [a, \infty) \to \mathbb{R}\), starting at a point \(a > 0\), is defined as:

\[
\mathcal{C}D^n g(t) = \frac{1}{\Gamma(n-q)} \int_a^t (t-s)^{n-q-1} g^{(n)}(s) ds, \quad n-1 < q < n, \quad n = [q] + 1,
\]

where \([q]\) denotes the integer part of the real number \(q\).

**Definition 2.** The Riemann–Liouville fractional integral of order \(q\) of a function \(g : [a, \infty) \to \mathbb{R}, a > 0\), is defined as:

\[
RLI^n g(t) = \frac{1}{\Gamma(q)} \int_a^t \frac{g(s)}{(t-s)^{1-q}} ds, \quad q > 0,
\]

provided the right side of an integral exists.

**Definition 3.** The Caputo-type Hadamard fractional derivative of order \(q\) for an at least \(n\)-times delta differentiable function \(g : [a, \infty) \to \mathbb{R}\), starting at a point \(a > 0\), is defined as

\[
H^j D^n g(t) = \frac{1}{\Gamma(n-q)} \int_a^t \left( \log \frac{t}{s} \right)^{n-q-1} \delta^n g(s) ds, \quad n-1 < q < n, \quad n = [q] + 1,
\]
where the delta derivative is defined by $\delta = t \frac{d}{dt}$ and the natural logarithm $\log(\cdot) = \log_a(\cdot)$.

**Definition 4.** The Hadamard fractional integral of order $q$ is defined as

$$H^q \mathcal{I} g(t) = \frac{1}{\Gamma(q)} \int_a^t \left( \log \frac{t}{s} \right)^{q-1} g(s) \frac{ds}{s}, \quad q > 0, \ a > 0,$$

provided the integral exists.

**Lemma 1.** The general solution of homogeneous fractional differential equation $\mathcal{C} D^q u(t) = 0, q > 0$ is given by

$$u(t) = c_0 + c_1 (t - a) + \ldots + c_{n-1} (t - a)^{n-1},$$

where $c_i \in \mathbb{R}, \ i = 0, 1, 2, \ldots, n - 1 \ (n = [q] + 1)$.

In view of Lemma 1, we have

$$H^q \mathcal{I} \mathcal{C} D^q u(t) = u(t) + c_0 + c_1 (t - a) + \ldots + c_{n-1} (t - a)^{n-1}, \quad (4)$$

for some constants $c_i \in \mathbb{R}, \ i = 0, 1, 2, \ldots, n - 1 \ (n = [q] + 1)$.

**Lemma 2** ([21]). Let $AC^n[a, b] = \{ g : [a, b] \to \mathbb{C} : \delta^{n-1} g(t) \in AC[a, b] \}$ and $u \in AC^n[a, b]$ or $C^n[a, b]$ and $q \in \mathbb{C}$. Then, the following formula holds

$$H^q \mathcal{I} (H^q \mathcal{I}) u(t) = u(t) - \sum_{k=0}^{n-1} c_k (\log(t/a))^k,$$

where $c_i \in \mathbb{R}, \ i = 0, 1, 2, \ldots, n - 1 \ (n = [q] + 1)$.

Next, we transform Problem (2) to integral equations by using a linear variant of Problem (2). For convenience, we put constants

$$\Omega_1 = \frac{\beta_3 (\log(b/a))^{q_2}}{\Gamma(q_1 + 1)}, \ \Omega_2 = \beta_1 - \frac{\alpha_1}{\alpha_2} \beta_2, \ \Omega_3 = \beta_4 - \frac{\alpha_4}{\alpha_3} \beta_3, \ \Omega_4 = \frac{\alpha_1 \beta_3 (b - a)^{p_2}}{\alpha_2 \Gamma(p_2 + 1)},$$

and $\Omega = \Omega_1 \Omega_4 + \Omega_2 \Omega_3 \neq 0$.

**Lemma 3.** Let $\omega, \phi \in C([a, b], \mathbb{R})$. Then, the linear system of sequential Caputo and Hadamard fractional differential equations with coupled separated boundary value problem

$$\begin{cases}
\mathcal{C} D^{p_1} \mathcal{H} D^{q_1} x(t) = \omega(t), \quad t \in [a, b], \\
\mathcal{H} D^{q_2} \mathcal{C} D^{p_2} y(t) = \phi(t), \quad t \in [a, b], \\
a_1 x(a) + a_2 \mathcal{C} D^{p_2} y(a) = 0, \quad \beta_1 x(b) + \beta_2 \mathcal{C} D^{p_2} y(b) = 0, \\
\alpha_3 y(a) + \alpha_4 \mathcal{H} D^{q_1} x(a) = 0, \quad \beta_3 y(b) + \beta_4 \mathcal{H} D^{q_1} x(b) = 0,
\end{cases} \quad (5)$$
can be written as integral equations

\[
x(t) = -\frac{1}{\Omega} \left( \Omega_3 + \Omega_4 \frac{(\log(t/a))^q_1}{\Gamma(q_1 + 1)} \right) \left( \beta_1^H I^{q_1}\Omega_1 \omega(b) + \beta_2^H I^{q_2}\phi(b) \right) \\
+ \frac{1}{\Omega} \left( \Omega_1 - \Omega_2 \frac{(\log(t/a))^q_1}{\Gamma(q_1 + 1)} \right) \left( \beta_3^{RL} I^{p_2} H^{p_2} \phi(b) + \beta_4^{RL} I^{p_1} \omega(b) \right)
\]

(6)

and

\[
y(t) = \frac{1}{\Omega} \left( \frac{d_4}{d_3} \Omega_4 + \frac{d_2}{d_3} \Omega_3 \frac{(t - a)^{p_2}}{\Gamma(p_2 + 1)} \right) \left( \beta_1^H I^{q_1} p_2 H^{p_2} \phi(b) + \beta_2^H I^{q_2} \phi(b) \right) \\
+ \frac{1}{\Omega} \left( \frac{d_4}{d_3} \Omega_2 - \frac{d_2}{d_3} \Omega_1 \frac{(t - a)^{p_2}}{\Gamma(p_2 + 1)} \right) \left( \beta_3^{RL} I^{p_2} H^{p_2} \phi(b) + \beta_4^{RL} I^{p_1} \omega(b) \right)
\]

(7)

**Proof.** Taking the Riemann–Liouville fractional integral of order \( p_1, p_1 \in (0, 1) \), to the first equation of Problem (5) and applying Problem (4), we obtain for \( t \in [a, b] \)

\[
^H D^{q_1} x(t) = c_1 + ^{RL} I^{p_1} \omega(t), \quad c_1 \in \mathbb{R}.
\]

(8)

In the above equation, we apply the Hadamard fractional integral of order \( q_1, q_1 \in (0, 1) \), with (4) for \( t \in [a, b] \) and obtain

\[
x(t) = c_2 + c_1 \frac{(\log(t/a))^q_1}{\Gamma(q_1 + 1)} + ^H I^{q_1} RL^{I^{p_1}} \omega(t), \quad c_2 \in \mathbb{R}.
\]

(9)

Considering the second equation of Problem (5), and by using the Hadamard fractional integral of order \( q_2 \), we get

\[
^C D^{p_2} y(t) = c_3 + ^H I^{p_2} \phi(t), \quad c_3 \in \mathbb{R}.
\]

(10)

By taking the Riemann–Liouville fractional integral operator of order \( p_2 \), we have

\[
y(t) = c_4 + c_3 \frac{(t - a)^{p_2}}{\Gamma(p_2 + 1)} + ^{RL} I^{p_2} H^{p_2} \phi(t), \quad c_4 \in \mathbb{R}.
\]

(11)

In particular, for \( t = a \) in Equations (9) and (10), and applying the first condition of Problem (5), one has

\[
\alpha_1 c_2 + \alpha_2 c_3 = 0.
\]

(12)

For \( t = b \) in Equations (9) and (10), it obtains by applying the second condition of Problem (5) as

\[
\beta_1 c_1 \frac{(\log(b/a))^q_1}{\Gamma(q_1 + 1)} + \beta_1 c_2 + \beta_2 c_3 = -\beta_1^H I^{q_1} RL^{I^{p_1}} \omega(b) - \beta_2^H I^{q_2} \phi(b) := \Omega_3.
\]

(13)

Substituting \( t = a \) in Equations (8) and (11) and applying the third condition of Problem (5), it leads to

\[
\alpha_4 c_1 + \alpha_3 c_4 = 0.
\]

(14)

The fourth condition of Problem (5) can be applied when \( t = b \) in Equations (8) and (11) as

\[
\beta_4 c_1 + \beta_3 c_3 \frac{(b - a)^{p_2}}{\Gamma(p_2 + 1)} + \beta_3 c_4 = -\beta_3^{RL} I^{p_2} H^{p_2} \phi(b) - \beta_4^{RL} I^{p_1} \omega(b) := \Omega_6.
\]

(15)
Reduce the above Equations (12)–(15) in a system of constants by
\[
\Omega_4 c_1 + \Omega_5 c_2 = \Omega_5, \quad \Omega_3 c_1 - \Omega_4 c_2 = \Omega_6.
\]

Computing for constants \( c_1 \) and \( c_2 \) and substituting it into Equations (12) and (14) for \( c_3 \) and \( c_4 \), we have
\[
c_1 = \frac{\Omega_4}{\Omega} \Omega_5 + \frac{\Omega_2}{\Omega} \Omega_6, \quad c_2 = \frac{\Omega_3}{\Omega} \Omega_5 - \frac{\Omega_1}{\Omega} \Omega_6,
\]
\[
c_3 = -\frac{a_1 \Omega_3}{a_2 \Omega} \Omega_5 + \frac{a_1 \Omega_3}{a_2 \Omega} \Omega_6, \quad c_4 = -\frac{a_4 \Omega_4}{a_3 \Omega} \Omega_5 - \frac{a_4 \Omega_4}{a_3 \Omega} \Omega_6.
\]

Substituting all obtained constants in Equations (9) and (11), we obtain integral Equations (6) and (7). By direct computation we can obtain the the converse. The proof is completed. \( \square \)

**Remark 1.** System (5) is well-defined because four constants \( a_i \in \mathbb{R} \setminus \{0\} \), \( i = 1, 2, 3, 4 \), make meaningful property for Caputo and Hadamard (Caputo-type) fractional derivatives, which lead to solve the system of linear equations.

3. Main Results

Let \( C = C([a, b], \mathbb{R}) \), \( a > 0 \), be the Banach space of all continuous functions form \([a, b]\) to \( \mathbb{R} \). Space \( X = \{ x(t) : x(t) \in C^2([a, b], \mathbb{R}) \} \) endowed with the norm \( \| x \| = \sup \{ |x(t)|, t \in [a, b] \} \) is a Banach space. In addition, let \( Y = \{ y(t) : y(t) \in C^2([a, b], \mathbb{R}) \} \) with the norm \( \| y \| = \sup \{ |y(t)|, t \in [a, b] \} \). It is obvious that product space \( (X \times Y, \| (x, y) \|) \) is a Banach space with the norm \( \| (x, y) \| = \| x \| + \| y \| \).

Now, for brevity, we use the notations:
\[
h_{x,y}(t) = h(t, x(t), y(t)), \quad h \in \{ f, g \},
\]
\[
H^p_{RL} f_{x,y}(\phi) = \frac{1}{\Gamma(q) \Gamma(p)} \int_a^\phi \int_a^s \left( \log \frac{\phi}{s} \right)^{q-1} (s-r)^{p-1} f_{x,y}(r) dr ds,
\]
and
\[
RL^p_{H} f_{x,y}(\phi) = \frac{1}{\Gamma(p) \Gamma(q)} \int_a^\phi \int_a^s (\phi - s)^{p-1} \left( \log \frac{\phi}{s} \right)^{q-1} f_{x,y}(r) dr ds,
\]
where \( \phi \in \{ t, b \} \). We also use this one for a single fractional integral operator of the Riemann–Liouville and Hadamard types of orders \( p_1 \) and \( q_2 \), respectively.

In view of Lemma 3, we define two operators \( \mathcal{K} : X \times Y \rightarrow X \times Y \) by
\[
\mathcal{K}(x, y)(t) = \begin{pmatrix} \mathcal{K}_1(x, y)(t) \\ \mathcal{K}_2(x, y)(t) \end{pmatrix},
\]
where
\[
\mathcal{K}_1(x, y)(t) = -\frac{1}{\Omega} \left( \Omega_3 + \Omega_4 \frac{(\log(t/a))^{q_1}}{\Gamma(q_1 + 1)} \right) \left( \beta_1 H^p_{RL} f_{x,y}(b) + \beta_2 H^p_{RL} g_{x,y}(b) \right) + \frac{1}{\Omega} \left( \Omega_1 - \Omega_2 \frac{(\log(t/a))^{q_1}}{\Gamma(q_1 + 1)} \right) \left( \beta_3 RL^p_{H} f_{x,y}(b) + \beta_4 RL^p_{H} g_{x,y}(b) \right) + H^p_{RL} f_{x,y}(t),
\]
and
\[
\mathcal{K}_2(x, y)(t) = \frac{1}{\Omega} \left( \Omega_4 \frac{(\log(t/a))^{q_1}}{\Gamma(q_1 + 1)} \right) \left( \beta_1 H^p_{RL} f_{x,y}(b) + \beta_2 H^p_{RL} g_{x,y}(b) \right).
\]
and
\[
K_2(x,y)(t) = \frac{1}{\Omega} \left( \frac{a_4}{a_3} \Omega_4 + \frac{a_1}{a_2} \Omega_3 \frac{(t-a)^{p_2}}{\Gamma(p_2+1)} \right) \left( \beta_1 H^{q_1, RL} I^{p_1}(f_{x,y})(b) + \beta_2 H^{q_2, RL} I^{p_1}(g_{x,y})(b) \right) \\
+ \frac{1}{\Omega} \left( \frac{a_4}{a_3} \Omega_2 - \frac{a_1}{a_2} \Omega_1 \frac{(t-a)^{p_2}}{\Gamma(p_2+1)} \right) \left( \beta_3 RL I^{p_2, H} I^{q_2}(g_{x,y})(b) + \beta_4 RL I^{p_1}(f_{x,y})(b) \right) \\
+ RL I^{p_2, H} I^{q_2}(g_{x,y})(t).
\]

For computational convenience, we set
\[
M_1 = \frac{|\beta_1|}{\Omega} \left( |\Omega_3| + |\Omega_4| \frac{(\log(b/a))^{q_1}}{\Gamma(q_1+1)} \right) H^{q_1, RL} I^{p_1}(1)(b) \\
+ \frac{|\beta_1|}{\Omega} \left( |\Omega_1| + |\Omega_2| \frac{(\log(b/a))^{q_1}}{\Gamma(q_1+1)} \right) \frac{(b-a)^{p_1}}{\Gamma(p_1+1)} + H^{q_1, RL} I^{p_1}(1)(b),
\]
(19)
\[
M_2 = \frac{|\beta_2|}{\Omega} \left( |\Omega_3| + |\Omega_4| \frac{(\log(b/a))^{q_1}}{\Gamma(q_1+1)} \right) \frac{(\log(b/a))^{q_2}}{\Gamma(q_2+1)} \frac{(b-a)^{p_1}}{\Gamma(p_1+1)} + RL I^{p_2, H} I^{q_2}(1)(b),
\]
(20)
\[
M_3 = \frac{|\beta_1|}{\Omega} \left( \frac{a_4}{a_3} |\Omega_4| + \frac{a_1}{a_2} |\Omega_3| \frac{(b-a)^{p_2}}{\Gamma(p_2+1)} \right) \left( H^{q_1, RL} I^{p_1}(1)(b) \right) \\
+ \frac{|\beta_1|}{\Omega} \left( \frac{a_4}{a_3} |\Omega_2| + \frac{a_1}{a_2} |\Omega_1| \frac{(b-a)^{p_2}}{\Gamma(p_2+1)} \right) \frac{(b-a)^{p_1}}{\Gamma(p_1+1)},
\]
(21)
\[
M_4 = \frac{|\beta_2|}{\Omega} \left( \frac{a_4}{a_3} |\Omega_4| + \frac{a_1}{a_2} |\Omega_3| \frac{(b-a)^{p_2}}{\Gamma(p_2+1)} \right) \frac{(\log(b/a))^{q_2}}{\Gamma(q_2+1)} \frac{(b-a)^{p_1}}{\Gamma(p_1+1)} \\
+ \frac{|\beta_2|}{\Omega} \left( \frac{a_4}{a_3} |\Omega_2| + \frac{a_1}{a_2} |\Omega_1| \frac{(b-a)^{p_2}}{\Gamma(p_2+1)} \right) RL I^{p_2, H} I^{q_2}(1)(b) + RL I^{p_2, H} I^{q_2}(1)(b).
\]
(22)

Note that all information of Problem (2) is contained in constants $M_i$, $i = 1, 2, 3, 4$, which are used to establish the following existence theorems. Banach’s contraction mapping principle is applied in the first result to prove the existence and uniqueness of solutions of System (2).

**Theorem 1.** Suppose that $f, g : [a, b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions. In addition, we assume that:

(H₁) there exist constants $m_i, n_i, i = 1, 2$, such that for all $t \in [a, b]$ and $x_i, y_i \in \mathbb{R}, i = 1, 2$
\[
|f(t, x_1, y_1) - f(t, x_2, y_2)| \leq m_1|x_1 - x_2| + m_2|y_1 - y_2|
\]
and
\[
|g(t, x_1, y_1) - g(t, x_2, y_2)| \leq n_1|x_1 - x_2| + n_2|y_1 - y_2|.
\]

Then, System (2) has a unique solution on $[a, b]$, if
\[
(M_1 + M_3)(m_1 + m_2) + (M_2 + M_4)(n_1 + n_2) < 1.
\]
(23)

**Proof.** Define $\sup_{t \in [a, b]} f(t, 0, 0) = N_1 < \infty$ and $\sup_{t \in [a, b]} g(t, 0, 0) = N_2 < \infty$, such that
\[
r > \frac{(M_1 + M_3)N_1 + (M_2 + M_4)N_2}{1 - [(M_1 + M_3)(m_1 + m_2) + (M_2 + M_4)(n_1 + n_2)]}.
\]
Now, we show that the set \( KB_r \subset B_r \), where \( B_r = \{(x,y) \in X \times Y : \| (x,y) \| \leq r \} \). For \((x,y) \in B_r\), we have that

\[
|K_1(x,y)(t)| \\
\leq \frac{1}{|\Omega|} \left( |\Omega_3| + |\Omega_4| \frac{(\log(t/a))^{q_1}}{1(q_1 + 1)} \right) \left( |\beta_1| H_I p_1 \|p_1 \| f_{x,y}(b) \right) + |\beta_2| H_I p_2 H_I p_2 \|p_2 \| f_{x,y}(b) \right) + |\beta_3| H_I p_1 \|p_1 \| f_{x,y}(b) \right) + |\beta_4| H_I p_1 \|p_1 \| f_{x,y}(b) \right)
\]

\[
\leq \frac{1}{|\Omega|} \left( |\Omega_3| + |\Omega_4| \frac{(\log(t/a))^{q_1}}{1(q_1 + 1)} \right) \left( |\beta_1| H_I p_1 \|p_1 \| f_{x,y}(b) \right) + |\beta_2| H_I p_2 H_I p_2 \|p_2 \| f_{x,y}(b) \right) + |\beta_3| H_I p_1 \|p_1 \| f_{x,y}(b) \right) + |\beta_4| H_I p_1 \|p_1 \| f_{x,y}(b) \right)
\]

\[
|K_2(x,y)(t)| \\
\leq \frac{1}{|\Omega|} \left( |\Omega_4| + |\Omega_3| \frac{(b - a)^p_2}{1(p_2 + 1)} \right) \left( |\beta_1| (m_1 \|x\| + m_2 \|y\|) \right.
\]

\[
N_1 + 2^H p_1 \|p_1 \| (1(b) \right) + |\beta_2| (n_1 \|x\| + n_2 \|y\|) \right) + N_2 \|p_2 \| (1(b) \right)
\]

\[
+ \frac{1}{|\Omega|} \left( |\Omega_4| + |\Omega_3| \frac{(b - a)^p_2}{1(p_2 + 1)} \right) \left( |\beta_1| (m_1 \|x\| + m_2 \|y\|) \right.
\]

\[
N_1 + 2^H p_1 \|p_1 \| (1(b) \right) + |\beta_2| (n_1 \|x\| + n_2 \|y\|) \right) + N_2 \|p_2 \| (1(b) \right)
\]

\[
M_1 (m_1 \|x\| + m_2 \|y\| + N_1) + M_2 (n_1 \|x\| + n_2 \|y\| + N_2)
\]

\[
= (M_1 m_1 + M_2 n_1) \|x\| + (M_1 m_2 + M_2 n_2) \|y\| + M_1 N_1 + M_2 N_2
\]

\[
\leq [M_1 (m_1 + m_2) + M_2 (n_1 + n_2)] r + M_1 N_1 + M_2 N_2.
\]

Hence,

\[
|K_1(x,y)| \leq [M_1 (m_1 + m_2) + M_2 (n_1 + n_2)] r + M_1 N_1 + M_2 N_2.
\]

By direct computation, we get

\[
K_2(x,y)(t) \leq \frac{1}{|\Omega|} \left( |\Omega_4| + |\Omega_3| \frac{(b - a)^p_2}{1(p_2 + 1)} \right) \left( |\beta_1| (m_1 \|x\| + m_2 \|y\|) \right.
\]

\[
N_1 + 2^H p_1 \|p_1 \| (1(b) \right) + |\beta_2| (n_1 \|x\| + n_2 \|y\|) \right) + N_2 \|p_2 \| (1(b) \right)
\]

\[
+ \frac{1}{|\Omega|} \left( |\Omega_4| + |\Omega_3| \frac{(b - a)^p_2}{1(p_2 + 1)} \right) \left( |\beta_1| (m_1 \|x\| + m_2 \|y\|) \right.
\]

\[
N_1 + 2^H p_1 \|p_1 \| (1(b) \right) + |\beta_2| (n_1 \|x\| + n_2 \|y\|) \right) + N_2 \|p_2 \| (1(b) \right)
\]

\[
= M_3 (m_1 \|x\| + m_2 \|y\| + N_1) + M_4 (n_1 \|x\| + n_2 \|y\| + N_2),
\]

and

\[
|K_2(x,y)| \leq [M_3 (m_1 + m_2) + M_4 (n_1 + n_2)] r + M_3 N_1 + M_4 N_2.
\]

Consequently, it follows that

\[
|K(x,y)| \leq [M_1 (m_1 + m_2) + M_2 (n_1 + n_2)] r + M_1 N_1 + M_2 N_2
\]

\[
+ [M_3 (m_1 + m_2) + M_4 (n_1 + n_2)] r + M_3 N_1 + M_4 N_2 \leq r,
\]
which implies $KB_r \subset B_r$. Next, we show that operator $K$ is contraction mapping. For any $(x_1, y_1), (x_2, y_2) \in \mathbb{X} \times \mathbb{Y}$, we obtain

\[
\begin{align*}
|\mathcal{K}(x_1, y_1)(t) - \mathcal{K}(x_2, y_2)(t)| &\leq \frac{1}{|\Omega|} \left( |\Omega_3| + |\Omega_4| \frac{\log((t/a)^{\eta_1})}{\Gamma(q_1 + 1)} \right) \left( |\beta_1| H_{p_1} H_{p_1} |f_{x_1, y_1} - f_{x_2, y_2}|(b) + |\beta_2| H_{p_1} H_{p_1} |g_{x_1, y_1} - g_{x_2, y_2}|(b) \right) \\
&\quad + |\beta_3| H_{p_1} H_{p_1} |g_{x_1, y_1} - g_{x_2, y_2}|(b) + \frac{1}{|\Omega|} \left( |\Omega_1| + |\Omega_2| \frac{\log((t/a)^{\eta_1})}{\Gamma(q_1 + 1)} \right) \\
&\quad \times \left( |\beta_3| H_{p_2} H_{p_2} |g_{x_1, y_1} - g_{x_2, y_2}|(b) + |\beta_4| H_{p_1} |f_{x_1, y_1} - f_{x_2, y_2}|(b) \right) \\
&\quad + H_{p_1} H_{p_1} |f_{x_1, y_1} - f_{x_2, y_2}|(b) \\
&\leq \frac{1}{|\Omega|} \left( |\Omega_3| + |\Omega_4| \frac{\log((b/a)^{\eta_1})}{\Gamma(q_1 + 1)} \right) \left( |\beta_1| (m_1 ||x_1 - x_2|| + m_2 ||y_1 - y_2||) \right) \\
&\quad \times H_{p_1} H_{p_1} (1)(b) + |\beta_2| (n_1 ||x_1 - x_2|| + n_2 ||y_1 - y_2||) H_{p_1} (1)(b) \\
&\quad + \frac{1}{|\Omega|} \left( |\Omega_1| + |\Omega_2| \frac{\log((b/a)^{\eta_1})}{\Gamma(q_1 + 1)} \right) \left( |\beta_3| (n_1 ||x_1 - x_2|| + n_2 ||y_1 - y_2||) \right) \\
&\quad \times H_{p_2} H_{p_2} (1)(b) + |\beta_4| (m_1 ||x_1 - x_2|| + m_2 ||y_1 - y_2||) H_{p_1} (1)(b) \\
&\quad + (m_1 ||x_1 - x_2|| + m_2 ||y_1 - y_2||) H_{p_1} H_{p_1} (1)(b) \\
&= M_1 (m_1 ||x_1 - x_2|| + m_2 ||y_1 - y_2||) + M_2 (n_1 ||x_1 - x_2|| + n_2 ||y_1 - y_2||) \\
&= (M_1 m_1 + M_2 n_1) ||x_1 - x_2|| + (M_1 m_2 + M_2 n_2) ||y_1 - y_2||.
\end{align*}
\]

Therefore, we get the following inequality:

\[
||\mathcal{K}(x_1, y_1) - \mathcal{K}(x_2, y_2)|| \leq M_1 (m_1 + m_2) + M_2 (n_1 + n_2) (||x_1 - x_2|| + ||y_1 - y_2||). \tag{24}
\]

In addition, we obtain

\[
||\mathcal{K}(x_1, y_1) - \mathcal{K}(x_2, y_2)|| \leq M_3 (m_1 + m_2) + M_4 (n_1 + n_2) (||x_1 - x_2|| + ||y_1 - y_2||). \tag{25}
\]

From Inequalities (24) and (25), it yields

\[
||\mathcal{K}(x_1, y_1) - \mathcal{K}(x_2, y_2)|| \leq [(M_1 + M_3)(m_1 + m_2) + (M_2 + M_4)(n_1 + n_2)] \times (||x_1 - x_2|| + ||y_1 - y_2||).
\]

As $(M_1 + M_3)(m_1 + m_2) + (M_2 + M_4)(n_1 + n_2) < 1$, therefore $\mathcal{K}$ is a contraction operator. By applying Banach’s fixed-point theorem, operator $\mathcal{K}$ has a unique fixed point in $B_r$. Hence, there exists a unique solution of Problem (2) on $[a, b]$. The proof is completed. \(\square\)

Now, we prove our second existence result via the Leray–Schauder alternative.

**Lemma 4.** (Leray-Schauder alternative) [22] Let $F : E \to E$ be a completely continuous operator. Let

\[\chi(F) = \{x \in E : x = \lambda F(x) \text{ for some } 0 < \lambda < 1\} .\]

Then, either set $\chi(F)$ is unbounded, or $F$ has at least one fixed point.

**Theorem 2.** Assume that there exist real constants $u_i, v_i \geq 0$ for $i = 1, 2$ and $u_0, v_0 > 0$, such that for any $x_i \in \mathbb{R}$, $(i = 1, 2)$ we have

\[
|f(t, x_1, x_2)| \leq u_0 + u_1 |x_1| + u_2 |x_2|, \\
|g(t, x_1, x_2)| \leq v_0 + v_1 |x_1| + v_2 |x_2| .
\]
If \((M_1 + M_3)u_1 + (M_2 + M_4)v_1 < 1\) and \((M_1 + M_3)u_2 + (M_2 + M_4)v_2 < 1\), where \(M_1, M_2, M_3, M_4\) are given in Equations (19)–(22), then Problem (2) has at least one solution on \([a, b]\).

**Proof.** By continuity of functions \(f, g\) on \([a, b] \times \mathbb{R} \times \mathbb{R}\), operator \(K\) is continuous. Now, we show that the operator \(K : X × Y → X × Y\) is completely continuous. Let \(Φ ⊂ X × Y\) be bounded. Then, there exist two positive constants, \(L_1\) and \(L_2\), such that

\[
|f(t, x, y)| \leq L_1, \quad |g(t, x, y)| \leq L_2, \quad \forall (x, y) \in Φ.
\]

Then, for any \((x, y) \in Φ\), we have

\[
|K_1(x, y)(t)| \leq \frac{1}{|Ω|} \left( |Ω_3| + |Ω_4| \frac{(\log(t/a))^{q_1}}{Γ(q_1 + 1)} \right) \left( |β_1| |L_1|^{H} |R_1|^{P_1} |f_x, y|(b) \right. \\
\left. + |β_2| |H|^{P_2} |g_x, y|(b) \right) + \frac{1}{|Ω|} \left( |Ω_1| + |Ω_2| \frac{(\log(t/a))^{q_1}}{Γ(q_1 + 1)} \right) \\
× \left( |β_3| |L_2|^{RL} |P_2|^{P_1} |g_x, y|(b) + |β_4| |L_4|^{RL} |P_1|^{P_1} |f_x, y|(b) \right) \\
\leq \frac{1}{|Ω|} \left( |Ω_3| + |Ω_4| \frac{(\log(b/a))^{q_1}}{Γ(q_1 + 1)} \right) \left( |β_1| |L_1|^{H} |R_1|^{P_1} (1)(b) \right. \\
\left. + |β_2| |L_2|^{H} |P_2|^{P_1} (1)(b) \right) + \frac{1}{|Ω|} \left( |Ω_1| + |Ω_2| \frac{(\log(b/a))^{q_1}}{Γ(q_1 + 1)} \right) \\
× \left( |β_3| |L_2|^{RL} |P_2|^{P_1} (1)(b) + |β_4| |L_4|^{RL} |P_1|^{P_1} (1)(b) \right) \\
+
L_1 M_1 + L_2 M_2 \\
]

which yields

\[
\|K_1(x, y)\| \leq L_1 M_1 + L_2 M_2.
\]

In addition, we obtain that

\[
\|K_2(x, y)\| \leq \frac{1}{|Ω|} \left( \frac{|a_4|}{|a_3|} |Ω_4| + \frac{|a_1|}{|a_2|} |Ω_3| \frac{(b - a)^{P_2}}{Γ(p_2 + 1)} \right) \left( |β_1| |L_1|^{H} |R_1|^{P_1} (1)(b) \right. \\
\left. + |β_2| |L_2|^{H} |P_2|^{P_1} (1)(b) \right) + \frac{1}{|Ω|} \left( \frac{|a_4|}{|a_3|} |Ω_2| + \frac{|a_1|}{|a_2|} |Ω_1| \frac{(b - a)^{P_2}}{Γ(p_2 + 1)} \right) \\
× \left( |β_3| |L_2|^{RL} |P_2|^{P_1} (1)(b) + |β_4| |L_4|^{RL} |P_1|^{P_1} (1)(b) \right) \\
+ L_1 M_3 + L_2 M_4.
\]

Hence, from the above inequalities, we get that set \(KΦ\) is uniformly bounded. Next, we prove that set \(KΦ\) is equicontinuous. For any \((x, y) \in Φ\), and \(τ_1, τ_2 \in [a, b]\) such that \(τ_1 < τ_2\), we have
\[
\begin{align*}
|K_1(x, y)(\tau_2) - K_1(x, y)(\tau_1)| & \leq \left( \frac{\Omega_1}{\Omega_1 \Gamma(q_1 + 1)} \right) |(\log(\tau_2/a))^{\eta_1} - (\log(\tau_1/a))^{\eta_1}| \\
& \times \left( |\beta_1| H_{p_1} H_{p_1} P_1 f_{\tau_1}(b) + |\beta_2| H_{p_2} f_{\tau_2}(b) \right) \\
& + \left( \frac{\Omega_2}{\Omega_1 \Gamma(q_1 + 1)} \right) |(\log(\tau_2/a))^{\eta_1} - (\log(\tau_1/a))^{\eta_1}| \\
& \times \left( |\beta_3| RL_{p_1} H_{p_1} P_1 f_{\tau_1}(b) + |\beta_4| RL_{p_1} f_{\tau_2}(b) \right) \\
& + \left( H_{p_1} RL_{p_1} P_1 f_{\tau_1}(b) - H_{p_1} RL_{p_1} P_1 f_{\tau_2}(b) \right) \\
& \leq \left( \frac{\Omega_1}{\Omega_1 \Gamma(q_1 + 1)} \right) |(\log(\tau_2/a))^{\eta_1} - (\log(\tau_1/a))^{\eta_1}| \\
& \times \left( L_1 |\beta_1| H_{p_1} H_{p_1} P_1 (b) + L_2 |\beta_2| H_{p_2} f_{\tau_2}(b) \right) \\
& + \left( \frac{\Omega_2}{\Omega_1 \Gamma(q_1 + 1)} \right) |(\log(\tau_2/a))^{\eta_1} - (\log(\tau_1/a))^{\eta_1}| \\
& \times \left( L_2 |\beta_3| RL_{p_1} H_{p_1} P_1 (b) + L_1 |\beta_4| RL_{p_1} f_{\tau_2}(b) \right) \\
& + \frac{L_1}{\Gamma(q_1 + 1) \Gamma(p_1 + 1)} \int_{\tau_1}^{\tau_2} \left[ \left( \log \frac{\tau_2}{s} \right)^{\eta_1} - (\log \frac{\tau_1}{s})^{\eta_1} \right] \\
& \times (s - a)^{p_1} ds + \int_{\tau_1}^{\tau_2} \left( \log \frac{\tau_2}{s} \right)^{\eta_1} (s - a)^{p_1} ds \\
& = \left( \frac{\Omega_1}{\Omega_1 \Gamma(q_1 + 1)} \right) |(\log(\tau_2/a))^{\eta_1} - (\log(\tau_1/a))^{\eta_1}| \\
& \times \left( L_1 |\beta_1| H_{p_1} H_{p_1} P_1 (b) + L_2 |\beta_2| H_{p_2} f_{\tau_2}(b) \right) \\
& + \left( \frac{\Omega_2}{\Omega_1 \Gamma(q_1 + 1)} \right) |(\log(\tau_2/a))^{\eta_1} - (\log(\tau_1/a))^{\eta_1}| \\
& \times \left( L_2 |\beta_3| RL_{p_1} H_{p_1} P_1 (b) + L_1 |\beta_4| RL_{p_1} f_{\tau_2}(b) \right) \\
& + \frac{L_1}{\Gamma(q_1 + 1) \Gamma(p_1 + 1)} \left[ 2 \left( \log \frac{\tau_2}{\tau_1} \right)^{\eta_1} + \left( \log \frac{\tau_2}{a} \right)^{\eta_1} - \left( \log \frac{\tau_1}{a} \right)^{\eta_1} \right].
\end{align*}
\]

Therefore, we obtain
\[
|K_1(x, y)(\tau_2) - K_1(x, y)(\tau_1)| \rightarrow 0, \quad \text{as} \quad \tau_1 \rightarrow \tau_2.
\]

Analogously, we can get the following inequality:
\[
|K_2(x, y)(\tau_2) - K_2(x, y)(\tau_1)| \rightarrow 0, \quad \text{as} \quad \tau_1 \rightarrow \tau_2.
\]

Hence, set \(K\Phi\) is equicontinuous. By applying the Arzelà–Ascoli theorem, set \(K\Phi\) is relatively compact, which implies that operator \(K\) is completely continuous. Lastly, we show that set \(\xi = \{(x, y) \in X \times Y : (x, y) = \lambda K(x, y), 0 \leq \lambda \leq 1\}\) is bounded. Now, let \((x, y) \in \xi\), then we obtain \((x, y) = \lambda K(x, y)\), which yields, for any \(t \in [a, b]\),
\[
x(t) = \lambda K_1(x, y)(t), \quad y(t) = \lambda K_2(x, y)(t).
\]
Then, we have
\[
\|x\| \leq (u_0 + u_1\|x\| + u_2\|y\|)M_1 + (v_0 + v_1\|x\| + v_2\|y\|)M_2,
\]
\[
\|y\| \leq (u_0 + u_1\|x\| + u_2\|y\|)M_3 + (v_0 + v_1\|x\| + v_2\|y\|)M_4,
\]
which imply that
\[
\|x\| + \|y\| \leq (M_1 + M_3)u_0 + (M_2 + M_4)v_0 + [(M_1 + M_3)u_1 + (M_2 + M_4)v_1]\|x\|
+ [(M_1 + M_3)u_2 + (M_2 + M_4)v_2]\|y\|.
\]
Thus, we get the inequality
\[
\|(x, y)\| \leq \frac{(M_1 + M_3)u_0 + (M_2 + M_4)v_0}{M^*},
\]
where \(M^* = \min\{1 - (M_1 + M_3)u_1 - (M_2 + M_4)v_1, 1 - (M_1 + M_3)u_2 - (M_2 + M_4)v_2\}\), which shows that set \(\xi\) is bounded. Therefore, by applying Lemma 4, operator \(K\) has at least one fixed point in \(\Phi\). Therefore, we deduce that Problem (2) has at least one solution on \([a, b]\). The proof is complete. \(\square\)

The last-existence theorem is based on Krasnoselskii’s fixed-point theorem.

**Lemma 5.** (Krasnoselskii’s fixed-point theorem) [23] Let \(M\) be a closed, bounded, convex, and nonempty subset of a Banach space \(X\). Let \(A, B\) be operators, such that (i) \(Ax + By \in M\) where \(x, y \in M\), (ii) \(A\) is compact and continuous, and (iii) \(B\) is a contraction mapping. Then, there exists \(z \in M\), such that \(z = Az + Bz\).

**Theorem 3.** Assume that \(f, g : [a, b] \times \mathbb{R} \to \mathbb{R}\) are continuous functions satisfying assumption (H1) in Theorem 1. In addition we suppose and there exist two positive constants \(P, Q\) such that for all \(t \in [a, b]\) and \(x_i, y_i \in \mathbb{R}, i = 1, 2,\)
\[
|f(t, x_1, x_2)| \leq P \text{ and } |g(t, x_1, x_2)| \leq Q.
\]
If
\[
\left((\frac{m_1}{H^{p_1}R^{p_1}(1)(b)} + \frac{m_2}{R^{p_2}H^{p_2}(1)(b)}) + (\frac{n_1}{p_1} + \frac{n_2}{p_2})\right) < 1,
\]
then the problem (2) has at least one solution on \([a, b]\).

**Proof.** Let \(B_\delta = \{(x, y) \in X \times Y : \|(x, y)\| \leq \delta\}\) be a ball, where a constant \(\delta \geq \max\{M_1P + M_2Q, M_3P + M_4Q\}\). To apply Lemma 5, we decompose operator \(K\) into four operators \(K_{1,1}, K_{1,2}, K_{2,1},\) and \(K_{2,2}\) on \(B_\delta\) as
\[ \mathcal{K}_{1,1}(x, y)(t) = -\frac{1}{\Omega} \left( \Omega_3 + \Omega_4 \frac{(\log(t/a))^\delta t}{\Gamma(q_1 + 1)} \right) \times \left( \beta_1 H^{p_1 H_{RL} P_1(f_{x,y})(b)} + \beta_2 H^{p_2 H_{RL} P_1(g_{x,y})(b)} \right) + \frac{1}{\Omega} \left( \Omega_1 - \Omega_2 \frac{(\log(t/a))^\delta t}{\Gamma(q_1 + 1)} \right) \times \left( \beta_3 H^{RL P_2 H_{RL} P_2(g_{x,y})(b)} + \beta_4 H^{RL P_1(f_{x,y})(b)} \right), \]

\[ \mathcal{K}_{1,2}(x, y)(t) = H^{p_1 H_{RL} P_1(f_{x,y})(t)}, \]

\[ \mathcal{K}_{2,1}(x, y)(t) = \frac{1}{\Omega} \left( \frac{\alpha_4}{\alpha_3} \Omega_4 + \frac{\alpha_1}{\alpha_2} \Omega_3 \frac{(t - a)^{p_2}}{\Gamma(p_2 + 1)} \right) \times \left( \beta_1 H^{p_1 H_{RL} P_1(f_{x,y})(b)} + \beta_2 H^{p_2 H_{RL} P_1(g_{x,y})(b)} \right) + \frac{1}{\Omega} \left( \frac{\alpha_4}{\alpha_3} \Omega_2 - \frac{\alpha_1}{\alpha_2} \Omega_1 \frac{(t - a)^{p_2}}{\Gamma(p_2 + 1)} \right) \times \left( \beta_3 H^{RL P_2 H_{RL} P_2(g_{x,y})(b)} + \beta_4 H^{RL P_1(f_{x,y})(b)} \right), \]

\[ \mathcal{K}_{2,2}(x, y)(t) = RL P_2 H^{p_2 H_{RL} P_2(g_{x,y})(t)}. \]

Note that \( \mathcal{K}_3(x, y)(t) = \mathcal{K}_{1,1}(x, y)(t) + \mathcal{K}_{1,2}(x, y)(t) \) and \( \mathcal{K}_2(x, y)(t) = \mathcal{K}_{2,1}(x, y)(t) + \mathcal{K}_{2,2}(x, y)(t) \). In addition, observe that ball \( B_\delta \) is a closed, bounded, and convex subset of Banach space \( C \). Now, we show that \( \mathcal{K} B_\delta \subset B_\delta \) for satisfying condition (i) of Lemma 5. Setting \( x = (x_1, x_2), y = (y_1, y_2) \in B_\delta \), and using Condition (27), we then have

\[
\left| \mathcal{K}_{1,1}(x_1, x_2) + \mathcal{K}_{1,2}(y_1, y_2) \right| \\
\leq \frac{1}{\Omega} \left( |\Omega_3| + |\Omega_4| \frac{(\log(t/a))^\delta t}{\Gamma(q_1 + 1)} \right) \left( |\beta_1| H^{p_1 H_{RL} P_1} f_{x_1, x_2}(b) + |\beta_2| H^{p_2 H_{RL} P_1} g_{x_1, x_2}(b) \right) + \frac{1}{\Omega} \left( |\Omega_1| + |\Omega_2| \frac{(\log(t/a))^\delta t}{\Gamma(q_1 + 1)} \right) \left( |\beta_3| H^{RL P_2 H_{RL} P_2} f_{x_1, x_2}(b) + |\beta_4| H^{RL P_1} f_{x_1, x_2}(b) \right) + H^{p_1 H_{RL} P_1} f_{y_1, y_2}(t) \\
\leq \frac{1}{\Omega} \left( |\Omega_3| + |\Omega_4| \frac{(\log(t/a))^\delta t}{\Gamma(q_1 + 1)} \right) \left( P |\beta_1| H^{p_1 H_{RL} P_1} (1)(b) + Q |\beta_2| H^{p_2 H_{RL} P_1} (1)(b) \right) + \frac{1}{\Omega} \left( |\Omega_1| + |\Omega_2| \frac{(\log(t/a))^\delta t}{\Gamma(q_1 + 1)} \right) \left( Q |\beta_3| H^{RL P_2 H_{RL} P_2} (1)(b) + P |\beta_4| H^{RL P_1} (1)(b) \right) + H^{p_1 H_{RL} P_1} (1)(b) \\
= M_1 P + M_2 Q \leq \delta.
\]

Furthermore, we can find that

\[
\left| \mathcal{K}_{2,1}(x_1, x_2) + \mathcal{K}_{2,2}(y_1, y_2) \right| \\
\leq \frac{1}{\Omega} \left( \frac{\alpha_4}{\alpha_3} |\Omega_4| + \frac{\alpha_1}{\alpha_2} |\Omega_3| \frac{(b - a)^{p_2}}{\Gamma(p_2 + 1)} \right) \left( P |\beta_1| H^{p_1 H_{RL} P_1} (1)(b) + Q |\beta_2| H^{p_2 H_{RL} P_1} (1)(b) \right) + \frac{1}{\Omega} \left( \frac{\alpha_4}{\alpha_3} |\Omega_2| + \frac{\alpha_1}{\alpha_2} |\Omega_1| \frac{(b - a)^{p_2}}{\Gamma(p_2 + 1)} \right) \\
\times \left( Q |\beta_3| H^{RL P_2 H_{RL} P_2} (1)(b) + P |\beta_4| H^{RL P_1} (1)(b) \right) + Q^{RL P_2 H_{RL} P_2} (1)(b) \\
= M_3 P + M_4 Q \leq \delta.
\]

That yields \( (K_{1,1}, K_{2,1}) x + (K_{1,2}, K_{2,2}) y \in B_\delta \). To show that operator \( (K_{1,2}, K_{2,2}) \) is a contraction mapping satisfying condition (iii) of Lemma 5, for \( (x_1, y_1), (x_2, y_2) \in B_\delta \), we have
\[ |K_{1,2}(x_1, y_1)(t) - K_{1,2}(x_2, y_2)(t)| \]
\[ \leq H^{p_1 RL} I^{p_1} |f_{x_1 y_1} - f_{x_2 y_2}|(t) \]
\[ \leq (m_1 \|x_1 - x_2\| + m_2 \|y_1 - y_2\|) H^{p_1 RL} I^{p_1}(b) \]
\[ \leq (m_1 + m_2) H^{p_1 RL} I^{p_1}(1)(b)(\|x_1 - x_2\| + \|y_1 - y_2\|), \quad (29) \]

and

\[ |K_{2,2}(x_1, y_1)(t) - K_{2,2}(x_2, y_2)(t)| \]
\[ \leq RL I^{p_2 H} I^{p_2} |\delta_{x_1 y_1} - \delta_{x_2 y_2}|(b) \]
\[ \leq (n_1 + n_2) RL I^{p_2 H} I^{p_2}(1)(b)(\|x_1 - x_2\| + \|y_1 - y_2\|). \quad (30) \]

It follows Form (29) and (30) that

\[ \| (K_{1,2}, K_{2,2})(x_1, y_1) - (K_{1,2}, K_{2,2})(x_2, y_2) \| \]
\[ \leq \left( (m_1 + m_2) H^{p_1 RL} I^{p_1}(1)(b) + (n_1 + n_2) RL I^{p_2 H} I^{p_2}(1)(b) \right)(\|x_1 - x_2\| + \|y_1 - y_2\|), \]

which is a contraction by inequality in (28). Therefore, condition (iii) of Lemma 5 is satisfied. Finally we show that operator \((K_{1,1}, K_{2,1})\) satisfied the condition (ii) of Lemma 5. By applying the continuity of functions \(f, g\) on \([a, b] \times \mathbb{R} \times \mathbb{R}\), we can conclude that operator \((K_{1,1}, K_{2,1})\) is continuous. For each \((x, y) \in B_\delta\), one has

\[ \frac{|K_{1,1}(x, y)(t)|}{|\Omega|} \]
\[ \leq \frac{1}{|\Omega|} \left( |\Omega_3| + |\Omega_4| \frac{(\log(b/a))^{p_1}}{\Gamma(q_1 + 1)} \right) \left( P |\beta_1| H^{p_1 RL} I^{p_1}(1)(b) + Q |\beta_2| H^{p_2}(1)(b) \right) \]
\[ + \frac{1}{|\Omega|} \left( |\Omega_1| + |\Omega_2| \frac{(\log(b/a))^{p_2}}{\Gamma(q_2 + 1)} \right) \left( Q |\beta_3| RL I^{p_2 H} I^{p_2}(1)(b) + P |\beta_4| RL I^{p_1}(1)(b) \right) \]
\[ := P^*, \]

and

\[ \frac{|K_{2,1}(x, y)(t)|}{|\Omega|} \]
\[ \leq \frac{1}{|\Omega|} \left( \frac{|a_4|}{|a_3|} |\Omega_4| + \frac{|a_1|}{|a_2|} |\Omega_3| \frac{(b-a)^{p_2}}{\Gamma(p_2 + 1)} \right) \left( P |\beta_1| H^{p_1 RL} I^{p_1}(1)(b) + Q |\beta_2| H^{p_2}(1)(b) \right) \]
\[ + \frac{1}{|\Omega|} \left( \frac{|a_4|}{|a_3|} |\Omega_2| + \frac{|a_1|}{|a_2|} |\Omega_1| \frac{(b-a)^{p_2}}{\Gamma(p_2 + 1)} \right) \left( Q |\beta_3| RL I^{p_2 H} I^{p_2}(1)(b) + P |\beta_4| RL I^{p_1}(1)(b) \right) \]
\[ := Q^*. \]

Then, we obtain the following fact

\[ \| (K_{1,1}, K_{2,1})(x, y) \| \leq P^* + Q^*, \]

which implies that set \((K_{1,1}, K_{2,1})B_\delta\) is uniformly bounded. Next, we show that set \((K_{1,1}, K_{2,1})B_\delta\) is equicontinuous. For \(\tau_1, \tau_2 \in [a, b]\), such that \(\tau_1 < \tau_2\), and for any \((x, y) \in B_\delta\), we prove that
Consider the following coupled system of sequential Caputo and Hadamard fractional differential equations with coupled separated boundary conditions:

\[
|\mathcal{K}_{1,1}(x,y)(\tau_2) - \mathcal{K}_{1,1}(x,y)(\tau_1)| \\
\leq \left( \frac{|\Omega_4|}{\Omega_1 \Gamma(q_1 + 1)} \right) |\log(\tau_2/a)\log(\tau_1/a)| \\
\times \left( |\beta_1| H p_1 q_1 RL P_{q_1} f_{x,y}(b) + |\beta_2| H p_2 q_2 RL P_{q_2} g_{x,y}(b) \right) \\
+ \left( \frac{|\Omega_2|}{\Omega_1 \Gamma(q_1 + 1)} \right) |\log(\tau_2/a)| |\log(\tau_1/a)| \\
\times \left( |\beta_3| RL P_{q_2} H p_2 q_2 RL P_{q_2} g_{x,y}(b) + |\beta_4| RL P_{q_1} f_{x,y}(b) \right).
\]

Indeed, we can show that

\[
|\mathcal{K}_{2,1}(x,y)(\tau_2) - \mathcal{K}_{2,1}(x,y)(\tau_1)| \\
\leq \left( \frac{|\alpha_1||\Omega_3|}{|\alpha_2||\Omega_2 \Gamma(p_2 + 1)} \right) |\tau_2 - a|^{p_2} - (\tau_1 - a)^{p_2} \\
\times \left( P|\beta_1| H q_1 RL P_{q_1} f_{1}(b) + Q|\beta_2| H q_2 RL P_{q_2} f_{1}(b) \right) \\
+ \left( \frac{|\alpha_1||\Omega_1|}{|\alpha_2||\Omega_2 \Gamma(p_2 + 1)} \right) |\tau_2 - a|^{p_2} - (\tau_1 - a)^{p_2} \\
\times \left( Q|\beta_3| RL P_{q_2} H p_2 q_2 RL P_{q_2} f_{1}(b) + P|\beta_4| RL P_{q_1} f_{1}(b) \right).
\]

Thus, \(|(K_{1,1}, K_{2,1})(x,y)(\tau_2) - (K_{1,1}, K_{2,1})(x,y)(\tau_1)|\) tends to zero as \(\tau_1 \to \tau_2\). Therefore, set \((K_{1,1}, K_{2,1})B_\delta\) is equicontinuous. By applying the Arzelà–Ascoli theorem, operator \((K_{1,1}, K_{2,1})\) is compact on \(B_\delta\). By application of Lemma 5, there exists \(z = (z_1, z_2) \in B_\delta\) such that \(z = (K_{1,1}, K_{2,1})z + (K_{1,2}, K_{2,2})z\). Therefore, Problem (2) has at least one solution on \([a,b]\). This completes the proof. \(\square\)

**Example 1.** Consider the following coupled system of sequential Caputo and Hadamard fractional differential equations with coupled separated boundary conditions:

\[
\begin{align*}
C D^{\delta} H D^{\delta} x(t) &= f(t, x(t), y(t)), \quad t \in [2,5], \\
H D^{\delta} C D^{\delta} y(t) &= g(t, x(t), y(t)), \quad t \in [2,5], \\
x(2) + 0.2 C D^{\delta} y(2) &= 0, \quad 0.42 x(5) + 0.5 C D^{\delta} y(5) = 0, \\
0.4 y(2) - 0.17 H D^{\delta} x(2) &= 0, \quad 0.46 y(5) + 3.5 H D^{\delta} x(5) = 0.
\end{align*}
\tag{31}
\]

Here \(p_1 = 1/2, p_2 = 3/4, q_1 = 1/3, q_2 = 2/3, a = 2, b = 5, \alpha_1 = 1, \alpha_2 = 0.2, \alpha_3 = 0.4, \alpha_4 = -0.17, \beta_1 = 0.42, \beta_2 = 0.5, \beta_3 = 0.46, \beta_4 = 0.35\). From the given information, we find that \(H^{1/3} RL^{1/2}(1)(5) = 10.2601, RL^{3/4} H^{2/3}(1)(5) = 3.7354\) and \(|\Omega| = 2.2846\), which lead to \(M_1 = 20.3479, M_2 = 3.5125, M_3 = 10.1477, M_4 = 9.6538\).
(i) Let two nonlinear functions \( f, g : [2, 5] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) be given by

\[
\begin{align*}
  f(t, x, y) &= \frac{\sin^2(t/\pi)}{(14 + t)^2} \left( \frac{x^2 + 2|x|}{|x| + 1} \right) + \frac{|y|}{t + 1} + \frac{1}{2} t \tag{32} \\
  g(t, x, y) &= \frac{\sin(|x|)}{156 + t^3} + \frac{\tan^{-1}(y)e^{-\sin t}}{6t^4 + 2} + \frac{3}{4} t. \tag{33}
\end{align*}
\]

Since

\[
|f(t, x_1, y_1) - f(t, x_2, y_2)| \leq \frac{1}{128} |x_1 - x_2| + \frac{1}{148} |y_1 - y_2|
\]

and

\[
|g(t, x_1, y_1) - g(t, x_2, y_2)| \leq \frac{1}{164} |x_1 - x_2| + \frac{1}{98} |y_1 - y_2|
\]

we obtain \((M_1 + M_3)(1/128 + 1/148) + (M_2 + M_4)(1/164 + e/98) = 0.8867 < 1\). By the conclusion of Theorem 1, Problem (31) with Problems (32) and (33) have a unique solution on \([2, 5]\).

(ii) Now consider functions \( f, g : [2, 5] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) defined by

\[
\begin{align*}
  f(t, x, y) &= e^{-t} + \frac{x^4\sin^2 t}{33(1 + |x|^3)} + \frac{|y|^3 \cos^2 t}{33(1 + y^4)}, \tag{34} \\
  g(t, x, y) &= \frac{2}{(t + 1)} + \frac{\sin x}{25(t + 6)} + \frac{e^{-t^2} \tan^{-1} y}{90 + 25t^2}. \tag{35}
\end{align*}
\]

It is easy to verify that \(|f(t, x, y)| \leq (1/11) + (1/33)|x| + (1/33)|y|\) and \(|g(t, x, y)| \leq (2/3) + (1/200)|x| + (1/190)|y|\). As \((M_1 + M_3)v_1 + (M_2 + M_4)v_2 = 0.9938 < 1\) and \((M_1 + M_3)u_1 + (M_2 + M_4)u_2 = 0.9973 < 1\), by applying Theorem 2, we get that System (31) with Systems (34) and (35) have at least one solution on \([2, 5]\).

(iii) Define functions \( f, g : [2, 5] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) by

\[
\begin{align*}
  f(t, x, y) &= \frac{1}{4} \sin^2 t + \frac{|x| e^{-(t-2)}}{40(1 + |x|)} + \frac{1}{40} \sin y, \tag{36} \\
  g(t, x, y) &= \frac{1}{2} \cos^2 t + \frac{1}{16} \tan^{-1} x + \frac{|y|}{16(1 + |y|)}. \tag{37}
\end{align*}
\]

We have \(|f(t, x, y)| \leq 3/10\), \(|g(t, x, y)| \leq 5/8\) and

\[
|f(t, x_1, y_1) - f(t, x_2, y_2)| \leq \frac{1}{40} |x_1 - x_2| + \frac{1}{40} |y_1 - y_2|
\]

and

\[
|g(t, x_1, y_1) - g(t, x_2, y_2)| \leq \frac{1}{16} |x_1 - x_2| + \frac{1}{16} |y_1 - y_2|
\]

Then we obtain \(\left( (m_1 + n_2)^{H^{RL}} p_1 (1)(5) + (n_1 + n_2)^{RL} p_2 (1)(5) \right) = 0.9799 < 1\). Using Theorem 3, the problem (31) with (36) and (37) has at least one solution on \([2, 5]\). Observe that the inequality \((M_1 + M_3)(m_1 + m_2) + (M_2 + M_4)(n_1 + n_2) = 3.1706 > 1\) and, thus, Condition (23) is not satisfied. Therefore, Theorem 1 cannot be applied for this case.

4. Conclusions

We have proven the existence and uniqueness of solutions for a coupled system of sequential Caputo and Hadamard fractional differential equations with coupled separated boundary conditions by applying the Banach fixed-point theorem, Leray–Schauder nonlinear alternative, and Krasnoselakki fixed-point theorem. We also provided examples to clarify our results.
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