symmetry

Article

Some Inequalities of Čebyšev Type for Conformable k-Fractional Integral Operators

Feng Qi [1,2,3], Gauhar Rahman [4], Sardar Muhammad Hussain [4], Wei-Shih Du [5,*] and Kottakkaran Sooppy Nisar [6]

1 Institute of Mathematics, Henan Polytechnic University, Jiaozuo 454010, China; qifeng618@gmail.com
2 College of Mathematics, Inner Mongolia University for Nationalities, Tongliao 028043, China
3 School of Mathematical Sciences, Tianjin Polytechnic University, Tianjin 300387, China
4 Department of Mathematical Science, Balochistan University of Information Technology, Engineering and Management Sciences, Quetta, Pakistan; gauhar55uom@gmail.com (G.R.); smhussain01@gmail.com (S.M.H.)
5 Department of Mathematics, National Kaohsiung Normal University, Kaohsiung 82444, Taiwan
6 Department of Mathematics, College of Arts and Science at Wadi Al Dawaser, Prince Sattam bin Abdulaziz University, Wadi Al Dawaser 11991, Saudi Arabia; n.sooppy@psau.edu.sa
* Correspondence: wsdu@mail.nknu.edu.tw

Received: 24 October 2018; Accepted: 6 November 2018; Published: 8 November 2018

Abstract: In the article, the authors present several inequalities of the Čebyšev type for conformable k-fractional integral operators.

Keywords: inequality; fractional integral; k-fractional integral; conformable k-fractional integral; operator

MSC: 26A33; 26D10; 26D15; 90C23; 33B20

1. Introduction

The Čebyšev inequality [1] reads that

$$\frac{1}{b-a}\int_a^b f(x)g(x)\,\mathrm{d}\,x \ge \left[\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right]\left[\frac{1}{b-a}\int_a^b g(x)\,\mathrm{d}\,x\right], \tag{1}$$

where f and g are two integrable and synchronous functions on $[a,b]$ and two functions f and g are called synchronous on $[a,b]$ if

$$[f(x)-f(y)][g(x)-g(y)] \ge 0, \quad x,y \in [a,b].$$

The inequality (1) has many applications in diverse research subjects such as numerical quadrature, transform theory, probability, existence of solutions of differential equations, and statistical problems (see ([2], Chapter IX) and the paper [3]). Many authors have investigated, generalized, and applied the Čebyšev inequality (1). For detailed information, please refer to [4,5] and closely related references.

In [6,7], the Riemann–Liouville fractional integrals $\mathfrak{I}_{a+}^{\alpha}$ and $\mathfrak{I}_{b-}^{\alpha}$ of order $\alpha > 0$ are defined respectively by

$$\mathfrak{I}_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)\,\mathrm{d}\,t, \quad x > a, \quad \Re(\alpha) > 0 \tag{2}$$

and

$$\mathfrak{I}_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)\,\mathrm{d}\,t, \quad x < b, \quad \Re(\alpha) > 0, \tag{3}$$

where Γ is the classical Euler gamma function [8–10].

In [11], Belarbi and Dahmani presented the following theorems related to the Čebyšev inequality (1) for the Riemann–Liouville fractional integral operators [12–14].

Theorem 1 ([11], Theorem 3.1)**.** *Let f and g be two synchronous functions on $[0,\infty)$. Then, for $t,\alpha > 0$, we have*

$$J^{\alpha}(fg) \geq \frac{\Gamma(\alpha+1)}{t^{\alpha}} J^{\alpha} f(t) J^{\alpha} g(t).$$

Theorem 2 ([11], Theorem 3.2)**.** *Let f and g be two synchronous functions on $[0,\infty)$. Then, for all $t,\alpha,\beta > 0$, we have*

$$\frac{t^{\alpha}}{\Gamma(\alpha+1)} J^{\beta}(fg)(t) + \frac{t^{\beta}}{\Gamma(\beta+1)} J^{\alpha}(fg)(t) \geq J^{\alpha} f(t) J^{\beta} g(t) + J^{\beta} f(t) J^{\alpha} g(t).$$

Theorem 3 ([11], Theorem 3.3)**.** *Let f_i for $1 \leq i \leq n$ be n positive and increasing functions on $[0,\infty)$. Then, for $t,\alpha > 0$, we have*

$$J^{\alpha}\left(\prod_{i=1}^{n} f_i\right)(t) \geq [J^{\alpha}(1)]^{1-n} \prod_{i=1}^{n} J^{\alpha} f_i(t).$$

Theorem 4 ([11], Theorem 3.4)**.** *Let f and g be two functions defined on $[0,\infty)$, such that f is increasing, g is differentiable, and there exists a real number $m = \inf_{t\geq 0} g'(t)$. Then, the inequality*

$$J^{\alpha}(fg)(t) \geq \frac{1}{J^{\alpha}(1)} J^{\alpha} f(t) J^{\alpha} g(t) - \frac{mt}{\alpha+1} J^{\alpha} f(t) + m J^{\alpha}(t f(t))$$

is valid for $t,\alpha > 0$.

In [15], the Riemann–Liouville k-fractional integrals are respectively defined by

$$\mathfrak{J}^{\alpha}_{k,a^+} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_a^x (x-t)^{\alpha/k-1} f(t)\,\mathrm{d}\,t, \quad x > a, \quad \Re(\alpha) > 0$$

and

$$\mathfrak{J}^{\alpha}_{k,b^-} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_x^b (t-x)^{\alpha/k-1} f(t)\,\mathrm{d}\,t, \quad x < b, \quad \Re(\alpha) > 0,$$

where Γ_k is the gamma k-function [16,17].

In [18], the left and right sided fractional conformable integral operators are respectively defined by

$$^{\beta}\mathfrak{J}^{\alpha}_{a+} f(x) = \frac{1}{\Gamma(\beta)} \int_a^x \left[\frac{(x-a)^{\alpha} - (\tau-a)^{\alpha}}{\alpha}\right]^{\beta-1} \frac{f(\tau)}{(\tau-a)^{1-\alpha}}\,\mathrm{d}\,\tau \tag{4}$$

and

$$^{\beta}\mathfrak{J}^{\alpha}_{b-} f(x) = \frac{1}{\Gamma(\beta)} \int_a^x \left[\frac{(b-x)^{\alpha} - (b-\tau)^{\alpha}}{\alpha}\right]^{\beta-1} \frac{f(\tau)}{(b-\tau)^{1-\alpha}}\,\mathrm{d}\,\tau, \tag{5}$$

where $\Re(\beta) > 0$. Obviously, if taking $a = 0$ and $\alpha = 1$, then the Equations (4) and (5) reduce to the Riemann–Liouville fractional integrals (2) and (3), respectively.

In [19], one sided conformable fractional integral operator was defined as

$$^{\beta}\mathfrak{J}^{\alpha} f(x) = \frac{1}{\Gamma(\beta)} \int_0^x \left(\frac{x^{\alpha} - \tau^{\alpha}}{\alpha}\right)^{\beta-1} \frac{f(\tau)}{\tau^{1-\alpha}}\,\mathrm{d}\,\tau. \tag{6}$$

Recently, conformable k-fractional integrals were defined [20] by

$$ {}_{k}^{\beta}\mathfrak{J}_{a+}^{\alpha}f(x) = \frac{1}{k\Gamma_k(\beta)}\int_a^x \left[\frac{(x-a)^\alpha-(\tau-a)^\alpha}{\alpha}\right]^{\beta/k-1}\frac{f(\tau)}{(\tau-a)^{1-\alpha}}\,\mathrm{d}\tau \tag{7} $$

and

$$ {}_{k}^{\beta}\mathfrak{J}_{b-}^{\alpha}f(x) = \frac{1}{k\Gamma_k(\beta)}\int_a^x \left[\frac{(b-x)^\alpha-(b-\tau)^\alpha}{\alpha}\right]^{\beta/k-1}\frac{f(\tau)}{(b-\tau)^{1-\alpha}}\,\mathrm{d}\tau, $$

where $\Re(\beta) > 0$.

In this paper, we introduce the conformable k-fractional integral operator

$$ {}^{\beta}\mathfrak{J}_{k}^{\alpha}f(x) = \frac{1}{k\Gamma_k(\beta)}\int_0^x \left(\frac{x^\alpha-\tau^\alpha}{\alpha}\right)^{\beta/k-1}\frac{f(\tau)}{\tau^{1-\alpha}}\,\mathrm{d}\tau. \tag{8} $$

When $k = 1$, the Equations (7) to (8) reduces to the Equations (4) to (6), respectively.

2. Main Results

In this section, we present several Čebyšev type inequalities for conformable k-fractional integral operators defined in the Equation (8).

Theorem 5. *Let f and g be two integrable functions which are synchronous on $[0,\infty)$. Then,*

$$ ({}^{\beta}J_k^{\alpha}fg)(x) \geq \frac{\Gamma_k(\beta+k)\alpha^{\beta/k}}{x^{\alpha\beta/k}}({}^{\beta}J_k^{\alpha}f)(x)({}^{\beta}J_k^{\alpha}g)(x), $$

where $\alpha, \beta > 0$.

Proof. Since f and g are synchronous on $[0,\infty)$, we have

$$ f(u)g(u) + f(v)g(v) \geq f(u)g(v) + f(v)g(u). \tag{9} $$

Multiplying both sides of the Equation (9) by

$$ \frac{1}{k\Gamma_k(\beta)u^{1-\alpha}}\left(\frac{x^\alpha-u^\alpha}{\alpha}\right)^{\beta/k-1}, \quad x\in\mathbb{R}, \quad 0<u<x $$

results in

$$ \begin{aligned} &\frac{1}{k\Gamma_k(\beta)u^{1-\alpha}}\left(\frac{x^\alpha-u^\alpha}{\alpha}\right)^{\beta/k-1} f(u)g(u) + \frac{1}{k\Gamma_k(\beta)u^{1-\alpha}}\left(\frac{x^\alpha-u^\alpha}{\alpha}\right)^{\beta/k-1} f(v)g(v) \\ &\quad\geq \frac{1}{k\Gamma_k(\beta)u^{1-\alpha}}\left(\frac{x^\alpha-u^\alpha}{\alpha}\right)^{\beta/k-1} f(u)g(v) + \frac{1}{k\Gamma_k(\beta)u^{1-\alpha}}\left(\frac{x^\alpha-u^\alpha}{\alpha}\right)^{\beta/k-1} f(v)g(u). \end{aligned} $$

Further integrating both sides with respect to u over $(0,x)$ gives

$$ \begin{aligned} &\frac{1}{k\Gamma_k(\beta)}\int_0^x\left(\frac{x^\alpha-u^\alpha}{\alpha}\right)^{\beta/k-1}\frac{f(u)g(u)}{u^{1-\alpha}}\,\mathrm{d}u + \frac{1}{k\Gamma_k(\beta)}\int_0^x\left(\frac{x^\alpha-u^\alpha}{\alpha}\right)^{\beta/k-1}\frac{f(v)g(v)}{u^{1-\alpha}}\,\mathrm{d}u \\ &\quad\geq \frac{1}{k\Gamma_k(\beta)}\int_0^x\left(\frac{x^\alpha-u^\alpha}{\alpha}\right)^{\beta/k-1}\frac{f(u)g(v)}{u^{1-\alpha}}\,\mathrm{d}u + \frac{1}{k\Gamma_k(\beta)}\int_0^x\left(\frac{x^\alpha-u^\alpha}{\alpha}\right)^{\beta/k-1}\frac{f(v)g(u)}{u^{1-\alpha}}\,\mathrm{d}u. \end{aligned} $$

Consequently, it follows that

$$({}^{\beta}J_k^{\alpha} fg)(x) + f(v)g(v)\frac{1}{k\Gamma_k(\beta)}\int_0^x \left(\frac{x^{\alpha}-u^{\alpha}}{\alpha}\right)^{\beta/k-1}\frac{\mathrm{d}\,u}{u^{1-\alpha}} \geq g(v)({}^{\beta}J_k^{\alpha} f)(x) + f(v)({}^{\beta}J_k^{\alpha} g)(x)$$

and

$$({}^{\beta}J_k^{\alpha} fg)(x) + \frac{x^{\alpha\beta/k}}{\Gamma_k(\beta+k)\alpha^{\beta/k}}f(v)g(v) \geq g(v)({}^{\beta}J_k^{\alpha} f)(x) + f(v)({}^{\beta}J_k^{\alpha} g)(x), \tag{10}$$

where

$$\int_0^x \left(\frac{x^{\alpha}-u^{\alpha}}{\alpha}\right)^{\beta/k-1}\frac{\mathrm{d}\,u}{u^{1-\alpha}} = \frac{kx^{\alpha\beta/k}}{\beta\alpha^{\beta/k}}.$$

Multiplying both sides of the Equation (10) by

$$\frac{1}{k\Gamma_k(\beta)v^{1-\alpha}}\left(\frac{x^{\alpha}-v^{\alpha}}{\alpha}\right)^{\beta/k-1}$$

arrives at

$$\begin{aligned}\frac{({}^{\beta}J_k^{\alpha} fg)(x)}{k\Gamma_k(\beta)v^{1-\alpha}}\left(\frac{x^{\alpha}-v^{\alpha}}{\alpha}\right)^{\beta/k-1} &+ \frac{f(v)g(v)}{k\Gamma_k(\beta)v^{1-\alpha}}\left(\frac{x^{\alpha}-v^{\alpha}}{\alpha}\right)^{\beta/k-1}\frac{x^{\alpha\beta/k}}{\Gamma_k(\beta+k)\alpha^{\beta/k}}\\ &\geq \frac{g(v)({}^{\beta}J_k^{\alpha} f)(x)}{k\Gamma_k(\beta)v^{1-\alpha}}\left(\frac{x^{\alpha}-v^{\alpha}}{\alpha}\right)^{\beta/k-1} + \frac{f(v)({}^{\beta}J_k^{\alpha} g)(x)}{k\Gamma_k(\beta)v^{1-\alpha}}\left(\frac{x^{\alpha}-v^{\alpha}}{\alpha}\right)^{\beta/k-1}.\end{aligned}$$

Now, integrating over $(0, x)$ reveals

$$\begin{aligned}&({}^{\beta}J_k^{\alpha} fg)(x)\frac{1}{k\Gamma_k(\beta)}\int_0^x \left(\frac{x^{\alpha}-v^{\alpha}}{\alpha}\right)^{\beta/k-1}\frac{\mathrm{d}\,v}{v^{1-\alpha}}\\ &\quad+\frac{x^{\alpha\beta/k}}{\Gamma_k(\beta+k)\alpha^{\beta/k}}\frac{1}{k\Gamma_k(\beta)}\int_0^x \left(\frac{x^{\alpha}-v^{\alpha}}{\alpha}\right)^{\beta/k-1}\frac{f(v)g(v)}{v^{1-\alpha}}\,\mathrm{d}\,v\\ &\quad\geq ({}^{\beta}J_k^{\alpha} f)(x)\frac{1}{k\Gamma_k(\beta)}\int_0^x \left(\frac{x^{\alpha}-v^{\alpha}}{\alpha}\right)^{\beta/k-1}\frac{g(v)}{v^{1-\alpha}}\,\mathrm{d}\,v\\ &\quad\quad+({}^{\beta}J_k^{\alpha} g)(x)\frac{1}{k\Gamma_k(\beta)}\int_0^x \left(\frac{x^{\alpha}-v^{\alpha}}{\alpha}\right)^{\beta/k-1}\frac{f(v)}{v^{1-\alpha}}\,\mathrm{d}\,v.\end{aligned}$$

Therefore, we have

$$\frac{x^{\alpha\beta/k}}{\Gamma_k(\beta+k)\alpha^{\beta/k}}({}^{\beta}J_k^{\alpha} fg)(x) + \frac{x^{\alpha\beta/k}}{\Gamma_k(\beta+k)\alpha^{\beta/k}}({}^{\beta}J_k^{\alpha} fg)(x) \geq ({}^{\beta}J_k^{\alpha} f)(x)({}^{\beta}J_k^{\alpha} g)(x) + ({}^{\beta}J_k^{\alpha} f)(x)({}^{\beta}J_k^{\alpha} g)(x).$$

The proof of Theorem 5 is complete. □

Corollary 1. *Let f and g be two integrable functions which are synchronous on $[0,\infty)$. Then,*

$$({}^{\beta}J_k fg)(x) \geq \frac{\Gamma_k(\beta+k)}{x^{\beta/k}}({}^{\beta}J_k f)(x)({}^{\beta}J_k g)(x), \quad \alpha,\beta > 0.$$

Proof. This follows from taking $\alpha = 1$ in Theorem 5. □

Theorem 6. *Let f and g be two integrable functions which are synchronous on $[0,\infty)$. Then,*

$$\frac{x^{\alpha\tau/k}}{\Gamma_k(\tau+k)\alpha^{\tau/k}}({}^{\beta}J_k^{\alpha} fg)(x) + \frac{x^{\alpha\beta/k}}{\Gamma_k(\beta+k)\alpha^{\beta/k}}({}^{\tau}J_k^{\alpha} fg)(x) \geq ({}^{\beta}J_k^{\alpha} f)(x)({}^{\tau}J_k^{\alpha} g)(x) + ({}^{\tau}J_k^{\alpha} f)(x)({}^{\beta}J_k^{\alpha} g)(x)$$

for $\alpha,\beta,\tau > 0$.

Proof. Multiplying both sides of the equality (10) by

$$\frac{1}{k\Gamma_k(\tau)v^{1-\alpha}}\left(\frac{x^\alpha - v^\alpha}{\alpha}\right)^{\tau/k-1}$$

yields

$$\begin{aligned}&\frac{1}{k\Gamma_k(\tau)v^{1-\alpha}}\left(\frac{x^\alpha - v^\alpha}{\alpha}\right)^{\tau/k-1}({}^{\beta}J_k^{\alpha}fg)(x) + \frac{f(v)g(v)}{k\Gamma_k(\tau)v^{1-\alpha}}\left(\frac{x^\alpha - v^\alpha}{\alpha}\right)^{\tau/k-1}\frac{x^{\alpha\beta/k}}{\Gamma_k(\beta+k)\alpha^{\beta/k}}\\ &\quad\geq \frac{1}{k\Gamma_k(\tau)v^{1-\alpha}}\left(\frac{x^\alpha - v^\alpha}{\alpha}\right)^{\tau/k-1} g(v)({}^{\beta}J_k^{\alpha}f)(x) + \frac{1}{k\Gamma_k(\tau)v^{1-\alpha}}\left(\frac{x^\alpha - v^\alpha}{\alpha}\right)^{\tau/k-1} f(v)({}^{\beta}J_k^{\alpha}g)(x).\end{aligned}$$

Further integrating both sides with respect to v over $(0, x)$ leads to

$$\begin{aligned}&\frac{({}^{\beta}J_k^{\alpha}fg)(x)}{k\Gamma_k(\tau)}\int_0^x\left(\frac{x^\alpha - v^\alpha}{\alpha}\right)^{\tau/k-1}\frac{\mathrm{d}\,v}{v^{1-\alpha}} + \frac{x^{\alpha\beta/k}}{\Gamma_k(\beta+k)\alpha^{\beta/k}}\frac{1}{k\Gamma_k(\tau)}\int_0^x\left(\frac{x^\alpha - v^\alpha}{\alpha}\right)^{\tau/k-1}\frac{f(v)g(v)}{v^{1-\alpha}}\,\mathrm{d}\,v\\ &\quad\geq \frac{({}^{\beta}J_k^{\alpha}f)(x)}{k\Gamma_k(\tau)}\int_0^x\left(\frac{x^\alpha - v^\alpha}{\alpha}\right)^{\tau/k-1}\frac{g(v)}{v^{1-\alpha}}\,\mathrm{d}\,v + \frac{({}^{\beta}J_k^{\alpha}g)(x)}{k\Gamma_k(\tau)}\int_0^x\left(\frac{x^\alpha - v^\alpha}{\alpha}\right)^{\tau/k-1}\frac{f(v)}{v^{1-\alpha}}\,\mathrm{d}\,v.\end{aligned}$$

Therefore, we have

$$\frac{x^{\alpha\tau/k}}{\Gamma_k(\tau+k)\alpha^{\tau/k}}({}^{\beta}J_k^{\alpha}fg)(x) + \frac{x^{\alpha\beta/k}}{\Gamma_k(\beta+k)\alpha^{\beta/k}}({}^{\tau}J_k^{\alpha}fg)(x) \geq ({}^{\beta}J_k^{\alpha}f)(x)({}^{\tau}J_k^{\alpha}g)(x) + ({}^{\tau}J_k^{\alpha}f)(x)({}^{\beta}J_k^{\alpha}g)(x).$$

Further integrating with respect to v over $(0, x)$, as did in the proof of Theorem 5, concludes Theorem 6. □

Remark 1. *Applying Theorem 6 to $\tau = \beta$ results in Theorem 5.*

Corollary 2. *Let f and g be two integrable functions which are synchronoms on $[0, \infty)$. Then*

$$\frac{x^{\tau/k}}{\Gamma_k(\tau+k)}({}^{\beta}J_k fg)(x) + \frac{x^{\beta/k}}{\Gamma_k(\beta+k)}({}^{\tau}J_k fg)(x) \geq ({}^{\beta}J_k f)(x)({}^{\tau}J_k g)(x) + ({}^{\tau}J_k f)(x)({}^{\beta}J_k g)(x)$$

for $\alpha, \beta, \tau > 0$.

Proof. This follows from taking $\alpha = 1$ in Theorem 6. □

Theorem 7. *Let f_i for $1 \leq i \leq n$ be positive and increasing functions on $[a, b]$. For $\alpha, \beta > 0$, we have*

$$\left({}^{\beta}J_k^{\alpha}\prod_{i=1}^{n} f_i\right)(x) \geq \left[\frac{\Gamma_k(\beta+k)\alpha^{\beta/k}}{x^{\alpha\beta/k}}\right]^{n-1}\prod_{i=1}^{n}({}^{\beta}J_k^{\alpha}f_i)(x). \tag{11}$$

Proof. We prove this theorem by induction on $n \in \mathbb{N}$. Obviously, the case $n = 1$ of (11) holds.

For $n = 2$, since f_1 and f_2 are increasing, we have

$$[f_1(x) - f_1(y)][f_2(x) - f_2(y)] \geq 0.$$

Now, the left proof of the inequality (11) for $n = 2$ is the same as that of Theorem 5.

Assume that the inequality (11) is true for some $n \geq 3$. We observe that, since f_i is increasing, $f = \prod_{i=1}^{n} f_i$ is increasing. Let $g = f_{n+1}$. Then, applying the case $n = 2$ to the functions f and g yields

$$\left({}^{\beta}J_k^{\alpha}\prod_{i=1}^{n} f_i f_{n+1}\right)(x) \geq \left[\frac{\Gamma_k(\beta+k)\alpha^{\beta/k}}{x^{\alpha\beta/k}}\right]\left({}^{\beta}J_k^{\alpha}\prod_{i=1}^{n} f_i\right)\left({}^{\beta}J_k^{\alpha} f_{n+1}\right)(x)$$
$$\geq \left(\frac{\Gamma_k(\beta+k)\alpha^{\beta/k}}{x^{\alpha\beta/k}}\right)^{n}\prod_{i=1}^{n+1}\left({}^{\beta}J_k^{\alpha} f_i\right)(x),$$

where the induction hypothesis for n is used in the deduction of the second inequality. The proof of Theorem 7 is complete. □

Corollary 3. *Let f_i for $1 \leq i \leq n$ be positive and increasing functions on $[a,b]$. For $\alpha, \beta > 0$, we have*

$$\left({}^{\beta}J_k\prod_{i=1}^{n} f_i\right)(x) \geq \left[\frac{\Gamma_k(\beta+k)}{x^{\beta/k}}\right]^{n-1}\prod_{i=1}^{n}\left({}^{\beta}J_k f_i\right)(x).$$

Proof. This follows from taking $\alpha = 1$ in Theorem 7. □

Theorem 8. *Let $\alpha, \beta > 0$ and the functions $f, g : [0,\infty) \to \mathbb{R}$ be such that f is increasing, g is differentiable, and g' has a lower bound $m = \inf_{t\in[0,\infty)} g'(t)$. Then,*

$$\left({}^{\beta}J_k^{\alpha} fg\right)(x) \geq \frac{\Gamma_k(\beta+k)\alpha^{\beta/k}}{x^{\alpha\beta/k}}\left({}^{\beta}J_k^{\alpha} f\right)(x)\left({}^{\beta}J_k^{\alpha} g\right)(x) - \frac{kmx}{(\beta+k)}\left({}^{\beta}J_k^{\alpha} f\right)(x) + m\left({}^{\beta}J_k^{\alpha} if\right)(x),$$

where $i(x)$ is the identity function.

Proof. Let $h(x) = g(x) - mx$. We find that h is differentiable and increasing on $[0,\infty)$. As in the proof of Theorem 7, for clarity, let $p(x) = mx$, we obtain

$$\left({}^{\beta}J_k^{\alpha} f(g-p)\right)(x) \geq \frac{\Gamma_k(\beta+k)\alpha^{\beta/k}}{x^{\alpha\beta/k}}\left({}^{\beta}J_k^{\alpha} f\right)(x)\left({}^{\beta}J_k^{\alpha}(g-p)\right)(x)$$
$$= \frac{\Gamma_k(\beta+k)\alpha^{\beta/k}}{x^{\alpha\beta/k}}\left({}^{\beta}J_k^{\alpha} f\right)(x)\left({}^{\beta}J_k^{\alpha} g\right)(x) - \frac{\Gamma_k(\beta+k)\alpha^{\beta/k}}{x^{\alpha\beta/k}}\left({}^{\beta}J_k^{\alpha} f\right)(x)\left({}^{\beta}J_k^{\alpha} p\right)(x), \quad (12)$$

where

$$\left({}^{\beta}J_k^{\alpha} f(g-p)\right)(x) = \left({}^{\beta}J_k^{\alpha} fg\right)(x) - m\left({}^{\beta}J_k^{\alpha} if\right)(x) \quad (13)$$

and

$$\left({}^{\beta}J_k^{\alpha} p\right)(x) = \frac{mx^{\alpha\beta/k+1}\Gamma_k(2k)}{\Gamma_k(\beta+2k)\alpha^{\beta/k}}.$$

Since $\Gamma_k(k) = 1$, see ([16], p. 183), then $\Gamma_k(2k) = k$. Therefore, we derive

$$\left({}^{\beta}J_k^{\alpha} p\right)(x) = \frac{kmx^{\alpha\beta/k+1}}{\Gamma_k(\beta+2k)\alpha^{\beta/k}}. \quad (14)$$

Substituting the Equations (13) and (14) into the Equation (12) leads to the desired result. □

Corollary 4. *Under conditions of Theorem 8, we have*

$$\left({}^{\beta}J_k fg\right)(x) \geq \frac{\Gamma_k(\beta+k)}{x^{\beta/k}}\left({}^{\beta}J_k f\right)(x)\left({}^{\beta}J_k g\right)(x) - \frac{kmx}{(\beta+k)}\left({}^{\beta}J_k f\right)(x) + m\left({}^{\beta}J_k if\right)(x),$$

where $i(x)$ is the identity function.

Proof. This follows from taking $\alpha = 1$ in Theorem 8. □

3. Conclusions

In this paper, we established several Čebyšev type inequalities for conformable k-fractional integral operators. We observed that, if allowing $k = 1$, inequalities obtained in this paper will reduce to those inequalities in [21]. Similarly, if letting $\alpha = k = 1$, inequalities obtained in this paper will reduce to those inequalities in [11].

Author Contributions: The authors contributed equally to this work. All authors have read and approved the final manuscript.

Funding: The fourth author was supported by Grant No. MOST 107-2115-M-017-004-MY2 of the Ministry of Science and Technology of the Republic of China.

Acknowledgments: The authors are thankful to the anonymous referees for their careful corrections to and valuable comments on the original version of this paper.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Čebyšev, P.L. Sur les expressions approximatives des intégrales définies par les autres prises entre les mêmes limites. *Proc. Math. Soc. Charkov* **1882**, *2*, 93–98.
2. Mitrinović, D.S.; Pečarić, J.E.; Fink, A.M. *Classical and New Inequalities in Analysis*; Kluwer Academic Publishers: Dordrecht, The Netherlands, 1993; doi:10.1007/978-94-017-1043-5.
3. Qi, F.; Cui, L.-H.; Xu, S.-L. Some inequalities constructed by Tchebysheff's integral inequality. *Math. Inequal. Appl.* **1999**, *2*, 517–528. [CrossRef]
4. Özdemir, M.E.; Set, E.; Akdemir, A.O.; Sarıkaya, M.Z. Some new Chebyshev type inequalities for functions whose derivatives belongs to L_p spaces. *Afr. Mat.* **2015**, *26*, 1609–1619. [CrossRef]
5. Set, E.; Dahmani, Z.; Mumcu, I. New extensions of Chebyshev type inequalities using generalized Katugampola integrals via Pólya-Szegö inequality. *Int. J. Optim. Control. Theor. Appl. IJOCTA* **2018**, *8*, 137–144. [CrossRef]
6. Kilbas, A.A. Hadamard-type fractional calculus. *J. Korean Math. Soc.* **2001**, *38*, 1191–1204,
7. Samko, S.G.; Kilbas, A.A.; Marichev, O.I. *Fractional Integrals and Derivatives: Theory and Applications*; Edited and with a Foreword by S. M. Nikol'skiĭ; Translated from the 1987 Russian Original; Revised by the Authors; Gordon and Breach Science Publishers: Yverdon, Switzerland, 1993.
8. Nisar, K.S.; Qi, F.; Rahman, G.; Mubeen, S.; Arshad, M. Some inequalities involving the extended gamma function and the Kummer confluent hypergeometric k-function. *J. Inequal. Appl.* **2018**, *2018*, 135. [CrossRef] [PubMed]
9. Qi, F.; Guo, B.-N. Integral representations and complete monotonicity of remainders of the Binet and Stirling formulas for the gamma function. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM* **2017**, *111*, 425–434. [CrossRef]
10. Srivastava, H.M.; Choi, J. *Zeta and q-Zeta Functions and Associated Series and Integrals*; Elsevier: Amsterdam, The Netherlands, 2012; doi:10.1016/B978-0-12-385218-2.00001-3.
11. Belarbi, S.; Dahmani, Z. On some new fractional integral inequalities. *J. Inequal. Pure Appl. Math.* **2009**, *10*, 86.
12. Shi, D.-P.; Xi, B.-Y.; Qi, F. Hermite–Hadamard type inequalities for (m, h_1, h_2)-convex functions via Riemann–Liouville fractional integrals. *Turkish J. Anal. Number Theory* **2014**, *2*, 22–27. [CrossRef]
13. Shi, D.-P.; Xi, B.-Y.; Qi, F. Hermite–Hadamard type inequalities for Riemann–Liouville fractional integrals of (α, m)-convex functions. *Fract. Differ. Calc.* **2014**, *4*, 33–43. [CrossRef]
14. Wang, S.-H.; Qi, F. Hermite–Hadamard type inequalities for s-convex functions via Riemann–Liouville fractional integrals. *J. Comput. Anal. Appl.* **2017**, *22*, 1124–1134.
15. Mubeen, S.; Habibullah, G.M. k-fractional integrals and application. *Int. J. Contemp. Math. Sci.* **2012**, *7*, 89–94.
16. Díaz, R.; Pariguan, E. On hypergeometric function and Pochhammer k-symbol. *Divulg. Mat.* **2007**, *15*, 179–192.
17. Qi, F.; Akkurt, A.; Yildirim, H. Catalan numbers, k-gamma and k-beta functions, and parametric integrals. *J. Comput. Anal. Appl.* **2018**, *25*, 1036–1042.
18. Jarad, F.; Uğurlu, E.; Abdeljawad, T.; Băleanu, D. On a new class of fractional operators. *Adv. Differ. Equ.* **2017**, *2017*, 247. [CrossRef]

19. Set, E.; Mumcu, İ.; Özdemir, M.E. Grüss Type Inequalities Involving New Conformable Fractional Integral Operators. ResearchGate Preprint. 2018. Available online: https://www.researchgate.net/publication/323545750 (accessed on 8 October).
20. Habib, S.; Mubeen, S.; Naeem, M.N.; Qi, F. Generalized *k*-Fractional Conformable Integrals and Related Inequalities. HAL Archives. 2018. Available online: https://hal.archives-ouvertes.fr/hal-01788916 (accessed on 8 October 2018).
21. Set, E.; Mumcu, İ.; Demirbaş, S. Chebyshev Type Inequalities Involving New Conformable Fractional Integral Operators. ResearchGate Preprint. 2018. Available online: https://www.researchgate.net/publication/323880498 (accessed on 8 October 2018).