Article

Some Fixed-Point Theorems in b -Dislocated Metric Space and Applications

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Abstract: In this article, we prove some fixed-point theorems in b -dislocated metric space. Thereafter, we propose a simple and efficient solution for a non-linear integral equation and non-linear fractional differential equations of Caputo type by using the technique of fixed point.

Keywords: fixed point; b -dislocated metric space (simply b -dislocated metric space); F -contraction; non-linear fractional differential equations of Caputo type

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1. Introduction

The F -contraction is widely renowned, and has provided research contributions on various aspects in fixed-point theory for over 10 years.

Definition 1 ([1]). Let (X, d) be a metric space. A mapping $T : X \rightarrow X$ is said to be a F -contraction on (X, d) if there exist F along with C_1, C_2, C_3 and $\tau > 0$ such that

$$d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) \leq F(d(x, y)). \quad (*)$$

From above we have:

- every F -contraction is necessarily continuous;
- every F -contraction is need not be a Banach contraction.

Now we present a list of certain F -contractive conditions for a self-map on a metric space:

$$(E1) \quad \frac{1}{2}d(x, Tx) < d(x, y) \Rightarrow \tau + F(d(Tx, Ty)) \leq F(d(x, y)),$$

$$(E2) \quad \frac{1}{2}d(x, Tx) < d(x, y) \Rightarrow \tau + F(d(Tx, Ty)) \leq F\left(d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Tx) + d(y, Ty)}{4}\right),$$

$$(E3) \quad d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) \leq F\left\{\max\left(d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\right)\right\}$$

$$(E4) \quad d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) \leq F(d(x, y) + Ld(y, Tx)) \text{ and}$$

$$d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) \leq F(d(x, y) + Ld(x, Ty))$$

$$(E5) \quad \omega(d(x, y)) + F(d(Tx, Ty)) \leq F(f_1d(x, y) + f_2d(x, Tx) + f_3d(y, Ty) + f_4d(x, Ty) + f_5d(y, Tx))$$

$$(E6) \quad \omega(d(x, y)) + F(d(Tx, Ty)) \leq F(d(x, y))$$

for all $x, y \in X$, where $\tau > 0$, $T : X \rightarrow X$, $F : \mathbb{R}_0^+ \rightarrow \mathbb{R}$ and $\omega : (0, \infty) \rightarrow (0, \infty)$.

Below is the list of necessary conditions which helps authors to establish the statements of the certain F -contractions and treated as main tool to obtain fixed points:

- (C₁). F is strictly increasing;
- (C₂). $\lim_{n \rightarrow \infty} t_{1_n} = 0$ if and only if $\lim_{n \rightarrow \infty} F(t_{1_n}) = -\infty$;
- (C₃). There exists $k \in (0, 1)$ such that $\lim_{t_1 \rightarrow 0^+} t_1^k F(t_1) = 0$;
- (C₄). $\inf F = -\infty$;
- (C₅). F is continuous on $(0, \infty)$;
- (C₆). $\liminf_{u \rightarrow v^+} \omega(u) > 0$ for all $v \geq 0$;
- (C₇). $\lim_{v \rightarrow 0^+} F(v) = -\infty$.

We present a list of F -contractions and we named these conditions with their respective authors:

- 1°. Secelean type F -contraction [2] if T satisfies $(\star)$ along with (C₁), (C₄) and (C₅);
- 2°. Piri type F -contraction [3] if T satisfies (E1) along with (C₁), (C₄) and (C₅);
- 3°. Karapinar type F -contraction [4] if T satisfies (E2) along with (C₁), (C₄) and (C₅);
- 4°. Wardowski-I type F -contraction [5] if T satisfies (E3) along with (C₁), (C₂) and (C₃);
- 5°. Minak type F -contraction [6] if T satisfies (E3) along with (C₁), (C₂) and (C₃);
- 6°. Minak-I type F -contraction [6] if T satisfies (E4) along with (C₁), (C₂) and (C₃);
- 7°. Vetro type F -contraction [7] if T satisfies (E5) along with (C₁), (C₂), (C₃) and (C₆);
- 8°. Wardowski-II type F -contraction [8] if T satisfies (E6) along with (C₁), (C₆) and (C₇);
- 9°. A. Lukacs and S. Kajanto type F -contraction [9] if T satisfies $(\star)$ along with (C₁) and (C₃);

If we consider the F -contraction in the theory of fixed point, we can see various developments in it. More particularly, the techniques used to obtain fixed points in [1] have attracted several authors. In this scenario, many authors imposed various restrictions to obtain the existence of a fixed point (see for example [10–15]).

On the other hand, there are many generalizations on the concept of metric spaces in the literature. In particular, Matthews [16] introduced the notion of dislocated metric space under the name 'metric domains'. Later, in 2000, Hitzler and Seda [17] renamed these spaces as 'dislocated metric spaces' as below:

Definition 2 ([17]). A dislocated metric on a nonempty set X is a function $d : X \times X \rightarrow [0, +\infty)$ such that for all $x, y, z \in X$. Then the following conditions hold good:

- 1. $d(x, y) = 0 \Rightarrow x = y$;
- 2. $d(x, y) = d(y, x)$;
- 3. $d(x, y) \leq d(x, z) + d(z, y)$.

The pair (X, d) is called a dislocated metric space.

Very recently, Alghamdi et al. [18] introduced b -dislocated metric space as below.

Definition 3 ([18]). A b -dislocated metric on a nonempty set X is a function $d : X \times X \rightarrow [0, +\infty)$ such that for all $x, y, z \in X$ and a constant $b \geq 1$ the following conditions hold good:

- 1. $d(x, y) = 0 \Rightarrow x = y$;
- 2. $d(x, y) = d(y, x)$;
- 3. $d(x, y) \leq k[d(x, z) + d(z, y)]$.

The pair (X, d) is called a b -dislocated metric space. Moreover, if $k = 1$ then b -dislocated metric space becomes dislocated metric space as in [16,17].

The structure properties of convergent, Cauchy and complete of b -dislocated metric space can be found in [18] as below:

Definition 4 ([18]). Let (X, d) be a b -dislocated metric space, and let $\{x_n\}$ be a sequence of points of X . Then $\{x_n\}$ called,

- Convergent to x if and only if $\lim_{n \rightarrow +\infty} d(x, x_n) = 0$
- Cauchy if and only if $\lim_{m, n \rightarrow \infty} d(x_n, x_m)$ exists and is tends to finite.
- Complete if and only if every Cauchy sequence $\{x_n\}$ in X converges to $x \in X$ this gives

$$\lim_{m, n \rightarrow \infty} d(x_n, x_m) = 0 = \lim_{n \rightarrow \infty} d(x_n, x).$$

The study of dislocated metric space and its generalizations was very interesting and expanded very extensively. For more info, the reader can refer to [19–26].

Motivated by Wardowski [1], we introduce the notions of an extended F -contraction and weak-generalized F -contraction. Furthermore, we establish some fixed-point results for given named contractions. Thereafter, we propose a simple and efficient solution for a non-linear integral equation by using the technique of fixed point in the setting of b -dislocated metric space.

2. Extended F -Contraction

Now we start this section by introducing below definition.

Definition 5. Let (X, d) be a b -dislocated metric space. A mapping $T : X \rightarrow X$ is said to be an Extended F -contraction if there exists $F \in \mathcal{F}$ and $\tau > 0$ such that for all $x, y \in X$ with $d(Tx, Ty) > 0$,

$$\frac{1}{2k} d(x, Tx) < d(x, y) \Rightarrow \tau + F(d(Tx, Ty)) \leq F(\mathcal{A}(x, y))$$

where,

$$\begin{aligned} \mathcal{A}(x, y) = & \frac{\mathcal{A}_1(k+1)}{k} d(x, y) + \mathcal{A}_2[d(x, Tx) + d(y, Ty)] + \mathcal{A}_3 \left[\frac{d(x, Ty) + d(y, Tx)}{3k} \right] \\ & + \mathcal{A}_4 \left[\frac{d(x, x) + d(y, y)}{4k} \right] \end{aligned}$$

here $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4 \geq 0$, $\frac{\mathcal{A}_1(k+1)}{k} + 2(\mathcal{A}_2 + \mathcal{A}_3 + \mathcal{A}_4) < 1$ and $F : \mathbb{R}^+ \rightarrow \mathbb{R}$ is a mapping satisfying the following conditions:

1. F is an order embedding. i.e., for all $t_1, t_2 \in \mathbb{R}^+$ we have $t_1 \leq t_2 \Rightarrow F(t_1) \leq F(t_2)$.
2. F is sub-additive, i.e., for $t_1, t_2, s_1, s_2 \in \mathbb{R}^+$ we have

$$F(s_1 t_1 + s_2 t_2) \leq s_1 F(t_1) + s_2 F(t_2)$$

3. For every sequence $\{t_{1_n}\}_{n \in \mathbb{N}}$ of positive numbers

$$\lim_{n \rightarrow \infty} t_{1_n} = 0 \text{ if and only if } \lim_{n \rightarrow \infty} F(t_{1_n}) = -\infty;$$

Let $\mathcal{F}$ denote the family of all functions $F : \mathbb{R}^+ \rightarrow \mathbb{R}$ which satisfy conditions (1), (2) and (3).

Under this new scenario, we will prove below theorem.

Theorem 1. Let (X, d) be a complete b -dislocated metric space such that d is a continuous function. If T is an Extended F -contraction then T has a unique fixed point.

Proof. Now, take $x \in X$ and build a sequence $\{x_n\}$ as follows:

$$x_n = Tx_{n-1} = T^n x, \quad \text{for all } n \in \mathbb{N} \text{ where } x_0 = x.$$

If there exists $n_* \in \mathbb{N}$ such that $d(x_{n_*}, Tx_{n_*}) = 0$ then $x_* = x_{n_*}$ turn into a fixed point which completes the proof. As a result, we assume for every $n \in \mathbb{N}$,

$$0 < d(x_n, Tx_n). \quad (1)$$

Therefore, from (1), we have

$$\frac{1}{2k}d(x_n, Tx_n) < d(x_n, Tx_n) = d(x_n, x_{n+1}), \quad \text{for all } n \in \mathbb{N} \quad (2)$$

Since T is Extended F -contraction,

$$\begin{aligned} \tau + F(d(Tx_n, Tx_{n+1})) &\leq F(\mathcal{A}(x_n, x_{n+1})) \\ &\leq F\left\{ \frac{\mathcal{A}_1(k+1)}{k}d(x_n, x_{n+1}) + \mathcal{A}_2[d(x_n, Tx_n) + d(x_{n+1}, Tx_{n+1})] \right. \\ &\quad \left. + \mathcal{A}_3\left[\frac{d(x_n, Tx_{n+1}) + d(x_{n+1}, Tx_n)}{3k}\right] + \mathcal{A}_4\left[\frac{d(x_n, x_n) + d(x_{n+1}, x_{n+1})}{4k}\right] \right\} \\ &\leq F\left\{ \frac{\mathcal{A}_1(k+1)}{k}d(x_n, x_{n+1}) + \mathcal{A}_2[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})] \right. \\ &\quad \left. + \mathcal{A}_3\left[\frac{d(x_n, x_{n+2}) + d(x_{n+1}, x_{n+1})}{3k}\right] + \mathcal{A}_4\left[\frac{d(x_n, x_n) + d(x_{n+1}, x_{n+1})}{4k}\right] \right\} \end{aligned} \quad (3)$$

From triangle inequality, we have,

$$d(x_n, x_{n+2}) \leq k[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})] \quad (4)$$

and

$$\begin{aligned} d(x_{n+1}, x_{n+1}) &\leq 2kd(x_{n+2}, x_{n+1}) \\ &\leq 2k[d(x_{n+1}, x_{n+2}) + d(x_n, x_{n+1})] \end{aligned} \quad (5)$$

By using (4) and (5),

$$\frac{d(x_n, x_{n+2}) + d(x_{n+1}, x_{n+1})}{3k} \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}). \quad (6)$$

Also,

$$\begin{aligned} d(x_n, x_n) &\leq 2kd(x_n, x_{n+1}) \\ &\leq 2k[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})] \end{aligned} \quad (7)$$

From (5) and (7),

$$\frac{d(x_n, x_n) + d(x_{n+1}, x_{n+1})}{4k} \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}). \quad (8)$$

From (3), (7) and (8) we get

$$\begin{aligned} \tau + F(d(Tx_n, Tx_{n+1})) \leq F \left\{ \frac{\mathcal{A}_1(k+1)}{k} d(x_n, x_{n+1}) + \mathcal{A}_2[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})] \right. \\ \left. + \mathcal{A}_3 \left[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) \right] \right. \\ \left. + \mathcal{A}_4 \left[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) \right] \right\} \end{aligned} \quad (9)$$

Since $F \in \mathcal{F}$,

$$\begin{aligned} \tau + F(d(x_{n+1}, x_{n+2})) \leq \frac{\mathcal{A}_1(k+1)}{k} F(d(x_n, x_{n+1})) + \mathcal{A}_2 F(d(x_n, x_{n+1})) + \mathcal{A}_2 F(d(x_{n+1}, x_{n+2})) \\ + \mathcal{A}_3 F(d(x_n, x_{n+1})) + \mathcal{A}_3 F(d(x_{n+1}, x_{n+2})) \\ + \mathcal{A}_4 F(d(x_n, x_{n+1})) + \mathcal{A}_4 F(d(x_{n+1}, x_{n+2})) \end{aligned} \quad (10)$$

$$\leq \left[\frac{\mathcal{A}_1(k+1)}{k} + \mathcal{A}_2 + \mathcal{A}_3 + \mathcal{A}_4 \right] F(d(x_n, x_{n+1})) + (\mathcal{A}_2 + \mathcal{A}_3 + \mathcal{A}_4) F(d(x_{n+1}, x_{n+2}))$$

$$\begin{aligned} \Rightarrow (1 - \mathcal{A}_2 - \mathcal{A}_3 - \mathcal{A}_4) F(d(x_{n+1}, x_{n+2})) \leq \left[\frac{\mathcal{A}_1(k+1)}{k} + \mathcal{A}_2 + \mathcal{A}_3 + \mathcal{A}_4 \right] F(d(x_n, x_{n+1})) - \tau \\ \Rightarrow F(d(x_{n+1}, x_{n+2})) \leq \frac{\left[\frac{\mathcal{A}_1(k+1)}{k} + \mathcal{A}_2 + \mathcal{A}_3 + \mathcal{A}_4 \right]}{(1 - \mathcal{A}_2 - \mathcal{A}_3 - \mathcal{A}_4)} F(d(x_n, x_{n+1})) - \frac{\tau}{1 - (\mathcal{A}_2 + \mathcal{A}_3 + \mathcal{A}_4)} \\ F(d(x_{n+1}, x_{n+2})) \leq Y F(d(x_n, x_{n+1})) - \frac{\tau}{1 - (\mathcal{A}_2 + \mathcal{A}_3 + \mathcal{A}_4)} \end{aligned} \quad (11)$$

$$\text{where } Y = \frac{\left[\frac{\mathcal{A}_1(k+1)}{k} + \mathcal{A}_2 + \mathcal{A}_3 + \mathcal{A}_4 \right]}{(1 - \mathcal{A}_2 - \mathcal{A}_3 - \mathcal{A}_4)} < 1.$$

Thus,

$$F(d(x_{n+1}, x_{n+2})) \leq F(d(x_n, x_{n+1})) - \frac{\tau}{1 - (\mathcal{A}_2 + \mathcal{A}_3 + \mathcal{A}_4)} \quad (12)$$

If we continue same scenario, we get,

$$\begin{aligned} F(d(x_n, Tx_n)) &= F(d(x_n, x_{n+1})) \\ &\leq F(d(x_{n-1}, x_n)) - \frac{\tau}{1 - (\mathcal{A}_2 + \mathcal{A}_3 + \mathcal{A}_4)} \\ &\leq F(d(x_{n-2}, x_{n-1})) - \frac{2\tau}{1 - (\mathcal{A}_2 + \mathcal{A}_3 + \mathcal{A}_4)} \\ &\leq F(d(x_{n-3}, x_{n-2})) - \frac{3\tau}{1 - (\mathcal{A}_2 + \mathcal{A}_3 + \mathcal{A}_4)} \\ &\vdots \\ &\leq F(d(x_0, x_1)) - n\tau \\ &\leq F(d(x_0, Tx_0)) - n\tau \end{aligned} \quad (13)$$

From (13), we obtain,

$$\lim_{n \rightarrow \infty} F(d(x_n, Tx_n)) = -\infty \quad (14)$$

From (15) and from the hypothesis, we have

$$\lim_{n \rightarrow \infty} F(d(x_n, Tx_n)) = \lim_{n \rightarrow \infty} F(d(x_n, x_{n+1})) = 0. \quad (15)$$

Now, we claim that

$$\lim_{n \rightarrow \infty} F(d(x_n, x_m)) = 0. \quad (16)$$

i.e., $\{x_n\}$ is a Cauchy sequence. Let us suppose that $\{x_n\}$ is not a Cauchy sequence, then there exist $\epsilon > 0$ and sequence $\{a(n)\}_{n=1}^{\infty}$ and $\{b(n)\}_{n=1}^{\infty}$ of natural numbers such that $a(n) > b(n) > n$, $d(x_{a(n)}, x_{b(n)}) \geq \epsilon$,

$$d(x_{a(n)-1}, x_{b(n)}) < \frac{\epsilon}{k}, \quad \text{for all } n \in \mathbb{N}. \quad (17)$$

From the triangle inequality, we get,

$$\begin{aligned} \epsilon &\leq d(x_{a(n)}, x_{b(n)}) \leq k[d(x_{a(n)}, x_{a(n)-1}) + d(x_{a(n)-1}, x_{b(n)})] \\ &\leq kd(x_{a(n)}, x_{a(n)-1}) + \epsilon \\ &= kd(x_{a(n)-1}, Tx_{a(n)-1}) + \epsilon \end{aligned} \quad (18)$$

From (15) and (18) and the sandwich theorem, we get,

$$\lim_{n \rightarrow \infty} d(x_{a(n)}, x_{b(n)}) = \epsilon, \quad \text{for all } n \in \mathbb{N}. \quad (19)$$

From Extended F -contraction, we have

$$\begin{aligned} \tau + F(d(Tx_{a(n)}, Tx_{b(n)})) &\leq F\left\{\frac{\mathcal{A}_1(k+1)}{k}d(x_{a(n)}, x_{b(n)}) + \mathcal{A}_2[d(x_{a(n)}, Tx_{a(n)}) + d(x_{b(n)}, Tx_{b(n)})]\right. \\ &\quad + \mathcal{A}_3\left[\frac{d(x_{a(n)}, Tx_{b(n)}) + d(x_{b(n)}, Tx_{a(n)})}{3k}\right] \\ &\quad \left. + \mathcal{A}_4\left[\frac{d(x_{a(n)}, x_{a(n)}) + d(x_{b(n)}, x_{b(n)})}{4k}\right]\right\} \\ &\leq F\left\{\frac{\mathcal{A}_1(k+1)}{k}d(x_{a(n)}, x_{b(n)}) + \mathcal{A}_2[d(x_{a(n)}, x_{a(n)+1}) + d(x_{b(n)}, x_{b(n)+1})]\right. \\ &\quad + \mathcal{A}_3\left[\frac{d(x_{a(n)}, x_{b(n)+1}) + d(x_{b(n)}, x_{a(n)+1})}{3k}\right] \\ &\quad \left. + \mathcal{A}_4\left[\frac{d(x_{a(n)}, x_{a(n)}) + d(x_{b(n)}, x_{b(n)})}{4k}\right]\right\} \end{aligned} \quad (20)$$

Letting $n \rightarrow \infty$,

$$\begin{aligned} \tau + F(\epsilon) &\leq F\left[\frac{\mathcal{A}_1(k+1)}{k}\epsilon + \mathcal{A}_3\left(\frac{k(\epsilon) + k(\epsilon)}{3k}\right)\right] \\ &\leq F\left(\frac{\mathcal{A}_1(k+1)}{k}\epsilon + \frac{2\mathcal{A}_3\epsilon}{3}\right), \end{aligned} \quad (21)$$

which is a contradiction.

Hence $\lim_{m, n \rightarrow \infty} d(x_n, x_m) = 0$. Thus, the sequence $\{x_n\}_{n=1}^{\infty}$ is a Cauchy in X .

Since (X, d) is a Complete b -dislocated metric space, there exists $\omega \in X$ such that

$$d(\omega, \omega) = \lim_{n \rightarrow \infty} d(x_n, \omega) = \lim_{m, n \rightarrow \infty} d(x_n, x_m) = 0. \quad (22)$$

From (22)

$$d(x_{n+1}, T\omega) \leq k[d(x_{n+1}, \omega) + d(\omega, T\omega)]$$

letting $n \rightarrow \infty$,

$$\lim_{n \rightarrow \infty} d(x_{n+1}, T\omega) = kb_d(\omega, T\omega) \quad (23)$$

We now prove that for every $n \in \mathbb{N}$

$$\begin{aligned} \frac{1}{2k}d(x_n, Tx_n) &< d(x_n, \omega) \quad \text{or} \\ \frac{1}{2k}d(Tx_n, T^2x_n) &< d(Tx_n, \omega) \quad \forall n \in \mathbb{N} \end{aligned} \quad (24)$$

Arguing by contradiction, we assume that there exists $e \in \mathbb{N}$ such that

$$\begin{aligned} \frac{1}{2k}d(x_e, Tx_e) &\geq d(x_e, \omega) \quad \text{and} \\ \frac{1}{2k}d(Tx_e, T^2x_e) &\geq d(Tx_e, \omega) \end{aligned} \quad (25)$$

Now from (12),

$$d(Tx_e, T^2x_e) < d(x_e, Tx_e) \quad (26)$$

From (25) and (26),

$$\begin{aligned} d(x_e, Tx_e) &\leq k[d(x_e, \omega) + d(\omega, Tx_e)] \\ &\leq k\left[\frac{1}{2k}d(x_e, Tx_e) + \frac{1}{2k}d(Tx_e, T^2x_e)\right] \\ &< k\left[\frac{1}{2k}d(x_e, Tx_e) + \frac{1}{2k}d(x_e, Tx_e)\right] \\ &= d(x_e, Tx_e), \end{aligned} \quad (27)$$

which is a contradiction. Hence (24) holds.

Suppose $\frac{1}{2k}d(x_n, Tx_n) < d(x_n, \omega)$ is satisfied and $d(\omega, T\omega) > 0$. From (23) there exists $N_1 \in \mathbb{N}$ such that $d(Tx_n, T\omega) = d(x_{n+1}, T\omega) > 0$ for $n \geq N_1$. Then from Extended F -contraction (with $n \geq N_1$), we have,

$$\begin{aligned} \tau + F(d(x_{n+1}, T\omega)) &= \tau + F(d(Tx_n, T\omega)) \\ &\leq F\left\{\frac{\mathcal{A}_1(k+1)}{k}d(x_n, \omega) + \mathcal{A}_2[d(x_n, Tx_n) + d(\omega, T\omega)]\right. \\ &\quad \left.+ \mathcal{A}_3\left[\frac{d(x_n, T\omega) + d(\omega, Tx_n)}{3k}\right]\right. \\ &\quad \left.+ \mathcal{A}_4\left[\frac{d(x_n, x_n) + d(\omega, \omega)}{4k}\right]\right\} \\ &\leq F\left\{\frac{\mathcal{A}_1(k+1)}{k}d(x_n, \omega) + \mathcal{A}_2[d(x_n, x_{n+1}) + d(\omega, T\omega)]\right. \\ &\quad \left.+ \mathcal{A}_3\left[\frac{d(x_n, T\omega) + d(\omega, x_{n+1})}{3k}\right]\right. \\ &\quad \left.+ \mathcal{A}_4\left[\frac{d(x_n, x_n) + d(\omega, \omega)}{4k}\right]\right\} \end{aligned} \quad (28)$$

From (23) there exists $N_k \in \mathbb{N}$ with $(N_k \geq N_1)$ such that for all $n \geq N_k$.

$$\begin{aligned} \tau + F(d(x_{n+1}, T\omega)) &\leq F\left\{\frac{\mathcal{A}_1(k+1)}{k}d(x_{n+1}, \omega) + \mathcal{A}_2d(\omega, T\omega) \right. \\ &\quad \left. + \mathcal{A}_3\left[k\frac{d(x_{n+1}, T\omega) + d(\omega, x_{n+1})}{3k}\right] \right. \\ &\quad \left. + \mathcal{A}_4\left[2s\frac{d(\omega, T\omega)}{4k}\right]\right\} \\ &\leq F\left\{\mathcal{A}_2[d(\omega, T\omega) + \frac{\mathcal{A}_3}{3}d(x_{n+1}, T\omega) + \mathcal{A}_4\frac{d(\omega, T\omega)}{2}]\right\}, \end{aligned} \quad (29)$$

which implies, $(1 - \frac{\mathcal{A}_3}{3})F(d(x_{n+1}, T\omega)) \leq (\mathcal{A}_3 + \frac{\mathcal{A}_4}{2})F(d(\omega, T\omega)) - \tau$

Letting $n \rightarrow \infty$,

$$\begin{aligned} F(d(\omega, T\omega)) &\leq \frac{\mathcal{A}_2 + \frac{\mathcal{A}_4}{2}}{1 - \frac{\mathcal{A}_3}{3}}F(d(\omega, T\omega)) - \frac{\tau}{1 - \frac{\mathcal{A}_3}{3}} \\ &< \frac{\mathcal{A}_2 + \frac{\mathcal{A}_4}{2}}{1 - \frac{\mathcal{A}_3}{3}}F(d(\omega, T\omega)) \\ &< F(d(\omega, T\omega)) \quad \forall n \geq N_k, \end{aligned} \quad (30)$$

which is a contradiction.

Now suppose $\frac{1}{2k}d(Tx_n, T^2x_n) < d(Tx_n, \omega)$ of (24) is true with $d(\omega, T\omega) > 0$. Noted from (23), there exists $N_p \in \mathbb{N}$ such that $d(T(Tx_n), T\omega) = d(x_{n+2}, T\omega) > 0$ for $n \geq N_p$. Then from our assumption with $n \geq N_p$, we have

$$\begin{aligned} \tau + F(d(x_{n+2}, T\omega)) &= \tau + F(d(T^2x_n, T\omega)) \\ &\leq F\left\{\frac{\mathcal{A}_1(k+1)}{k}d(Tx_n, \omega) + \mathcal{A}_2[d(Tx_n, T^2x_n) + d(\omega, T\omega)] \right. \\ &\quad \left. + \mathcal{A}_3\left[\frac{d(Tx_n, T\omega) + d(\omega, T^2x_n)}{3k}\right] \right. \\ &\quad \left. + \mathcal{A}_4\left[\frac{d(Tx_n, Tx_n) + d(\omega, \omega)}{4k}\right]\right\} \\ &\leq F\left\{\frac{\mathcal{A}_1(k+1)}{k}d(x_{n+1}, \omega) + \mathcal{A}_2[d(x_{n+1}, x_{n+2}) + d(\omega, T\omega)] \right. \\ &\quad \left. + \mathcal{A}_3\left[\frac{d(x_{n+1}, T\omega) + d(\omega, x_{n+2})}{3k}\right] \right. \\ &\quad \left. + \mathcal{A}_4\left[\frac{d(x_{n+1}, x_{n+1}) + d(\omega, \omega)}{4k}\right]\right\} \\ &\leq F(\mathcal{A}_2d(\omega, T\omega) + \frac{\mathcal{A}_3}{3k}kb_d(\omega, T\omega)) \\ &\leq \mathcal{A}_2F(d(\omega, T\omega)) + \frac{\mathcal{A}_3}{3}F(d(\omega, T\omega)) \end{aligned} \quad (31)$$

Letting $n \rightarrow \infty$,

$$\tau + F(d(\omega, T\omega)) \leq \left[\mathcal{A}_2 + \frac{\mathcal{A}_3}{3}\right]F(d(\omega, T\omega))$$

$$\begin{aligned}
 F(d(\omega, T\omega)) &\leq \left[\mathcal{A}_2 + \frac{\mathcal{A}_3}{3} \right] F(d(\omega, T\omega)) - \tau \\
 &< \left[\mathcal{A}_2 + \frac{\mathcal{A}_3}{3} \right] F(d(\omega, T\omega)) \\
 &< F(d(\omega, T\omega)),
 \end{aligned} \tag{32}$$

which is a contradiction.

Hence we conclude that $d(\omega, T\omega) = 0$, which gives $\omega = T\omega$. Therefore, ω is a fixed point of T .

To prove uniqueness, let ω and ω^* are two fixed points of T such that $\omega \neq \omega^*$, which means,

$$\begin{aligned}
 \frac{1}{2k} d(\omega, T\omega^*) &< d(\omega, T\omega^*) \\
 &= d(\omega, \omega^*)
 \end{aligned} \tag{33}$$

Thus, from, Extended F -contraction,

$$\begin{aligned}
 \tau + F(d(T\omega, T\omega^*)) &\leq F(\mathcal{A}(\omega, \omega^*)) \\
 &\leq F \left\{ \frac{\mathcal{A}_1(k+1)}{k} d(\omega, \omega^*) + \mathcal{A}_2[d(\omega, T\omega) + d(\omega^*, T\omega^*)] \right. \\
 &\quad \left. + \mathcal{A}_3 \left[\frac{d(\omega, T\omega^*) + d(\omega^*, T\omega)}{3k} \right] + \mathcal{A}_4 \left[\frac{d(\omega, \omega^*) + d(\omega^*, \omega^*)}{4k} \right] \right\} \\
 &\leq F \left\{ \frac{\mathcal{A}_1(k+1)}{k} d(\omega, \omega^*) + \mathcal{A}_2[d(\omega, \omega) + d(\omega^*, \omega^*)] \right. \\
 &\quad \left. + \mathcal{A}_3 \left[\frac{d(\omega, \omega^*) + d(\omega^*, \omega)}{3k} \right] \right\}
 \end{aligned} \tag{34}$$

From (22), $d(\omega, \omega) = d(\omega^*, \omega^*) = 0$. Therefore (34) becomes:

$$\begin{aligned}
 \tau + F(d(\omega, \omega^*)) &\leq F \left\{ \frac{\mathcal{A}_1(k+1)}{k} d(\omega, \omega^*) + 2\mathcal{A}_3 \frac{d(\omega, \omega^*)}{3k} \right\} \\
 &\leq \left(\frac{\mathcal{A}_1(k+1)}{k} + \frac{2\mathcal{A}_3}{3k} \right) F(d(\omega, \omega^*)) \\
 &= \left(\frac{3\mathcal{A}_1(k+1) + 2\mathcal{A}_3k}{k} \right) F(d(\omega, \omega^*)) \\
 &< F(d(\omega, \omega^*))
 \end{aligned} \tag{35}$$

Thus, $F(d(\omega, \omega^*)) < F(d(\omega, \omega^*)) - \tau$, this gives a contradiction. Hence $\omega = \omega^*$. $\square$

Example 1. Consider $X = \{0, 1, 2\}$. Let $d : X \times X \rightarrow [0, \infty)$ be a mapping defined by

$$d(0, 0) = d(1, 1) = 0, d(2, 2) = 2.5;$$

$$d(0, 2) = d(2, 0) = 2, d(1, 2) = d(2, 1) = 3;$$

$$d(0, 1) = d(1, 0) = 1.5.$$

We can easily prove that (X, d) is a b -dislocated metric space with $k = 2$. Clearly (X, d) is a complete b -dislocated metric space.

Let $T : X \rightarrow X$ be given by $T0 = T1 = 0$ and $T2 = 1$. Suppose that $F(v) = \frac{-1}{v} + v$.

Case-1: Let $x = 0$, now $d(T0, T0) = d(T0, T1) = d(0, 0) = 0$.

So, we need to consider $y = 2$. Then we have $0.25d(0, T0) < d(0, 2)$.

Consider,

$$\begin{aligned}
 \mathcal{A}(0, 2) &= \mathcal{A}_1 1.5d(0, 2) + \mathcal{A}_2[d(0, T0) + d(2, T2)] + \mathcal{A}_3 \left[\frac{d(0, T2) + d(2, T0)}{6} \right] \\
 &\quad + \mathcal{A}_4 \left[\frac{d(0, 0) + d(2, 2)}{8} \right] \\
 &= \mathcal{A}_1 1.5 \times 2 + \mathcal{A}_2[d(0, 0) + d(2, 1)] + \mathcal{A}_3 \left[\frac{d(0, 1) + d(2, 0)}{6} \right] \\
 &\quad + \mathcal{A}_4 \left[\frac{d(0, 0) + d(2, 2)}{8} \right] \\
 &= 3\mathcal{A}_1 + \mathcal{A}_2[0 + 3] + \mathcal{A}_3 \left[\frac{1.5 + 2}{6} \right] + \mathcal{A}_4 \left[\frac{0 + 2.5}{8} \right] \\
 &= 3\mathcal{A}_1 + 3\mathcal{A}_2 + 0.58\mathcal{A}_3 + 0.31\mathcal{A}_4
 \end{aligned} \tag{36}$$

Let $\mathcal{A}_1 = 0.56, \mathcal{A}_2 = \mathcal{A}_3 = \mathcal{A}_4 = 0$ such that $1.5\mathcal{A}_1 + 2(\mathcal{A}_2 + \mathcal{A}_3 + \mathcal{A}_4) < 1$.

Now, $\mathcal{A}(0, 2) = 1.68$.

Consider,

$$\begin{aligned}
 F(d(T0, T2)) - F(\mathcal{A}(0, 2)) &= F(d(0, 1)) - F(\mathcal{A}(0, 2)) \\
 &= F(1.5) - F(1.68) \\
 &= -0.66 + 1.5 + 0.59 - 1.68 \\
 &= -0.25 \\
 &= -\tau
 \end{aligned} \tag{37}$$

In this case, for $\tau = 0.25$, T satisfies all the conditions of Theorem 1 and 0 is the unique fixed point.

Case-2: Let $x = 1$, now $d(T0, T1) = d(T1, T1) = d(0, 0) = 0$. So, we must consider $y = 2$.

$$d(1, T1) = d(1, 0) = 1.5, d(1, 2) = 3.$$

Therefore $\frac{1}{4}d(1, T1) < d(1, 2)$.

$$\begin{aligned}
 \mathcal{A}(1, 2) &= \mathcal{A}_1 1.5d(1, 2) + \mathcal{A}_2[d(1, T1) + d(2, T2)] + \mathcal{A}_3 \left[\frac{d(1, T2) + d(2, T1)}{6} \right] \\
 &\quad + \mathcal{A}_4 \left[\frac{d(1, 1) + d(2, 2)}{8} \right] \\
 &= \mathcal{A}_1 1.5 \times 3 + \mathcal{A}_2[d(1, 0) + d(2, 1)] + \mathcal{A}_3 \left[\frac{d(1, 1) + d(2, 0)}{6} \right] \\
 &\quad + \mathcal{A}_4 \left[\frac{d(1, 1) + d(2, 2)}{8} \right] \\
 &= 4.5\mathcal{A}_1 + \mathcal{A}_2[1.5 + 3] + \mathcal{A}_3 \left[\frac{0 + 2}{6} \right] + \mathcal{A}_4 \left[\frac{0 + 2.5}{8} \right] \\
 &= 4.5\mathcal{A}_1 + 4.5\mathcal{A}_2 + 0.33\mathcal{A}_3 + 0.31\mathcal{A}_4
 \end{aligned} \tag{38}$$

Let $\mathcal{A}_1 = 0.56, \mathcal{A}_2 = \mathcal{A}_3 = \mathcal{A}_4 = 0$ such that $1.5\mathcal{A}_1 + 2(\mathcal{A}_2 + \mathcal{A}_3 + \mathcal{A}_4) < 1$.

Consider,

$$\begin{aligned}
 F(d(T1, T2)) - F(\mathcal{A}(1, 2)) &= F(d(0, 1)) - F(\mathcal{A}(1, 2)) \\
 &= F(1.5) - F(2.52) \\
 &= -1.29 \\
 &= -\tau
 \end{aligned} \tag{39}$$

In this case, for $\tau = 1.29$, T satisfies all the conditions of Theorem 1 and 0 is the unique fixed point.

Case-3: Let $x = 2$, now $d(T2, T2) = 0$. So, we need only consider the case $y \in \{0, 1\}$.

Subcase-I: If $y = 0$. Please note that $0.25d(2, T2) < d(2, 0)$ as $0.75 = 0.25d(2, 1) < d(2, 0) = 2$.

$$\tau + F(d(T2, T0)) \leq F(\mathcal{A}(2, 0)),$$

follows as in Case-1.

Subcase-II: If $y = 1$, note that $0.25d(2, T2) < d(2, 1)$ as $0.75 = 0.25d(2, 1) < d(2, 1) = 3$.

$$\tau + F(d(T2, T1)) \leq F(\mathcal{A}(2, 1)),$$

follows as in Case-2.

Hence T is an extended F -contraction and it is clear that 0 is the fixed point of T .

If we take $k = 1$, then the above theorem reduces to below corollary.

Corollary 1. Let (X, d) be a dislocated metric space such that d is a continuous function. If T is an extended F -contraction, i.e., there exists $F \in \mathcal{F}$ and $\tau > 0$ such that for all $x, y \in X$ with $d(Tx, Ty) > 0$,

$$\frac{1}{2}d(x, Tx) < d(x, y) \Rightarrow \tau + F(d(Tx, Ty)) \leq F(\mathcal{A}(x, y))$$

where,

$$\begin{aligned}
 \mathcal{A}(x, y) &= 2\mathcal{A}_1d(x, y) + \mathcal{A}_2[d(x, Tx) + d(y, Ty)] + \mathcal{A}_3\left[\frac{d(x, Ty) + d(y, Tx)}{3}\right] \\
 &\quad + \mathcal{A}_4\left[\frac{d(x, x) + d(y, y)}{4}\right]
 \end{aligned}$$

here $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4 \geq 0, 2\mathcal{A}_1 + 2(\mathcal{A}_2 + \mathcal{A}_3 + \mathcal{A}_4) < 1$. Then T has a unique fixed point.

If we take $\mathcal{A}_2 = \mathcal{A}_3 = \mathcal{A}_4 = 0$ and $\mathcal{A}_1 < \frac{1}{2}$ in above theorem, we get below corollary.

Corollary 2. Let (X, d) be a b -dislocated metric space. If T is an extended F -contraction, i.e., there exists $F \in \mathcal{F}$ and $\tau > 0$ such that for all $x, y \in X$ with $d(Tx, Ty) > 0, \frac{1}{2}d(x, Tx) < d(x, y) \Rightarrow \tau + F(d(Tx, Ty)) \leq F(\mathcal{A}(x, y))$, here, $\mathcal{A}(x, y) = 2\mathcal{A}_1d(x, y)$. Then T has a unique fixed point.

3. Weak-Generalized F -Contraction

Definition 6. Let (X, d) be a b -dislocated metric space and $T : X \rightarrow X$ be a mapping. Then T is said to be a Weak-generalized F -contraction if F along with C_1, C_2, C_3 and there exists $\tau > 0$ such that $\forall x, y \in X$,

$$d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) \leq F(\mathcal{S}(x, y)) \tag{40}$$

where

$$\begin{aligned} S(x, y) = \max \left\{ d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Tx) + d(y, Ty)}{2}, \right. \\ \left. \frac{d(x, Tx)d(y, Ty)}{1 + d(x, y)}, \frac{d(y, Ty)[1 + d(x, Tx)]}{1 + d(x, y)}, \right. \\ \left. \frac{d^2(x, Tx)d(y, Ty)}{1 + d^2(x, y)} \right\} \end{aligned} \quad (41)$$

Theorem 2. Let (X, d) be a b -dislocated metric space such that d is continuous functional. If T is a Weak-generalized F -contraction, then T has a unique fixed point.

Proof. Choose an arbitrary point $x_0 \in X$ and define a sequence $\{x_n\}$ in X by $x_n = Tx_{n-1}$ for $n \in \{1, 2, 3, \dots\}$.

i.e., $x_1 = Tx_0$, $x_2 = Tx_1 = T(Tx_0) = T^2x_0, \dots, x_n = Tx_{n-1} = T^nx_0$. If $d(x_{n_0}, x_{n_0+1}) = 0$ for some $n_0 \in \{0, 1, 2, \dots\}$, then $x_{n_0} = x_{n_0+1}$, which yields $x_{n_0} = Tx_{n_0}$, and so T has a fixed point. Then there is nothing left to prove, and our proof is complete.

Thus, let us assume $d(x_n, x_{n+1}) > 0$ for every $n \in \{0, 1, 2, \dots\}$. i.e., $d(Tx_{n-1}, Tx_n) > 0$.

Now using (36), we get,

$$\tau + F(kb_d(Tx_{n-1}, Tx_n)) \leq F(S(x_{n-1}, x_n)) \quad (42)$$

where

$$\begin{aligned} S(x_{n-1}, x_n) &= \max \left\{ d(x_{n-1}, x_n), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n), \frac{d(x_{n-1}, Tx_{n-1}) + d(x_n, Tx_n)}{2}, \right. \\ &\quad \frac{d(x_n, Tx_n)[1 + d(x_{n-1}, Tx_{n-1})]}{1 + d(x_{n-1}, x_n)}, \frac{d(x_{n-1}, Tx_{n-1})d(x_n, Tx_n)}{1 + d(x_{n-1}, x_n)}, \\ &\quad \left. \frac{d^2(x_{n-1}, Tx_{n-1})d(x_n, Tx_n)}{1 + d^2(x_{n-1}, x_n)} \right\} \\ &= \max \left\{ d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{2}, \right. \\ &\quad \frac{d(x_n, x_{n+1})[1 + d(x_{n-1}, x_n)]}{1 + d(x_{n-1}, x_n)}, \frac{d(x_{n-1}, x_n)d(x_{n+1}, x_n)}{1 + d(x_{n-1}, x_n)}, \\ &\quad \left. \frac{d^2(x_{n-1}, x_n)d(x_n, x_{n+1})}{1 + d^2(x_{n-1}, x_n)} \right\} \\ &\leq \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} \end{aligned} \quad (43)$$

This yields,

$$\tau + F(kB_d(x_n, x_{n+1})) \leq F(\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}) \quad (44)$$

If $\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = d(x_n, x_{n+1})$ then $\tau + F(kb_d(x_n, x_{n+1})) \leq F(d(x_n, x_{n+1}))$, which is a contradiction, since $\tau > 0$.

Hence $\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = d(x_{n-1}, x_n)$. Thus, (40) reduces to $\tau + F(kB_d(x_n, x_{n+1})) \leq F(d(x_{n-1}, x_n))$. Therefore, we obtain,

$$\begin{aligned} F(k^n d(x_n, x_{n+1})) &\leq F(k^{n-1} d(x_{n-1}, x_n)) - \tau \\ &\leq F(k^{n-2} d(x_{n-2}, x_{n-1})) - 2\tau \\ &\vdots \\ &\leq F(k^0 d(x_0, x_1)) - n\tau \end{aligned} \quad (45)$$

Applying $\lim_{n \rightarrow \infty}$ in above, we get,

$$\lim_{n \rightarrow \infty} F(k^n d(x_n, x_{n+1})) = -\infty.$$

Thus, by using (F_2) of Wardowski, we get,

$$\lim_{n \rightarrow \infty} k^n d(x_n, x_{n+1}) = 0.$$

Thus, from (F_3) there exists $\vartheta \in (0, 1)$ such that

$$\lim_{n \rightarrow \infty} [k^n d(x_n, x_{n+1})]^\vartheta F(d(x_n, x_{n+1})) = 0.$$

By (41), the following holds good for all $n \in \{1, 2, 3, \dots\}$,

$$\begin{aligned} [k^n d(x_n, x_{n+1})]^\vartheta F(d(x_n, x_{n+1})) - [k^n d(x_n, x_{n+1})]^\vartheta F(d(x_0, x_1)) &\leq -[k^n d(x_n, x_{n+1})]^\vartheta n\tau \\ &\leq 0 \end{aligned} \quad (46)$$

Letting $n \rightarrow \infty$ in (42), we get

$$\lim_{n \rightarrow \infty} n[k^n d(x_n, x_{n+1})]^\vartheta = 0. \quad (47)$$

From (43), there exists $N_1 \in \{1, 2, 3, \dots\}$ such that $n[d(x_n, x_{n+1})]^\vartheta \leq 1$ for all $n \geq N_1$. Therefore, we have for all $n \geq N_1$.

$$k^n d(x_n, x_{n+1}) \leq \frac{1}{n^{\frac{1}{\vartheta}}} \quad (48)$$

Now to prove that $\{x_n\}$ is a Cauchy sequence, consider $m, n \in N$ such that $m > n \geq N_1$. Using the triangle inequality and from (44), we have,

$$\begin{aligned} d(x_n, x_m) &\leq k[d(x_n, x_{n+1}) + d(x_{n+1}, x_m)] \\ &\leq kd(x_n, x_{n+1}) + k[k(d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_m))] \\ &\leq kb_d(x_n, x_{n+1}) + k^2 d(x_{n+1}, x_{n+2}) + k^2 d(x_{n+2}, x_m) \\ &\vdots \\ &\leq kb_d(x_n, x_{n+1}) + k^2 d(x_{n+1}, x_{n+2}) + k^3 d(x_{n+2}, x_{n+3}) + \dots + k^{m-1} d(x_{m-1}, x_m) \end{aligned} \quad (49)$$

Thus,

$$\begin{aligned} d(x_n, x_m) &\leq \sum_{e=n}^{m-1} k^e d(x_e, x_{e+1}) \\ &\leq \sum_{e=n}^{\infty} k^e d(x_e, x_{e+1}) \\ &\leq \sum_{e=n}^{\infty} \frac{1}{e^{\frac{1}{\theta}}} \end{aligned} \quad (50)$$

which is convergent, passing to limit $n \rightarrow \infty$, then we get $\lim_{n \rightarrow \infty} d(x_n, x_m) = 0$. This yields that $\{x_n\}$ is a Cauchy sequence in (X, d) .

Since (X, d) is a complete b -dislocated metric space, the sequence $\{x_n\}$ converges to some point $\varrho \in X$, that is $\lim_{n \rightarrow \infty} x_n = \varrho$, and $d(\varrho, \varrho) = 0$. As $d(\varrho, \varrho) = \lim_{n \rightarrow \infty} d(x_n, \varrho) = \lim_{n, m \rightarrow \infty} d(x_n, x_m) = 0$.

Case-I: If T is continuous. Then we have,

$$\varrho = \lim_{n \rightarrow \infty} x_{n+1} = \lim_{n \rightarrow \infty} Tx_n = T(\lim_{n \rightarrow \infty} x_n) = T\varrho$$

and so, ϱ is a fixed point of T .

Case-II: If F is continuous. In this case, we will prove that $\varrho = T\varrho$. Let us assume that $\varrho \neq T\varrho$.

In this case, there exists an $n_0 \in \mathbb{N}$ and a subsequence $\{x_{n_e}\}$ of $\{x_n\}$ such that $d(Tx_{n_e}, T\varrho) > 0$ for all $n_e \geq n_0$.

Otherwise, if not, there exist $n_1 \in \mathbb{N}$ such that $x_n = T\varrho$ for all $n \geq n_1$, which yields that $x_n \rightarrow T\varrho$. This is a contradiction, since $\varrho \neq T\varrho$. Being $d(Tx_{n_e}, T\varrho) > 0$ for all $n_e \geq n_0$, then from hypothesis, we have,

$$\begin{aligned} \tau + F(d(x_{n_e+1}, T\varrho)) &= \tau + F(d(Tx_{n_e+1}, T\varrho)) \\ &\leq F(\mathcal{A}(x_{n_e}, \varrho)) \end{aligned} \quad (51)$$

where,

$$\begin{aligned} \mathcal{A}(x_{n_e}, \varrho) = \max \left\{ d(x_{n_e}, \varrho), d(x_{n_e}, Tx_{n_e}), d(\varrho, T\varrho), \frac{d(x_{n_e}, Tx_{n_e}) + d(\varrho, T\varrho)}{2}, \right. \\ \left. \frac{d(\varrho, T\varrho)[1 + d(x_{n_e}, Tx_{n_e})]}{1 + d(x_{n_e}, \varrho)}, \frac{d(x_{n_e}, Tx_{n_e})d(\varrho, T\varrho)}{1 + d(x_{n_e}, \varrho)}, \right. \\ \left. \frac{d^2(x_{n_e}, Tx_{n_e})d(\varrho, T\varrho)}{1 + d^2(x_{n_e}, \varrho)} \right\} \end{aligned} \quad (52)$$

From (47),

$$\begin{aligned} \tau + F(d(x_{n_e+1}, T\varrho)) &= \max \left\{ d(x_{n_e}, \varrho), d(x_{n_e}, x_{n_e+1}), d(\varrho, T\varrho), \frac{d(x_{n_e}, x_{n_e+1}) + d(\varrho, T\varrho)}{2}, \right. \\ &\quad \frac{d(\varrho, T\varrho)[1 + d(x_{n_e}, x_{n_e+1})]}{1 + d(x_{n_e}, \varrho)}, \frac{d(x_{n_e}, x_{n_e+1})d(\varrho, T\varrho)}{1 + d(x_{n_e}, \varrho)}, \\ &\quad \left. \frac{d^2(x_{n_e}, x_{n_e+1})d(\varrho, T\varrho)}{1 + d^2(x_{n_e}, \varrho)} \right\} \end{aligned} \quad (53)$$

Taking the limit $n_e \rightarrow \infty$ and using the continuity of F , we get,

$$\begin{aligned} \tau + F(d(\varrho, T\varrho)) &\leq \max \left\{ d(\varrho, \varrho), d(\varrho, T\varrho), \frac{d(\varrho, \varrho) + d(\varrho, T\varrho)}{2} \right\} \\ &\leq \max \{ d(\varrho, \varrho), d(\varrho, T\varrho) \} \end{aligned} \quad (54)$$

Case-A: If $\max \{ d(\varrho, \varrho), d(\varrho, T\varrho) \} = d(\varrho, T\varrho)$.

Then from (50), we get,

$$\tau + F(d(\varrho, T\varrho)) \leq d(\varrho, T\varrho),$$

which is a contradiction as $\tau > 0$.

Case-B: If $\max \{ d(\varrho, \varrho), d(\varrho, T\varrho) \} = d(\varrho, \varrho)$.

Then from (50), we get,

$$\begin{aligned} \tau + F(d(\varrho, T\varrho)) &\leq d(\varrho, \varrho), \\ \Rightarrow \tau + F(d(\varrho, T\varrho)) &\leq F(0), \end{aligned}$$

which is a contradiction.

Thus, in the both cases, we get the contradiction due to $d(x_{n_e+1}, T\varrho) > 0$. Therefore, $x_{n_e+1} = T\varrho$, which implies $Tx_{n_e} = T\varrho$. Thus, $T\varrho = \varrho$. This gives, ϱ is a fixed point of T . To prove uniqueness, let ϱ and ϱ^* are two fixed points of T , i.e. $T\varrho = \varrho$ and $T\varrho^* = \varrho^*$. Let us assume that $\varrho \neq \varrho^*$.

$$\begin{aligned} \Rightarrow d(\varrho, \varrho^*) &> 0 \\ \Rightarrow d(T\varrho, T\varrho^*) &> 0 \end{aligned}$$

From hypothesis, we get,

$$\tau + F(d(T\varrho, T\varrho^*)) \leq F(\mathcal{A}(\varrho, \varrho^*)) \quad (55)$$

where,

$$\begin{aligned} \mathcal{A}(\varrho, \varrho^*) &= \max \left\{ d(\varrho, \varrho^*), d(\varrho, T\varrho), d(\varrho^*, T\varrho^*), \frac{d(\varrho, T\varrho) + d(\varrho^*, T\varrho^*)}{2}, \right. \\ &\quad \frac{d(\varrho^*, T\varrho^*)[1 + d(\varrho, T\varrho)]}{1 + d(\varrho, \varrho^*)}, \frac{d(\varrho, T\varrho)d(\varrho^*, T\varrho^*)}{1 + d(\varrho, \varrho^*)}, \\ &\quad \left. \frac{d^2(\varrho, T\varrho)d(\varrho^*, T\varrho^*)}{1 + d^2(\varrho, \varrho^*)} \right\} \\ &= \max \left\{ d(\varrho, \varrho^*), d(\varrho, \varrho), d(\varrho^*, \varrho^*), \frac{d(\varrho, \varrho) + d(\varrho^*, \varrho^*)}{2}, \right. \\ &\quad \frac{d(\varrho^*, \varrho^*)[1 + d(\varrho, \varrho)]}{1 + d(\varrho, \varrho^*)}, \frac{d(\varrho, \varrho)d(\varrho^*, \varrho^*)}{1 + d(\varrho, \varrho^*)}, \\ &\quad \left. \frac{d^2(\varrho, \varrho)d(\varrho^*, \varrho^*)}{1 + d^2(\varrho, \varrho^*)} \right\} \\ &= \max \{ d(\varrho, \varrho^*) \} \\ &= d(\varrho, \varrho^*) \end{aligned} \quad (56)$$

From hypothesis, we get,

$$\tau + F(d(Tq, Tq^*)) \leq F(d(q, q^*))$$

$$\Rightarrow \tau + F(d(q, q^*)) \leq F(d(q, q^*)),$$

which is a contradiction. Hence $q = q^*$, this completes the proof of the theorem. $\square$

Corollary 3. Let (X, d) be a dislocated metric space such that d is continuous functional. If T is a Weak-generalized F -contraction, i.e., there exist F along with C_1, C_2, C_3 and $\tau > 0$ such that $\forall x, y \in X$,

$$d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) \leq F(\mathcal{O}(x, y))$$

where

$$\mathcal{O}(x, y) = \max \left\{ d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Tx) + d(y, Ty)}{2}, \right. \\ \left. \frac{d(x, Tx)d(y, Ty)}{1 + d(x, y)}, \frac{d(y, Ty)[1 + d(x, Tx)]}{1 + d(x, y)}, \right. \\ \left. \frac{d^2(x, Tx)d(y, Ty)}{1 + d^2(x, y)} \right\}.$$

Then T has a unique fixed point.

4. Application to Non-linear Integral Equation

Now we study the existence of unique solutions for non-linear integral equations as an application using the Theorem 1.

Let us consider the non-linear integral equation:

$$\Xi(i) = \int_0^t \Lambda(i, j) \Gamma(j, \Xi(j)) dj \quad (57)$$

for all $i, j \in [0, T]$, where the unknown function $\Xi(i)$ takes real values.

Let $X = \mathbb{C}([0, T])$ be the space of all real continuous functions defined on $[0, T]$. We endow X with the d -metric $d : X \times X \rightarrow \mathbb{R}$ by

$$d(\Xi, \Theta) = \sup_{t \in [0, T]} (|\Xi(i) + \Theta(i)|)^2$$

for all $\Xi, \Theta \in X$.

Clearly $(X, d, 2)$ is a complete b -dislocated metric space.

Let for all $j \in [0, T]$, $\Gamma(j, \cdot)$ be a decreasing function, that is:

$$x, y \in \mathbb{R}, \quad x \geq y \Rightarrow \Gamma(j, x) \leq \Gamma(j, y)$$

Define a mapping $T : X \rightarrow X$ by

$$T\Xi(i) = \int_0^i \Lambda(i, j) \Gamma(j, \Xi(j)) dj; \text{ for all } i \in [0, T].$$

Furthermore, we assume the following conditions:

N1: $\Lambda(i, j) \leq e^{\frac{j\tau}{2}}$ where $\tau > 0$ and $j \in [0, T]$

N2: For all $j \in [0, T]$ and $\Xi(j), \Theta(j) \in X$

$$|\Gamma(j, \Xi(j))| + |\Gamma(j, \Theta(j))| \leq \left[\frac{3\mathcal{A}_1}{2}(\Xi(j) + \Theta(j))^2 + \mathcal{A}_2[(\Xi(j) + T\Xi(j))^2 + (\Theta(j) + T\Theta(j))^2] \right. \\ \left. + \mathcal{A}_3 \left(\frac{(\Xi(j) + T\Xi(j))^2 + (\Theta(j) + T\Theta(j))^2}{6} \right) \right. \\ \left. + \mathcal{A}_4 \left(\frac{(2\Xi(j))^2 + (2\Theta(j))^2}{8} \right) \right]^{\frac{1}{2}} \frac{\tau}{2} e^{-\frac{\tau}{2}}$$

where $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4 \geq 0$ and $\frac{3\mathcal{A}_1}{2} + 2(\mathcal{A}_2 + \mathcal{A}_3 + \mathcal{A}_4) < 1$.

For $\Xi \in X$, we define a norm $\|\Xi\|_\tau = \sup_{i \in [0, T]} |\Xi(i)| e^{-\tau i}$. It is easy to check that $\|\cdot\|_\tau$ is equivalent to the supremum norm $\|\cdot\|$ in X , and X is endowed with the d -metric d defined by

$$d(\Xi, \Theta) = \|\Xi + \Theta\|_\tau \\ = \sup_{i \in [0, T]} \{(|\Xi(i)| + |\Theta(i)|)^2 e^{-i\tau}\}; \quad \text{for } \Xi, \Theta \in X. \quad (58)$$

Then (X, d) is a complete b -dislocated metric space.

Now we will prove the existence of unique solution of the non-linear integral Equation (53).

Theorem 3. Let (X, d) be a b -dislocated metric space as defined above with assuming the above conditions N1 and N2. Then the non-linear integral Equation (53) have a unique solution.

Proof: Let $\Xi, \Theta \in X$ and $i \in [0, T]$ then consider,

$$\begin{aligned}
[|T\Xi(i)| + |T\Theta(i)|]^2 &= \left(\left| \int_0^i \Lambda(i, j) \Gamma(j, \Xi(j)) dj \right| + \left| \int_0^i \Lambda(i, j) \Gamma(j, \Theta(j)) dj \right| \right)^2 \\
&= \left(\int_0^i \Lambda(i, j) |\Gamma(j, \Xi(j))| dj + \int_0^i \Lambda(i, j) |\Gamma(j, \Theta(j))| dj \right)^2 \\
&= \left(\int_0^i \Lambda(i, j) (|\Gamma(j, \Xi(j))| + |\Gamma(j, \Theta(j))|) dj \right)^2 \\
&\leq \left(\int_0^i \left[\frac{3\mathcal{A}_1}{2} (\Xi(j) + \Theta(j))^2 + \mathcal{A}_2 [(\Xi(j) + T\Xi(j))^2 + (\Theta(j) + T\Theta(j))^2] \right. \right. \\
&\quad \left. \left. + \mathcal{A}_3 \left(\frac{(\Xi(j) + T\Xi(j))^2 + (\Theta(j) + T\Theta(j))^2}{6} \right) \right. \right. \\
&\quad \left. \left. + \mathcal{A}_4 \left(\frac{(2\Xi(j))^2 + (2\Theta(j))^2}{8} \right) \right] \frac{\tau}{2} e^{-\frac{\tau}{2}j} dj \right)^2 \\
&\leq \left(\int_0^i \Lambda(i, j) \left[\frac{3\mathcal{A}_1}{2} (|\Xi(j)| + |\Theta(j)|)^2 + \mathcal{A}_2 [(|\Xi(j)| + |T\Xi(j)|)^2 + (|\Theta(j)| + |T\Theta(j)|)^2] \right. \right. \\
&\quad \left. \left. + \mathcal{A}_3 \left(\frac{(|\Xi(j)| + |T\Xi(j)|)^2 + (|\Theta(j)| + |T\Theta(j)|)^2}{6} \right) \right. \right. \\
&\quad \left. \left. + \mathcal{A}_4 \left(\frac{(2|\Xi(j)| + 2|\Theta(j)|)^2}{8} \right) \right] \frac{\tau}{2} e^{-\frac{\tau}{2}j} dj \right)^2 \\
&\leq \left(\int_0^i \Lambda(i, j) e^{j\tau} \left(e^{-j\tau} \left[\frac{3\mathcal{A}_1}{2} (|\Xi(j)| + |\Theta(j)|)^2 + \mathcal{A}_2 [(|\Xi(j)| + |T\Xi(j)|)^2 + (|\Theta(j)| + |T\Theta(j)|)^2] \right. \right. \right. \\
&\quad \left. \left. + \mathcal{A}_3 \left(\frac{(|\Xi(j)| + |T\Xi(j)|)^2 + (|\Theta(j)| + |T\Theta(j)|)^2}{6} \right) \right. \right. \\
&\quad \left. \left. + \mathcal{A}_4 \left(\frac{(2|\Xi(j)| + 2|\Theta(j)|)^2}{8} \right) \right] \frac{\tau}{2} e^{-\frac{\tau}{2}j} dj \right) \right)^2 \tag{59} \\
&\leq \left(\int_0^i \Lambda(i, j) e^{j\tau} \left[\frac{3\mathcal{A}_1}{2} d(\Xi, \Theta) + \mathcal{A}_2 [d(\Xi, T\Xi) + d(\Theta, T\Theta)] \right. \right. \\
&\quad \left. \left. + \mathcal{A}_3 \left(\frac{d(\Xi, T\Theta) + d(\Theta, T\Xi)}{6} \right) \right. \right. \\
&\quad \left. \left. + \mathcal{A}_4 \left(\frac{d(\Xi, \Xi) + d(\Theta, \Theta)}{8} \right) \right] \frac{\tau}{2} e^{-\frac{\tau}{2}j} dj \right)^2 \\
&\leq \frac{\tau^2}{4} e^{-\tau} \left(\int_0^i \Lambda(i, j) \left[\frac{3\mathcal{A}_1}{2} d(\Xi, \Theta) + \mathcal{A}_2 [d(\Xi, T\Xi) + d(\Theta, T\Theta)] \right. \right. \\
&\quad \left. \left. + \mathcal{A}_3 \left(\frac{d(\Xi, T\Theta) + d(\Theta, T\Xi)}{6} \right) \right. \right. \\
&\quad \left. \left. + \mathcal{A}_4 \left(\frac{d(\Xi, \Xi) + d(\Theta, \Theta)}{8} \right) \right] \frac{\tau}{2} dj \right)^2 \\
&= \frac{\tau^2}{4} e^{-\tau} \left[\frac{3\mathcal{A}_1}{2} d(\Xi, \Theta) + \mathcal{A}_2 [d(\Xi, T\Xi) + d(\Theta, T\Theta)] \right. \\
&\quad \left. + \mathcal{A}_3 \left(\frac{d(\Xi, T\Theta) + d(\Theta, T\Xi)}{6} \right) \right. \\
&\quad \left. + \mathcal{A}_4 \left(\frac{d(\Xi, \Xi) + d(\Theta, \Theta)}{8} \right) \right] \left(\int_0^i \Lambda(i, j) dj \right)^2
\end{aligned}$$

$$\begin{aligned}
(|T\Xi(i)| + |T\Theta(i)|)^2 &\leq \frac{\tau^2}{4} e^{-\tau} \left[\frac{3\mathcal{A}_1}{2} d(\Xi, \Theta) + \mathcal{A}_2 [d(\Xi, T\Xi) + d(\Theta, T\Theta)] \right. \\
&\quad + \mathcal{A}_3 \left(\frac{d(\Xi, T\Theta) + d(\Theta, T\Xi)}{6} \right) \\
&\quad \left. + \mathcal{A}_4 \left(\frac{d(\Xi, \Xi) + d(\Theta, \Theta)}{8} \right) \right] \left(\int_0^i e^{\frac{j\tau}{2}} dj \right)^2 \\
&= \frac{\tau^2}{4} e^{-\tau} \left[\frac{3\mathcal{A}_1}{2} d(\Xi, \Theta) + \mathcal{A}_2 [d(\Xi, T\Xi) + d(\Theta, T\Theta)] \right. \\
&\quad + \mathcal{A}_3 \left(\frac{d(\Xi, T\Theta) + d(\Theta, T\Xi)}{6} \right) \\
&\quad \left. + \mathcal{A}_4 \left(\frac{d(\Xi, \Xi) + d(\Theta, \Theta)}{8} \right) \right] \left(\left(\frac{e^{\frac{j\tau}{2}}}{\frac{\tau}{2}} \right)_0^i \right)^2 \\
&< \frac{\tau^2}{4} e^{-\tau} \left[\frac{3\mathcal{A}_1}{2} d(\Xi, \Theta) + \mathcal{A}_2 [d(\Xi, T\Xi) + d(\Theta, T\Theta)] \right. \\
&\quad + \mathcal{A}_3 \left(\frac{d(\Xi, T\Theta) + d(\Theta, T\Xi)}{6} \right) \\
&\quad \left. + \mathcal{A}_4 \left(\frac{d(\Xi, \Xi) + d(\Theta, \Theta)}{8} \right) \right] \left(e^{\frac{i\tau}{2}} \times \frac{2}{\tau} \right)^2 \\
&= \frac{\tau^2}{4} e^{-\tau} \left[\frac{3\mathcal{A}_1}{2} d(\Xi, \Theta) + \mathcal{A}_2 [d(\Xi, T\Xi) + d(\Theta, T\Theta)] \right. \\
&\quad + \mathcal{A}_3 \left(\frac{d(\Xi, T\Theta) + d(\Theta, T\Xi)}{6} \right) \\
&\quad \left. + \mathcal{A}_4 \left(\frac{d(\Xi, \Xi) + d(\Theta, \Theta)}{8} \right) \right] \left(e^{i\tau} \frac{4}{\tau^2} \right) \\
&= e^{-\tau(1-i)} \left[\frac{3\mathcal{A}_1}{2} d(\Xi, \Theta) + \mathcal{A}_2 [d(\Xi, T\Xi) + d(\Theta, T\Theta)] \right. \\
&\quad + \mathcal{A}_3 \left(\frac{d(\Xi, T\Theta) + d(\Theta, T\Xi)}{6} \right) \\
&\quad \left. + \mathcal{A}_4 \left(\frac{d(\Xi, \Xi) + d(\Theta, \Theta)}{8} \right) \right]
\end{aligned} \tag{60}$$

which implies,

$$\begin{aligned}
(|T\Xi(i)| + |T\Theta(i)|)^2 &\leq e^{-\tau(1-i)} \left[\frac{3\mathcal{A}_1}{2} d(\Xi, \Theta) + \mathcal{A}_2 [d(\Xi, T\Xi) + d(\Theta, T\Theta)] \right. \\
&\quad + \mathcal{A}_3 \left(\frac{d(\Xi, T\Theta) + d(\Theta, T\Xi)}{6} \right) \\
&\quad \left. + \mathcal{A}_4 \left(\frac{d(\Xi, \Xi) + d(\Theta, \Theta)}{8} \right) \right]
\end{aligned} \tag{61}$$

$$\begin{aligned}
\Rightarrow (|T\Xi(i)| + |T\Theta(i)|)^2 e^{-i\tau} &\leq e^{-\tau} \left[\frac{3\mathcal{A}_1}{2} d(\Xi, \Theta) + \mathcal{A}_2 [d(\Xi, T\Xi) + d(\Theta, T\Theta)] \right. \\
&\quad + \mathcal{A}_3 \left(\frac{d(\Xi, T\Theta) + d(\Theta, T\Xi)}{6} \right) \\
&\quad \left. + \mathcal{A}_4 \left(\frac{d(\Xi, \Xi) + d(\Theta, \Theta)}{8} \right) \right]
\end{aligned} \tag{62}$$

this gives,

$$\begin{aligned}
 d(T\Xi, T\Theta) \leq e^{-\tau} & \left[\frac{3\mathcal{A}_1}{2} d(\Xi, \Theta) + \mathcal{A}_2 [d(\Xi, T\Xi) + d(\Theta, T\Theta)] \right. \\
 & + \mathcal{A}_3 \left(\frac{d(\Xi, T\Theta) + d(\Theta, T\Xi)}{6} \right) \\
 & \left. + \mathcal{A}_4 \left(\frac{d(\Xi, \Xi) + d(\Theta, \Theta)}{8} \right) \right]
 \end{aligned} \tag{63}$$

Applying logarithms on both sides, we get,

$$\begin{aligned}
 \ln(d(T\Xi, T\Theta)) & \leq \ln \left(e^{-\tau} \left[\frac{3\mathcal{A}_1}{2} d(\Xi, \Theta) + \mathcal{A}_2 [d(\Xi, T\Xi) + d(\Theta, T\Theta)] \right. \right. \\
 & \quad \left. + \mathcal{A}_3 \left(\frac{d(\Xi, T\Theta) + d(\Theta, T\Xi)}{6} \right) \right. \\
 & \quad \left. \left. + \mathcal{A}_4 \left(\frac{d(\Xi, \Xi) + d(\Theta, \Theta)}{8} \right) \right] \right) \\
 & \leq \ln e^{-\tau} + \ln \left(\left[\frac{3\mathcal{A}_1}{2} d(\Xi, \Theta) + \mathcal{A}_2 [d(\Xi, T\Xi) + d(\Theta, T\Theta)] \right. \right. \\
 & \quad \left. + \mathcal{A}_3 \left(\frac{d(\Xi, T\Theta) + d(\Theta, T\Xi)}{6} \right) \right. \\
 & \quad \left. \left. + \mathcal{A}_4 \left(\frac{d(\Xi, \Xi) + d(\Theta, \Theta)}{8} \right) \right] \right) \\
 & \leq -\tau + \ln \left(\left[\frac{3\mathcal{A}_1}{2} d(\Xi, \Theta) + \mathcal{A}_2 [d(\Xi, T\Xi) + d(\Theta, T\Theta)] \right. \right. \\
 & \quad \left. + \mathcal{A}_3 \left(\frac{d(\Xi, T\Theta) + d(\Theta, T\Xi)}{6} \right) \right. \\
 & \quad \left. \left. + \mathcal{A}_4 \left(\frac{d(\Xi, \Xi) + d(\Theta, \Theta)}{8} \right) \right] \right)
 \end{aligned} \tag{64}$$

$$\begin{aligned}
 \tau + \ln(d(T\Xi, T\Theta)) & \leq \ln \left(\left[\frac{3\mathcal{A}_1}{2} d(\Xi, \Theta) + \mathcal{A}_2 [d(\Xi, T\Xi) + d(\Theta, T\Theta)] \right. \right. \\
 & \quad \left. + \mathcal{A}_3 \left(\frac{d(\Xi, T\Theta) + d(\Theta, T\Xi)}{6} \right) \right. \\
 & \quad \left. \left. + \mathcal{A}_4 \left(\frac{d(\Xi, \Xi) + d(\Theta, \Theta)}{8} \right) \right] \right)
 \end{aligned} \tag{65}$$

Define $F : \mathbb{R}^+ \rightarrow \mathbb{R}$ by $F(\alpha) = \ln(\alpha)$, $\alpha > 0$. Then from (61), we get

$$\begin{aligned}
 \tau + F(d(T\Xi, T\Theta)) & \leq F \left(\left[\frac{3\mathcal{A}_1}{2} d(\Xi, \Theta) + \mathcal{A}_2 [d(\Xi, T\Xi) + d(\Theta, T\Theta)] \right. \right. \\
 & \quad \left. + \mathcal{A}_3 \left(\frac{d(\Xi, T\Theta) + d(\Theta, T\Xi)}{6} \right) \right. \\
 & \quad \left. \left. + \mathcal{A}_4 \left(\frac{d(\Xi, \Xi) + d(\Theta, \Theta)}{8} \right) \right] \right)
 \end{aligned} \tag{66}$$

Thus, T is an Extended F -contraction and satisfied all the conditions of the Theorem 1. Thus, T has a unique fixed point which is the unique solution of the non-linear integral equation.

5. Application to Fractional Calculus

In this section, we will start with the existence of solution for the non-linear fractional differential equation.

$$C_{D^\omega}(x(\kappa)) = \Lambda(\kappa, x(\kappa)), \quad (0 < \kappa < 1, 1 < \omega \leq 2) \quad (67)$$

via the integral boundary conditions

$$x(0) = 0, \quad x(1) = \int_0^\xi x(s)ds, \quad (0 < \xi < 1) \quad (68)$$

where C_{D^ω} denotes the Caputo fractional derivative of order ω and $\Lambda : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function. Let the space $X = C([0, 1], \mathbb{R})$ be the set of real continuous functions on $[0, 1]$. We confer X with the dislocated metric

$$D_{\mathcal{M}(\infty)}(x, y) = \sup_{\kappa \in [0, 1]} (|x(\kappa)| + |y(\kappa)|), \quad \text{for all } x, y \in X.$$

Then $(X, D_{\mathcal{M}(\infty)})$ is a complete dislocated metric space. It is very well known that, for a continuous function $\Theta : [0, \infty] \rightarrow \mathbb{R}$, the Caputo derivative of fractional order ω is defined as:

$$C_{D^\omega}(\Theta(\kappa)) = \frac{1}{\Gamma(n - \omega)} \int_0^\kappa (\kappa - s)^{n - \omega - 1} \Theta^n(s) ds; \quad n - 1 < \omega < n, \quad n = [\omega] + 1,$$

where $[\omega]$ is represented as integer part of the real number ω and Γ is a gamma function.

Now, we consider the following conditions:

$\mathcal{C}_1$: For $u, v \in \mathbb{R}, \tau > 0$ and $\mathcal{A}_1 < \frac{1}{2}$

$$(|\Lambda(\kappa, u)| + |\Lambda(\kappa, v)|) \leq e^{-\tau} 2\mathcal{A}_1 \frac{\Gamma(\omega + 1)}{5} (|u| + |v|)$$

$\mathcal{C}_2$: Define $T : X \rightarrow X$ by

$$\begin{aligned} Tx(\kappa) = & \frac{1}{\Gamma\omega} \int_0^\kappa (\kappa - s)^{\omega-1} \Lambda(s, x(s)) ds \\ & - \frac{2\kappa}{(2 - \xi^2)\Gamma(\omega)} \int_0^1 (1 - s)^{\omega-1} \Lambda(s, x(s)) ds \\ & + \frac{2\kappa}{(2 - \xi^2)\Gamma(\omega)} \int_0^\xi \left(\int_0^s (s - m)^{\omega-1} \Lambda(\ell, x(\ell)) d\ell \right) ds; \quad \text{for } \kappa \in [0, 1]. \end{aligned}$$

Theorem 4. If $\mathcal{C}_1$ & $\mathcal{C}_2$ are satisfied then the problem (67) has at least one solution.

Proof. It is very well known that $x \in X$ is a solution of (67) if and only if $x \in X$ is a solution of the integral equation.

$$\begin{aligned} x(\kappa) = & \frac{1}{\Gamma(\omega)} \int_0^\kappa (\kappa - s)^{\omega-1} \Lambda(s, x(s)) ds \\ & - \frac{2\kappa}{(2 - \xi^2)\Gamma(\omega)} \int_0^1 (1 - s)^{\omega-1} \Lambda(s, \omega(s)) ds \\ & + \frac{2\kappa}{(2 - \xi^2)\Gamma(\omega)} \int_0^\xi \left(\int_0^s (s - m)^{\omega-1} \Lambda(\ell, x(\ell)) d\ell \right) ds; \quad \text{for } \kappa \in [0, 1]. \end{aligned}$$

Then, the problem (67) is equivalent to find $x^* \in X$ which is a fixed point of T .

Now, let $x, y \in X$ for all $t \in [0, 1]$.

Consider,

$$\begin{aligned}
 |Tx(\kappa)| + |Ty(\kappa)| &= \left| \frac{1}{\Gamma(\omega)} \int_0^\kappa (\kappa - s)^{\omega-1} \Lambda(s, x(s)) ds \right. \\
 &\quad - \frac{2\kappa}{(2 - \xi^2)\Gamma(\omega)} \int_0^1 (1 - s)^{\omega-1} \Lambda(s, x(s)) ds \\
 &\quad + \frac{2\kappa}{(2 - \xi^2)\Gamma(\omega)} \int_0^\xi \left(\int_0^s (s - m)^{\omega-1} \Lambda(\ell, x(\ell)) d\ell \right) ds \Big| \\
 &\quad + \left| \frac{1}{\Gamma(\omega)} \int_0^\kappa (\kappa - s)^{\omega-1} \Lambda(s, y(s)) ds \right. \\
 &\quad - \frac{2\kappa}{(2 - \xi^2)\Gamma(\omega)} \int_0^1 (1 - s)^{\omega-1} \Lambda(s, y(s)) ds \\
 &\quad + \frac{2\kappa}{(2 - \xi^2)\Gamma(\omega)} \int_0^\xi \left(\int_0^s (s - m)^{\omega-1} \Lambda(\ell, y(\ell)) d\ell \right) ds \Big| \\
 &= \left| \frac{1}{\Gamma(\omega)} \int_0^\kappa (\kappa - s)^{\omega-1} [\Lambda(s, x(s)) + \Lambda(s, y(s))] ds \right. \\
 &\quad - \frac{2\kappa}{(2 - \xi^2)\Gamma(\omega)} \int_0^1 (1 - s)^{\omega-1} [\Lambda(s, x(s)) + \Lambda(s, y(s))] ds \\
 &\quad + \frac{2\kappa}{(2 - \xi^2)\Gamma(\omega)} \int_0^\xi \left(\int_0^s (s - m)^{\omega-1} [\Lambda(\ell, x(\ell)) + \Lambda(\ell, y(\ell))] d\ell \right) ds \Big| \\
 &\leq \frac{1}{\Gamma(\omega)} \int_0^\kappa (\kappa - s)^{\omega-1} [|\Lambda(s, x(s))| + |\Lambda(s, y(s))|] ds \\
 &\quad - \frac{2\kappa}{(2 - \xi^2)\Gamma(\omega)} \int_0^1 (1 - s)^{\omega-1} [|\Lambda(s, x(s))| + |\Lambda(s, y(s))|] ds \\
 &\quad + \frac{2\kappa}{(2 - \xi^2)\Gamma(\omega)} \int_0^\xi \left(\int_0^s (s - m)^{\omega-1} [|\Lambda(\ell, x(\ell))| + |\Lambda(\ell, y(\ell))|] d\ell \right) ds \\
 &\leq \frac{1}{\Gamma(\omega)} \int_0^\kappa (\kappa - s)^{\omega-1} [|\Lambda(s, x(s))| + |\Lambda(s, y(s))|] ds \\
 &\quad - \frac{2\kappa}{(2 - \xi^2)\Gamma(\omega)} \int_0^1 (1 - s)^{\omega-1} [|\Lambda(s, x(s))| + |\Lambda(s, y(s))|] ds \\
 &\quad + \frac{2\kappa}{(2 - \xi^2)\Gamma(\omega)} \int_0^\xi \left(\int_0^s (s - m)^{\omega-1} [|\Lambda(\ell, x(\ell))| + |\Lambda(\ell, y(\ell))|] d\ell \right) ds \\
 &\leq \frac{\Gamma\omega + 1}{5} [|x(s)| + |y(s)|] e^{-\tau} \sup \left\{ \frac{1}{\Gamma\omega} \int_0^\kappa (\kappa - s)^{\omega-1} ds \right. \\
 &\quad \left. - \frac{2\kappa}{(2 - \xi^2)\Gamma(\omega)} \int_0^1 (1 - s)^{\omega-1} ds + \frac{2\kappa}{(2 - \xi^2)\Gamma(\omega)} \int_0^\xi \left(\int_0^s (s - m)^{\omega-1} d\ell \right) ds \right\} \\
 &\leq e^{-\tau} 2\mathcal{A}_1 |x(s)| + |y(s)|
 \end{aligned}$$

By taking supremum on both sides, we get,

$$D_{\mathcal{M}(\infty)}(Tx, Ty) \leq e^{-\tau} 2\mathcal{A}_1 D_{\mathcal{M}(\infty)}(x, y).$$

Taking logarithms on both sides, $\ln(D_{\mathcal{M}(\infty)}(Tx, Ty)) \leq -\tau + \ln(2\mathcal{A}_1 D_{\mathcal{M}(\infty)}(x, y))$, which implies, $F(D_{\mathcal{M}(\infty)}(Tx, Ty)) + \tau \leq F(2\mathcal{A}_1 D_{\mathcal{M}(\infty)}(x, y))$.

By taking $F(\alpha) = \ln \alpha$ for $\alpha > 0$ and $\mathcal{A}_1 < \frac{1}{2}$, $\mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4 = 0$, all the conditions of Corollary 2 satisfied. Hence T has a unique fixed point. i.e., $Tx^* = x^*$. Hence x^* is a solution of (67). $\square$

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