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Radial Symmetry for Weak Positive Solutions of Fractional Laplacian with a Singular Nonlinearity

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Abstract: This paper is concerned with the radial symmetry weak positive solutions for a class of singular fractional Laplacian. The main results in the paper demonstrate the existence and multiplicity of radial symmetry weak positive solutions by Schwarz spherical rearrangement, constrained minimization, and Ekeland's variational principle. It is worth pointing out that our results extend the previous works of T. Mukherjee and K. Sreenadh to a setting in which the testing functions need not have a compact support. Moreover, we weakened one of the conditions used in their papers. Our results improve on existing studies on radial symmetry solutions of nonlocal boundary value problems.

Keywords: radial symmetry; fractional Laplacian; Schwarz symmetry rearrangement; weak solutions; non-differentiable functional

1. Introduction

In this paper, we focus on radial symmetry positive solutions to a singular elliptic problem involving a nonlocal operator: the fractional powers of the Laplacian in a bounded sphere domain in $B_R(0) \subseteq R^N (N \geq 3)$. Nonlinear equations with fractional powers of the Laplacian are actively studied. The fractions of the Laplacian are the infinitesimal generators of Lévy stable diffusion processes and appear in anomalous diffusion in plasmas, population dynamics, American options in finances, and geophysical fluid dynamics. For more details, we refer the reader to [1,2]. To circumvent the nonlocal nature of the fractional Laplacian operator, Caffarelli, Salsa, and Silvestre [3,4] introduced the s-harmonic extension, which turns the nonlocal problem into a local one in higher dimensions. In our paper, we are interested in the existence of radial symmetry weak positive solutions that satisfy the singular fractional Laplacian boundary value problem,

$$\begin{cases} (-\Delta)^s u = \lambda u^\beta + a(x) u^{-\gamma}, & in \quad B_R(0), \\ u > 0, & in \quad B_R(0), \\ u = 0, & on \ \partial B_R(0) \end{cases} \tag{1}$$

where $B_R(0) \subset R^N (N \geq 3)$ is the ball centered at 0 with radius R, $N > 2s (0 < s < 1)$, $0 < \gamma < 1 < \beta < 2_s^* - 1 = \frac{N+2s}{N-2s}$, $\lambda > 0$ is a real parameter, and the nonnegative real function $a : \Omega \longrightarrow R$ is integrable.

In order to introduce our results, we start by recalling some functional spaces (see, e.g., [5–9]).

Let $\Omega \subset R^N (N \geq 3)$ be a bounded domain with a smooth boundary. For $0 < s < 1$, the fractional Laplacian $(-\Delta)^s u$ is defined as

$$(-\Delta)^s u(x) := C_{N,\,s} P.V. \int_{R^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}} dy$$
$$= C_{N,\,s} \lim_{\varepsilon \to 0^+} \int_{R^N \setminus B(x,\,\varepsilon)} \frac{u(x)-u(y)}{|x-y|^{N+2s}} dy$$

where $B(x,\ \varepsilon)$ is the ball centered at $x \in R^N$ with radius ε, and $C_{N,\,s} = \pi^{-\frac{N}{2}} 2^{2s} \frac{\Gamma(\frac{N+2s}{2})}{\Gamma(1-s)}$ is a normalization constant. The fractional Sobolev space $H^s(\Omega)$ is as follows:

$$H^s(\Omega) := \left\{ u \in L^2(\Omega) \Big| \frac{|u(x)-u(y)|}{|x-y|^{\frac{N}{2}+s}} \in L^2(\Omega \times \Omega) \right\},$$

endowed with the natural norm

$$\|u\|_{H^s} = \left(\int_\Omega |u(x)|^2 dx + \int_{\Omega \times \Omega} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dx\, dy \right)^{\frac{1}{2}}.$$

As in the classical case, we denote by $H_0^s(\Omega)$ the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{H^s}$. The inequalities about a Sobolev space and the embedding of the spaces $H_0^s(\Omega)$ into the Lebesgue spaces L^p have been exhaustively researched in [6,9,10]. Moreover, there is another norm $\|\cdot\|_{H_0^s}$ endowed in $H_0^s(\Omega)$ which is equivalent to the natural norm $\|\cdot\|_{H^s}$; that is,

$$\|u\|_{H_0^s} = \left(\int_{\Omega \times \Omega} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}} dx \right)^{\frac{1}{2}}.$$

In recent years, elliptic problems with a singular nonlinearity have attracted many researchers who study partial differential equations. Firstly, for the local operator ($s = 1$), the pioneering work by Crandall, Rabinowitz, and Tartar [11] starts the following singular Laplacian Equation (2).

$$\begin{cases} -\Delta u = \lambda u^\beta + a(x) u^{-\gamma}, & in \quad \Omega, \\ \quad u > 0, & in \quad \Omega, \\ \quad u = 0, & on \quad \partial\Omega \end{cases} \tag{2}$$

where Ω is a bounded domain with a smooth boundary, $0 < \gamma < 1 < \beta < 2^* - 1 = \frac{N+2}{N-2}$, $\lambda > 0$ is a real parameter, and the real function $a : \Omega \longrightarrow R$ is integrable in domain Ω. In [11], the authors proved that Equation (2) has a unique class solution $u \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ when $\lambda = 0$, $a(x) = 1$. By means of the Ekeland's variational principle, Sun [12] proved that Equation (2) has at least two weak positive solutions in $H_0^1(\Omega)$ when the parameter λ is sufficiently small and $\beta < 2^* - 1$. By using the geometry of the Nehari manifold and the concentration-compactness method, our previous work [13] achieved results similar to [12]. Furthermore, our results improved on the existing research on the power exponent $\beta = 2^* - 1$. Local Equation (1) and some other versions of it have been extensively studied over the past decades; for further information, one can refer to [11–20] and references therein.

In the nonlocal setting $(0 < s < 1)$, the existence of weak solutions and various properties of solutions have been considered for the fractional Laplacian with a singular nonlinearity, Equation (1), by many authors in recent years. In [21], the author stated that $u \in H_0^s(\Omega)$ is a weak solution of Equation (1) with $\lambda = 0$ if the identity

$$\int_\Omega (-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} \varphi dx = \int_\Omega u^{-\gamma} \varphi dx, \quad \forall\, \varphi \in H_0^s(\Omega)$$

holds. By using the sub-supersolution method, the author proved the existence and uniqueness of a weak positive solution of Equation (1) with $\lambda = 0$. In [22], using variational methods, the authors proved that Equation (1) has at least two distinct weak positive solutions $u,\ v \in L^\infty(\Omega)$ when $\beta < 2_s^* - 1$, among other conditions.

Before stating the main results contained in this paper, we need to clarify the concept of weak positive solutions. We say that the function $u \in H_0^s(B_R(0))$ is a weak solution of Equation (1) if u satisfies Equation (1) weakly. More precisely, we are looking for a function u from $B_R(0)$ to R such that $u \in H_0^s(B_R(0)),\ u(x) > 0$ a.e. in $B_R(0)$, and

$$\begin{aligned} &\int_Q \frac{(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{N+2s}}\,dxdy \\ &= \int_{B_R(0)} \lambda u^\beta \phi + a(x)u^{-\gamma}\phi\,dx,\ \ \forall \phi \in H_0^s(B_R(0)). \end{aligned} \tag{3}$$

where $Q = R^{2N} \setminus (\mathcal{C}B_R(0) \times \mathcal{C}B_R(0))$ and $\mathcal{C}B_R(0) = R^N \setminus B_R(0)$.

We say that $u \in H_0^s(B_R(0))$ is a weak sub(super)solution of Equation (1) if $u > 0$ in $B_R(0)$ and

$$\begin{aligned} &\int_Q \frac{(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{N+2s}}\,dxdy \\ &\leq (\geq) \int_{B_R(0)} \lambda u^\beta \phi + a(x)u^{-\gamma}\phi\,dx,\ \ \forall \phi \in H_0^s(B_R(0)),\ \phi \geq 0. \end{aligned}$$

The greatest difficulty in this problem is that the vanish boundary value is such that the nonlinearity singular is at the boundary $\partial B_R(0)$. Therefore, the essence of this problem is determining which class of the testing function ϕ makes Equation (3) hold. It is worth emphasizing that since $0 < 1-\gamma < 1$, the natural associated functional

$$I_\lambda(u) = \frac{1}{2}\int_Q \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dxdy - \frac{\lambda}{1+\beta}\int_{B_R(0)} |u|^{1+\beta}\,dx - \frac{1}{1-\gamma}\int_{B_R(0)} a(x) \mid u \mid^{1-\gamma}\ dx$$

is not Frechet-differentiable. So, the fractional singular elliptic Equation (1) cannot be studied by directly using critical point theory. In recent years, the study of elliptic problems with a singular nonlinearity has attracted many researchers of partial differential equations ([23–25] and the references therein). In [24], the authors studied the existence, regularity, and multiplicity of weak solutions for fractional p-Laplacian equations with singular nonlinearities via fibering maps. The authors studied the existence and regularity of weak solutions to Equation (1).

A. Capella, J. Davila, L. Dupaigne, and Y. Sire [25] provided new results with respect to the existence and regularity of radial extremal solutions for some nonlocal problems with smooth nonlinearity by following the s-harmonic extension approach, as in [3]. Recently, W.X. Chen and C.M. Li [26] established radial symmetry and monotonicity for positive solutions to the fractional p-Laplacian by moving planes.

As far as we know, there are no published results with respect to the existence and multiplicity of radial symmetry weak solutions to Equation (1) in the sense of Equation (3). As we know, the moving plane method is one of the most effective strategies to establish radial symmetry for weak solutions of the classic Laplacian equations. However, in our case, because of the singular nature of our problem, we have to manage more difficulties. One way to overcome these difficulties is by using the variational principle combined with the Schwarz spherical rearrangement.

The structure of the paper is as follows. In Section 2, we give some preliminaries and basic facts, and we formulate our main results. In Section 3, we use variational methods, Nehari manifold, and Schwarz spherical rearrangement to prove our main results.

2. Preliminaries and Main Results

In this section, we present some collected preliminary facts for future reference. Before proceeding, we need some definitions for the spaces, results, and notations. Throughout this paper, we make use of the following notations. $C, C_0, C_1, C_2, \cdots$ denote (possibly different) positive constants. We denote $B_R(0)$ as B. Let $|\Omega|$ be the measure of domain Ω and let $\|\cdot\|_q$ be the $L^q(\Omega)(1 \leq q \leq \infty)$ normal. The norm in $H_0^s(\Omega)$ is denoted by $\|\cdot\| = \|\cdot\|_{H_0^s}$. By using the embedding theorems in [4], we derive $H_0^s(\Omega)$ so that it is compactly and continuously embedded in $L^p(\Omega)$ when $1 \leq p < 2_s^* = \frac{2N}{N-2s}(N > 2s)$, and the embedding is continuous but not compact if $p = 2_s^*$. We can let (see, for instance, [6,27,28])

$$S = \inf_{u \in H_0^s(B) \setminus \{0\}} \frac{\int_Q \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dxdy}{(\int_B |u|^{2_s^*})^{2/2_s^*}}. \tag{4}$$

Now, we can state the main results of our paper.

Theorem 1. *Let $B = B_R(0) \subset R^N(N \geq 3)$ be the ball centered at 0 with radius R. Let $\gamma \in (0, 1)$, $1 < \beta < 2_s^* - 1$, $a(x) \in L^2(\Omega)$ with $a(x) \geq 0$ in B. Then, there exists a real λ^* such that for all $\lambda \in (0, \lambda^*)$, Equation (1) has at least two radial symmetry weak positive solutions u_λ, $v_\lambda \in H_0^s(B)$ in the sense of Equation (3).*

Remark 1. *Since $\{\phi_n\} \subset C_0^\infty(B)$ such that $\phi_n \longrightarrow \phi$ in $H_0^s(B)$ topology, it is not true, in general, that*

$$\int_B a(x)u^{-\gamma}\phi_n \, dx \longrightarrow \int_B a(x)u^{-\gamma}\phi \, dx.$$

Therefore, one cannot replace $H_0^s(B)$ in (3.1) by $C_0^\infty(B)$. We point out that there is a requirement for the testing functions ϕ in Definition 2.1 in [24] and Definition 2.1 in [22], i.e., $\phi \in C_0^\infty(B)$. This indicates that our results include and extend their previous conclusions.

Remark 2. *It is worth pointing out that we only assume the coefficient $a(x) \geq 0$. This greatly relaxes the condition for $a(x)$ in [22]. Here, the authors require that $a(x)$ has a uniform positive lower bound and that there exists a positive constant $\theta > 0$ such that $a(x) \geq \theta$ for all $x \in \Omega$. Hence, the results reported in this paper are new in the area of singular fractional elliptic problems.*

3. Existence and Multiplicity of Weak Positive Solution of Equation (1)

We are now in a position to give the proof of Theorem (4). To start, let us define the Nehari manifold,

$$\Lambda = \left\{ u \in H_0^s(B) \middle| \int_Q \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} \, dxdy - \lambda \int_B u^{1+\beta} \, dx - \int_B a(x)u^{1-\gamma} \, dx = 0, u \geq 0 \right\}.$$

Notice that $u \in \Lambda$ if u is a weak positive solution of Equation (1). The fact suggests that we apply the following splitting for Λ.

$$\Lambda^+ = \left\{ u \in \Lambda \middle| (1+\gamma) \int_Q \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} \, dxdy - \lambda(\beta+\gamma) \int_B u^{1+\beta} \, dx > 0 \right\}$$

$$\Lambda^- = \left\{ u \in \Lambda \middle| (1+\gamma) \int_Q \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} \, dxdy - \lambda(\beta+\gamma) \int_B u^{1+\beta} \, dx < 0 \right\}$$

$$\Lambda^0 = \left\{ u \in \Lambda \middle| (1+\gamma) \int_Q \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} \, dxdy - \lambda(\beta+\gamma) \int_B u^{1+\beta} \, dx = 0 \right\}.$$

To obtain Theorem 1, we divide the proof into several preliminary lemmas.

Lemma 1. *There exists $L > 0$ such that $\|u\| \leq L, \forall u \in \Lambda^+$.*

Proof. Since $u \in \Lambda^+$, using the definition of Λ and Λ^+, we obtain

$$\|u\|^2 < \frac{1}{\beta - 1}\int_B a(x)u^{1-\gamma}dx$$

From the Hölder inequality, we derive the existence of the constant $C > 0$ such that

$$\int_B a(x)u^{1-\gamma}dx \leq \|a\|_2|B|^{\frac{\beta-1+2\gamma}{2(1+\beta)}}\|u\|_{1+\beta}^{1-\gamma} \leq C\|a\|_2|B|^{\frac{\beta-1+2\gamma}{2(1+\beta)}}\|u\|^{1-\gamma} \tag{5}$$

Thus, we obtain

$$\|u\|^2 < \frac{1}{\beta-1}C\|a\|_2|B|^{\frac{\beta-1+2\gamma}{2(1+\beta)}}\|u\|^{1-\gamma}.$$

Therefore, the result of Lemma 1 follows by letting $2 > 1-\gamma$. This completes the proof of Lemma 1. □

Lemma 2. *The functional I_λ is coercive and bounded below on Λ^+.*

Proof. Let $u \in \Lambda^+$. Combining the definition of Λ^+ and Equation (5), we have

$$\begin{aligned} I_\lambda(u) &= \Big(\frac{1}{2} - \frac{1}{1+\beta}\Big)\|u\|^2 + \Big(\frac{1}{1+\beta} - \frac{1}{1-\gamma}\Big)\int_B a(x)u^{1-\gamma}dx \\ &\geq \Big(\frac{1}{2} - \frac{1}{1+\beta}\Big)\|u\|^2 + \Big(\frac{1}{1+\beta} - \frac{1}{1-\gamma}\Big)C\|a\|_2|B|^{\frac{\beta-1+2\gamma}{2(1+\beta)}}\|u\|^{1-\gamma} \end{aligned}$$

i.e.,

$$I_\lambda(u) \geq C_1\|u\|^2 - C_2\|u\|^{1-\gamma}, \ \ \forall\, u \in \Lambda^+$$

for some positive constants C_1 and C_2. This implies that I_λ is coercive and bounded below on Λ^+. This completes the proof of Lemma 2. □

Lemma 3. *The minimal value $m_\lambda = \inf_{u\in\Lambda^+} I_\lambda(u) < 0$.*

Proof. By using the Hölder inequality and Equation (4), we get

$$\int_B |u|^{1+\beta}dx \leq \int_B |u|^{2_s^*}dx \leq S^{-\frac{2_s^*}{2}}\|u\|^{2_s^*}, \ \ \forall u \in \Lambda^+.$$

Applying the inequality in Equation (5), we deduce

$$I_\lambda(u) \geq \frac{1}{2}\|u\|^2 - \lambda C\|u\|^{2_s^*} - C\|u\|^{1-\gamma}, \ \ \forall u \in \Lambda^+. \tag{6}$$

Since $0 < 1-\gamma < 2 < 1+\beta < 2_s^*$, there exist $r > 0, \epsilon > 0$ such that

$$\frac{1}{2}\|u\|^2 - \frac{1}{1+\beta}\|u\|_{1+\beta}^{1+\beta} \geq \epsilon, \ \frac{1}{2}\|u\|^2 - C\|u\|^{1-\gamma} \geq \epsilon, \ \ \forall u \in \partial U_r \tag{7}$$

and

$$\frac{1}{2}\|u\|^2 - \frac{1}{1+\beta}\|u\|_{1+\beta}^{1+\beta} \geq 0, \ \ \forall u \in U_r \tag{8}$$

where $U_r = \{u \in \Lambda^+ | \parallel u \parallel \leq r\}$. Then, we can choose a small enough $\lambda_1 > 0$ such that for any fixed $\lambda \in (0,\ \lambda_1)$, it follows that

$$I_\lambda(u) \geq \frac{\epsilon}{2} > 0, \quad \forall u \in \partial U_r.$$

Furthermore, for any fixed $v > 0$, simple calculations show that

$$\begin{aligned} I_\lambda(tv) &= \frac{1}{2}\|tv\|^2 - \frac{\lambda}{1+\beta}\|tv\|_{1+\beta}^{1+\beta} - \frac{1}{1-\gamma}\int_\Omega a(x)|tv|^{1-\gamma}\,dx \\ &= -Ct^{1-\gamma} + o(t^2)(t \to 0) \end{aligned}$$

From $0 < 1-\gamma < 2 < 1+\beta$, we conclude that if $t > 0$ is sufficiently small, then $I_\lambda(tv) < 0$ for any fixed $v > 0$. This implies that $m_\lambda = \inf_{\Lambda^+} I_\lambda < 0$. This completes the proof of Lemma 3. □

For the reader's convenience, we are ready to describe Lemma 4 (below) on the embedding properties of $H_0^s(\Omega)$. We refer to [6,9] and their references for its proof.

Lemma 4. *Let $s \in (0,1)$ and $p \in [1,+\infty)$ such that $sp < N$. Let $\Omega \subset R^N$ be an extension domain for $W^{s,p}(\Omega)$. Then, there exists a positive constant $C(N,s,p,\Omega)$ such that, for any $u \in W^{s,p}(\Omega)$,*

$$\|u\|_q \leq C\|u\|_{W^{s,p}}$$

for any $q \in [p,\ p_s^* = \frac{Np}{N-sp}]$; that is, the space $W^{s,p}(\Omega)$ is continuously embedded in $L^q(\Omega)$ for any $q \in [p,\ p_s^*]$. If, in addition, Ω is bounded, then the space $W^{s,p}(\Omega)$ is compactly embedded in $L^q(\Omega)$ for any $q \in [p,\ p_s^*]$.

To state the next results, we need the next Lemma on Schwarz symmetrization and rearrangement, presented without proofs. One can refer to [29–31]. Assume u is a real function defined in R^N. The distribution function of u is defined as

$$\mu_u(t) = \Big|x \in R^N : | u(x) | > t\Big|.$$

Then, μ_u is non-increasing and right-continuous. The decreasing rearrangement of u is given by

$$u^*(s) = sup\{t \geq 0 : \mu_u(t) \geq s\}.$$

The function

$$u^\sharp(x) = u^*(\omega_n|x|^n)$$

is defined as the Schwarz symmetrization of u. The function $u^\sharp$ has the following basic properties.

Lemma 5. *Assume u, v are integral functions in R^N, and let $g : R^N \to R$ be non-decreasing nonnegative functions. Then, we conclude that*

(1) $\int_{R^N} g(|u(x)|)\,dx = \int_{R^N} g(|u^\sharp(x)|)\,dx$,
(2) If $u, v \in L^p(R^N)(p > 1)$, then $\parallel u^\sharp - v^\sharp \parallel_{L^p(R^N)} \leq \parallel u - v \parallel_{L^p(R^N)}$,
(3) If $u \in H^s(R^N)$, then $u^\sharp \in H^s(R^N)$. Furthermore,

$$\int_{R^N \times R^N} \frac{|u^\sharp(x) - u^\sharp(y)|^2}{|x-y|^{N+2s}}dxdy \leq \int_{R^N \times R^N} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}}dxdy.$$

Lemma 6. *For all $\lambda \in (0,\ \lambda_1)$, there exists a function $u_\lambda \in \Lambda^+$ which is radially symmetric about the origin such that $I_\lambda(u_\lambda) = m_\lambda = \inf_{\Lambda^+} I_\lambda$.*

Proof. The proof is inspired by [14]. Let $\{u_n\} \subseteq U_r$ be a minimizing sequence such that $I_\lambda(u_n) \to m_\lambda$ as $n \to \infty$. Using Lemma 1, the sequence $\{u_n\}$ is bounded in $H_0^s(B)$. Thus, we can claim that there exists a subsequence of $\{u_n\}$ (still denoted by $\{u_n\}$) such that $u_n \rightharpoonup \widetilde{u}_\lambda$ weakly in $H_0^s(B)$, strongly in $L^p(B)(1 \leq p < \frac{2N}{N-2s})$, and pointwise a.e. in B. According to Hölder's inequality, as $n \to \infty$,

$$\begin{aligned}\int_B a(x)u_n^{1-\gamma}dx &\leq \int_B a(x)\widetilde{u}_\lambda^{1-\gamma}dx + C\|u_n - \widetilde{u}_\lambda\|_2^{1-\gamma} \\ &= \int_B a(x)\widetilde{u}_\lambda^{1-\gamma}dx + o(1)\end{aligned}$$

and

$$\begin{aligned}\int_B a(x)\widetilde{u}_\lambda^{1-\gamma}dx &\leq \int_B a(x)u_n^{1-\gamma}dx + C\|u_n - \widetilde{u}_\lambda\|_2^{1-\gamma} \\ &= \int_B a(x)u_n^{1-\gamma}dx + o(1).\end{aligned}$$

Consequently, we obtain

$$\int_B a(x)u_n^{1-\gamma}dx = \int_B a(x)\widetilde{u}_\lambda^{1-\gamma}dx + o(1). \tag{9}$$

Using the Brezis–Lieb Lemma, we derive

$$\|u_n\|_{1+\beta}^{1+\beta} = \|\widetilde{u}_\lambda\|_{1+\beta}^{1+\beta} + \|u_n - \widetilde{u}_\lambda\|_{1+\beta}^{1+\beta} + o(1) \tag{10}$$

and

$$\|u_n\|^2 = \|\widetilde{u}_\lambda\|^2 + \|u_n - \widetilde{u}_\lambda\|^2 + o(1). \tag{11}$$

Recall Equation (7) and $m_\lambda < 0$; thus, $\|u_n\| \leq r_0 < r$ for some positive constant r independent of n. So, from Equation (11) and $\widetilde{u}_\lambda \in U_r$, we have $u_n - \widetilde{u}_\lambda \in U_r$ while n is large enough. By using Equation (8) again, we deduce

$$\frac{1}{2}\|u_n - \widetilde{u}_\lambda\|^2 - \frac{1}{1+\beta}\|u_n - \widetilde{u}_\lambda\|_{1+\beta}^{1+\beta} \geq 0.$$

Combining the above arguments with Equations (9)–(11), we have

$$\begin{aligned}m_\lambda &= I_\lambda(u_n) + o(1) \\ &= I_\lambda(\widetilde{u}_\lambda) + \frac{1}{2}\|u_n - \widetilde{u}_\lambda\|^2 - \frac{\lambda}{1+\beta}\|u_n - \widetilde{u}_\lambda\|_{1+\beta}^{1+\beta} + o(1) \\ &\geq I_\lambda(\widetilde{u}_\lambda) + o(1) \\ &\geq m_\lambda + o(1)(n \to \infty),\end{aligned}$$

namely, $0 \geq I_\lambda(\widetilde{u}_\lambda) - m_\lambda + o(1) \geq o(1)$. Letting $n \to \infty$, we conclude $I_\lambda(\widetilde{u}_\lambda) = m_\lambda$.

Next, we show that $\widetilde{u}_\lambda \in \Lambda^+$. It is sufficient to prove $u_n \to u_\lambda$ strongly in $H_0^s(B)$. From $I_\lambda(\widetilde{u}_\lambda) = m_\lambda$ and $m_\lambda = I_\lambda(\widetilde{u}_\lambda) + \frac{1}{2}\|u_n - \widetilde{u}_\lambda\|^2 - \frac{\lambda}{1+\beta}\|u_n - \widetilde{u}_\lambda\|_{1+\beta}^{1+\beta} + o(1)$, we have

$$0 = \frac{1}{2}\|u_n - \widetilde{u}_\lambda\|^2 - \frac{\lambda}{1+\beta}\|u_n - \widetilde{u}_\lambda\|_{1+\beta}^{1+\beta} + o(1).$$

Since $u_n \rightharpoonup \widetilde{u}_\lambda$ weakly in $H_0^s(B)$, by Lemma 4, we infer that $u_n \to \widetilde{u}_\lambda$ strongly in $L^{1+\beta}(B)$, thus, $\|u_n - \widetilde{u}_\lambda\|_{1+\beta}^{1+\beta} \to 0$ as $n \to \infty$. Consequently,

$$\|u_n - \widetilde{u}_\lambda\|^2 \to 0(n \to \infty),$$

i.e., $u_n \to u_\lambda$ strongly in $H_0^s(B)$. Hence, $\widetilde{u}_\lambda \in \Lambda^+$ is a minimizer of I_λ in Λ^+.

In order to apply the Schwarz symmetrization rearrangement of Lemma 5, we should extend $\widetilde{u}_\lambda$ to a function defined in R^N. In fact, by using the extension theorems, $\widetilde{u}_\lambda$ can be extended to a function in $H^s(R^N)$ by defining it as zero outside of $R^N \setminus B$. Since the functions $t^{1+\beta}$ and $t^{1-\gamma}$ are non-decreasing, by using Lemma 5, we have

$$\int_B |(\widetilde{u}_\lambda)^\sharp|^{1+\beta}dx = \int_{R^N} |(\widetilde{u}_\lambda)^\sharp|^{1+\beta}dx = \int_{R^N} |\widetilde{u}_\lambda|^{1+\beta}dx = \int_B |\widetilde{u}_\lambda|^{1+\beta}dx,$$

$$\begin{aligned}\int_B a(x) \mid (\widetilde{u}_\lambda)^\sharp \mid^{1-\gamma} dx &= \int_{R^N} a(x) \mid (\widetilde{u}_\lambda)^\sharp \mid^{1-\gamma} dx \\ = \int_{R^N} a(x) \mid \widetilde{u}_\lambda \mid^{1-\gamma} dx &= \int_B a(x) \mid \widetilde{u}_\lambda \mid^{1-\gamma} dx\end{aligned}$$

and

$$\int_Q \frac{|(\widetilde{u}_\lambda)^\sharp(x) - (\widetilde{u}_\lambda)^\sharp(y)|^2}{|x-y|^{N+2s}}dxdy \le \int_Q \frac{|(\widetilde{u}_\lambda)(x) - (\widetilde{u}_\lambda)(y)|^2}{|x-y|^{N+2s}}dxdy.$$

Consequently, we deduce that

$$I_\lambda((\widetilde{u}_\lambda)^\sharp) \le I_\lambda(\widetilde{u}_\lambda).$$

Therefore, the **radial symmetry** function $u_\lambda := (\widetilde{u}_\lambda)^\sharp \in \Lambda^+$ is also a minimizer of I_λ in Λ^+. This completes the proof of Lemma 6. □

Existence of radial symmetry weak positive solution u_λ.

Lemma 7. *The minimizer $u_\lambda(x) > 0$, $\forall x \in B$.*

Proof. For any $\phi \in H_0^s(B)$ with $\phi \ge 0$ and $t > 0$ small enough, since u_λ is a minimizer, we have

$$\begin{aligned}
&0 \le I_\lambda(u_\lambda + t\phi) - I_\lambda(u_\lambda) \\
&= \frac{1}{2}\int_Q \frac{|u_\lambda(x) + t\phi(x) - u_\lambda(y) - t\phi(y)|^2}{|x-y|^{N+2s}}\, dxdy - \frac{\lambda}{1+\beta}\int_B |u_\lambda + t\phi|^{1+\beta}\, dx \\
&\quad - \frac{1}{1-\gamma}\int_B a(x) \mid u_\lambda + t\phi \mid^{1-\gamma}\, dx - \frac{1}{2}\int_Q \frac{|u_\lambda(x) - u_\lambda(y)|^2}{|x-y|^{N+2s}}\, dxdy \\
&\quad + \frac{\lambda}{1+\beta}\int_B |u_\lambda|^{1+\beta}\, dx + \frac{1}{1-\gamma}\int_B a(x) \mid u_\lambda \mid^{1-\gamma}\, dx \\
&\le \frac{1}{2}\int_Q \frac{|u_\lambda(x) - u_\lambda(y) + t(\phi(x) - \phi(y))|^2 - |u_\lambda(x) - u_\lambda(y)|^2}{|x-y|^{N+2s}}\, dxdy.
\end{aligned}$$

Dividing by $t > 0$ and letting $t \to 0$ therefore shows

$$\int_Q \frac{(u_\lambda(x) - u_\lambda(y))(\phi(x) - \phi(y))}{|x-y|^{N+2s}}\, dxdy \ge 0, \quad \forall \phi \in H_0^s(B),\ \phi \ge 0.$$

This means that $u_\lambda \in H_0^s(B)$ is a weak subsolution $(-\Delta)^s u \ge 0$, in B.

In the following, we prove that $u_\lambda > 0$, in B.

We need the following strong maximum principle for the nonlocal operator $(-\Delta)^s$ (Theorem 4.1 in [32]). For the convenience of the reader, we report the main result of Theorem 4.1 in [32]. If $u \in H_0^s(\Omega)$ satisfies, in a weak sense, that $(-\Delta)^s u \ge 0$ in Ω and $\int_\Omega \frac{|u(x)|}{1+|x|^{N+2s}}\, dx < \infty$, then u is lower semicontinuous in Ω, and $u(x) > \inf_\Omega u$, $\forall x \in \Omega$.

Now, since

$$\int_B \frac{|u_\lambda(x)|}{1+|x|^{N+2s}}\,dx \le \int_B |u_\lambda(x)|\,dx < \infty,$$

then,

$$u_\lambda > \inf_B u_\lambda \ge 0, \; in \; B.$$

This completes the proof of Lemma 6. □

From Lemma 3 in [12], we have the following Lemma 8 immediately below.

Lemma 8. *For any $u \in \Lambda^+$, there exists $\varepsilon > 0$ and a continuous function $f = f(\omega) > 0$, $\omega \in H_0^s(B)$, $\|\omega\| < \varepsilon$ satisfying that $f(0) = 1$, $f(\omega)(u+\omega) \in \Lambda^+$, $\forall \omega \in H_0^s(B)$, $\|\omega\| < \varepsilon$.*

Lemma 9. *For any given $\varphi \in H_0^s(B)$, $\varphi \ge 0$, there exists $T > 0$ such that $I_\lambda(u_\lambda + t\varphi) \ge I_\lambda(u_\lambda)$ for all $t \in [0,\ T]$.*

Proof. Let

$$g(t) = \|u_\lambda + t\varphi\|^2 - \lambda\beta\|u_\lambda + t\varphi\|_{1+\beta}^{1+\beta} + \gamma\int_B a(x)|u_\lambda + t\varphi|^{1-\gamma}\,dx,\ \forall\, t > 0.$$

Using the continuity of g, $g(0) > 0$, and $u_\lambda \in \Lambda^+$, we deduce that there exists $T > 0$ such that $g(t) > 0$ for all $t \in [0,\ T]$. On the other hand, applying Lemma 8, for each $t > 0$ there exists $t' > 0$ such that $t'(u_\lambda + t\varphi) \in \Lambda^+$. Therefore, $t' \to 1$ as $t \to 0$, and for each $t \in [0,\ T]$, we obtain $I_\lambda(u_\lambda + t\varphi) \ge I_\lambda[t'(u_\lambda + t\varphi)] \ge \inf_{\Lambda^+} I_\lambda = I_\lambda(u_\lambda)$. This completes the proof of Lemma 9. □

Lemma 10. *The minimizer $u_\lambda \in \Lambda^+$ is a weak positive $H_0^s(B)-$ solution of Equation (1), i.e., $u_\lambda \in H_0^s(B)$ satisfying*

$$\int_Q \frac{(u_\lambda(x) - u_\lambda(y))(\phi(x) - \phi(y))}{|x-y|^{N+2s}}\,dxdy = \int_B \lambda u_\lambda^\beta \phi + a(x)u_\lambda^{-\gamma}\phi\,dx, \quad \forall \phi \in H_0^s(B).$$

Proof. The novelty of Equation (1) lies not only in the non-differentiability of the corresponding functional $I_\lambda(u)$ but also in the singularity of Equation (1). There seem to be difficulties to get that the minimizer u_λ is a weak solution of Equation (1) directly from critical point theory. Inspired by Y.J. Sun [12], using direct and detailed computations, we still proved that minimizer u_λ is a weak solution of Equation (1).

Recall that in Lemma 9, we infer that for any $\varphi \in H_0^s(B)$, $\varphi \ge 0$, and $t \in [0,\ T]$, there is $I_\lambda(u_\lambda + t\varphi) - I_\lambda(u_\lambda) \ge 0$. Hence, easy computations show that

$$\begin{aligned}
&\frac{1}{1-\gamma}\int_B a(x)[|\ u_\lambda + t\phi\ |^{1-\gamma} - |\ u_\lambda\ |^{1-\gamma}]\,dx \\
\le\ &\frac{1}{2}\int_Q \frac{|u_\lambda(x) - u_\lambda(y) + t(\phi(x) - \phi(y))|^2 - |u_\lambda(x) - u_\lambda(y)|^2}{|x-y|^{N+2s}}\,dxdy \\
&-\frac{\lambda}{1+\beta}\int_B [(u_\lambda + t\phi)^{1+\beta} - (u_\lambda)^{1+\beta}]\,dx, \qquad \forall \varphi \in H_0^s(B),\ \varphi \ge 0.
\end{aligned}$$

Dividing $t > 0$ and letting $t \to 0^+$ implies that

$$\frac{1}{1-\gamma}\liminf_{t\to 0^+}\int_B \frac{a(x)[|\ u_\lambda + t\phi\ |^{1-\gamma} - |\ u_\lambda\ |^{1-\gamma}]}{t}\,dx$$

$$\leq \int_Q \frac{(u_\lambda(x) - u_\lambda(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+2s}}\, dxdy - \lambda \int_B u_\lambda^\beta \varphi\, dx, \ \ \forall \varphi \in H_0^s(B),\ \varphi \geq 0.$$

From simple arguments and Fatou's Lemma, we can get

$$\int_B a(x) u_\lambda^{-\gamma} \varphi\, dx \leq \frac{1}{1-\gamma} \liminf_{t \to 0^+} \int_B \frac{a(x)[\mid u_\lambda + t\phi \mid^{1-\gamma} - \mid u_\lambda \mid^{1-\gamma}]}{t}\, dx.$$

Combining these relations, we conclude that

$$\begin{aligned} &\lambda \int_B u_\lambda^\beta \varphi\, dx - \int_B a(x) u_\lambda^{-\gamma} \varphi\, dx \\ &\leq \int_Q \frac{(u_\lambda(x) - u_\lambda(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+2s}}\, dxdy \quad \forall \varphi \in H_0^s(B),\ \varphi \geq 0. \end{aligned} \tag{12}$$

For any given $\phi \in H_0^s(B)$, taking

$$\varphi = (u_\lambda + t\phi)^+ \in H_0^s(B),\ \ \varphi \geq 0$$

into Equation (12), we have

$$\begin{aligned} 0 &\leq \int_Q \frac{(u_\lambda(x) - u_\lambda(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+2s}}\, dxdy - \lambda \int_B u_\lambda^\beta \varphi\, dx - \int_B a(x) u_\lambda^{-\gamma} \varphi\, dx \\ &= \int_{Q'} \frac{(u_\lambda(x) - u_\lambda(y))((u_\lambda + t\phi)(x) - (u_\lambda + t\phi)(y))}{|x-y|^{N+2s}}\, dxdy \\ &\quad - \lambda \int_{\{x|(u_\lambda + t\phi) \geq 0\}} u_\lambda^\beta (u_\lambda + t\phi)\, dx - \int_{\{x|(u_\lambda + t\phi) \geq 0\}} a(x) u_\lambda^{-\gamma} (u_\lambda + t\phi)\, dx \\ &= \|u_\lambda\|^2 - \lambda \|u_\lambda\|_{1+\beta}^{1+\beta} - \int_B a(x) u_\lambda^{1-\gamma} \\ &\quad + t \left(\int_Q \frac{(u_\lambda(x) - u_\lambda(y))(\phi(x) - \phi(y))}{|x-y|^{N+2s}}\, dxdy - \lambda \int_B u_\lambda^\beta \phi\, dx - \int_B a(x) u_\lambda^{-\gamma} \phi\, dx \right) \\ &\quad - \int_{Q''} \frac{(u_\lambda(x) - u_\lambda(y))((u_\lambda + t\phi)(x) - (u_\lambda + t\phi)(y))}{|x-y|^{N+2s}}\, dxdy \\ &\quad - \lambda \int_{\{x|(u_\lambda + t\phi) < 0\}} u_\lambda^\beta (u_\lambda + t\phi)\, dx - \int_{\{x|(u_\lambda + t\phi) < 0\}} a(x) u_\lambda^{-\gamma} (u_\lambda + t\phi)\, dx \\ &\leq t \left(\int_Q \frac{(u_\lambda(x) - u_\lambda(y))(\phi(x) - \phi(y))}{|x-y|^{N+2s}}\, dxdy - \lambda \int_B u_\lambda^\beta \phi\, dx - \int_B a(x) u_\lambda^{-\gamma} \phi\, dx \right) \\ &\quad - t \int_{Q''} \frac{(u_\lambda(x) - u_\lambda(y))(\phi(x) - \phi(y))}{|x-y|^{N+2s}}\, dxdy \end{aligned}$$

where $Q' = R^{2N} \setminus (\mathcal{C}\Omega' \times \mathcal{C}\Omega')$, $\mathcal{C}\Omega' = R^N \setminus \{x|(u_\lambda + t\phi) \geq 0\}$ and $Q'' = R^{2N} \setminus (\mathcal{C}\Omega'' \times \mathcal{C}\Omega'')$, $\mathcal{C}\Omega'' = R^N \setminus \{x|(u_\lambda + t\phi) < 0\}$. Since the measure of the set $\{x|(u_\lambda + t\phi) < 0\}$ tends to 0 as $t \to 0^+$, it means $\int_{Q''} \frac{(u_\lambda(x) - u_\lambda(y))(\phi(x) - \phi(y))}{|x-y|^{N+2s}}\, dxdy \to 0$ when $t \to 0^+$. Thus, dividing by $t > 0$, we infer that

$$\int_Q \frac{(u_\lambda(x) - u_\lambda(y))(\phi(x) - \phi(y))}{|x-y|^{N+2s}}\, dxdy - \lambda \int_B u_\lambda^\beta \phi\, dx - \int_B a(x) u_\lambda^{-\gamma} \phi\, dx \geq 0, \ \ \forall \phi \in H_0^s(B).$$

Observe that $\phi \in H_0^s(B)$ is arbitrary. Replacing ϕ by $-\phi$ in the above inequality, one gets

$$\int_Q \frac{(u_\lambda(x) - u_\lambda(y))(\phi(x) - \phi(y))}{|x-y|^{N+2s}}\, dxdy - \lambda \int_B u_\lambda^\beta \phi\, dx - \int_B a(x) u_\lambda^{-\gamma} \phi\, dx \leq 0, \ \ \forall \phi \in H_0^s(B).$$

Hence,

$$\int_Q \frac{(u_\lambda(x) - u_\lambda(y))(\phi(x) - \phi(y))}{|x-y|^{N+2s}}\, dxdy - \lambda \int_B u_\lambda^\beta \phi\, dx - \int_B a(x) u_\lambda^{-\gamma} \phi\, dx = 0, \quad \forall \phi \in H_0^s(B)$$

and the conclusion follows. The proof of this lemma is completed. □

Existence of a weak positive $H_0^s(B)-$ solution v_λ.

Lemma 11. *There exists $\lambda_2 > 0$ such that Λ^- is closed in $H_0^s(B)$ for all $\lambda \in (0,\ \lambda_2)$.*

Proof. We claim $\Lambda^0 = \{0\}$. Suppose, by contradiction, that there exists an $h \in \Lambda^0$ with $h \neq 0$. From the definitions of Λ^0 and Λ, it follows that

$$(1+\gamma)\|h\|^2 = \lambda(\beta+\gamma)\|h\|_{1+\beta}^{1+\beta}, \ \Big(\frac{\beta-1}{\beta+\gamma}\Big)\|h\|^2 = \int_B a(x) h^{1-\gamma}\, dx.$$

Therefore, we find that

$$\Gamma = \|h\|^2\Big(\frac{\beta-1}{\beta+\gamma}\Big)\Big\{\Big[\frac{(1+\gamma)\|h\|^2}{\lambda(\beta+\gamma)\|h\|_{1+\beta}^{1+\beta}}\Big]^{\frac{1+\gamma}{\beta-1}} - 1\Big\} \equiv 0.$$

On the other hand, by using Equation (5) and fractional Sobolev inequality, we infer that

$$\begin{aligned}\Gamma &= \Big(\frac{1+\gamma}{\lambda(\beta+\gamma)}\Big)^{\frac{1+\gamma}{\beta-1}}\Big(\frac{\beta-1}{\beta+\gamma}\Big)\Big[\frac{\|h\|^{2(\beta+\gamma)}}{\|h\|_{1+\beta}^{(1+\beta)(\gamma+1)}}\Big]^{\frac{1}{\beta-1}} - \int_B a(x) h^{1-\gamma}\, dx \\ &\geq \Big(\frac{1+\gamma}{\lambda(\beta+\gamma)}\Big)^{\frac{1+\gamma}{\beta-1}}\Big(\frac{\beta-1}{\beta+\gamma}\Big) C\|h\|_{1+\beta}^{1-\gamma} - \int_B a(x) h^{1-\gamma}\, dx \\ &\geq \Big(\frac{1+\gamma}{\lambda(\beta+\gamma)}\Big)^{\frac{1+\gamma}{\beta-1}}\Big(\frac{\beta-1}{\beta+\gamma}\Big) C\|h\|_{1+\beta}^{1-\gamma} - \|a\|_2 |B|^{\frac{\beta-1+2\gamma}{2(1+\beta)}}\|h\|_{1+\beta}^{1-\gamma}\end{aligned}$$

where the constant $C > 0$ is independent of λ. Since $\lim\limits_{\lambda\to 0}\frac{1+\gamma}{\lambda(\beta+\gamma)} = +\infty$, it means that there exists $\lambda_2 > 0$ small enough to satisfy

$$\Big(\frac{1+\gamma}{\lambda(\beta+\gamma)}\Big)^{\frac{1+\gamma}{\beta-1}}\Big(\frac{\beta-1}{\beta+\gamma}\Big) C - \|a\|_2 |B|^{\frac{\beta-1+2\gamma}{2(1+\beta)}} > 1, \quad \forall\, \lambda \in (0,\ \lambda_2)$$

and, consequently,

$$\Gamma \geq \|h\|_{1+\beta}^{1-\gamma} > 0$$

which yields a contraction. So, the set $\Lambda^0 = \{0\}$.

Assume $\{u_n\} \subseteq \Lambda^-$ is a sequence satisfying $u_n \to u$ in $H_0^s(B)$. Using the Sobolev inequalities and continuous compact embedding, we have $u_n \to u$ in $L^{1+\beta}(B)$ and $u \in \Lambda^- \cup \Lambda^0$. Recalling the definition of Λ^- once more, we infer that

$$\|u\|_{1+\beta} \geq \Big(\frac{1+\gamma}{\lambda(\beta+\gamma)}\Big)^{\frac{1}{\beta-1}} S^{\frac{1}{\beta-1}} |B|^{-2\big[\frac{2_s^*-1-\beta}{2_s^*(1+\beta)(\beta-1)}\big]} > 0, \quad \forall\, u \in \Lambda^- \tag{13}$$

thus, $u \neq 0$, i.e., $u \in \Lambda^-$. This completes the proof of Lemma 11. □

Lemma 12. *There exists $\lambda_3 > 0$ such that $I_\lambda(u) \geq 0$ for all $u \in \Lambda^-$ while $\lambda \in (0,\ \lambda_3)$.*

Proof. Suppose, by contradiction, there is a $v \in \Lambda^-$ such that $I_\lambda(v) < 0$, that is,

$$\frac{1}{2}\|v\|^2 - \frac{\lambda}{1+\beta}\|v\|_{1+\beta}^{1+\beta} - \frac{1}{1-\gamma}\int_B a(x)v^{1-\gamma}\,dx < 0.$$

By the definition of Λ^-, it follows that

$$\lambda\Big(\frac{1}{2} - \frac{1}{1+\beta}\Big)\|v\|_{1+\beta}^{1+\beta} - \Big(\frac{1}{1-\gamma} - \frac{1}{2}\Big)\int_B a(x)v^{1-\gamma}\,dx < 0,$$

and, combining Equation (5), we have

$$\|v\|_{1+\beta}^{\gamma+\beta} < \frac{(\gamma+1)(1+\beta)}{\lambda(1-\gamma)(\beta-1)}\|a\|_2|B|^{\frac{\beta-1+2\gamma}{2(1+\beta)}},$$

$$\text{i.e.,}\quad \|v\|_{1+\beta} < \Big(\frac{1}{\lambda}\Big)^{\frac{1}{\beta+\gamma}}\Big(\frac{(\gamma+1)(1+\beta)}{(1-\gamma)(\beta-1)}\Big)^{\frac{1}{\beta+\gamma}}\|a\|_2^{\frac{1}{\beta+\gamma}}|\Omega|^{\frac{\beta-1+2\gamma}{2(1+\beta)(\beta+\gamma)}}, \forall\, v \in \Lambda^-. \tag{14}$$

Combining inequalities in Equations (13) and (14), we deduce that

$$\begin{aligned}&\Big(\frac{1+\gamma}{\beta+\gamma}\Big)^{\frac{1}{\beta-1}} S^{\frac{1}{\beta-1}}|B|^{-2[\frac{2_s^*-1-\beta}{2_s^*(1+\beta)(\beta-1)}]}\Big(\frac{1}{\lambda}\Big)^{\frac{1}{\beta-1}}\\ &\le \|u\|_{1+\beta}\\ &< \Big(\frac{(\gamma+1)(1+\beta)}{(1-\gamma)(\beta-1)}\Big)^{\frac{1}{\beta+\gamma}}\|a\|_2^{\frac{1}{\beta+\gamma}}|B|^{\frac{\beta-1+2\gamma}{2(1+\beta)(\beta+\gamma)}}\Big(\frac{1}{\lambda}\Big)^{\frac{1}{\beta+\gamma}},\ \ \forall\, u \in \Lambda^-.\end{aligned}$$

Direct calculations show that

$$0 < C = \frac{\Big(\frac{1+\gamma}{\beta+\gamma}\Big)^{\frac{1}{\beta-1}} S^{\frac{1}{\beta-1}}|B|^{-2[\frac{2_s^*-1-\beta}{2_s^*(1+\beta)(\beta-1)}]}}{\Big(\frac{(\gamma+1)(1+\beta)}{(1-\gamma)(\beta-1)}\Big)^{\frac{1}{\beta+\gamma}}\|a\|_2^{\frac{1}{\beta+\gamma}}|B|^{\frac{\beta-1+2\gamma}{2(1+\beta)(\beta+\gamma)}}} < \lambda^{\frac{1+\gamma}{(\beta+\gamma)(\beta-1)}},$$

which contradicts the fact that $\lambda^{\frac{1+\gamma}{(\beta+\gamma)(\beta-1)}}$ tends to 0 as $\lambda \to 0$. This completes the proof of Lemma 12. □

By Lemma 12, the definition $\bar{m}_\lambda = \inf\limits_{u\in\Lambda^-} I_\lambda(u)$ is well defined.

Lemma 13. *There exists $\lambda_4 > 0$ small enough such that for all $\lambda \in (0,\ \lambda_4)$, there exists a **radial symmetry** function $v_\lambda \in \Lambda^-$ satisfying $I_\lambda(v_\lambda) = \bar{m}_\lambda = \inf\limits_{u\in\Lambda^-} I_\lambda(u)$. Moreover, v_λ is a weak positive $H_0^s(\Omega)-$ solution of Equation (1).*

Proof. We start by claiming that I_λ is coercive on Λ. In fact, for any $v \in \Lambda$, we get

$$\|v\|^2 - \lambda\|v\|_{1+\beta}^{1+\beta} - \int_\Omega a(x)v^{1-\gamma}\,dx = 0,$$

which yields

$$\begin{aligned}I_\lambda(v) &= \frac{1}{2}\|v\|^2 - \frac{\lambda}{1+\beta}\|v\|_{1+\beta}^{1+\beta} - \frac{1}{1-\gamma}\int_\Omega a(x)v^{1-\gamma}\,dx\\ &= \Big(\frac{1}{2} - \frac{1}{1+\beta}\Big)\|v\|^2 - \Big(\frac{1}{1-\gamma} - \frac{1}{1+\beta}\Big)\int_\Omega a(x)v^{1-\gamma}\,dx\end{aligned}$$

$$\geq \left(\frac{1}{2} - \frac{1}{1+\beta}\right)\|v\|^2 - \left(\frac{1}{1-\gamma} - \frac{1}{1+\beta}\right)C\|a\|_2|\Omega|^{\frac{\beta-1+2\gamma}{2(1+\beta)}}\|v\|^{1-\gamma}$$

where, in the last step, we have used the inequality in Equation (5). Thus, I_λ is coercive on Λ, and it is also true for Λ^-. Assume the sequence $\{v_n\} \subseteq \Lambda^-$ that satisfies $I_\lambda(v_n) \to \bar{m}_\lambda = \inf_{u\in\Lambda^-} I_\lambda(u)$ as $n \to \infty$. Using the coercive of I_λ, we derive that $\{v_n\}$ is bonded in Λ^-. Thus, we can assume that $v_n \rightharpoonup \widetilde{v}_\lambda$ weakly as $n \to \infty$ in Λ^-. Recall Λ^- is completed in $H_0^s(B)$ (Lemma 11); following the same arguments as in those proving the existence of the minimizer u_λ (Lemma 6) and the compactness of the embedding $H_0^s(B) \to L^{1+\beta}(B)(\beta < 2_s^* - 1)$, we obtain $\widetilde{v}_\lambda \in \Lambda^-$ as the minimizer of I_λ. Similar to the proof in Lemma 5, denoting

$$v_\lambda := (\widetilde{v}_\lambda)^\sharp$$

as the Schwarz spherical rearrangement of $\widetilde{v}_\lambda$, we also have the **radial symmetry** function $v_\lambda \in \Lambda^-$ as the minimizer of I_λ. Moreover, arguing exactly as in the proof of the weak positive solution u_λ (Lemma 10), one can prove that $v_\lambda \in H_0^s(B)$ is also a weak positive solution for Equation (1).

This completes the proof of Lemma 13. □

Proof of Theorem 1. Letting $\lambda^* = \min\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$, it is easy to verify directly that Lemmas 1–13 are true for all $\lambda \in (0,\ \lambda^*)$. Therefore, it follows from Lemma 10 and Lemma 13 that u_λ and v_λ are the **radial symmetry** weak positive solutions of Equation (1). This completes the proof of Theorem 1. □

4. Conclusions

This paper is concerned with the radial symmetry weak positive solutions for a class of singular fractional Laplacian. The most difficulty with this problem is that the vanish boundary value is such that the nonlinearity singular is at the boundary $\partial B_R(0)$. Therefore, the essence of the problem is determining for which class of the testing function ϕ makes Equation (3) hold. It is worth emphasizing that the natural associated functional $I_\lambda(u)$ is not Frechet-differentiable. So, fractional singular elliptic Equation (1) cannot be studied by directly using critical point theory. In order to solve this problem, we used Ekeland's variational principle. It is worth pointing out that we weakened one of the conditions $a(x) \geq \theta > 0$ stated in the previous works of T. Mukherjee and K. Sreenadh. Our results improve on studies on the radial symmetry solutions of nonlocal boundary value problems.

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