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Coupled Systems of Sequential Caputo and Hadamard Fractional Differential Equations with Coupled Separated Boundary Conditions

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Abstract: This paper studies the existence and uniqueness of solutions for a new coupled system of nonlinear sequential Caputo and Hadamard fractional differential equations with coupled separated boundary conditions, which include as special cases the well-known symmetric boundary conditions. Banach's contraction principle, Leray–Schauder's alternative, and Krasnoselskii's fixed-point theorem were used to derive the desired results, which are well-illustrated with examples.

Keywords: Caputo fractional derivative; Hadamard fractional derivative; coupled system; separated boundary conditions; existence

MSC: 34A08; 34B10

1. Introduction

Fractional differential equations appear in the mathematical modeling of many real-world phenomena occurring in engineering and scientific disciplines, for instance, see References [1–6]. Mathematical models based on fractional-order integral and differential operators yield more insight into the characteristics of the associated phenomena, as such operators are nonlocal in nature, in contrast to classical ones. In particular, coupled systems of fractional-order differential equations have received great attention in view of their great utility in handling and comprehending practical issues, such as the synchronization of chaotic systems [7,8], anomalous diffusion [9], and ecological effects [10]. For recent theoretical results on the topic, we refer the reader to a series of papers [11–18] and the references cited therein.

Recently, in Reference [19], the authors discussed existence and the uniqueness of solutions for sequential Caputo and Hadamard fractional differential equations subject to separated boundary conditions as

$$\begin{cases} {}^{C}D^{p}({}^{H}D^{q}x)(t) = f(t, x(t)), \qquad t \in [a, b], \\ \alpha_1 x(a) + \alpha_2 ({}^{H}D^{q}x)(a) = 0, \quad \beta_1 x(b) + \beta_2 ({}^{H}D^{q}x)(b) = 0, \end{cases} \tag{1}$$

where ${}^{C}D^{p}$ and ${}^{H}D^{q}$ are the Caputo and Hadamard fractional derivatives of orders p and q, respectively, $0 < p, q \le 1$, starting at a point $a > 0$, $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ is a continuous function and given constants $\alpha_i, \beta_i \in \mathbb{R}$, $i = 1, 2$.

In this paper, we established the existence criteria for a coupled system of sequential Caputo and Hadamard fractional differential equations with coupled separated boundary conditions as:

$$\begin{cases} {}^C D^{p_1} {}^H D^{q_1} x(t) = f(t, x(t), y(t)), \qquad t \in [a,b], \\ {}^H D^{q_2} {}^C D^{p_2} y(t) = g(t, x(t), y(t)), \qquad t \in [a,b], \\ \alpha_1 x(a) + \alpha_2 {}^C D^{p_2} y(a) = 0, \quad \beta_1 x(b) + \beta_2 {}^C D^{p_2} y(b) = 0, \\ \alpha_3 y(a) + \alpha_4 {}^H D^{q_1} x(a) = 0, \quad \beta_3 y(b) + \beta_4 {}^H D^{q_1} x(b) = 0, \end{cases} \tag{2}$$

where ${}^C D^{p_i}$ and ${}^H D^{q_i}$ are notations of the Caputo and Hadamard fractional derivatives of orders p_i and q_i, respectively, $0 < p_i, q_i \leq 1$, $i = 1, 2$, the nonlinear continuous functions $f, g : [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $a > 0$, $\alpha_i \in \mathbb{R} \setminus \{0\}$, $\beta_i \in \mathbb{R}$, $i = 1, \ldots, 4$. Meanwhile, the different definitions of Caputo and Hadamard fractional derivatives that appeared in System (2) are proposed to study the existence theory of solutions of a fractional differential system using a variety of fixed-point theorems. A special case, when $p_i = q_i = 1$, $i = 1, 2$, in differential Equation (2) can be presented as:

$$\begin{cases} tx'' + x' = f(t, x, y), \qquad ty'' = g(t, x, y), \quad t \in [a,b], \\ \alpha_1 x(a) + \alpha_2 y'(a) = 0, \quad \beta_1 x(b) + \beta_2 y'(b) = 0, \\ \alpha_3 y(a) + \alpha_4 (tx')(a) = 0, \quad \beta_3 y(b) + \beta_4 (tx')(b) = 0, \end{cases} \tag{3}$$

which is mixed type of ordinary differential equations and boundary conditions.

The rest of this paper is organized as follows: Section 2 aims to recall basic definitions and lemmas used in this paper. Section is devoted to the main results concerning the existence and uniqueness of solutions for for System (2). The Leray–Schauder alternative and Krasnoselskii's fixed-point theorem were applied to prove existence, while the uniqueness result was obtained via the Banach contraction mapping principle. Some illustrative examples are presented in Section 4.

2. Preliminaries

To ensure that readers can easily understand the results, we recall some notations and definitions of fractional calculus [3,20].

Definition 1. *The Caputo fractional derivative of order q for an at least n-times differentiable function $g : [a, \infty) \to \mathbb{R}$, starting at a point $a > 0$, is defined as:*

$${}^C D^q g(t) = \frac{1}{\Gamma(n-q)} \int_a^t (t-s)^{n-q-1} g^{(n)}(s) ds, \;\; n-1 < q < n, \;\; n = [q]+1,$$

where $[q]$ denotes the integer part of the real number q.

Definition 2. *The Riemann–Liouville fractional integral of order q of a function $g : [a, \infty) \to \mathbb{R}$, $a > 0$, is defined as:*

$${}^{RL} I^q g(t) = \frac{1}{\Gamma(q)} \int_a^t \frac{g(s)}{(t-s)^{1-q}} ds, \;\; q > 0,$$

provided the right side of an integral exists.

Definition 3. *The Caputo-type Hadamard fractional derivative of order q for an at least n-times delta differentiable function $g : [a, \infty) \to \mathbb{R}$, starting at a point $a > 0$, is defined as*

$${}^H D^q g(t) = \frac{1}{\Gamma(n-q)} \int_a^t \left(\log \frac{t}{s} \right)^{n-q-1} \delta^n g(s) \frac{ds}{s}, \;\; n-1 < q < n, \;\; n = [q]+1,$$

where the delta derivative is defined by $\delta = t\frac{d}{dt}$ *and the natural logarithm* $\log(\cdot) = \log_e(\cdot)$.

Definition 4. *The Hadamard fractional integral of order q is defined as*

$$^{H}I^{q}g(t) = \frac{1}{\Gamma(\alpha)}\int_a^t \left(\log\frac{t}{s}\right)^{q-1} g(s)\frac{ds}{s}, \ \ q > 0, \ a > 0,$$

provided the integral exists.

Lemma 1. *The general solution of homogeneous fractional differential equation* $^{C}D^{q}u(t) = 0, q > 0$ *is given by*

$$u(t) = c_0 + c_1(t-a) + \ldots + c_{n-1}(t-a)^{n-1},$$

where $c_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, n-1$ $(n = [q] + 1)$.

In view of Lemma 1, we have

$$^{RL}I^{q\,C}D^{q}u(t) = u(t) + c_0 + c_1(t-a) + \ldots + c_{n-1}(t-a)^{n-1}, \tag{4}$$

for some constants $c_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, n-1$ $(n = [q] + 1)$.

Lemma 2 ([21])**.** *Let* $AC^n_\delta[a,b] = \{g : [a,b] \to \mathbb{C} : \delta^{n-1}g(t) \in AC[a,b]\}$ *and* $u \in AC^n_\delta[a,b]$ *or* $C^n_\delta[a,b]$ *and* $q \in \mathbb{C}$. *Then, the following formula holds*

$$^{H}I^{q}(^{H}D^{q})u(t) = u(t) - \sum_{k=0}^{n-1} c_k \left(\log(t/a)\right)^k,$$

where $c_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, n-1$ $(n = [q] + 1)$.

Next, we transform Problem (2) to integral equations by using a linear variant of Problem (2). For convenience, we put constants

$$\Omega_1 = \frac{\beta_1(\log(b/a))^{q_1}}{\Gamma(q_1+1)}, \quad \Omega_2 = \beta_1 - \frac{\alpha_1}{\alpha_2}\beta_2,$$
$$\Omega_3 = \beta_4 - \frac{\alpha_4}{\alpha_3}\beta_3, \quad \Omega_4 = \frac{\alpha_1\beta_3(b-a)^{p_2}}{\alpha_2\Gamma(p_2+1)},$$

and $\Omega = \Omega_1\Omega_4 + \Omega_2\Omega_3 \neq 0$.

Lemma 3. *Let* $\omega, \phi \in C([a,b],\mathbb{R})$. *Then, the linear system of sequential Caputo and Hadamard fractional differential equations with coupled separated boundary value problem*

$$\begin{cases} ^{C}D^{p_1\,H}D^{q_1}x(t) = \omega(t), & t \in [a,b], \\ ^{H}D^{q_2\,C}D^{p_2}y(t) = \phi(t), & t \in [a,b], \\ \alpha_1 x(a) + \alpha_2{}^{C}D^{p_2}y(a) = 0, & \beta_1 x(b) + \beta_2{}^{C}D^{p_2}y(b) = 0, \\ \alpha_3 y(a) + \alpha_4{}^{H}D^{q_1}x(a) = 0, & \beta_3 y(b) + \beta_4{}^{H}D^{q_1}x(b) = 0, \end{cases} \tag{5}$$

can be written as integral equations

$$x(t) = -\frac{1}{\Omega}\left(\Omega_3 + \Omega_4 \frac{(\log(t/a))^{q_1}}{\Gamma(q_1+1)}\right)\left(\beta_1 {}^{H}I^{q_1}\,{}^{RL}I^{p_1}\omega(b) + \beta_2 {}^{H}I^{q_2}\phi(b)\right) + \frac{1}{\Omega}\left(\Omega_1 - \Omega_2 \frac{(\log(t/a))^{q_1}}{\Gamma(q_1+1)}\right)\left(\beta_3 {}^{RL}I^{p_2}\,{}^{H}I^{q_2}\phi(b) + \beta_4 {}^{RL}I^{p_1}\omega(b)\right) + {}^{H}I^{q_1}\,{}^{RL}I^{p_1}\omega(t), \tag{6}$$

and

$$y(t) = \frac{1}{\Omega}\left(\frac{\alpha_4}{\alpha_3}\Omega_4 + \frac{\alpha_1}{\alpha_2}\Omega_3 \frac{(t-a)^{p_2}}{\Gamma(p_2+1)}\right)\left(\beta_1 {}^{H}I^{q_1}\,{}^{RL}I^{p_1}\omega(b) + \beta_2 {}^{H}I^{q_2}\phi(b)\right) + \frac{1}{\Omega}\left(\frac{\alpha_4}{\alpha_3}\Omega_2 - \frac{\alpha_1}{\alpha_2}\Omega_1 \frac{(t-a)^{p_2}}{\Gamma(p_2+1)}\right)\left(\beta_3 {}^{RL}I^{p_2}\,{}^{H}I^{q_2}\phi(b) + \beta_4 {}^{RL}I^{p_1}\omega(b)\right) + {}^{RL}I^{p_2}\,{}^{H}I^{q_2}\phi(t). \tag{7}$$

Proof. Taking the Riemann–Liouville fractional integral of order p_1, $p_1 \in (0,1]$, to the first equation of Problem (5) and applying Problem (4), we obtain for $t \in [a,b]$

$${}^{H}D^{q_1}x(t) = c_1 + {}^{RL}I^{p_1}\omega(t), \quad c_1 \in \mathbb{R}. \tag{8}$$

In the above equation, we apply the Hadamard fractional integral of order q_1, $q_1 \in (0,1]$, with (4) for $t \in [a,b]$ and obtain

$$x(t) = c_2 + c_1 \frac{(\log(t/a))^{q_1}}{\Gamma(q_1+1)} + {}^{H}I^{q_1}\,{}^{RL}I^{p_1}\omega(t), \quad c_2 \in \mathbb{R}. \tag{9}$$

Considering the second equation of Problem (5), and by using the Hadamard fractional integral of order q_2, we get

$${}^{C}D^{p_2}y(t) = c_3 + {}^{H}I^{q_2}\phi(t), \quad c_3 \in \mathbb{R}. \tag{10}$$

By taking the Riemann–Liouville fractional integral operator of order p_2, we have

$$y(t) = c_4 + c_3 \frac{(t-a)^{p_2}}{\Gamma(p_2+1)} + {}^{RL}I^{p_2}\,{}^{H}I^{q_2}\phi(t), \quad c_4 \in \mathbb{R}. \tag{11}$$

In particular, for $t = a$ in Equations (9) and (10), and applying the first condition of Problem (5), one has

$$\alpha_1 c_2 + \alpha_2 c_3 = 0. \tag{12}$$

For $t = b$ in Equations (9) and (10), it obtains by applying the second condition of Problem (5) as

$$\beta_1 c_1 \frac{(\log(b/a))^{q_1}}{\Gamma(q_1+1)} + \beta_1 c_2 + \beta_2 c_3 = -\beta_1 {}^{H}I^{q_1}\,{}^{RL}I^{p_1}\omega(b) - \beta_2 {}^{H}I^{q_2}\phi(b) := \Omega_5. \tag{13}$$

Substituting $t = a$ in Equations (8) and (11) and applying the third condition of Problem (5), it leads to

$$\alpha_4 c_1 + \alpha_3 c_4 = 0. \tag{14}$$

The fourth condition of Problem (5) can be applied when $t = b$ in Equations (8) and (11) as

$$\beta_4 c_1 + \beta_3 c_3 \frac{(b-a)^{p_2}}{\Gamma(p_2+1)} + \beta_3 c_4 = -\beta_3 {}^{RL}I^{p_2}\,{}^{H}I^{q_2}\phi(b) - \beta_4 {}^{RL}I^{p_1}\omega(b) := \Omega_6. \tag{15}$$

Reduce the above Equations (12)–(15) in a system of constants by

$$\Omega_1 c_1 + \Omega_2 c_2 = \Omega_5, \quad \Omega_3 c_1 - \Omega_4 c_2 = \Omega_6.$$

Computing for constants c_1 and c_2 and substituting it into Equations (12) and (14) for c_3 and c_4, we have

$$c_1 = \frac{\Omega_4}{\Omega}\Omega_5 + \frac{\Omega_2}{\Omega}\Omega_6, \quad c_2 = \frac{\Omega_3}{\Omega}\Omega_5 - \frac{\Omega_1}{\Omega}\Omega_6,$$
$$c_3 = -\frac{\alpha_1\Omega_3}{\alpha_2\Omega}\Omega_5 + \frac{\alpha_1\Omega_1}{\alpha_2\Omega}\Omega_6, \quad c_4 = -\frac{\alpha_4\Omega_4}{\alpha_3\Omega}\Omega_5 - \frac{\alpha_4\Omega_2}{\alpha_3\Omega}\Omega_6.$$

Substituting all obtained constants in Equations (9) and (11), we obtain integral Equations (6) and (7). By direct computation we can obtain the the converse. The proof is completed. □

Remark 1. *System (5) is well-defined because four constants $\alpha_i \in \mathbb{R} \setminus \{0\}$, $i = 1,2,3,4$, make meaningful property for Caputo and Hadamard (Caputo-type) fractional derivatives, which lead to solve the system of linear equations.*

3. Main Results

Let $\mathcal{C} = C([a,b],\mathbb{R})$, $a > 0$, be the Banach space of all continuous functions form $[a,b]$ to $\mathbb{R}$. Space $X = \{x(t) : x(t) \in C^2([a,b],\mathbb{R})\}$ endowed with the norm $\|x\| = \sup\{|x(t)|, t \in [a,b]\}$ is a Banach space. In addition, let $Y = \{y(t) : y(t) \in C^2([a,b],\mathbb{R})\}$ with the norm $\|y\| = \sup\{|y(t)|, t \in [a,b]\}$. It is obvious that product space $(X \times Y, \|(x,y)\|)$ is a Banach space with the norm $\|(x,y)\| = \|x\| + \|y\|$.

Now, for brevity, we use the notations:

$$h_{x,y}(t) = h(t, x(t), y(t)), \;\; h \in \{f, g\}, \tag{16}$$

$${}^{H}I^{q\,RL}I^{p} f_{x,y}(\phi) = \frac{1}{\Gamma(q)\Gamma(p)} \int_a^{\phi} \int_a^{s} \left(\log\frac{\phi}{s}\right)^{q-1} (s-r)^{p-1} f_{x,y}(r) dr \frac{ds}{s}, \tag{17}$$

and

$${}^{RL}I^{p\,H}I^{q} f_{x,y}(\phi) = \frac{1}{\Gamma(p)\Gamma(q)} \int_a^{\phi} \int_a^{s} (\phi - s)^{p-1} \left(\log\frac{s}{r}\right)^{q-1} f_{x,y}(r) \frac{dr}{r} ds, \tag{18}$$

where $\phi \in \{t, b\}$. We also use this one for a single fractional integral operator of the Riemann–Liouville and Hadamard types of orders p_1 and q_2, respectively.

In view of Lemma 3, we define two operators $\mathcal{K} : X \times Y \to X \times Y$ by

$$\mathcal{K}(x,y)(t) = \begin{pmatrix} \mathcal{K}_1(x,y)(t) \\ \mathcal{K}_2(x,y)(t) \end{pmatrix},$$

where

$$\begin{aligned} \mathcal{K}_1(x,y)(t) \;\; = \;\; & -\frac{1}{\Omega}\left(\Omega_3 + \Omega_4 \frac{(\log(t/a))^{q_1}}{\Gamma(q_1+1)}\right)\left(\beta_1 {}^{H}I^{q_1\,RL}I^{p_1}(f_{x,y})(b) + \beta_2 {}^{H}I^{q_2}(g_{x,y})(b)\right) \\ & + \frac{1}{\Omega}\left(\Omega_1 - \Omega_2 \frac{(\log(t/a))^{q_1}}{\Gamma(q_1+1)}\right)\left(\beta_3 {}^{RL}I^{p_2\,H}I^{q_2}(g_{x,y})(b) + \beta_4 {}^{RL}I^{p_1}(f_{x,y})(b)\right) \\ & + {}^{H}I^{q_1\,RL}I^{p_1}(f_{x,y})(t), \end{aligned}$$

and

$$\begin{aligned}
\mathcal{K}_2(x,y)(t) &= \frac{1}{\Omega}\left(\frac{\alpha_4}{\alpha_3}\Omega_4+\frac{\alpha_1}{\alpha_2}\Omega_3\frac{(t-a)^{p_2}}{\Gamma(p_2+1)}\right)\left(\beta_1{}^{H}I^{q_1\,RL}I^{p_1}(f_{x,y})(b)+\beta_2{}^{H}I^{q_2}(g_{x,y})(b)\right)\\
&\quad+\frac{1}{\Omega}\left(\frac{\alpha_4}{\alpha_3}\Omega_2-\frac{\alpha_1}{\alpha_2}\Omega_1\frac{(t-a)^{p_2}}{\Gamma(p_2+1)}\right)\left(\beta_3{}^{RL}I^{p_2\,H}I^{q_2}(g_{x,y})(b)+\beta_4{}^{RL}I^{p_1}(f_{x,y})(b)\right)\\
&\quad+{}^{RL}I^{p_2\,H}I^{q_2}(g_{x,y})(t).
\end{aligned}$$

For computational convenience, we set

$$\begin{aligned}
M_1 &= \frac{|\beta_1|}{|\Omega|}\left(|\Omega_3|+|\Omega_4|\frac{(\log(b/a))^{q_1}}{\Gamma(q_1+1)}\right){}^{H}I^{q_1\,RL}I^{p_1}(1)(b)\\
&\quad+\frac{|\beta_4|}{|\Omega|}\left(|\Omega_1|+|\Omega_2|\frac{(\log(b/a))^{q_1}}{\Gamma(q_1+1)}\right)\frac{(b-a)^{p_1}}{\Gamma(p_1+1)}+{}^{H}I^{q_1\,RL}I^{p_1}(1)(b), \qquad (19)
\end{aligned}$$

$$\begin{aligned}
M_2 &= \frac{|\beta_2|}{|\Omega|}\left(|\Omega_3|+|\Omega_4|\frac{(\log(b/a))^{q_1}}{\Gamma(q_1+1)}\right)\frac{(\log(b/a))^{q_2}}{\Gamma(q_2+1)}\\
&\quad+\frac{|\beta_3|}{|\Omega|}\left(|\Omega_1|+|\Omega_2|\frac{(\log(b/a))^{q_1}}{\Gamma(q_1+1)}\right){}^{RL}I^{p_2\,H}I^{q_2}(1)(b), \qquad (20)
\end{aligned}$$

$$\begin{aligned}
M_3 &= \frac{|\beta_1|}{|\Omega|}\left(\frac{|\alpha_4|}{|\alpha_3|}|\Omega_4|+\frac{|\alpha_1|}{|\alpha_2|}|\Omega_3|\frac{(b-a)^{p_2}}{\Gamma(p_2+1)}\right)\left({}^{H}I^{q_1\,RL}I^{p_1}(1)(b)\right)\\
&\quad+\frac{|\beta_4|}{|\Omega|}\left(\frac{|\alpha_4|}{|\alpha_3|}|\Omega_2|+\frac{|\alpha_1|}{|\alpha_2|}|\Omega_1|\frac{(b-a)^{p_2}}{\Gamma(p_2+1)}\right)\frac{(b-a)^{p_1}}{\Gamma(p_1+1)}, \qquad (21)
\end{aligned}$$

$$\begin{aligned}
M_4 &= \frac{|\beta_2|}{|\Omega|}\left(\frac{|\alpha_4|}{|\alpha_3|}|\Omega_4|+\frac{|\alpha_1|}{|\alpha_2|}|\Omega_3|\frac{(b-a)^{p_2}}{\Gamma(p_2+1)}\right)\frac{(\log(b/a))^{q_2}}{\Gamma(q_2+1)}\\
&\quad+\frac{|\beta_3|}{|\Omega|}\left(\frac{|\alpha_4|}{|\alpha_3|}|\Omega_2|+\frac{|\alpha_1|}{|\alpha_2|}|\Omega_1|\frac{(b-a)^{p_2}}{\Gamma(p_2+1)}\right){}^{RL}I^{p_2\,H}I^{q_2}(1)(b)+{}^{RL}I^{p_2\,H}I^{q_2}(1)(b). \qquad (22)
\end{aligned}$$

Note that all information of Problem (2) is contained in constants M_i, $i = 1, 2, 3, 4$, which are used to establish the following existence theorems. Banach's contraction mapping principle is applied in the first result to prove the existence and uniqueness of solutions of System (2).

Theorem 1. *Suppose that $f, g : [a,b]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are continuous functions. In addition, we assume that:*

(H_1) *there exist constants $m_i, n_i, i = 1,2$, such that for all $t\in[a,b]$ and $x_i, y_i\in\mathbb{R}, i = 1,2$*

$$|f(t,x_1,y_1)-f(t,x_2,y_2)|\le m_1|x_1-x_2|+m_2|y_1-y_2|$$

and

$$|g(t,x_1,y_1)-g(t,x_2,y_2)|\le n_1|x_1-x_2|+n_2|y_1-y_2|.$$

Then, System (2) has a unique solution on $[a,b]$, if

$$(M_1+M_3)(m_1+m_2)+(M_2+M_4)(n_1+n_2)<1. \qquad (23)$$

Proof. Define $\sup_{t\in[a,b]} f(t,0,0) = N_1 < \infty$ and $\sup_{t\in[a,b]} g(t,0,0) = N_2 < \infty$, such that

$$r > \frac{(M_1+M_3)N_1+(M_2+M_4)N_2}{1-[(M_1+M_3)(m_1+m_2)+(M_2+M_4)(n_1+n_2)]}.$$

Now, we show that the set $\mathcal{K}B_r \subset B_r$, where $B_r = \{(x,y) \in X \times Y : \|(x,y)\| \leq r\}$. For $(x,y) \in B_r$, we have that

$$\begin{aligned}
& |\mathcal{K}_1(x,y)(t)| \\
\leq\ & \frac{1}{|\Omega|}\left(|\Omega_3| + |\Omega_4|\frac{(\log(t/a))^{q_1}}{\Gamma(q_1+1)}\right)\big(|\beta_1|\,{}^{H}I^{q_1\,RL}I^{p_1}|f_{x,y}|(b) \\
& + |\beta_2|\,{}^{H}I^{q_2}|g_{x,y}|(b)\big) + \frac{1}{|\Omega|}\left(|\Omega_1| + |\Omega_2|\frac{(\log(t/a))^{q_1}}{\Gamma(q_1+1)}\right) \\
& \times\Big(|\beta_3|\,{}^{RL}I^{p_2\,H}I^{q_2}|g_{x,y}|(b) + |\beta_4|\,{}^{RL}I^{p_1}|f_{x,y}|(b)\Big) + {}^{H}I^{q_1\,RL}I^{p_1}|f_{x,y}|(b) \\
\leq\ & \frac{1}{|\Omega|}\left(|\Omega_3| + |\Omega_4|\frac{(\log(b/a))^{q_1}}{\Gamma(q_1+1)}\right)\big(|\beta_1|\,{}^{H}I^{q_1\,RL}I^{p_1}(|f_{x,y}-f_{0,0}| + |f_{0,0}|)(b) \\
& + |\beta_2|\,{}^{H}I^{q_2}(|g_{x,y}-g_{0,0}| + |g_{0,0}|)(b)\big) + \frac{1}{|\Omega|}\left(|\Omega_1| + |\Omega_2|\frac{(\log(b/a))^{q_1}}{\Gamma(q_1+1)}\right) \\
& \times\big(|\beta_3|\,{}^{RL}I^{p_2\,H}I^{q_2}(|g_{x,y}-g_{0,0}| + |g_{0,0}|)(b) + |\beta_4|\,{}^{RL}I^{p_1}(|f_{x,y}-f_{0,0}| \\
& + |f_{0,0}|)(b)\big) + {}^{H}I^{q_1\,RL}I^{p_1}(|f_{x,y}-f_{0,0}| + |f_{0,0}|)(b) \\
\leq\ & \frac{1}{|\Omega|}\left(|\Omega_3| + |\Omega_4|\frac{(\log(b/a))^{q_1}}{\Gamma(q_1+1)}\right)\big(|\beta_1|\,{}^{H}I^{q_1\,RL}I^{p_1}(m_1\|x\| + m_2\|y\| \\
& + N_1)(b) + |\beta_2|\,{}^{H}I^{q_2}(n_1\|x\| + n_2\|y\| + N_2)(b)\big) + \frac{1}{|\Omega|}\Big(|\Omega_1| \\
& + |\Omega_2|\frac{(\log(b/a))^{q_1}}{\Gamma(q_1+1)}\Big)\big(|\beta_3|\,{}^{RL}I^{p_2\,H}I^{q_2}(n_1\|x\| + n_2\|y\| + N_2)(b) \\
& + |\beta_4|\,{}^{RL}I^{p_1}(m_1\|x\| + m_2\|y\| + N_1)(b)\big) + {}^{H}I^{q_1\,RL}I^{p_1}(m_1\|x\| \\
& + m_2\|y\| + N_1)(b) \\
=\ & M_1(m_1\|x\| + m_2\|y\| + N_1) + M_2(n_1\|x\| + n_2\|y\| + N_2) \\
=\ & (M_1m_1 + M_2n_1)\|x\| + (M_1m_2 + M_2n_2)\|y\| + M_1N_1 + M_2N_2 \\
\leq\ & [M_1(m_1+m_2) + M_2(n_1+n_2)]r + M_1N_1 + M_2N_2.
\end{aligned}$$

Hence,

$$\|\mathcal{K}_1(x,y)\| \leq [M_1(m_1+m_2) + M_2(n_1+n_2)]r + M_1N_1 + M_2N_2.$$

By direct computation, we get

$$\begin{aligned}
\mathcal{K}_2(x,y)(t) \leq\ & \frac{1}{|\Omega|}\left(\frac{|\alpha_4|}{|\alpha_3|}|\Omega_4| + \frac{|\alpha_1|}{|\alpha_2|}|\Omega_3|\frac{(b-a)^{p_2}}{\Gamma(p_2+1)}\right)\big(|\beta_1|(m_1\|x\| + m_2\|y\| \\
& + N_1)^{H}I^{q_1\,RL}I^{p_1}(1)(b) + |\beta_2|(n_1\|x\| + n_2\|y\| + N_2)^{H}I^{q_2}(1)(b)\big) \\
& + \frac{1}{|\Omega|}\left(\frac{|\alpha_4|}{|\alpha_3|}|\Omega_2| + \frac{|\alpha_1|}{|\alpha_2|}|\Omega_1|\frac{(b-a)^{p_2}}{\Gamma(p_2+1)}\right)\big(|\beta_3|(n_1\|x\| + n_2\|y\| \\
& + N_2)^{RL}I^{p_2\,H}I^{q_2}(1)(b) + |\beta_4|(m_1\|x\| + m_2\|y\| + N_1)^{RL}I^{p_1}(1)(b)\big) \\
& + (n_1\|x\| + n_2\|y\| + N_2)^{RL}I^{p_2\,H}I^{q_2}(1)(b) \\
=\ & M_3(m_1\|x\| + m_2\|y\| + N_1) + M_4(n_1\|x\| + n_2\|y\| + N_2),
\end{aligned}$$

and

$$\|\mathcal{K}_2(x,y)\| \leq [M_3(m_1+m_2) + M_4(n_1+n_2)]r + M_3N_1 + M_4N_2.$$

Consequently, it follows that

$$\begin{aligned}
\|\mathcal{K}(x,y)\| \leq\ & [M_1(m_1+m_2) + M_2(n_1+n_2)]r + M_1N_1 + M_2N_2 \\
& + [M_3(m_1+m_2) + M_4(n_1+n_2)]r + M_3N_1 + M_4N_2 \leq r,
\end{aligned}$$

which implies $\mathcal{K}B_r \subset B_r$. Next, we show that operator $\mathcal{K}$ is contraction mapping. For any $(x_1, y_1), (x_2, y_2) \in X \times Y$, we obtain

$$\begin{aligned}
& |\mathcal{K}_1(x_1,y_1)(t) - \mathcal{K}_1(x_2,y_2)(t)| \\
\leq\; & \frac{1}{|\Omega|}\left(|\Omega_3| + |\Omega_4|\frac{(\log(t/a))^{q_1}}{\Gamma(q_1+1)}\right)\left(|\beta_1|\, {}^{H}I^{q_1}\,{}^{RL}I^{p_1}|f_{x_1,y_1} - f_{x_2,y_2}|(b)\right. \\
& \left. + |\beta_2|\, {}^{H}I^{q_2}|g_{x_1,y_1} - g_{x_2,y_2}|(b)\right) + \frac{1}{|\Omega|}\left(|\Omega_1| + |\Omega_2|\frac{(\log(t/a))^{q_1}}{\Gamma(q_1+1)}\right) \\
& \times \left(|\beta_3|\, {}^{RL}I^{p_2}\,{}^{H}I^{q_2}|g_{x_1,y_1} - g_{x_2,y_2}|(b) + |\beta_4|\, {}^{RL}I^{p_1}|f_{x_1,y_1} - f_{x_2,y_2}|(b)\right) \\
& + {}^{H}I^{q_1}\,{}^{RL}I^{p_1}|f_{x_1,y_1} - f_{x_2,y_2}|(b) \\
\leq\; & \frac{1}{|\Omega|}\left(|\Omega_3| + |\Omega_4|\frac{(\log(b/a))^{q_1}}{\Gamma(q_1+1)}\right)\left(|\beta_1|\left(m_1\|x_1 - x_2\| + m_2\|y_1 - y_2\|\right)\right. \\
& \times {}^{H}I^{q_1}\,{}^{RL}I^{p_1}(1)(b) + |\beta_2|(n_1\|x_1 - x_2\| + n_2\|y_1 - y_2\|)\,{}^{H}I^{q_2}(1)(b)\Big) \\
& + \frac{1}{|\Omega|}\left(|\Omega_1| + |\Omega_2|\frac{(\log(b/a))^{q_1}}{\Gamma(q_1+1)}\right)\left(|\beta_3|\left(n_1\|x_1 - x_2\| + n_2\|y_1 - y_2\|\right)\right. \\
& \times {}^{RL}I^{p_2}\,{}^{H}I^{q_2}(1)(b) + |\beta_4|\left(m_1\|x_1 - x_2\| + m_2\|y_1 - y_2\|\right){}^{RL}I^{p_1}(1)(b)\Big) \\
& + (m_1\|x_1 - x_2\| + m_2\|y_1 - y_2\|)\,{}^{H}I^{q_1}\,{}^{RL}I^{p_1}(1)(b) \\
=\; & M_1(m_1\|x_1 - x_2\| + m_2\|y_1 - y_2\|) + M_2(n_1\|x_1 - x_2\| + n_2\|y_1 - y_2\|) \\
=\; & (M_1 m_1 + M_2 n_1)\|x_1 - x_2\| + (M_1 m_2 + M_2 n_2)\|y_1 - y_2\|.
\end{aligned}$$

Therefore, we get the following inequality:

$$\|\mathcal{K}_1(x_1,y_1) - \mathcal{K}_1(x_2,y_2)\| \leq M_1(m_1 + m_2) + M_2(n_1 + n_2)\left(\|x_1 - x_2\| + \|y_1 - y_2\|\right). \tag{24}$$

In addition, we obtain

$$\|\mathcal{K}_2(x_1,y_1) - \mathcal{K}_2(x_2,y_2)\| \leq M_3(m_1 + m_2) + M_4(n_1 + n_2)\left(\|x_1 - x_2\| + \|y_1 - y_2\|\right). \tag{25}$$

From Inequalities (24) and (25), it yields

$$\begin{aligned}
\|\mathcal{K}(x_1,y_1) - \mathcal{K}(x_2,y_2)\| \;\leq\; & [(M_1 + M_3)(m_1 + m_2) + (M_2 + M_4)(n_1 + n_2)] \\
& \times \left(\|x_1 - x_2\| + \|y_1 - y_2\|\right).
\end{aligned}$$

As $(M_1 + M_3)(m_1 + m_2) + (M_2 + M_4)(n_1 + n_2) < 1$, therefore $\mathcal{K}$ is a contraction operator. By applying Banach's fixed-point theorem, operator $\mathcal{K}$ has a unique fixed point in B_r. Hence, there exists a unique solution of Problem (2) on $[a, b]$. The proof is completed. □

Now, we prove our second existence result via the Leray–Schauder alternative.

Lemma 4. *(Leray-Schauder alternative) [22]. Let $F : E \to E$ be a completely continuous operator. Let*

$$\xi(F) = \{x \in E : x = \lambda F(x) \text{ for some } 0 < \lambda < 1\}.$$

Then, either set $\xi(F)$ is unbounded, or F has at least one fixed point.

Theorem 2. *Assume that there exist real constants $u_i, v_i \geq 0$ for $i = 1, 2$ and $u_0, v_0 > 0$, such that for any $x_i \in \mathbb{R}$, $(i = 1, 2)$ we have*

$$\begin{aligned}
|f(t, x_1, x_2)| &\leq u_0 + u_1|x_1| + u_2|x_2|, \\
|g(t, x_1, x_2)| &\leq v_0 + v_1|x_1| + v_2|x_2|.
\end{aligned}$$

If $(M_1+M_3)u_1+(M_2+M_4)v_1<1$ *and* $(M_1+M_3)u_2+(M_2+M_4)v_2<1$, *where* M_1, M_2, M_3, M_4 *are given in Equations* (19)–(22), *then Problem* (2) *has at least one solution on* $[a,b]$.

Proof. By continuity of functions f, g on $[a,b]\times\mathbb{R}\times\mathbb{R}$, operator $\mathcal{K}$ is continuous. Now, we show that the operator $\mathcal{K}: X\times Y\to X\times Y$ is completely continuous. Let $\Phi\subset X\times Y$ be bounded. Then, there exist two positive constants, L_1 and L_2, such that

$$|f(t,x,y)|\le L_1,\ |g(t,x,y)|\le L_2,\ \forall(x,y)\in\Phi.$$

Then, for any $(x,y)\in\Phi$, we have

$$\begin{aligned}
|\mathcal{K}_1(x,y)(t)| \le\ & \frac{1}{|\Omega|}\left(|\Omega_3|+|\Omega_4|\frac{(\log(t/a))^{q_1}}{\Gamma(q_1+1)}\right)\big(|\beta_1|\,{}^{H}I^{q_1}\,{}^{RL}I^{p_1}|f_{x,y}|(b)\\
&+|\beta_2|\,{}^{H}I^{q_2}|g_{x,y}|(b)\big)+\frac{1}{|\Omega|}\left(|\Omega_1|+|\Omega_2|\frac{(\log(t/a))^{q_1}}{\Gamma(q_1+1)}\right)\\
&\times\Big(|\beta_3|\,{}^{RL}I^{p_2}\,{}^{H}I^{q_2}|g_{x,y}|(b)+|\beta_4|\,{}^{RL}I^{p_1}|f_{x,y}|(b)\Big)+{}^{H}I^{q_1}\,{}^{RL}I^{p_1}|f_{x,y}|(b)\\
\le\ & \frac{1}{|\Omega|}\left(|\Omega_3|+|\Omega_4|\frac{(\log(b/a))^{q_1}}{\Gamma(q_1+1)}\right)\big(|\beta_1|\,L_1\,{}^{H}I^{q_1}\,{}^{RL}I^{p_1}(1)(b)\\
&+|\beta_2|\,L_2\,{}^{H}I^{q_2}(1)(b)\big)+\frac{1}{|\Omega|}\left(|\Omega_1|+|\Omega_2|\frac{(\log(b/a))^{q_1}}{\Gamma(q_1+1)}\right)\\
&\times\Big(|\beta_3|\,L_2\,{}^{RL}I^{p_2}\,{}^{H}I^{q_2}(1)(b)+|\beta_4|\,L_1\,{}^{RL}I^{p_1}(1)(b)\Big)+L_1\,{}^{H}I^{q_1}\,{}^{RL}I^{p_1}(1)(b),
\end{aligned}$$

which yields

$$\|\mathcal{K}_1(x,y)\|\le L_1M_1+L_2M_2.$$

In addition, we obtain that

$$\begin{aligned}
\|\mathcal{K}_2(x,y)\| \le\ & \frac{1}{|\Omega|}\left(\frac{|\alpha_4|}{|\alpha_3|}|\Omega_4|+\frac{|\alpha_1|}{|\alpha_2|}|\Omega_3|\frac{(b-a)^{p_2}}{\Gamma(p_2+1)}\right)\big(|\beta_1|L_1\,{}^{H}I^{q_1}\,{}^{RL}I^{p_1}(1)(b)\\
&+|\beta_2|L_2\,{}^{H}I^{q_2}(1)(b)\big)+\frac{1}{|\Omega|}\left(\frac{|\alpha_4|}{|\alpha_3|}|\Omega_2|+\frac{|\alpha_1|}{|\alpha_2|}|\Omega_1|\frac{(b-a)^{p_2}}{\Gamma(p_2+1)}\right)\\
&\times\big(|\beta_3|L_2\,{}^{RL}I^{p_2}\,{}^{H}I^{q_2}(1)(b)+|\beta_4|L_1\,{}^{RL}I^{p_1}(1)(b)\big)+L_2\,{}^{RL}I^{p_2}\,{}^{H}I^{q_2}(1)(b)\\
=\ & L_1M_3+L_2M_4.
\end{aligned}$$

Hence, from the above inequalities, we get that set $\mathcal{K}\Phi$ is uniformly bounded. Next, we prove that set $\mathcal{K}\Phi$ is equicontinuous. For any $(x,y)\in\Phi$, and $\tau_1,\tau_2\in[a,b]$ such that $\tau_1<\tau_2$, we have

$$
\begin{aligned}
& |\mathcal{K}_1(x,y)(\tau_2) - \mathcal{K}_1(x,y)(\tau_1)| \\
\leq\ & \left(\frac{|\Omega_4|}{|\Omega|\Gamma(q_1+1)}\right) |(\log(\tau_2/a))^{q_1} - (\log(\tau_1/a))^{q_1}| \\
& \times \Big(|\beta_1|\, {}^{H}I^{q_1\, RL}I^{p_1}|f_{x,y}|(b) + |\beta_2|\, {}^{H}I^{q_2}|g_{x,y}|(b)\Big) \\
& + \left(\frac{|\Omega_2|}{|\Omega|\Gamma(q_1+1)}\right) |(\log(\tau_2/a))^{q_1} - (\log(\tau_1/a))^{q_1}| \\
& \times \Big(|\beta_3|\, {}^{RL}I^{p_2\, H}I^{q_2}|g_{x,y}|(b) + |\beta_4|\, {}^{RL}I^{p_1}|f_{x,y}|(b)\Big) \\
& + \Big| {}^{H}I^{q_1\, RL}I^{p_1}|f_{x,y}|(\tau_2) - {}^{H}I^{q_1\, RL}I^{p_1}|f_{x,y}|(\tau_1) \Big| \\
\leq\ & \left(\frac{|\Omega_4|}{|\Omega|\Gamma(q_1+1)}\right) |(\log(\tau_2/a))^{q_1} - (\log(\tau_1/a))^{q_1}| \\
& \times \Big(L_1\, |\beta_1|\, {}^{H}I^{q_1\, RL}I^{p_1}(1)(b) + L_2\, |\beta_2|\, {}^{H}I^{q_2}(1)(b)\Big) \\
& + \left(\frac{|\Omega_2|}{|\Omega|\Gamma(q_1+1)}\right) |(\log(\tau_2/a))^{q_1} - (\log(\tau_1/a))^{q_1}| \\
& \times \Big(L_2\, |\beta_3|\, {}^{RL}I^{p_2\, H}I^{q_2}(1)(b) + L_1\, |\beta_4|\, {}^{RL}I^{p_1}(1)(b)\Big) \\
& + \frac{L_1}{\Gamma(q_1)\Gamma(p_1+1)} \Bigg| \int_a^{\tau_1} \left[\Big(\log\frac{\tau_2}{s}\Big)^{q_1-1} - \Big(\log\frac{\tau_1}{s}\Big)^{q_1-1}\right] \\
& \times (s-a)^{p_1}\frac{ds}{s} + \int_{\tau_1}^{\tau_2} \Big(\log\frac{\tau_2}{s}\Big)^{q_1-1}(s-a)^{p_1}\frac{ds}{s} \Bigg| \\
=\ & \left(\frac{|\Omega_4|}{|\Omega|\Gamma(q_1+1)}\right) |(\log(\tau_2/a))^{q_1} - (\log(\tau_1/a))^{q_1}| \\
& \times \Big(L_1\, |\beta_1|\, {}^{H}I^{q_1\, RL}I^{p_1}(1)(b) + L_2\, |\beta_2|\, {}^{H}I^{q_2}(1)(b)\Big) \\
& + \left(\frac{|\Omega_2|}{|\Omega|\Gamma(q_1+1)}\right) |(\log(\tau_2/a))^{q_1} - (\log(\tau_1/a))^{q_1}| \\
& \times \Big(L_2\, |\beta_3|\, {}^{RL}I^{p_2\, H}I^{q_2}(1)(b) + L_1\, |\beta_4|\, {}^{RL}I^{p_1}(1)(b)\Big) \\
& + \frac{L_1(b-a)^{p_1}}{\Gamma(q_1+1)\Gamma(p_1+1)} \left[2\Big(\log\frac{\tau_2}{\tau_1}\Big)^{q_1} + \left| \Big(\log\frac{\tau_2}{a}\Big)^{q_1} - \Big(\log\frac{\tau_1}{a}\Big)^{q_1} \right| \right].
\end{aligned}
$$

Therefore, we obtain

$$|\mathcal{K}_1(x,y)(\tau_2) - \mathcal{K}_1(x,y)(\tau_1)| \to 0, \quad \text{as } \tau_1 \to \tau_2.$$

Analogously, we can get the following inequality:

$$|\mathcal{K}_2(x,y)(\tau_2) - \mathcal{K}_2(x,y)(\tau_1)| \to 0, \quad \text{as } \tau_1 \to \tau_2.$$

Hence, set $\mathcal{K}\Phi$ is equicontinuous. By applying the Arzelá–Ascoli theorem, set $\mathcal{K}\Phi$ is relatively compact, which implies that operator $\mathcal{K}$ is completely continuous. Lastly, we show that set $\xi = \{(x,y) \in X \times Y : (x,y) = \lambda\mathcal{K}(x,y),\ 0 \leq \lambda \leq 1\}$ is bounded. Now, let $(x,y) \in \xi$, then we obtain $(x,y) = \lambda\mathcal{K}(x,y)$, which yields, for any $t \in [a,b]$,

$$x(t) = \lambda\mathcal{K}_1(x,y)(t), \quad y(t) = \lambda\mathcal{K}_2(x,y)(t).$$

Then, we have

$$\begin{aligned}\|x\| &\le (u_0 + u_1\|x\| + u_2\|y\|)M_1 + (v_0 + v_1\|x\| + v_2\|y\|)M_2,\\ \|y\| &\le (u_0 + u_1\|x\| + u_2\|y\|)M_3 + (v_0 + v_1\|x\| + v_2\|y\|)M_4,\end{aligned}$$

which imply that

$$\begin{aligned}\|x\| + \|y\| \le\ & (M_1+M_3)u_0 + (M_2+M_4)v_0 + [(M_1+M_3)u_1 + (M_2+M_4)v_1]\,\|x\|\\ & + [(M_1+M_3)u_2 + (M_2+M_4)v_2]\,\|y\|.\end{aligned}$$

Thus, we get the inequality

$$\|(x,y)\| \le \frac{(M_1+M_3)u_0 + (M_2+M_4)v_0}{M^*}, \tag{26}$$

where $M^* = \min\{1-(M_1+M_3)u_1-(M_2+M_4)v_1, 1-(M_1+M_3)u_2-(M_2+M_4)v_2\}$, which shows that set ξ is bounded. Therefore, by applying Lemma 4, operator $\mathcal{K}$ has at least one fixed point in Φ. Therefore, we deduce that Problem (2) has at least one solution on $[a,b]$. The proof is complete. □

The last-existence theorem is based on Krasnoselskii's fixed-point theorem.

Lemma 5. *(Krasnoselskii's fixed-point theorem) [23] Let M be a closed, bounded, convex, and nonempty subset of a Banach space X. Let A, B be operators, such that (i) $Ax + By \in M$ where $x, y \in M$, (ii) A is compact and continuous, and (iii) B is a contraction mapping. Then, there exists $z \in M$, such that $z = Az + Bz$.*

Theorem 3. *Assume that $f, g : [a,b] \times \mathbb{R} \to \mathbb{R}$ are continuous functions satisfying assumption (H_1) in Theorem 1. In addition we suppose and there exist two positive constants P, Q such that for all $t \in [a,b]$ and $x_i, y_i \in \mathbb{R}, i = 1, 2,$*

$$|f(t,x_1,x_2)| \le P \ \text{ and } \ |g(t,x_1,x_2)| \le Q. \tag{27}$$

If

$$\Big((m_1+m_2)^{H}I^{q_1\,RL}I^{p_1}(1)(b) + (n_1+n_2)^{RL}I^{p_2\,H}I^{q_2}(1)(b)\Big) < 1, \tag{28}$$

then the problem (2) has at least one solution on $[a,b]$.

Proof. Let $B_\delta = \{(x,y) \in X \times Y : \|(x,y)\| \le \delta\}$ be a ball, where a constant $\delta \ge \max\{M_1P + M_2Q, M_3P + M_4Q\}$. To apply Lemma 5, we decompose operator $\mathcal{K}$ into four operators $\mathcal{K}_{1,1}, \mathcal{K}_{1,2}, \mathcal{K}_{2,1}$, and $\mathcal{K}_{2,2}$ on B_δ as

$$
\begin{aligned}
\mathcal{K}_{1,1}(x,y)(t) &= -\frac{1}{\Omega}\left(\Omega_3+\Omega_4\frac{(\log(t/a))^{q_1}}{\Gamma(q_1+1)}\right)\\
&\quad\times\left(\beta_1{}^{H}I^{q_1\,RL}I^{p_1}(f_{x,y})(b)+\beta_2{}^{H}I^{q_2}(g_{x,y})(b)\right)\\
&\quad+\frac{1}{\Omega}\left(\Omega_1-\Omega_2\frac{(\log(t/a))^{q_1}}{\Gamma(q_1+1)}\right)\\
&\quad\times\left(\beta_3{}^{RL}I^{p_2\,H}I^{q_2}(g_{x,y})(b)+\beta_4{}^{RL}I^{p_1}(f_{x,y})(b)\right),\\
\mathcal{K}_{1,2}(x,y)(t) &= {}^{H}I^{q_1\,RL}I^{p_1}(f_{x,y})(t),\\
\mathcal{K}_{2,1}(x,y)(t) &= \frac{1}{\Omega}\left(\frac{\alpha_4}{\alpha_3}\Omega_4+\frac{\alpha_1}{\alpha_2}\Omega_3\frac{(t-a)^{p_2}}{\Gamma(p_2+1)}\right)\\
&\quad\times\left(\beta_1{}^{H}I^{q_1\,RL}I^{p_1}(f_{x,y})(b)+\beta_2{}^{H}I^{q_2}(g_{x,y})(b)\right)\\
&\quad+\frac{1}{\Omega}\left(\frac{\alpha_4}{\alpha_3}\Omega_2-\frac{\alpha_1}{\alpha_2}\Omega_1\frac{(t-a)^{p_2}}{\Gamma(p_2+1)}\right)\\
&\quad\times\left(\beta_3{}^{RL}I^{p_2\,H}I^{q_2}(g_{x,y})(b)+\beta_4{}^{RL}I^{p_1}(f_{x,y})(b)\right),\\
\mathcal{K}_{2,2}(x,y)(t) &= {}^{RL}I^{p_2\,H}I^{q_2}(g_{x,y})(t).
\end{aligned}
$$

Note that $\mathcal{K}_1(x,y)(t) = \mathcal{K}_{1,1}(x,y)(t)+\mathcal{K}_{1,2}(x,y)(t)$ and $\mathcal{K}_2(x,y)(t) = \mathcal{K}_{2,1}(x,y)(t)+\mathcal{K}_{2,2}(x,y)(t)$. In addition, observe that ball B_δ is a closed, bounded, and convex subset of Banach space $\mathcal{C}$. Now, we show that $\mathcal{K}B_\delta \subset B_\delta$ for satisfying condition (i) of Lemma 5. Setting $x=(x_1,x_2), y=(y_1,y_2)\in B_\delta$, and using Condition (27), we then have

$$
\begin{aligned}
&|\mathcal{K}_{1,1}(x_1,x_2)(t)+\mathcal{K}_{1,2}(y_1,y_2)(t)|\\
\leq\ & \frac{1}{|\Omega|}\left(|\Omega_3|+|\Omega_4|\frac{(\log(t/a))^{q_1}}{\Gamma(q_1+1)}\right)\left(|\beta_1|\,{}^{H}I^{q_1\,RL}I^{p_1}|f_{x_1,x_2}|(b)+|\beta_2|\,{}^{H}I^{q_2}|g_{x_1,x_2}|(b)\right)\\
&+\frac{1}{|\Omega|}\left(|\Omega_1|+|\Omega_2|\frac{(\log(t/a))^{q_1}}{\Gamma(q_1+1)}\right)\left(|\beta_3|\,{}^{RL}I^{p_2\,H}I^{q_2}|g_{x_1,x_2}|(b)+|\beta_4|\,{}^{RL}I^{p_1}|f_{x_1,x_2}|(b)\right)\\
&+{}^{H}I^{q_1\,RL}I^{p_1}|f_{y_1,y_2}|(t)\\
\leq\ & \frac{1}{|\Omega|}\left(|\Omega_3|+|\Omega_4|\frac{(\log(b/a))^{q_1}}{\Gamma(q_1+1)}\right)\left(P\,|\beta_1|\,{}^{H}I^{q_1\,RL}I^{p_1}(1)(b)+Q\,|\beta_2|\,{}^{H}I^{q_2}(1)(b)\right)\\
&+\frac{1}{|\Omega|}\left(|\Omega_1|+|\Omega_2|\frac{(\log(b/a))^{q_1}}{\Gamma(q_1+1)}\right)\left(Q\,|\beta_3|\,{}^{RL}I^{p_2\,H}I^{q_2}(1)(b)+P\,|\beta_4|\,{}^{RL}I^{p_1}(1)(b)\right)\\
&+P\,{}^{H}I^{q_1\,RL}I^{p_1}(1)(b)\\
=\ & M_1P+M_2Q\leq\delta.
\end{aligned}
$$

Furthermore, we can find that

$$
\begin{aligned}
&|\mathcal{K}_{2,1}(x_1,x_2)(t)+\mathcal{K}_{2,2}(y_1,y_2)(t)|\\
\leq\ & \frac{1}{|\Omega|}\left(\frac{|\alpha_4|}{|\alpha_3|}|\Omega_4|+\frac{|\alpha_1|}{|\alpha_2|}|\Omega_3|\frac{(b-a)^{p_2}}{\Gamma(p_2+1)}\right)\left(P|\beta_1|^{H}I^{q_1\,RL}I^{p_1}(1)(b)+Q|\beta_2|^{H}I^{q_2}(1)(b)\right)\\
&+\frac{1}{|\Omega|}\left(\frac{|\alpha_4|}{|\alpha_3|}|\Omega_2|+\frac{|\alpha_1|}{|\alpha_2|}|\Omega_1|\frac{(b-a)^{p_2}}{\Gamma(p_2+1)}\right)\\
&\times\left(Q|\beta_3|^{RL}I^{p_2\,H}I^{q_2}(1)(b)+P|\beta_4|^{RL}I^{p_1}(1)(b)\right)+Q\,{}^{RL}I^{p_2\,H}I^{q_2}(1)(b)\\
=\ & M_3P+M_4Q\leq\delta.
\end{aligned}
$$

That yields $(\mathcal{K}_{1,1},\mathcal{K}_{2,1})x+(\mathcal{K}_{1,2},\mathcal{K}_{2,2})y\in B_\delta$. To show that operator $(\mathcal{K}_{1,2},\mathcal{K}_{2,2})$ is a contraction mapping satisfying condition (iii) of Lemma 5, for $(x_1,y_1),(x_2,y_2)\in B_\delta$, we have

$$\begin{aligned} & |\mathcal{K}_{1,2}(x_1,y_1)(t) - \mathcal{K}_{1,2}(x_2,y_2)(t)| \\ \leq \;& {}^{H}I^{q_1}\,{}^{RL}I^{p_1}|f_{x_1,y_1} - f_{x_2,y_2}|(t) \\ \leq \;& (m_1\|x_1 - x_2\| + m_2\|y_1 - y_2\|)\,{}^{H}I^{q_1}\,{}^{RL}I^{p_1}(1)(b) \\ \leq \;& (m_1 + m_2)\,{}^{H}I^{q_1}\,{}^{RL}I^{p_1}(1)(b)(\|x_1 - x_2\| + \|y_1 - y_2\|), \end{aligned} \tag{29}$$

and

$$\begin{aligned} & |\mathcal{K}_{2,2}(x_1,y_1)(t) - \mathcal{K}_{2,2}(x_2,y_2)(t)| \\ \leq \;& {}^{RL}I^{p_2}\,{}^{H}I^{q_2}|g_{x_1,y_1} - g_{x_2,y_2}|(b) \\ \leq \;& (n_1 + n_2)\,{}^{RL}I^{p_2}\,{}^{H}I^{q_2}(1)(b)(\|x_1 - x_2\| + \|y_1 - y_2\|). \end{aligned} \tag{30}$$

It follows Form (29) and (30) that

$$\begin{aligned} & \|(\mathcal{K}_{1,2},\mathcal{K}_{2,2})(x_1,y_1) - (\mathcal{K}_{1,2},\mathcal{K}_{2,2})(x_2,y_2)\| \\ \leq \;& \Big((m_1+m_2)\,{}^{H}I^{q_1}\,{}^{RL}I^{p_1}(1)(b) + (n_1+n_2)\,{}^{RL}I^{p_2}\,{}^{H}I^{q_2}(1)(b)\Big)\,(\|x_1 - x_2\| + \|y_1 - y_2\|), \end{aligned}$$

which is a contraction by inequality in (28). Therefore, condition (iii) of Lemma 5 is satisfied. Finally we show that operator $(\mathcal{K}_{1,1},\mathcal{K}_{2,1})$ satisfied the condition (ii) of Lemma 5. By applying the continuity of functions f,g on $[a,b]\times\mathbb{R}\times\mathbb{R}$, we can conclude that operator $(\mathcal{K}_{1,1},\mathcal{K}_{2,1})$ is continuous. For each $(x,y)\in B_\delta$, one has

$$\begin{aligned} & |\mathcal{K}_{1,1}(x,y)(t)| \\ \leq \;& \frac{1}{|\Omega|}\left(|\Omega_3| + |\Omega_4|\frac{(\log(b/a))^{q_1}}{\Gamma(q_1+1)}\right)\Big(P\,|\beta_1|\,{}^{H}I^{q_1}\,{}^{RL}I^{p_1}(1)(b) + Q\,|\beta_2|\,{}^{H}I^{q_2}(1)(b)\Big) \\ & + \frac{1}{|\Omega|}\left(|\Omega_1| + |\Omega_2|\frac{(\log(b/a))^{q_1}}{\Gamma(q_1+1)}\right)\Big(Q\,|\beta_3|\,{}^{RL}I^{p_2}\,{}^{H}I^{q_2}(1)(b) + P\,|\beta_4|\,{}^{RL}I^{p_1}(1)(b)\Big) \\ := \;& P^*, \end{aligned}$$

and

$$\begin{aligned} & |\mathcal{K}_{2,1}(x,y)(t)| \\ \leq \;& \frac{1}{|\Omega|}\left(\frac{|\alpha_4|}{|\alpha_3|}|\Omega_4| + \frac{|\alpha_1|}{|\alpha_2|}|\Omega_3|\frac{(b-a)^{p_2}}{\Gamma(p_2+1)}\right)(P|\beta_1|\,{}^{H}I^{q_1}\,{}^{RL}I^{p_1}(1)(b) + Q|\beta_2|\,{}^{H}I^{q_2}(1)(b)) \\ & + \frac{1}{|\Omega|}\left(\frac{|\alpha_4|}{|\alpha_3|}|\Omega_2| + \frac{|\alpha_1|}{|\alpha_2|}|\Omega_1|\frac{(b-a)^{p_2}}{\Gamma(p_2+1)}\right)(Q|\beta_3|\,{}^{RL}I^{p_2}\,{}^{H}I^{q_2}(1)(b) + P|\beta_4|\,{}^{RL}I^{p_1}(1)(b)) \\ := \;& Q^*. \end{aligned}$$

Then, we obtain the following fact

$$\|(\mathcal{K}_{1,1},\mathcal{K}_{2,1})(x,y)\| \leq P^* + Q^*,$$

which implies that set $(\mathcal{K}_{1,1},\mathcal{K}_{2,1})B_\delta$ is uniformly bounded. Next, we show that set $(\mathcal{K}_{1,1},\mathcal{K}_{2,1})B_\delta$ is equicontinuous. For $\tau_1,\tau_2\in[a,b]$, such that $\tau_1<\tau_2$, and for any $(x,y)\in B_\delta$, we prove that

$$\begin{aligned}
&|\mathcal{K}_{1,1}(x,y)(\tau_2) - \mathcal{K}_{1,1}(x,y)(\tau_1)| \\
\leq\ & \left(\frac{|\Omega_4|}{|\Omega|\Gamma(q_1+1)}\right) |(\log(\tau_2/a))^{q_1} - (\log(\tau_1/a))^{q_1}| \\
&\times \Big(|\beta_1|\, {}^{H}I^{q_1\,RL}I^{p_1}|f_{x,y}|(b) + |\beta_2|\, {}^{H}I^{q_2}|g_{x,y}|(b)\Big) \\
&+ \left(\frac{|\Omega_2|}{|\Omega|\,\Gamma(q_1+1)}\right) |(\log(\tau_2/a))^{q_1} - (\log(\tau_1/a))^{q_1}| \\
&\times \Big(|\beta_3|\, {}^{RL}I^{p_2\,H}I^{q_2}|g_{x,y}|(b) + |\beta_4|\, {}^{RL}I^{p_1}|f_{x,y}|(b)\Big) \\
\leq\ & \left(\frac{|\Omega_4|}{|\Omega|\Gamma(q_1+1)}\right) |(\log(\tau_2/a))^{q_1} - (\log(\tau_1/a))^{q_1}| \\
&\times \Big(P\,|\beta_1|\, {}^{H}I^{q_1\,RL}I^{p_1}(1)(b) + Q\,|\beta_2|\, {}^{H}I^{q_2}(1)(b)\Big) \\
&+ \left(\frac{|\Omega_2|}{|\Omega|\,\Gamma(q_1+1)}\right) |(\log(\tau_2/a))^{q_1} - (\log(\tau_1/a))^{q_1}| \\
&\times \Big(Q\,|\beta_3|\, {}^{RL}I^{p_2\,H}I^{q_2}(1)(b) + P\,|\beta_4|\, {}^{RL}I^{p_1}(1)(b)\Big).
\end{aligned}$$

Indeed, we can show that

$$\begin{aligned}
&|\mathcal{K}_{2,1}(x,y)(\tau_2) - \mathcal{K}_{2,1}(x,y)(\tau_1)| \\
\leq\ & \left(\frac{|\alpha_1||\Omega_3|}{|\alpha_2|\,|\Omega|\,\Gamma(p_2+1)}\right) |(\tau_2-a)^{p_2} - (\tau_1-a)^{p_2}| \\
&\times \Big(P|\beta_1|^{H}I^{q_1\,RL}I^{p_1}(1)(b) + Q|\beta_2|^{H}I^{q_2}(1)(b)\Big) \\
&+ \left(\frac{|\alpha_1||\Omega_1|}{|\alpha_2||\Omega|\Gamma(p_2+1)}\right) |(\tau_2-a)^{p_2} - (\tau_1-a)^{p_2}| \\
&\times \Big(Q|\beta_3|^{RL}I^{p_2\,H}I^{q_2}(1)(b) + P|\beta_4|^{RL}I^{p_1}(1)(b)\Big).
\end{aligned}$$

Thus, $|(\mathcal{K}_{1,1},\mathcal{K}_{2,1})(x,y)(\tau_2) - (\mathcal{K}_{1,1},\mathcal{K}_{2,1})(x,y)(\tau_1)|$ tends to zero as $\tau_1 \to \tau_2$. Therefore, set $(\mathcal{K}_{1,1},\mathcal{K}_{2,1})B_\delta$ is equicontinuous. By applying the Arzelá–Ascoli theorem, operator $(\mathcal{K}_{1,1},\mathcal{K}_{2,1})$ is compact on B_δ. By application of Lemma 5, there exists $z = (z_1,z_2) \in B_\delta$, such that $z = (\mathcal{K}_{1,1},\mathcal{K}_{2,1})z + (\mathcal{K}_{1,2},\mathcal{K}_{2,2})z$. Therefore, Problem (2) has at least one solution on $[a,b]$. This completes the proof. □

Example 1. *Consider the following coupled system of sequential Caputo and Hadamard fractional differential equations with coupled separated boundary conditions:*

$$\begin{cases}
{}^{C}D^{\frac{1}{2}\,H}D^{\frac{1}{3}}x(t) = f(t,x(t),y(t)), \qquad t\in[2,5], \\
{}^{H}D^{\frac{2}{3}\,C}D^{\frac{3}{4}}y(t) = g(t,x(t),y(t)), \qquad t\in[2,5], \\
x(2) + 0.2^{C}D^{\frac{3}{4}}y(2) = 0, \quad 0.42x(5) + 0.5^{C}D^{\frac{3}{4}}y(5) = 0, \\
0.4y(2) - 0.17^{H}D^{\frac{1}{3}}x(2) = 0, \quad 0.46y(5) + 3.5^{H}D^{\frac{1}{3}}x(5) = 0.
\end{cases} \tag{31}$$

Here $p_1 = 1/2$, $p_2 = 3/4$, $q_1 = 1/3$, $q_2 = 2/3$, $a = 2$, $b = 5$, $\alpha_1 = 1$, $\alpha_2 = 0.2$, $\alpha_3 = 0.4$, $\alpha_4 = -0.17$, $\beta_1 = 0.42$, $\beta_2 = 0.5$, $\beta_3 = 0.46$, $\beta_4 = 0.35$. From the given information, we find that ${}^{H}I^{1/3\,RL}I^{1/2}(1)(5) = 10.2601$, ${}^{RL}I^{3/4\,H}I^{2/3}(1)(5) = 3.7354$ and $|\Omega| = 2.2846$, which lead to $M_1 = 20.3479$, $M_2 = 3.5125$, $M_3 = 10.1477$, $M_4 = 9.6538$.

(i) Let two nonlinear functions $f, g : [2,5] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ be given by

$$f(t,x,y) = \frac{\sin^2(t/\pi)}{(14+t)^2}\left(\frac{x^2+2|x|}{|x|+1}\right) + \frac{|y|}{(t+146)} + \frac{1}{2}, \tag{32}$$

$$g(t,x,y) = \frac{\sin(|x|)}{156+t^3} + \frac{\tan^{-1}(y)e^{-\sin t}}{6t^4+2} + \frac{3}{4}. \tag{33}$$

Since

$$|f(t,x_1,y_1) - f(t,x_2,y_2)| \le \frac{1}{128}|x_1-x_2| + \frac{1}{148}|y_1-y_2|$$

and

$$|g(t,x_1,y_1) - g(t,x_2,y_2)| \le \frac{1}{164}|x_1-x_2| + \frac{e}{98}|y_1-y_2|,$$

we obtain $(M_1+M_3)(1/128+1/148)+(M_2+M_4)(1/164+e/98) = 0.8867 < 1$. By the conclusion of Theorem 1, Problem (31) with Problems (32) and (33) have a unique solution on $[2,5]$.

(ii) Now consider functions $f, g : [2,5] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$f(t,x,y) = \frac{e^{-t}}{11} + \frac{x^4\sin^2 t}{33(1+|x|^3)} + \frac{|y|^5\cos^2 t}{33(1+y^4)}, \tag{34}$$

$$g(t,x,y) = \frac{2}{(t+1)} + \frac{\sin x}{25(t+6)} + \frac{e^{-t^2}\tan^{-1} y}{90+25t^2}. \tag{35}$$

It is easy to verify that $|f(t,x,y)| \le (1/11)+(1/33)|x|+(1/33)|y|$ and $|g(t,x,y)| \le (2/3) + (1/200)|x| + (1/190)|y|$. As $(M_1+M_3)u_1 + (M_2+M_4)v_1 = 0.9938 < 1$ and $(M_1+M_3)u_2 + (M_2+M_4)v_2 = 0.9973 < 1$, by applying Theorem 2, we get that System (31) with Systems (34) and (35) have at least one solution on $[2,5]$.

(iii) Define functions $f, g : [2,5] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ by

$$f(t,x,y) = \frac{1}{4}\sin^2 t + \frac{|x|e^{-(t-2)}}{40(1+|x|)} + \frac{1}{40}\sin y, \tag{36}$$

$$g(t,x,y) = \frac{1}{2}\cos^2 t + \frac{1}{16}\tan^{-1} x + \frac{|y|}{16(1+|y|)}. \tag{37}$$

We have $|f(t,x,y)| \le 3/10$, $|g(t,x,y)| \le 5/8$ and

$$|f(t,x_1,y_1) - f(t,x_2,y_2)| \le \frac{1}{40}|x_1-x_2| + \frac{1}{40}|y_1-y_2|$$

and

$$|g(t,x_1,y_1) - g(t,x_2,y_2)| \le \frac{1}{16}|x_1-x_2| + \frac{1}{16}|y_1-y_2|.$$

Then we obtain $\big((m_1+m_2)^H I^{q_1\,RL} I^{p_1}(1)(5) + (n_1+n_2)^{RL} I^{p_2\,H} I^{q_2}(1)(5)\big) = 0.9799 < 1$. Using Theorem 3, the problem (31) with (36) and (37) has at least one solution on $[2,5]$. Observe that the inequality $(M_1+M_3)(m_1+m_2)+(M_2+M_4)(n_1+n_2) = 3.1706 > 1$ and, thus, Condition (23) is not satisfied. Therefore, Theorem 1 cannot be applied for this case.

4. Conclusions

We have proven the existence and uniqueness of solutions for a coupled system of sequential Caputo and Hadamard fractional differential equations with coupled separated boundary conditions by applying the Banach fixed-point theorem, Leray–Schauder nonlinear alternative, and Krasnoselakki fixed-point theorem. We also provided examples to clarify our results.

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References

1. Torvik, P.J.; Bagley, R.L. On the appearance of the fractional derivative in the behavior of real materials. *J. Appl. Mech.* **1984**, *51*, 294–298. [CrossRef]
2. Magin, R.L. *Fractional Calculus in Bioengineering*; Begell House Publishers: Connecticut, CT, USA, 2006.
3. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; North-Holland Mathematics Studies, 204; Elsevier Science B.V.: Amsterdam, The Netherlands, 2006.
4. Klafter, J.; Lim, S.C.; Metzler, R. (Eds.) *Fractional Dynamics in Physics*; World Scientific: Singapore, 2011.
5. Carvalho, A.; Pinto, C.M.A. A delay fractional order model for the co-infection of malaria and HIV/AIDS. *Int. J. Dyn. Control* **2017**, *5*, 168–186. [CrossRef]
6. Ahmad, B.; Alsaedi, A.; Ntouyas, S.K.; Tariboon, J. *Hadamard-Type Fractional Differential Equations, Inclusions and Inequalities*; Springer: Berlin, Germany, 2017.
7. Faieghi, M.; Kuntanapreeda, S.; Delavari, H.; Baleanu, D. LMI-based stabilization of a class of fractional-order chaotic systems. *Nonlinear Dyn.* **2013**, *72*, 301–309. [CrossRef]
8. Zhang, F.; Li, G.C.C.; Kurths, J. Chaos synchronization in fractional differential systems. *Philos. Trans. R. Soc. A* **2013**, *371*, 20120155. [CrossRef] [PubMed]
9. Metzler, R.; Klafter, J. The random walk's guide to anomalous diffu sion: a fractional dynamics approach. *Phys. Rep.* **2000**, *339*, 1–77. [CrossRef]
10. Javidi, M.; Ahmad, B. Dynamic analysis of time fractional order phytoplankton-toxic phytoplankton–zooplankton system. *Ecol. Model.* **2015**, *318*, 8–18. [CrossRef]
11. Henderson, J.; Luca, R.; Tudorache, A. On a system of fractional differential equations with coupled integral boundary conditions. *Fract. Calc. Appl. Anal.* **2015**, *18*, 361–386. [CrossRef]
12. Wang, J.R.; Zhang, Y. Analysis of fractional order differential coupled systems. *Math. Methods Appl. Sci.* **2015**, *38*, 3322–3338. [CrossRef]
13. Zhang, L.; Ahmad, B.; Wang, G. Monotone iterative method for a class of nonlinear fractional differential equations on unbounded domains in Banach spaces. *Filomat* **2017**, *31*, 1331–1338. [CrossRef]
14. Assante, D.; Cesarano, C.; Fornaro, C.; Vazquez, L. Higher order and fractional diffusive equations. *J. Eng. Sci. Technol. Rev.* **2015**, *8*, 202–204. [CrossRef]
15. Thongsalee, N.; Laoprasittichok, S.; Ntouyas, S.K.; Tariboon, J. System of fractional differential equations with Erdélyi-Kober fractional integral conditions. *Open Math.* **2015**, *13*, 847–859. [CrossRef]
16. Ahmad, B.; Alsaedi, A.; Aljoudi, S.; Ntouyas, S.K. On a coupled system of sequential fractional differential equations with variable coeffcients and coupled integral boundary conditions. *Bull. Math. Soc. Sci. Math. Roum.* **2017**, *60*, 3–18.
17. Tariboon, J.; Ntouyas, S.K.; Asawasamrit, S.; Promsakon, C.H. Positive solutions for Hadamard differential systems with fractional integral conditions on an unbounded domain. *Open Math.* **2017**, *15*, 645–666
18. Ahmad, B.; Luca, R. Existence of solutions for a sequential fractional integro-differential system with coupled integral boundary conditions. *Chaos Solitons Fractals* **2017**, *104*, 378–388. [CrossRef]
19. Tariboon, J.; Cuntavepanit, A.; Ntouyas, S.K.; Nithiarayaphaks, W. Separated boundary value problems of sequential Caputo and Hadamard fractional differential equations. *J. Funct. Spaces* **2018**, *2018*. [CrossRef]
20. Podlubny, I. *Fractional Differential Equations*; Academic Press: San Diego, CA, USA, 1999.
21. Jarad, F.; Abdeljawad, T.; Baleanu, D. Caputo-type modification of the Hadamard fractional derivatives. *Adv. Differ. Equ.* **2012**, *2012*, 142. [CrossRef]

22. Granas, A.; Dugundji, J. *Fixed Point Theory*; Springer: New York, NY, USA, 2003.
23. Krasnoselskii, M.A. Two remarks on the method of successive approximations. *Uspekhi Mat. Nauk* **1955**, *10*, 123–127.