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Conservation Laws for a Delayed Hamiltonian System in a Time Scales Version

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Abstract: The theory of time scales which unifies differential and difference analysis provides a new perspective for scientific research. In this paper, we derive the canonical equations of a delayed Hamiltonian system in a time scales version and prove the Noether theorem by using the method of reparameterization with time. The results extend not only the continuous version of the Noether theorem with delayed arguments but also the discrete one. As an application of the results, we find a Noether-type conserved quantity of a delayed Emden-Fowler equation on time scales.

Keywords: Noether theorem; Hamiltonian system; time scale; time delay

1. Introduction

The influence of time delay becomes a very important factor in scientific research to achieve more accurate and more objective results. Since TÈl'sgol'c'sT work [1] in 1964, the dynamical equations in the framework of difference and differential have been investigated with delayed arguments extensively and the results proved to be effective in reflecting a better essence of things and development law [2–7]. Nonetheless, in reality, discrepancies still remain, sometimes even essential differences. Therefore, it's important and difficult to study the delayed dynamical equations in a time scales version.

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers. The differential calculus, difference calculus, and quantum calculus are three most popular examples of calculus on time scales, i.e., the time scales $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{N}$ and $\mathbb{T} = h^{\mathbb{N}_0} = \left\{h^i : i \in \mathbb{N}_0\right\}$, where $h > 1$. The theory of time scales, which unifies and extends continuous and discrete analysis, has been proved to be more accurate in modelling dynamic process, for example, the simulation of the current change rates of a simple electric circuit with resistance, inductance, and capacitance when discharging the capacitor periodically every time unit [8]. It not only reveals the discrepancies between the results concerning differential equations and difference equations, but also helps avoid proving results twice. Up to now, tremendous applications have been found in different dynamical models, such as population models, geometric models, and economic models [9,10]. Bohner and Hilscher [11] studied the calculus of variations in a time scales version. The method of symmetry plays an important role in finding an invariant solution or the first integral of dynamical equations. The famous Noether theorem which reveals a relation between symmetries and conserved quantities achieved some results in a time scales version [12,13]. Corresponding applications about constrained mechanical systems [14], Hamiltonian systems [15], Birkhoffian systems [16], and control problems [17] were discussed in a time scales version.

Until now, preliminary results in delayed optimal control systems on time scales [18], delayed neural networks on time scales [19], oscillation, and stability of delayed equations on time scales [20,21] have been carried out. The Noether symmetry theory has been applied to the delayed non-conservative mechanical systems in a version of time scales [22] successfully. However, the Noether symmetry theory for a delayed Hamiltonian system has not been investigated in version of time scales yet. It is very important to study this new problem.

2. Preliminaries on Time Scales

In this section, we remind some basic definitions and properties about calculus on time scales. For further discussion and proof, readers can refer to [9,10] and references therein.

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. For all $t \in \mathbb{T}$, the following operators are used:

(i) The forward jump operator $\sigma \in \mathbb{T} \to \mathbb{T}$, $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$;
(ii) The backward jump operator $\rho \in \mathbb{T} \to \mathbb{T}$, $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$;
(iii) The graininess function $\mu : \mathbb{T} \to [0, \infty)$, $\mu(t) := \sigma(t) - t$.

A point $t \in \mathbb{T}$ is called right-dense, right-scattered, left-dense, and left-scattered if $\sigma(t) = t$, $\sigma(t) > t$, $\rho(t) = t$ and $\rho(t) < t$, respectively. If $\mathbb{T}$ has a left-scattered maximum M, then we define $\mathbb{T}^k = \mathbb{T} - \{M\}$, otherwise $\mathbb{T}^k = \mathbb{T}$.

Definition 1. *Let $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^k$. Then the delta derivative $f^\Delta(t)$ is the number with the poverty that given any $\varepsilon > 0$, there exists a neighborhood $U = (t - \delta, t + \delta) \cap \mathbb{T}$ of t for some $\delta > 0$ such that*

$$\left| f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s) \right| \leq \varepsilon|\sigma(t) - s|, \text{ for all } s \in U$$

For delta differentiable f and g, the next formulae hold:

(i) $f^\sigma(t) = f(t) + \mu(t) f^\Delta(t)$,
(ii) $(af + bg)^\Delta = af^\Delta(t) + bg^\Delta(t)$,
(iii) $(fg)^\Delta(t) = f^\Delta(t) f^\sigma(t) + f(t) g^\Delta(t) = f^\Delta(t) g(t) + f^\sigma(t) g^\Delta(t)$,

where we denote $f \circ \sigma$ by f^σ.

Definition 2. *A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous if it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. The set of rd-continuous functions can be denoted by C_{rd}. The set of differentiable functions with rd-continuous derivative is denoted by $\mathrm{C}^1_{\mathrm{rd}}$.*

Assume that $\nu : \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\mathbb{T}^* := \nu(\mathbb{T})$ is a time scale, then the following results hold:

(i) Let $\omega : \mathbb{T}^* \to \mathbb{R}$. If $\nu^\Delta(t)$ and $\omega^{\Delta^*}(\nu(t))$ exist for $t \in \mathbb{T}^k$, then

$$(\omega \circ \nu)^\Delta = \left(\omega^{\Delta^*} \circ \nu\right) \nu^\Delta$$

(ii) If $f : \mathbb{T} \to \mathbb{R}$ is a C_{rd} function and ν is a $\mathrm{C}^1_{\mathrm{rd}}$ function, then for $a, b \in \mathbb{T}$,

$$\int_a^b f(\nu(t)) \nu^\Delta(t) \Delta t = \int_{\nu(a)}^{\nu(b)} \left(f \circ \nu^{-1}\right)(s) \Delta^* s$$

Lemma 1. *(Dubois-Reymond) Let $g \in \mathrm{C}_{\mathrm{rd}}$, $g : [\mathrm{a}, \mathrm{b}] \to \mathbb{R}^n$, for all $\eta \in \mathrm{C}^1_{\mathrm{rd}}$ with $\eta(a) = \eta(b) = 0$, then $\int_a^b fg^T(t) \eta^\Delta(t) \Delta t = 0$ holds if and only if $g(t) = c$ on $[a, b]^k$ for some $c \in \mathbb{R}^n$.*

3. Main Results

3.1. Time-Scale Canonical Equations

Integral

$$S[q_k(\cdot), q_{k\tau}(\cdot)] = \int_{t_1}^{t_2} L\Big(t, q_k^\sigma, q_k^\Delta, q_{k\tau}^\sigma, q_{k\tau}^\Delta\Big)\Delta t \tag{1}$$

can be called the time-scale Hamilton action with delayed arguments. The integrand $L\big(t, q_k^\sigma, q_k^\Delta, q_{k\tau}^\sigma, q_{k\tau}^\Delta\big)$ is the Lagrangian of the delayed system, where $t \in \mathbb{T}$, τ is a constant time delay, $\tau < t_2 - t_1$ and $t - \tau \in \mathbb{T}$, the generalized coordinates $q_k : [t_1, t_2]_{\mathbb{T}} \to \mathbb{R}^n$ are assumed to be $\mathrm{C}^1_{\mathrm{rd}}$, $q_{k\tau}^\sigma = q_k^\sigma(t-\tau)$, $q_{k\tau}^\Delta = q_k^\Delta(t-\tau)$, $k = 1, 2, \cdots, n$.

The isochronous variational principle

$$\delta S = 0 \tag{2}$$

with relationship [15]

$$\delta q_k^\Delta = (\delta q_k)^\Delta \tag{3}$$

and boundary conditions

$$q_k(t) = \varphi_k(t), t \in [t_1 - \tau, t_1] \tag{4}$$

$$q_k(t) = q_k(t_2), t = t_2, \tag{5}$$

can be called the time-scale Hamilton principle with delayed arguments, where $\varphi_k(t)$ are piecewise smooth functions.

We define the time-scale Hamiltonian of the delayed system as

$$H = H(t, q_k^\sigma(t), p_k(t), q_k^\sigma(t-\tau), p_k(t-\tau)) = p_k(t)q_k^\Delta(t) + p_k(t-\tau)q_k^\Delta(t-\tau) - L, \tag{6}$$

where

$$p_k(t) = \frac{\partial L}{\partial q_k^\Delta(t)},\ p_k(t-\tau) = \frac{\partial L}{\partial q_k^\Delta(t-\tau)} \tag{7}$$

are generalized momentum.

Thus, we have

$$S[q_k(\cdot), q_{k\tau}(\cdot), p_k(\cdot), p_{k\tau}(\cdot)] = \int_{t_1}^{t_2} \Big[p_k q_k^\Delta + p_{k\tau} q_{k\tau}^\Delta - H(t, q_k^\sigma, p_k, q_{k\tau}^\sigma, p_{k\tau})\Big]\Delta t \tag{8}$$

We obtain the time-scale canonical equations of the delayed Hamiltonian system,

$$\begin{gathered} q_k^\Delta(t) + q_{k\tau}^\Delta(t+\tau) = \tfrac{\partial H}{\partial p_k}(t) + \tfrac{\partial H}{\partial p_{k\tau}}(t+\tau), p_k^\Delta(t) + p_{k\tau}^\Delta(t+\tau) + \tfrac{\partial H}{\partial q_k^\sigma}(t) + \tfrac{\partial H}{\partial q_{k\tau}^\sigma}(t+\tau) = 0, t \in [t_1, t_2 - \tau], \\ q_k^\Delta(t) = \frac{\partial H}{\partial p_k}(t), p_k^\Delta(t) + \frac{\partial H}{\partial q_k^\sigma}(t) = 0, t \in (\,t_2 - \tau, t_2]\,, \end{gathered} \tag{9}$$

where $k = 1, 2, \cdots, n$.

Actually, from Formula (2), we have

$$\begin{aligned} \delta S \ &= \textstyle\int_{t_1}^{t_2-\tau}\Big[\Big(q_k^\Delta(t) - \frac{\partial H}{\partial p_k}(t) + q_{k\tau}^\Delta(t+\tau) - \frac{\partial H}{\partial p_{k\tau}}(t+\tau)\Big)\delta p_k \\ &\quad + (p_k(t) + p_{k\tau}(t+\tau))\delta q_k^\Delta - \Big(\tfrac{\partial H}{\partial q_k^\sigma}(t) + \tfrac{\partial H}{\partial q_{k\tau}^\sigma}(t+\tau)\Big)\delta q_k^\sigma\Big]\Delta t \\ &\quad + \textstyle\int_{t_2-\tau}^{t_2}\Big[\Big(q_k^\Delta(t) - \frac{\partial H}{\partial p_k}(t)\Big)\delta p_k + p_k(t)\delta q_k^\Delta - \frac{\partial H}{\partial q_k^\sigma}(t)\delta q_k^\sigma\Big]\Delta t \\ &= \textstyle\int_{t_1}^{t_2-\tau}\Big[\Big(p_k(t) + p_{k\tau}(t+\tau) - \int_t^{t_2-\tau}\Big(\frac{\partial H}{\partial q_k^\sigma}(\zeta) + \frac{\partial H}{\partial q_{k\tau}^\sigma}(\zeta+\tau)\Big)\Delta\zeta\Big)(\delta q_k)^\Delta \\ &\quad + \Big(q_k^\Delta(t) - \tfrac{\partial H}{\partial p_k}(t) + q_{k\tau}^\Delta(t+\tau) - \tfrac{\partial H}{\partial p_{k\tau}}(t+\tau)\Big)\delta p_k\Big]\Delta t \\ &\quad + \textstyle\int_{t_2-\tau}^{t_2}\Big[\Big(p_k(t) + \int_{t_2-\tau}^{t}\frac{\partial H}{\partial q_k^\sigma}(\zeta)\Delta\zeta\Big)(\delta q_k)^\Delta + \Big(q_k^\Delta(t) - \frac{\partial H}{\partial p_k}(t)\Big)\delta p_k\Big]\Delta t = 0. \end{aligned} \tag{10}$$

According to Formula (6) and Dubois-Reymond Lemma 1, we can derive the Equation (9).

Remark 1. *If the delay is not exist, Equation (9) becomes [16]*

$$q_k^{\Delta}(t) = \frac{\partial H}{\partial p_k}(t), p_k^{\Delta}(t) = -\frac{\partial H}{\partial q_k^{\sigma}}(t),\ (k = 1, 2, \cdots, n) \tag{11}$$

Furthermore, if $\mathbb{T} = \mathbb{R}$, functional (1) becomes the classical Hamilton action [23], and Equation (9) becomes the classical Hamilton canonical equations

$$\dot{q}_k(t) = \frac{\partial H}{\partial p_k}(t), \dot{p}_k(t) = -\frac{\partial H}{\partial q_k}(t),\ (k = 1, 2, \cdots, n) \tag{12}$$

3.2. Invariance under the Infinitesimal Transformations

The Noether symmetry under the one-parameter infinitesimal transformations for the delayed Hamiltonian system in a time scales version can be described as follows:

Definition 3. *A time-scale Hamilton action (8) is said to be invariant under the infinitesimal transformations*

$$\begin{aligned} t^* &= T(t, q_j, p_j, \varepsilon) = t + \varepsilon\xi_0(t, q_j, p_j), \\ q_k^* &= Q_k(t, q_j, p_j, \varepsilon) = q_k(t) + \varepsilon\xi_k(t, q_j, p_j), \\ p_k^* &= P_k(t, q_j, p_j, \varepsilon) = p_k(t) + \varepsilon\eta_k(t, q_j, p_j), (k, j = 1, 2, \cdots, n), \end{aligned} \tag{13}$$

if and only if

$$\begin{aligned} &\int_{t_1}^{t_2} [p_k q_k^{\Delta} + p_{k\tau} q_{k\tau}^{\Delta} - H(t, q_k^{\sigma}, p_k, q_{k\tau}^{\sigma}, p_{k\tau})]\Delta t \\ &= \int_{\beta(t_1)}^{\beta(t_2)} p_k^* q_k^{*\Delta^*} + p_{k\tau}^* q_{k\tau}^{*\Delta^*} - H\left(t^*, q_k^{*\sigma^*}, p_k^*, q_{k\tau}^{*\sigma^*}, p_{k\tau}^*\right)\Delta^* t^* \end{aligned} \tag{14}$$

holds.

Here, the map $t \in [t_1, t_2] \mapsto \beta(t) := T(t, q_j, p_j, \varepsilon) \in \mathbb{R}$ is considered as a strictly increasing C_{rd}^1 function. The new time scale $\mathbb{T}^*$ is the image of the map. We also assume $\sigma^* \circ \beta = \beta \circ \sigma$, where σ^* is the new forward jump operator.

According to Definition 3, we can obtain the necessary condition of the invariance:

Theorem 1. *If the time-scale Hamilton action (8) is invariant under the infinitesimal transformations (13), then*

$$\begin{aligned} &(p_k(t) + p_{k\tau}(t+\tau))\xi_k^{\Delta} - H(t)\xi_0^{\Delta} - \tfrac{\partial H}{\partial t}(t)\xi_0 - \left(\tfrac{\partial H}{\partial q_k^{\sigma}}(t) + \tfrac{\partial H}{\partial q_{k\tau}^{\sigma}}(t+\tau)\right)\xi_k^{\sigma} \\ &+\left(q_k^{\Delta}(t) - \tfrac{\partial H}{\partial p_k}(t) + q_{k\tau}^{\Delta}(t+\tau) - \tfrac{\partial H}{\partial p_{k\tau}}(t+\tau)\right)\eta_k = 0,\ t \in [t_1, t_2 - \tau], \\ &p_k(t)\xi_k^{\Delta} - H(t)\xi_0^{\Delta} - \tfrac{\partial H}{\partial t}(t)\xi_0 - \tfrac{\partial H}{\partial q_k^{\sigma}}(t)\xi_k^{\sigma} + \left(q_k^{\Delta}(t) - \tfrac{\partial H}{\partial p_k}(t)\right)\eta_k = 0,\ t \in (t_2 - \tau, t_2], \end{aligned} \tag{15}$$

where $\xi_{k=}^{\sigma}\xi_k(\sigma(t), q_j(\sigma(t)), p_j(\sigma(t)))$, $\xi_k^{\Delta} = \frac{\Delta}{\Delta t}\xi_k(t, q_j(t), p_j(t))$.

Proof. We have

$$\begin{aligned} &\int_{t_1}^{t_2} [p_k q_k^{\Delta} + p_{k\tau} q_{k\tau}^{\Delta} - H(t, q_k^{\sigma}, p_k, q_{k\tau}^{\sigma}, p_{k\tau})]\Delta t \\ &= \int_{t_1}^{t_2} \Bigg[(p_k^* \circ \beta)(t)\frac{(q_k^* \circ \beta)^{\Delta}(t)}{\beta^{\Delta}(t)} + (p_k^* \circ \beta)(t-\tau)\frac{((q_k^* \circ \beta)(t-\tau))^{\Delta}}{\beta^{\Delta}(t-\tau)} \\ &- H(\beta(t), (q_k^* \circ \sigma^* \circ \beta)(t), (p_k^* \circ \beta)(t), (q_k^* \circ \sigma^* \circ \beta)(t-\tau), (p_k^* \circ \beta)(t-\tau))\Bigg]\beta^{\Delta}\Delta t \end{aligned} \tag{16}$$

$$= \int_{t_1}^{t_2} \left[P_k \frac{Q_k^{\Delta}}{T^{\Delta}} + P_{k\tau}\frac{Q_{k\tau}^{\Delta}}{T_{\tau}^{\Delta}} - H(T, Q_k^{\sigma}, P_k, Q_{k\tau}^{\sigma}, P_{k\tau})\right]T^{\Delta}\Delta t, \tag{17}$$

where we denote

$T_\tau = T\big(t-\tau, q_j(t-\tau), p_j(t-\tau), \varepsilon\big), Q_{k\tau} = Q_k\big(t-\tau, q_j(t-\tau), p_j(t-\tau), \varepsilon\big)$ and
$P_{k\tau} = P_k\big(t-\tau, q_j(t-\tau), p_j(t-\tau), \varepsilon\big)$.

We yield the Formula (15) by taking derivative of Formula (17) with respect to ε and setting $\varepsilon = 0$. □

The Formula (15) can be called the time-scale Noether identity for the delayed Hamiltonian system.

3.3. Time-Scale Noether Theorem

The time-scale Noether theorem for the delayed Hamiltonian system can be described as follows:

Theorem 2. *If the time-scale Hamilton action (8) is invariant under Definition 3, then*

$$\begin{aligned} &I = (p_k(t) + p_{k\tau}(t+\tau))\xi_k - H(t)\xi_0 + \mu(t)\tfrac{\partial H}{\partial t}(t)\xi_0 = c,\ t \in [t_1, t_2-\tau], \\ &I = p_k(t)\xi_k - H(t)\xi_0 + \mu(t)\tfrac{\partial H}{\partial t}(t)\xi_0 = c,\ t \in (t_2-\tau, t_2], \end{aligned} \tag{18}$$

is a conserved quantity.

The proof is presented in Section 4.

Remark 2. *If $\mathbb{T} = \mathbb{R}$, then $\sigma(t) = t, \mu(t) = 0$. For the delayed Hamiltonian system, Formula (15) becomes*

$$\begin{aligned} &(p_k(t) + p_{k\tau}(t+\tau))\dot{\xi}_k - H(t)\dot{\xi}_0 - \tfrac{\partial H}{\partial t}(t)\xi_0 - \Big(\tfrac{\partial H}{\partial q_k}(t) + \tfrac{\partial H}{\partial q_{k\tau}}(t+\tau)\Big)\xi_k \\ &+\Big(\dot{q}_k(t) - \tfrac{\partial H}{\partial p_k}(t) + \dot{q}_{k\tau}(t+\tau) - \tfrac{\partial H}{\partial p_{k\tau}}(t+\tau)\Big)\eta_k = 0,\ t \in [t_1, t_2-\tau], \\ &p_k(t)\dot{\xi}_k - H(t)\dot{\xi}_0 - \tfrac{\partial H}{\partial t}(t)\xi_0 - \tfrac{\partial H}{\partial q_k}(t)\xi_k + \Big(\dot{q}_k(t) - \tfrac{\partial H}{\partial p_k}(t)\Big)\eta_k = 0, t \in (t_2-\tau, t_2] \end{aligned} \tag{19}$$

and Formula (18) gives

$$\begin{aligned} &I = (p_k(t) + p_{k\tau}(t+\tau))\xi_k - H(t)\xi_0 = c,\ t \in [t_1, t_2-\tau], \\ &I = p_k(t)\xi_k - H(t)\xi_0 = c,\ t \in (t_2-\tau, t_2]. \end{aligned} \tag{20}$$

Remark 3. *If $\mathbb{T} = h\mathbb{Z}$, then $\sigma(t) = t+h, \mu(t) = h$. For the discrete delayed Hamiltonian system, Formula (15) becomes*

$$\begin{aligned} &(p_k(t) + p_{k\tau}(t+\tau))\tfrac{\xi_k(t+h)-\xi_k(t)}{h} - H(t)\tfrac{\xi_0(t+h)-\xi_0(t)}{h} - \tfrac{\partial H}{\partial t}(t)\xi_0(t) \\ &-\Big(\tfrac{\partial H}{\partial q_k(t+h)}(t) + \tfrac{\partial H}{\partial q_{k\tau}(t+h)}(t+\tau)\Big)\xi_k(t+h) + \Big(\tfrac{q_k(t+h)-q_k(t)}{h} - \tfrac{\partial H}{\partial p_k}(t)\Big)\eta_k \\ &+\Big(\tfrac{q_{k\tau}(t+h+\tau)-q_{k\tau}(t+\tau)}{h} - \tfrac{\partial H}{\partial p_{k\tau}}(t+\tau)\Big)\eta_k = 0,\ t \in [t_1, t_2-\tau], \\ &p_k(t)\tfrac{\xi_k(t+h)-\xi_k(t)}{h} - H(t)\tfrac{\xi_0(t+h)-\xi_0(t)}{h} - \tfrac{\partial H}{\partial t}(t)\xi_0(t) \\ &-\tfrac{\partial H}{\partial q_k(t+h)}(t)\xi_k(t+h) + \Big(\tfrac{q_k(t+h)-q_k(t)}{h} - \tfrac{\partial H}{\partial p_k}(t)\Big)\eta_k = 0,\ t \in (t_2-\tau, t_2] \end{aligned} \tag{21}$$

and Formula (18) gives

$$\begin{aligned} &I = (p_k(t) + p_{k\tau}(t+\tau))\xi_k(t) - H(t)\xi_0(t) + h\tfrac{\partial H}{\partial t}(t)\xi_0(t) = c,\ t \in [t_1, t_2-\tau], \\ &I = p_k(t)\xi_k(t) - H(t)\xi_0(t) + h\tfrac{\partial H}{\partial t}(t)\xi_0(t) = c,\ t \in (t_2-\tau, t_2]. \end{aligned} \tag{22}$$

Remark 4. *If $\mathbb{T} = h^{\mathbb{N}_0} = \{h^i : i \in \mathbb{N}_0\}$, then $\sigma(t) = ht$, $\mu(t) = (h-1)t$. For the quantum delayed Hamiltonian system, Formula (15) becomes*

$$\begin{array}{l} (p_k(t) + p_{k\tau}(t+\tau))\frac{\xi_k(ht)-\xi_k(t)}{(h-1)t} - H(t)\frac{\xi_0(ht)-\xi_0(t)}{(h-1)t} - \frac{\partial H}{\partial t}(t)\xi_0(t) \\ -\left(\frac{\partial H}{\partial q_k(ht)}(t) + \frac{\partial H}{\partial q_{k\tau}(ht)}(t+\tau)\right)\xi_k(ht) + \left(\frac{q_k(ht)-q_k(t)}{(h-1)t} - \frac{\partial H}{\partial p_k}(t)\right)\eta_k \\ +\left(\frac{q_{k\tau}(ht+\tau)-q_{k\tau}(t+\tau)}{(h-1)t} - \frac{\partial H}{\partial p_{k\tau}}(t+\tau)\right)\eta_k = 0,\ t \in [t_1, t_2-\tau], \\ p_k(t)\frac{\xi_k(ht)-\xi_k(t)}{(h-1)t} - H(t)\frac{\xi_0(ht)-\xi_0(t)}{(h-1)t} - \frac{\partial H}{\partial t}(t)\xi_0(t), \\ -\frac{\partial H}{\partial q_k(ht)}(t)\xi_k(ht) + \left(\frac{q_k(ht)-q_k(t)}{(h-1)t} - \frac{\partial H}{\partial p_k}(t)\right)\eta_k = 0,\ t \in (t_2-\tau, t_2] \end{array} \tag{23}$$

and Formula (18) gives

$$\begin{array}{l} I = (p_k(t) + p_{k\tau}(t+\tau))\xi_k - H(t)\xi_0(t) + (h-1)t\frac{\partial H}{\partial t}(t)\xi_0(t) = c,\ t \in [t_1, t_2-\tau], \\ I = p_k(t)\xi_k - H(t)\xi_0(t) + (h-1)t\frac{\partial H}{\partial t}(t)\xi_0(t) = c,\ t \in (t_2-\tau, t_2]. \end{array} \tag{24}$$

4. Proof of the Time-Scale Noether Theorem

We prove the Theorem 2 by using the method of reparameterization with time. The proof is divided into two steps.

First, we give the time-scale Noether theorem in terms of the special transformations

$$t^* = t, q_k^* = q_k(t) + \varepsilon\xi_k(t, q_j, p_j), p_k^* = p_k(t) + \varepsilon\eta_k(t, q_j, p_j), (k, j = 1, 2, \cdots, n), \tag{25}$$

where the time variable is not changing. Therefore, in terms of the transformations (25), the invariance of the action (8) is presented as

$$\begin{array}{l} \int_{t_1}^{t_2}\left[p_k q_k^\Delta + p_{k\tau}q_{k\tau}^\Delta - H(t, q_k^\sigma, p_k, q_{k\tau}^\sigma, p_{k\tau})\right]\Delta t \\ = \int_{t_1}^{t_2}\left[p_k^* q_k^{*\Delta} + p_{k\tau}^* q_{k\tau}^{*\Delta} - H(t, q_k^{*\sigma}, p_k^*, q_{k\tau}^{*\sigma}, p_{k\tau}^*)\right]\Delta t. \end{array} \tag{26}$$

Theorem 3. *If the time-scale Hamilton action (8) is invariant under transformations (25), that is, the condition*

$$\begin{array}{l} (p_k(t) + p_{k\tau}(t+\tau))\xi_k^\Delta - \left(\frac{\partial H}{\partial q_k^\sigma}(t) + \frac{\partial H}{\partial q_{k\tau}^\sigma}(t+\tau)\right)\xi_k^\sigma \\ +\left(q_k^\Delta(t) - \frac{\partial H}{\partial p_k}(t) + q_{k\tau}^\Delta(t+\tau) - \frac{\partial H}{\partial p_{k\tau}}(t+\tau)\right)\eta_k = 0,\ t \in [t_1, t_2-\tau], \\ p_k(t)\xi_k^\Delta - \frac{\partial H}{\partial q_k^\sigma}(t)\xi_k^\sigma + \left(q_k^\Delta(t) - \frac{\partial H}{\partial p_k}(t)\right)\eta_k = 0,\ t \in (t_2-\tau, t_2] \end{array} \tag{27}$$

holds, then

$$\begin{array}{l} I = (p_k(t) + p_{k\tau}(t+\tau))\xi_k = c,\ t \in [t_1, t_2-\tau], \\ I = p_k(t)\xi_k = c,\ t \in (t_2-\tau, t_2] \end{array} \tag{28}$$

is a conserved quantity.

Proof. With noting Formula (27), we have

$$\begin{array}{l} \frac{\Delta}{\Delta t}[(p_k(t) + p_{k\tau}(t+\tau))\xi_k] \\ = (p_k(t) + p_{k\tau}(t+\tau))\xi_k^\Delta - \left(\frac{\partial H}{\partial q_k^\sigma}(t) + \frac{\partial H}{\partial q_{k\tau}^\sigma}(t+\tau)\right)\xi_k^\sigma,\ t \in [t_1, t_2-\tau], \\ \frac{\Delta}{\Delta t}[p_k(t)\xi_k] = p_k(t)\xi_k^\Delta - \frac{\partial H}{\partial q_k^\sigma}(t)\xi_k^\sigma,\ t \in (t_2-\tau, t_2], \end{array}$$

with considering Equation(9) and Formula (27), we obtain the conserved quantity (28). □

Now, by using a reparameterization with time, the Lagrangian becomes

$$\begin{aligned}&\overline{L}\left(t; s^{\sigma}, q_k^{\sigma}; s^{\Delta}, q_k^{\Delta}; p_k; s_{\tau}^{\sigma}, q_{k\tau}^{\sigma}; s_{\tau}^{\Delta}, q_{k\tau}^{\Delta}; p_{k\tau}\right)\\ &\triangleq \left[p_k \frac{q_k^{\Delta}}{s^{\Delta}} + p_{k\tau}\frac{q_{k\tau}^{\Delta}}{s_{\tau}^{\Delta}} - H\left(s^{\sigma} - \mu(t)s^{\Delta}, q_k^{\sigma}, p_k, q_{k\tau}^{\sigma}, p_{k\tau}\right)\right]s^{\Delta}.\end{aligned} \tag{29}$$

For the invariance of S in terms of the transformations (13), with setting

$$s(t) = t, s(t-\tau) = t-\tau, \tag{30}$$

we get the equality

$$\begin{aligned}&\overline{S}[s(\cdot), s_{\tau}(\cdot), q_k(\cdot), q_{k\tau}(\cdot), p_k(\cdot), p_{k\tau}(\cdot)] = S[q_k(\cdot), q_{k\tau}(\cdot), p_k(\cdot), p_{k\tau}(\cdot)]\\ &= \int_{t_1}^{t_2}\Bigg[\left(p_k^*\circ\beta\right)(t)\frac{\left(q_k^*\circ\beta\right)^{\Delta}(t)}{\beta^{\Delta}(t)} + \left(p_k^*\circ\beta\right)(t-\tau)\frac{\left(\left(q_k^*\circ\beta\right)(t-\tau)\right)^{\Delta}}{\beta^{\Delta}(t-\tau)}\\ &-H\left(\beta^{\sigma}(t) - \mu(t)\beta^{\Delta}(t), \left(q_k^*\circ\sigma^*\circ\beta\right)(t), \left(p_k^*\circ\beta\right)(t), \left(q_k^*\circ\sigma^*\circ\beta\right)(t-\tau),\ \left(p_k^*\circ\beta\right)(t-\tau)\right)\Big]\beta^{\Delta}\Delta t\\ &= \int_{t_1}^{t_2}\overline{L}\Big(t; \beta^{\sigma}(t), \left(q_k^*\circ\beta\right)^{\sigma}(t); \beta^{\Delta}(t), \left(q_k^*\circ\beta\right)^{\Delta}(t); \left(p_k^*\circ\beta\right)(t);\end{aligned} \tag{31}$$

$$\begin{aligned}&\beta^{\sigma}(t-\tau), \left(q_k^*\circ\beta\right)^{\sigma}(t-\tau); \beta^{\Delta}(t-\tau), \left(q_k^*\circ\beta\right)^{\Delta}(t-\tau); \left(p_k^*\circ\beta\right)(t-\tau)\Big)\Delta t\\ &= \overline{S}\left[\beta(\cdot), \beta_{\tau}(\cdot), \left(q_k^*\circ\beta\right)(\cdot), \left(q_k^*\circ\beta\right)_{\tau}(\cdot), \left(p_k^*\circ\beta\right)(\cdot), \left(p_k^*\circ\beta\right)_{\tau}(\cdot)\right].\end{aligned} \tag{32}$$

Noting that for (30),

$$\begin{aligned}&\left(\beta(t), \beta(t-\tau), \left(q_k^*\circ\beta\right)(t), \left(q_k^*\circ\beta\right)(t-\tau), \left(p_k^*\circ\beta\right)(t), \left(p_k^*\circ\beta\right)(t-\tau)\right)\\ &= \big(T\left(t, q_j(t), p_j(t), \varepsilon\right), T\left(t-\tau, q_j(t-\tau), p_j(t-\tau), \varepsilon\right), Q_k\left(t, q_j(t), p_j(t), \varepsilon\right),\\ &\quad Q_k\left(t-\tau, q_j(t-\tau), p_j(t-\tau), \varepsilon\right), P_k\left(t, q_j(t), p_j(t), \varepsilon\right), P_k\left(t-\tau, q_j(t-\tau), p_j(t-\tau), \varepsilon\right)\big),\\ &= \big(T\left(s(t), q_j(t), p_j(t), \varepsilon\right), T\left(s(t-\tau), q_j(t-\tau), p_j(t-\tau), \varepsilon\right),\\ &\quad Q_k\left(s(t), q_j(t), p_j(t), \varepsilon\right),\ Q_k\left(s(t-\tau), q_j(t-\tau), p_j(t-\tau), \varepsilon\right),\\ &\quad P_k\left(s(t), q_j(t), p_j(t), \varepsilon\right), P_k\left(s(t-\tau), q_j(t-\tau), p_j(t-\tau), \varepsilon\right)\big).\end{aligned}$$

Formula (32) shows that $\overline{S}[s(\cdot), s_{\tau}(\cdot), q_k(\cdot), q_{k\tau}(\cdot), p_k(\cdot), p_{k\tau}(\cdot)]$ corresponds to the invariance in the sense of (26).

By a linear change of time, Formula (31) becomes

$$\begin{aligned}&= \int_{t_1}^{t_2-\tau}\left[\left(p_k^*\circ\beta\right)(t)\frac{\left(q_k^*\circ\beta\right)^{\Delta}(t)}{\beta^{\Delta}(t)} + \left(p_k^*\circ\beta\right)_{\tau}(t+\tau)\frac{\left(\left(q_k^*\circ\beta\right)_{\tau}(t+\tau)\right)^{\Delta}}{\beta_{\tau}^{\Delta}(t+\tau)} - H\right]\beta^{\Delta}\Delta t\\ &+ \int_{t_2-\tau}^{t_2}\left[\left(p_k^*\circ\beta\right)(t)\frac{\left(q_k^*\circ\beta\right)^{\Delta}(t)}{\beta^{\Delta}(t)} - H\right]\beta^{\Delta}\Delta t.\end{aligned}$$

Applying Theorem 3, we have the conserved quantity

$$I = \frac{\partial\overline{L}}{\partial q_k^{\Delta}}(t)\xi_k + \frac{\partial\overline{L}}{\partial s^{\Delta}}(t)\xi_0 = c,\ t\in[t_1, t_2-\tau],$$

$$I = \frac{\partial\overline{L}}{\partial q_k^{\Delta}}(t)\xi_k + \frac{\partial\overline{L}}{\partial s^{\Delta}}(t)\xi_0 = c,\ t\in(t_2-\tau, t_2].$$

Since

$$\begin{aligned}&\frac{\partial\overline{L}}{\partial q_k^{\Delta}}(t) = \frac{\partial}{\partial q_k^{\Delta}}\left(\left(p_k\frac{q_k^{\Delta}}{s^{\Delta}} + p_{k\tau}(t+\tau)\frac{q_{k\tau}^{\Delta}}{s_{\tau}^{\Delta}}(t+\tau) - H\right)s^{\Delta}\right)(t), t\in[t_1, t_2-\tau],\\ &\frac{\partial\overline{L}}{\partial s^{\Delta}}(t) = \frac{\partial}{\partial s^{\Delta}}\left(\left(p_k\frac{q_k^{\Delta}}{s^{\Delta}} + p_{k\tau}(t+\tau)\frac{q_{k\tau}^{\Delta}}{s_{\tau}^{\Delta}}(t+\tau) - H\right)s^{\Delta}\right)(t), t\in[t_1, t_2-\tau],\\ &\frac{\partial\overline{L}}{\partial q_k^{\Delta}}(t) = \frac{\partial}{\partial q_k^{\Delta}}\left(\left(p_k\frac{q_k^{\Delta}}{s^{\Delta}} - H\right)s^{\Delta}\right)(t),\ t\in(t_2-\tau, t_2],\\ &\frac{\partial\overline{L}}{\partial s^{\Delta}}(t) = \frac{\partial}{\partial s^{\Delta}}\left(\left(p_k\frac{q_k^{\Delta}}{s^{\Delta}} - H\right)s^{\Delta}\right)(t),\ t\in(t_2-\tau, t_2],\end{aligned}$$

where $H\left(s^{\sigma}-\mu(t)s^{\Delta},q_k^{\sigma},p_k,q_{k\tau}^{\sigma},p_{k\tau}\right)$.

For (30), we obtain

$$\begin{array}{l}\frac{\partial \overline{L}}{\partial q_k^{\Delta}}(t)=p_k(t)+p_{k\tau}(t+\tau),t\in[t_1,t_2-\tau],\\ \frac{\partial \overline{L}}{\partial q_k^{\Delta}}(t)=p_k(t),\ t\in(t_2-\tau,t_2],\\ \frac{\partial \overline{L}}{\partial s^{\Delta}}(t)=-H(t)+\mu(t)\frac{\partial H}{\partial t}(t),t\in[t_1,t_2],\end{array}$$

hence the conserved quantity (18) are obtained. The proof is complete.

5. Example of a Delayed Emden-Fowler Equation on Time Scales

We assume that the time-scale Lagrangian of a delayed system is

$$L=t\left(q^{\Delta}(t)\right)^2+\frac{1}{t}(q^{\sigma}(t-\tau))^2,t\in\mathbb{T}. \tag{33}$$

Formulae (6) and (7) give

$$p(t)=\frac{\partial L}{\partial q^{\Delta}}(t)=2tq^{\Delta}(t),p(t-\tau)=0, \tag{34}$$

$$H=\frac{1}{4t}(p(t))^2-\frac{1}{t}(q^{\sigma}(t-\tau))^2. \tag{35}$$

Thus, we obtain the time-scale canonical equations of the system,

$$\begin{array}{l}q^{\Delta}(t)=\frac{1}{2t}p(t),t\in[t_1,t_2],\\ p^{\Delta}(t)=\frac{2}{t}q_{\tau}^{\sigma}(t+\tau),t\in[t_1,t_2-\tau],\\ p^{\Delta}(t)=0,t\in(t_2-\tau,t_2].\end{array} \tag{36}$$

Equation (36) can also be presented as

$$\begin{array}{l}\left(tq^{\Delta}(t)\right)^{\Delta}-\frac{1}{t}q_{\tau}^{\sigma}(t+\tau)=0,t\in[t_1,t_2-\tau],\\ \left(2tq^{\Delta}(t)\right)^{\Delta}=0,t\in(t_2-\tau,t_2].\end{array} \tag{37}$$

Equation (37) is a kind of delayed Emden-Fowler equations on time scales.

If the delay is not exist, Equation (36) turns to be

$$q^{\Delta\Delta}(t)+\frac{1}{t}q^{\Delta\sigma}(t)-\frac{1}{t^2}q^{\sigma}(t)=0. \tag{38}$$

This kind of delayed Emden-Fowler damped dynamic equations has been widely discussed, see [20] and the references therein.

For Equation (35), the Noether identity (15) gives

$$\begin{array}{l}p(t)\xi^{\Delta}-\left(\frac{1}{4t}(p(t))^2-\frac{1}{t}(q_{\tau}^{\sigma}(t))^2\right)\xi_0^{\Delta}+\frac{1}{t^2}\left(\frac{1}{4}(p(t))^2-(q_{\tau}^{\sigma}(t))^2\right)\xi_0\\ \frac{2}{t}(q_{\tau}^{\sigma}(t+\tau))^2\xi^{\sigma}+\left(q^{\Delta}(t)-\frac{1}{2t}p(t)\right)\eta=0,\ t\in[t_1,t_2-\tau],\\ p(t)\xi^{\Delta}-\left(\frac{1}{4t}(p(t))^2-\frac{1}{t}(q_{\tau}^{\sigma}(t))^2\right)\xi_0^{\Delta}+\frac{1}{t^2}\left(\frac{1}{4}(p(t))^2-(q_{\tau}^{\sigma}(t))^2\right)\xi_0\\ +\left(q^{\Delta}(t)-\frac{1}{2t}p(t)\right)\eta=0,t\in(t_2-\tau,t_2].\end{array} \tag{39}$$

Equation (39) has the solution

$$\xi_0=t,\xi=0,\eta=0. \tag{40}$$

Thus, a conserved quantity can be generate from Theorem 2,

$$I = (-1 - \mu(t)\frac{1}{t})(\frac{1}{4}(p(t))^2 - (q^{\sigma}_{\tau}(t))^2) = \text{const.}, t \in [t_1, t_2]. \tag{41}$$

The time-scale Emden model not only contains both continuous case and discrete case, but also more general case.

If $\mathbb{T} = \mathbb{R}$, then $\sigma(t) = t, \mu(t) = 0$, Formula (41) becomes

$$I = -\frac{1}{4}(p(t))^2 + (q^{\sigma}_{\tau}(t))^2 = \text{const.}, t \in [t_1, t_2]. \tag{42}$$

If $\mathbb{T} = h^{\mathbb{N}_0} = \{h^i : i \in \mathbb{N}_0\}$, then $\sigma(t) = ht, \mu(t) = (h-1)t$, Formula (41) becomes

$$I = -h(\frac{1}{4}(p(t))^2 - (q(h(t-\tau)))^2) = \text{const.}, t \in [t_1, t_2]. \tag{43}$$

More potential applications for the Emden model on time scales in the fields of mechanics, symmetries, oscillations and control, stabilities, astrophysics etc. are worth looking forward to.

6. Conclusions

This paper gives a delayed Hamiltonian system in version of time scales and the Noether-type theorem. Our formulation not only allows the discrete result and the continuous result into a single model, but also achieves the more general model. We derived the time-scale canonical Equation (9) and by using the method of reparameterization with time, we discussed the Noether symmetries for the system and obtained a Noether-type conserved quantity (18). Because of the universality of the time scales, our results are more suitable in describing complex processing and also avoid some repetitive works between difference equations and differential equations.

The classical Hamilton canonical equations turn to be a kind of general dynamic equation in the sense of a non-canonical transformation, that is, Birkhoff's equation which is richer in content than Hamilton canonical equations. Thus, it's desirable to discuss the delayed Birkhoffian system [6,7] on time scales.

The symmetry theory is really important in scientific research. It's also a fertile area to study not only the famous Noether-type symmetry but also Lie symmetry and Mei symmetry in a time scales version. Some geometric notions are trying to research on the time scales [24–26]. From a geometrical point of view, further works about finding the integral of dynamical equations on time scales are still worth doing, for example, the Poisson theory on time scales and the Hamilton-Jacobi theory on time scales.

Recent work about the fractional calculus on time scales [27] potentiates research not only in the fractional calculus but also in solving fractional dynamical equations. The fractional action-like variational approach [28] was proposed to model nonconservative dynamical systems. This important approach is also worth to discuss in a time scales version. What's more, because of the freshness and difficulty, it needs efficient numerical methods to solve the equations on time scales and those important problems.

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