Article

Cyclicity in *EL*–Hypergroups

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Abstract: In the algebra of single-valued structures, *cyclicity* is one of the fundamental properties of groups. Therefore, it is natural to study it also in the algebra of multivalued structures (algebraic hyperstructure theory). However, when one considers the nature of generalizing this property, at least two (or rather three) approaches seem natural. Historically, all of these had been introduced and studied by 1990. However, since most of the results had originally been published in journals without proper international impact and later—without the possibility to include proper background and context-synthetized in books, the current way of treating the concept of cyclicity in the algebraic hyperstructure theory is often rather confusing. Therefore, we start our paper with a rather long introduction giving an overview and motivation of existing approaches to the cyclicity in algebraic hyperstructures. In the second part of our paper, we relate these to *EL*-hyperstructures, a broad class of algebraic hyperstructures constructed from (pre)ordered (semi)groups, which were defined and started to be studied much later than sources discussed in the introduction were published.

1. Introduction, Preliminaries and Historical Context

The notion of *cyclicity* in group theory is one of the first notions that all university students learn in their introductory courses of algebra. Usually, the additive group of integers serves as an example of an infinite cyclic group while its decomposition modulo *n* serves as an example of a finite cyclic group. In the algebraic hyperstructure theory, which was initiated by Marty [1], the notion of *cyclicity* was one of the very first topics to be noticed. In fact, only three years after Marty (or rather a year and a half; see "Received" date), Wall touched this problem in one of the concluding remarks of his paper [2]. His definition uses plain words yet the mathematization of his idea is an obvious analogy of group cyclicity. After Wall, the notion of cyclicity was studied only by a small number of authors (to a great depth, though). Until the 1990s, we can observe two schools which picked up this topic: an *Italian* one represented by De Salvo, Freni, Corsini and Bonansinga and a *Greek* one represented by Vougiouklis, Konguetsof, Kessoglides and Spartalis.

1.1. The Origins by Wall

Prior to including Wall's definition, we recall the well-known definition of the (algebraic) *hypergroup*.

Definition 1. *By a* hypergroupoid, *we mean a pair* $(H, *)$, *where "$*$" is a mapping* $H \times H \to \mathcal{P}^*(H)$, *where* $\mathcal{P}^*(H)$ *is the set of all non-empty subsets of* H. *By a* hypergroup, *we mean a hypergroupoid which is* associative, *i.e., for all* $a, b, c \in H$, *there is* $a * (b * c) = (a * b) * c$, *and* reproductive, *i.e., for all* $a \in H$, *there*

is $a * H = H = H * a$. An associative hypergroupoid is called semihypergroup; *a reproductive hypergroupoid is called* quasihypergroup.

Definition 2 ([2], p. 96). *If a hypergroup H is generated by a single element a of H, then H will be called a* cyclic hypergroup.

After defining cyclic hypergroups, Wall observed that: "In a cyclic hypergroup the commutative law does not necessarily hold. We may write

$$aa = a^2, \quad aaa = a^3, \dots$$

and clearly $a^m a^n = a^{m+n}$. It is evident that the set of elements generated by a single element of a hypergroup is closed, . . . , but I am unable to say whether or not it forms a hypergroup", and then he sought some analogy of the representation of a cyclic group as the group of rotations of a regular polygon.

1.2. The Original Approach of De Salvo and Freni

The Italian school linked the concept of cyclicity to the concept of *completeness* in hypergroups—see, e.g., De Salvo and Freni [3]—because *complete hypergroups* were being studied by Freni, Corsini, De Salvo, Romeo and others [3–6] at around that time. It is important to notice that the first papers of the Italian school concerning cyclicity such as Freni [7] or De Salvo and Freni [3,8] do not mention Wall's paper at all. Also worth mentioning is the fact that most of the papers of the Italian school were written in Italian (and published in Italian university journals) and some in French. Unlike Wall, the Italian school studied the notion of *cyclicity* not only for hypergroups, but also for *semihypergroups* De Salvo and Freni [8] and later even for *hypergroupoids* Freni [9].

Even though not mentioning Wall, the Italian school originally viewed cyclicity in hypergroups in the same way as he did: *a (hyper)group (or, rather, a semihypergroup in their case) is called cyclic, if we obtain the whole (hyper)group when a (hyper)operation is repeatedly applied on a certain element called a generator*.

Definition 3 ([3,8]). *We call $P \neq \emptyset$, $P \subseteq H$, a* cyclic part *of a semihypergroup H if there exists an element $x \in P$ such that, for all $a \in P$, there exists $n \in \mathbb{N}$ such that $a \in x^n$. The element $x \in P$ is called* generator *of P. If H is a cyclic part, we call H a* cyclic semihypergroup.

Notice that this $n \in \mathbb{N}$ need not be the same for all $a \in P$, which becomes important when one considers the below Definition 5. Of course, when considering cyclic (hyper)structures, it is useful to consider the concept of *cyclicity of an element*. This can be found already in De Salvo and Freni [3,8].

Definition 4 ([3,8]). *Let H be a cyclic semihypergroup with generator h. We call* cyclicity *(Italian: ciclità) of an element $a \in H$ the minimum $m \in \mathbb{N} \setminus \{1\}$ such that $a \in h^m$; we write $cicl(a) = m$. We call* cyclicity *of H and denote $cicl(H) = \max\{cicl(a) \mid a \in H\}$.*

1.3. The Approach of Vougiouklis

Not knowing of the Italian school, Vougiouklis [10] picked up the original Wall's definition and—in a paper published in English at around the same time as Freni [7], De Salvo and Freni [3,8] and other notable papers of the Italian school—he initiated the study of *P-cyclic hypergroups*, or rather cyclicity in hypergroups related to *P–hyperoperations*. The definition of Vougiouklis is rather wide in order to describe as many types of cyclicity as possible. In the same way as Wall (and unlike the Italians who started from *semihypergroups*), Vougiouklis considered cyclicity in *hypergroups*. However, one can see that the Italian Definition 3 is, for hypergroups, a special case of his definition. Moreover, notice that the concept of period n is different from $cicl(H)$. Indeed, if two generators h_1, h_2 of a cyclic hypergroup H have a different period, then H has no *period*, yet its *cyclicity* can be determined.

Definition 5 ([10]). *A hypergroup $(H, \circ)$ is called cyclic if, for some $h \in H$, there is*

$$H = h^1 \cup h^2 \cup \ldots \cup h^n \cup \ldots, \tag{1}$$

where $h^1 = \{h\}$ and $h^m = \underbrace{h \circ \ldots \circ h}_{m}$. If there exists $n \in \mathbb{N}$ such that Formula (1) is finite, we say that H is a cyclic hypergroup with finite period; *otherwise, H is a cyclic hypergroup with* infinite period. *The element $h \in H$ in Formula (1) is called* generator *of H, the smallest power n for which Formula (1) is valid is called period of h. If all generators of H have the same period n, then H is called cyclic with period n. If, for a given generator h, Formula (1) is valid but no such n exists (i.e., Formula (1) cannot be finite), then H is called cyclic with infinite period. If we can, for some $h \in H$, write*

$$H = h^n. \tag{2}$$

Then, the hypergroup H is called single-power cyclic with a generator h. If Formula (1) is valid and for all $n \in \mathbb{N}$ and, for a fixed $n_0 \in \mathbb{N}$, $n \geq n_0$ there is

$$h^1 \cup h^2 \cup \ldots \cup h^{n-1} \subsetneq h^n, \tag{3}$$

then we say that H is a single-power cyclic hypergroup with an infinite period for h.

By a *P–hyperoperation*, Vougiouklis means a hypergroupoid $(H, *^P)$ constructed from a commutative group $(H, \cdot)$ and its arbitrary subset P using the hyperoperation

$$x *^P y = \{x \cdot y \cdot z \mid z \in \{e\} \cup P\} \tag{4}$$

defined for all $x, y \in H$, where e is the unit element of $(H, \cdot)$. In [10], Vougiouklis shows that $(H, *^P)$ is a hypergroup which is—if we start with a cyclic group–cyclic in the sense of Definition 5. Later in the paper and also in Konguetsof et al. [11] or Vougiouklis [12], some criteria for elements of H and its generators are established; a great number of other results in this direction can be found in the PhD thesis of Vougiouklis defended in 1980 and called "Cyclicity of hypergroups".

1.4. An Alternative Approach of the Italian School

In a series of papers published mainly in Italian between 1980 and 1986 (notice that many of the Italian papers were written simultaneously or their "in print" versions are cross referenced), members of the Italian school, which apart from De Salvo and Freni included also Corsini, Romeo or Bonansinga, provided (under the influence of Koskas, Mittas and Sureau) a great number of results on notions such as *completeness* of hypergroups, *complete closures*, *complete parts*, *subhypergroups generated by some elements of a hypergroup*, etc. In [13] (published in 1986 but cited already in De Salvo [14], which was received in 1984), De Salvo and Freni studied subhypergroups *generated* by one element, where by "generating" we mean the intersection of all subhypergroups which are complete parts and contain the element in question. They called such hypergroups *strongly cyclic* ("fortemente ciclici") and observed that there exists a connection between this kind of generating and the property of *cyclicity of the group* H/β^*, where $\beta^\star$ is the fundamental relation of H. The way of finding this well-known relation of the algebraic hyperstructure theory is described in Antampoufis and Hošková-Mayerová [15], where also a thorough discussion on two possible approaches to this topic is given.

Definition 6 ([15,16]). *Let $(H, *)$ be a (semi)hypergroup and $n > 1$ a natural number. We define relation β_n as follows:*

$$x \beta_n y \text{ if there exist } a_1, a_2, \ldots, a_n \in H \text{ such that } \{x, y\} \subseteq \prod_{i=1}^{n} a_i \tag{5}$$

and define $\beta = \bigcup_{n \geq 1} \beta_n$, *where* $\beta_1 = \{(x, x) \mid x \in H\}$ *is the diagonal relation on* H. *We denote* β^* *the transitive closure of* β *and call it the* fundamental relation *on* H. *The quotient* H/β^* *is called the* fundamental (semi)group. *The mapping* $\varphi_H : H \to H/\beta$ *is called* canonical projection. *On hypergroups, the set* $\omega_H = \{x \in H \mid \varphi_H(x) = 1\}$, *where* 1 *is the identity of the group* H/β, *is called the* heart *of the hypergroup* H. *Finally, by* $\beta_n(y)$, *we denote the set of elements* $x \in H$ *which are in relation* β_n *with* $y \in H$.

Before we proceed, it is to be noted that one of the well established facts of the algebraic hyperstructure theory is that, in a hypergroup, $\beta^* = \beta$ (see Freni [17]). The following is a reworded remark from De Salvo and Freni [13], which—in a form of definition—can be found in Corsini [18] and Corsini and Leoreanu [19].

Theorem 1 ([13], Osservazione 1.3). *A hypergroup* H *is strongly cyclic, generated by* x, *if and only if* H/β^* *is a cyclic group generated by* $\varphi_H(x)$.

Definition 7 ([18], p. 276). *A hypergroup* H *is called* cyclic *with a generator* x *if* $\varphi_H(H)$ *is a cyclic group generated from* $\varphi_H(x)$.

1.5. Corsini's Book: A Synthesis of Approaches

In 1986, Corsini published (in Italian) his book on hyperstructure theory, which first appeared in English in 1993 as [18]. It is important to notice that, in his book, he mentions that Vougiouklis encouraged him to write it. In other words, what we see in [18] is a synthesis of the approaches existing so far.

Now, in order to distinguish between the original definition of De Salvo and Freni on one hand and the definition using the canonical projection (i.e., *strong cyclicity*), Corsini [18] called the original approach, in the case of *hypergroups*, where the names would overlap, *s-cyclic* while, in the case of *semihypergroup*, where no overlapping was possible (because Definition 7 is based on *hypergroups*), he left in use the original name; notice that the first part of Definition 8 is equivalent to Definition 3 of De Salvo and Freni [3,8]. Moreover, *strong cyclicity* of De Salvo and Freni [13] is referred to as *cyclicity*; see the above Definition 7.

Definition 8 ([18], p. 278). *A semihypergroup* H *is called* cyclic *if there exists* $h \in H$ *such that, for all* $x \in H$, *there exists* $n \in \mathbb{N}$ *such that* $x \in h^n$. *We call* h *the* s–generator *of* H. *A hypergroup is called* s-cyclic *if it is a cyclic semihypergroup.*

The relation between cyclic (= strongly cyclic) and *s*-cyclic *hypergroups* is the following.

Theorem 2 ([18], p. 281). *Every hypergroup* H, *if it is s-cyclic, is cyclic, and is generated by its s–generator* h.

Notice that the Theorem 2 originated in De Salvo and Freni [13] as Proposizione 2.5, where the result is given using the original terminology ("if a hypergroup is cyclic, then it is strongly cyclic").

The following theorem is valid both for "cyclic" in the sense of canonical projection and for *s*-cyclicity (in this case even for semihypergroups as follows from [8], Proposizione 1). Notice that, while all cyclic *groups* are commutative, it was already Wall [2] who observed that commutativity is not always granted for cyclic *hypergroups*.

Theorem 3 ([19], p. 18). *If* H *is a cyclic and complete hypergroup, then it is commutative.*

If we realize that only one definition of cyclicity in *semihypergroups* exists, no confusion can arise from the following theorem.

Theorem 4 ([19], p. 18). *A cyclic and complete semihypergroup is a join space.*

1.6. Current State of the Art

Even though a great many important results have been achieved by the Italian and the Greek schools, the concept of cyclicity in the algebraic hyperstructure theory is not a "closed issue". Recently, *single power cyclicity* has been especially studied. Al Tahan and Davvaz [20] studied some properties of single power cyclic hypergroups and links of these to regular relations. They also applied their results in the study of *braid groups* (which have important applications in *knot theory*). Moreover, in [21], Al Tahan and Davvaz implicitly use the idea of *EL*–hyperstructures (i.e., hyperstructures that we discuss further on) to construct a cyclic hypergroup of an arbitrary braid group B_n of n strands. Finally, e.g., Karimian and Davvaz [22] recently studied γ-*cyclic hypergroups*, which are based on Freni's [23] *relation* γ, which is a generalization of the β *relation* used throughout this paper.

1.7. A Remark Concluding the Introduction

Regardless of the evolution of terminology, one can observe the following:

1. There exist *two main approaches* towards cyclicity in the hyperstructure theory: through the fact that we get a hypergroup by repeatedly applying the hyperoperation on a generator, or through the canonical projection. Both were introduced by the Italian school. The former was independently introduced by Vougiouklis and is related to the original Wall's definition. The latter is related to the notion of completeness and the results of Koskas.
2. Vougiouklis defined numerous cases of cyclicity and periods such as *single power* (one generator only) and finite/infinite period. The purpose of his definition was to study a specific class of cyclic hypergroups, those based on P–hyperoperations.
3. The original definition of Wall, the definition of Vougiouklis and the definition using canonical projection were provided for *hypergroups*. The original definition of De Salvo and Freni was provided for semihypergroups.
4. *Cyclicity* of a (semi)hypergroup (defined by the Italian schools) and *period* of a hypergroup (defined by Vougiouklis) are, in a general case, different.

Finally, let us point out that the class of *cyclic hypergroups*, i.e., the one using canonical projection, *is the broadest one. Some cyclic hypergroups are s-cyclic* and *some s-cyclic hypergroups are single power cyclic*; see Figure 1.
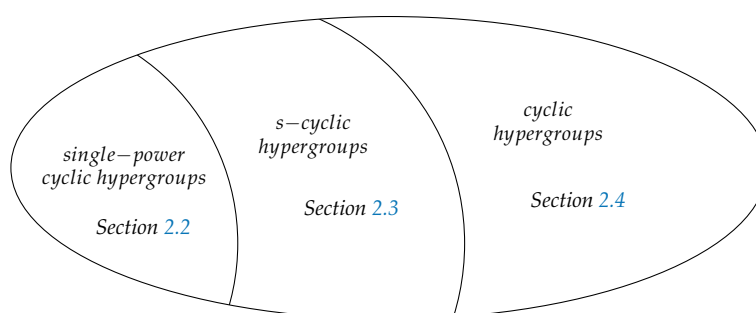

Figure 1. Various notions of cyclicity in hypergroups and their relations; from the most special case (single-power cyclic hypergroups) to the most general one (cyclic hypergroups).

Notation. From now on, we are going to use only the following names: *cyclic*, *s-cyclic* and *single power cyclic*.

2. Examples

Unfortunately, the papers of both the Italian and the Greek schools contain examples only very occasionally (if at all). Therefore, in this section, we try to demonstrate various notions of cyclicity using as many various examples as possible.

2.1. Theoretical Background

Before including some examples of cyclic hypergroups, recall three very useful lemmas. Notice that, in the following definition, the name "extensive" is used e.g., by Chvalina [24,25] while e.g., Massouros [26] uses a geometrically motivated name "closed".

Definition 9. *A hyperoperation "$*$" on H is called* extensive *(or* closed*) if for all $a, b \in H$ there is $\{a, b\} \subseteq a * b$. A hypergroupoid $(H, *)$ with an extensive hyperoperation is called an* extensive hypergroupoid.

Lemma 1. *Every extensive semihypergroup is a hypergroup.*

Proof. Obvious because $a * H = \bigcup_{b \in H} a * b$ and "$*$" is extensive. □

Lemma 2. *Let $(H, *)$ be an extensive semihypergroup (i.e., by Lemma 1 a hypergroup). Then, relation β is a total relation, $\omega_H = H$ and, as a result, H / β is a one-element group.*

Proof. Obvious because in this case $\beta_2 = H \times H$. □

In other words, since every one-element group is cyclic, every hypergroup $(H, *)$ such that H / β is a one-element group, is cyclic, which—as included e.g., in [20] is the case of all single power cyclic hypergroups.

Lemma 3 ([20]). *Every single power cyclic hypergroup $(H, *)$ has a trivial fundamental group H / β.*

Most of the examples included below are based upon what is known as "Ends lemma", which was proved by Chvalina [25] and studied by Novák [27,28], later with Cristea [29] and Křehlík [30,31] and Chvalina [32]. Semihypergroups of the below type are called *EL–semihypergroups*. Notice that Novák or Chvalina have actually not "invented" the concept of *EL*–hyperstructures as they have rather provided a name for a concept implicitely present in a number of papers dated as early as (to our best knowledge) Pickett [33] (1967) or even—if *preordering* is considered—Marty [1]. In addition, by Remark 1, most such results are valid for *preordered semigroups*, i.e., also in the context of *equivalence relations*. For some further reading on this topic, see Novák [34].

Lemma 4 ([25], Theorem 1.3, p. 146). *Let $(S, \cdot, \leq)$ be a partially ordered semigroup. Binary hyperoperation $* : S \times S \to \mathcal{P}^*(S)$ defined by*

$$a * b = [a \cdot b)_{\leq} = \{x \in S \mid a \cdot b \leq x\} \tag{6}$$

is associative. The semihypergroup $(S, *)$ is commutative if and only if the semigroup $(S, \cdot)$ is commutative.

Lemma 5 ([25], Theorem 1.4, p. 147). *Let $(S, \cdot, \leq)$ be a partially ordered semigroup. The following conditions are equivalent:*

1. *For any pair $a, b \in S$, there exists a pair $c, c' \in S$ such that $b \cdot c \leq a$ and $c' \cdot b \leq a$.*
2. *The semi-hypergroup $(S, *)$ defined by Formula (6) is a hypergroup.*

Remark 1. *In groups, condition 1^0 of Lemma 5 holds trivially as it is sufficient to put $c = b^{-1} \cdot a$ and $c' = a \cdot b^{-1}$. In addition, when one verifies the proofs, one can see that downgrading from partially ordered (semi)groups* to *preordered* ones *is possible for all implications except for "commutativity of '$*$' implies commutativity of '$\cdot$'.*

2.2. Single Power Cyclic Hypergroups

The following example presents a class of single power cyclic hypergroups with infinite periods, i.e., hypergroups which are also *s*-cyclic and cyclic. With respect to *single power cyclicity* and *s-cyclicity*, only some of its elements are generators while, with respect to *cyclicity*, each of its elements is a generator.

Example 1. *Suppose* $(\mathbb{Z}, +, \leq)$, *with the usual addition and ordering of integers. Since* $(\mathbb{Z}, +, \leq)$ *is a partially ordered group,* $(\mathbb{Z}, *)$, *where* $a * b = \{x \in \mathbb{Z} \mid a + b \leq x\}$ *is a hypergroup. Let us examine its cyclicity with respect to the different approaches mentioned above.*

In the context of Definition 5 of Vougiouklis, we can see that, for an arbitrary negative $a \in \mathbb{Z}$ *and* $k > 1$, *we have* $a^k = \underbrace{a * \ldots * a}_{k} = \{x \in \mathbb{Z} \mid ka \leq x\}$, *i.e.,* $\mathbb{Z} = a \cup a^2 \cup a^3 \cup \ldots \cup a^n \cup \ldots$ *and* $(\mathbb{Z}, *)$ *is a single-power cyclic hypergroup with infinite periods for infinitely many generators (yet not all because only negative integers generate it).*

In order to relate this hypergroup to the Definition 7 of Corsini, we first need to establish the relation β and the canonical projection φ. However, by Lemma 3, we know that $\varphi_{\mathbb{Z}}(\mathbb{Z})$ is a one-element set, i.e., a trivial cyclic group; it is easy to show that $\varphi_{\mathbb{Z}}(\mathbb{Z}) = \{\mathbb{Z}\}$, i.e., $\varphi_{\mathbb{Z}}(x) = \mathbb{Z}$ for all $x \in \mathbb{Z}$. Thus, $(\mathbb{Z}, *)$ is a cyclic hypergroup.

As regards generators, by Theorem 2, any negative integer, i.e., any s–generator of $(\mathbb{Z}, *)$, is a generator. However, if we regard Definition 7, every integer is a generator.

In the following example, we again have a single power cyclic hypergroup with infinite period. However, in this case, an arbitrary element is a generator under all definitions.

Example 2. *Regard multiplication on the open interval* $I = (0, 1)$. *Then* $(I, *)$, *where* $a * b = [a \cdot b]_{\leq} = \{x \in I \mid a \cdot b \leq x\}$, *is a semihypergroup, which—due to extensivity—is a hypergroup. Now, for an arbitrary* $a \in I$ *there is* $a^n \subsetneq a^{n+1}$ *and, obviously,* $(I, *)$ *is a single power cyclic hypergroup with infinite period for an arbitrary* $a \in I$.

In the following example we have a class of single power cyclic hypergroups with finite period and only one generator. As was the case in Example 1, the set of generators will be different under *cyclicity* because the hyperoperation is extensive.

Example 3. *Regard* $(\mathbb{N}, \gcd, |)$, *where "gcd" stands for the greatest common divisor of natural numbers* $\mathbb{N} = \{1, 2, \ldots\}$ *and "|" is the divisibility relation. Since* $(\mathbb{N}, \gcd, |)$ *is a partially ordered semigroup,* $(\mathbb{N}, *)$, *where* $a * b = \{x \in \mathbb{N} \mid \gcd\{a, b\} | x\}$, *for all* $a, b \in \mathbb{N}$, *is a semihypergroup. Since* $\{a, b\} \subseteq a * b$ *for all* $a, b \in \mathbb{N}$, $(\mathbb{N}, *)$ *is, by Lemma 1, a hypergroup.*

Now, obviously $1 * 1 = [\gcd\{1, 1\}]_{\leq} = [1]_{\leq} = \mathbb{N}$. *Thus,* $(\mathbb{N}, *)$ *is single power cyclic with period 2 and one generator 1. Of course, instead of $\mathbb{N}$, we can regard the set of divisors of an arbitrary $n \in \mathbb{N}$ and get a class of finite single power cyclic hypergroups with period 2.*

2.3. s–Cyclic Hypergroups Which Are Not Single Power Cyclic

Now, we present an example of classes of hypergroups which are *s*-cyclic, i.e., also cyclic, yet not single power cyclic. For reasons of clarity, we choose a set of a small cardinality.

Example 4. *Denote* S_3 *the group of permutations of three elements, i.e.,*

$$S_3 = \{e = (1, 2, 3), p_1 = (1, 3, 2), p_2 = (3, 2, 1), p_3 = (2, 1, 3), p_4 = (2, 3, 1), p_5 = (3, 1, 2)\}.$$

Obviously, $\mathrm{sgn}(p_0) = \mathrm{sgn}(p_4) = \mathrm{sgn}(p_5) = 1$ *and* $\mathrm{sgn}(p_1) = \mathrm{sgn}(p_2) = \mathrm{sgn}(p_3) = -1$. *Now, since, for an arbitrary pair of permutations* σ, τ *there is* $\mathrm{sgn}(\sigma \circ \tau) = \mathrm{sgn}(\sigma) \cdot \mathrm{sgn}(\tau)$, *where "$\circ$" is the usual operation*

of composition of permutations, it is easy to verify that if we set $\tau \sim \sigma$ *whenever* $\mathrm{sgn}(\sigma) = \mathrm{sgn}(\tau)$, *we get that* $(S_3, \circ, \sim)$ *is a preordered group. Therefore, if we define a hyperoperation on* S_3 *by*

$$\sigma * \tau = \{\pi \in S_3 \mid \tau \circ \sigma \sim \pi\} = \{\pi \in S_3 \mid \mathrm{sgn}(\tau) \cdot \mathrm{sgn}(\sigma) = \mathrm{sgn}(\pi)\}$$

for all $\sigma, \tau \in S_3$, *then* $(S_3, *)$ *is an EL–hypergroup which is not extensive. Now, in order to construct the canonical projection, we construct the (commutative) Cayley table of the hyperoperation. In it, we will use the following notation: "+" will stand for the set* $\{e, p_4, p_5\}$, *i.e., the set of permutations with positive parity, and "−" will stand for the set* $\{p_1, p_2, p_3\}$, *i.e., the set of permutations with negative parity.*

$*$	e	p_1	p_2	p_3	p_4	p_5
e	$+$	$-$	$-$	$-$	$+$	$+$
p_1		$+$	$+$	$+$	$-$	$-$
p_2			$+$	$+$	$-$	$-$
p_3				$+$	$-$	$-$
p_4					$+$	$+$
p_5						$+$

Thus, we can see that "$\sigma \beta_2 \tau$ whenever there exist permutations $\pi_1, \pi_2 \in S_3$ such that $\tau, \sigma \in \pi_1 * \pi_2$" means that $\beta_2(e) = \{e, p_4, p_5\}$ and $\beta_2(p_1) = \{p_1, p_2, p_3\}$. In addition, since e.g., $p_1 * p_2 * p_4 = \{e, p_4, p_5\}$, we never obtain a cross-combination between sets denoted "+" and "−". Therefore, e.g., $\beta(e) = \{e, p_4, p_5\}$ and e.g., $\beta(p_1) = \{p_1, p_2, p_3\}$ and we have that

$$(S_3, *)/\beta = \{\beta(e), \beta(p_1)\},$$

which is isomorphic to the cyclic group $(\mathbb{Z}_2, +)$ generated by $\beta(p_1)$. Therefore, $(S_3, *)$ is cyclic with generators p_1, p_2, p_3.

Obviously, $p_2 * p_2 = \{e, p_4, p_5\}$ and $p_2 * p_2 * p_2 = \{p_1, p_2, p_3\}$ (and likewise for p_1 and p_3), we get that $(S_3, *)$ is s-cyclic with period 3 (because $S_3 = p_2^2 \cup p_2^3$); $\mathrm{cicl}(S_3) = 3$. However, since for no element of S_3 there exists any $n \in \mathbb{N}$ such that $h^n = H$ (notice that $e^n = \{e, p_4, p_5\}$ for all $n \in \mathbb{N}$—and likewise for p_4, p_5), $(S_3, *)$ is not single power cyclic.

2.4. Cyclic Hypergroups Which Are Not s–Cyclic

Finally, we include an example of cylic hypergroups which are not *s*-cyclic, i.e., which cannot be single power cyclic either. Notice that the hyperoperation is again not extensive, yet we get that β is a total relation.

Example 5. *Let $(H, *)$ be a hypergroup, where "$*$" is defined by the following table.*

$*$	1	2	3	4
1	$\{1\}$	$\{1\}$	$\{1,2,3\}$	$\{1,2,4\}$
2	$\{1\}$	$\{1\}$	$\{1,2,3\}$	$\{1,2,4\}$
3	$\{1,2,3\}$	$\{1,2,3\}$	$\{1,2,3\}$	$\{3,4\}$
4	$\{1,2,4\}$	$\{1,2,4\}$	$\{3,4\}$	$\{1,2,4\}$

Now, it is easy to see that $1^n = 2^n = \{1\}$ for all $n \in \mathbb{N}$, $3^n = \{1,2,3\}$ for all $n \in \mathbb{N}$ and $4^n = \{1,2,4\}$ for all $n \in \mathbb{N}$. Therefore, $(H, *)$ is neither s-cyclic nor single power cyclic. However, since $\beta_2(x) = H$ for all $x \in H$, i.e., β is a total relation, we have that H/β is a one-element group (which is trivially–cyclic), i.e., $(H, *)$ is a cyclic hypergroup.

3. New Results

In this section, we present several new results concerning various kind of cyclicity, especially with respect to *EL*–hyperstructures. We choose this class of hyperstructures because the papers of both the Italian and the Greek schools contain a great many results on various types of cyclicity, yet none of them are specifically linked to *EL*–hyperstructures—for the simple reason that this concept was defined several years later.

First of all, if we rewrite Definition 5 (or rather, Definition 3) in the context of the *EL*–hyperoperation Formula (6), we immediately have the following. (Notice that, as has been pointed out above, in the case of hypergroups, "cyclic" of Definition 5 is "*s*-cyclic" of Definition 3.)

Theorem 5. *Let* $(H, *)$ *be the EL–semihypergroup of a partially-ordered semigroup* $(H, \cdot, \leq)$*. Then,* $(H, *)$ *is a cyclic semihypergroup (i.e., in case of hypergroups, an s-cyclic hypergroup) if and only if there exists an element* $h \in H$ *such that all minimal elements of H can be expressed as* $h^n = \underbrace{h \cdot \ldots \cdot h}_{n}$ *for some* $n \in \mathbb{N}$*. In such a case, h is a generator of* $(H, *)$ *and* $cicl(H)$ *is the greatest of these exponents.*

Example 6. *Denote* $\mathbb{N}_2 = \{(a, b) \mid a \in \mathbb{N}, b \in \mathbb{N} \cup \{0\}\}$*. Define addition of the pairs component-wise and the relation "$\leq$" in such a way that* $(a_1, b_1) \leq (a_2, b_2)$ *whenever* $a_1 = a_2$ *and* $b_1 \leq b_2$*. Obviously,* $(\mathbb{N}_2, +, \leq)$ *is a partially-ordered semigroup which has infinitely many minimal elements—pairs* $(a, 0)$*, where* $a \in \mathbb{N}$*. All of these elements can be expressed as powers of the pair* $(1, 0)$*, which makes this pair a generator of this cyclic semihypergroup. However, cyclicity of* $(H, *)$ *is infinite.*

In the "Ends lemma", we often regard *idempotent operations* such as "min", "max", "$\cup$", "$\cap$", etc. For these, the following corollary can be given.

Corollary 1. *Let* $(H, *)$ *be an EL–semihypergroup constructed from a partially ordered semigroup* $(H, \cdot, \leq)$ *such that "$*$" is extensive and "$\cdot$" is idempotent. Then,* $(H, *)$ *has at most one generator—the smallest element of* $(H, \leq)$*. Moreover, in this case,* $(H, *)$ *is a single power cyclic hypergroup with period (and cyclicity) 2.*

Proof. In the case described in Theorem 5, we get, for an arbitrary $h \in H$ and arbitrary $k \geq 2$,

$$h^k = \underbrace{h * \ldots * h}_{k} = [\underbrace{h \cdot \ldots \cdot h}_{k})_{\ldots} = [h)_{\leq} = \{x \in H \mid h \leq x\}, \tag{7}$$

which means that the only chance that Formula (1) becomes valid is in case that $[h)_{\leq} = H$ (notice that, due to reflexivity of "$\leq$", there is always $h \in [h)_{\leq}$). In other words, if $(H, *)$ is a hypergroup (and, by Lemma 1, extensivity of the hyperoperation turns the semihypergroup into a hypergroup), then H is single power cyclic with period 2. Of course, the only generator of H is the smallest element of H. $\square$

The following example can be compared to De Salvo and Freni [8], Proposizione 12 and Corollario 4; notice that, in our example, $cicl(\mathbb{N}) = 2$ and "$*$" is extensive. In addition, see Example 3.

Example 7. *Regard the preordered semigroup* $(\mathbb{N}, \min, \leq)$ *and its EL–hypergroup* $(\mathbb{N}, *)$*, where* $a * b = \{x \in \mathbb{N} \mid \min\{a, b\} \leq x\}$ *for all* $a, b \in \mathbb{N}$*. Since* $[1)_{\leq} = \mathbb{N}$*, we get that* $1 * 1 = \mathbb{N}$ *and 1 is the only generator of the single power cyclic hypergroup* $(\mathbb{N}, *)$ *with period 2. (or 0 instead of 1 if we consider* $\mathbb{N} \cup \{0\}$*). If we change* $\mathbb{N}$ *to* $\mathbb{Z}$*, we get that the hypergroup* $(\mathbb{Z}, *)$ *is not single power cyclic because* $(\mathbb{Z}, \leq)$ *has no smallest element, i.e., there is no integer* $a \in \mathbb{Z}$ *such that* $[a)_{\leq} = \mathbb{Z}$*.*

The following lemma is obvious yet very useful in our future considerations; notice that, by $lb\{a, b\}$, we mean a lower bound of a and b.

Lemma 6. *Let* $(H, *)$ *be an EL–hypergroup of a partially ordered group* $(H, \cdot, \leq)$ *and* $a, b \in H$ *arbitrary.*

1. If $a \leq b$, then $a\beta_2 b$, i.e., $a\beta b$.
2. If $a, b \in H$ are such that $lb\{a, b\}$ exists, then $a\beta_2 b$, i.e., $a\beta b$.

Proof. If $a \leq b$, then obviously also $a \cdot e \leq a$, i.e., also $a \cdot e \leq a \leq b$, i.e., $\{a, b\} \subseteq a * e$. In other words, $a\beta_2 b$. If $a, b \in H$ are such that $lb\{a, b\}$ exists, then, since $lb\{a, b\} \leq a$ and $lb\{a, b\} \leq b$, we have that $\{a, b\} \subseteq lb\{a, b\} * e$. In other words, $a\beta_2 b$. □

Thus, we can see, as an immediate corollary, that:

Theorem 6. *Suppose that $(H, *)$ is an EL–hypergroup of a partially ordered group $(H, \cdot, \leq)$ such that $(H, \leq)$ has the smallest element or is well ordered. Then, H/β is a one-element group.*

Corollary 2. *Let $(H, *)$ be an EL–hypergroup of a partially ordered group $(H, \cdot, \leq)$. If there are at least two distinct minimal elements in $(H, \leq)$, then H/β is not trivial.*

Proof. Obvious because the relation "$\leq$" is reflexive. Thus, if a, b are minimal, then $a \in [a]_\leq = \{x \in H \mid a \leq x\}$, $b \in [b]_\leq = \{x \in H \mid b \leq x\}$ and $a \notin [b]_\leq$, $b \notin [a]_\leq$. Therefore, we can never find two (or more) elements $z_1, \ldots, z_n$ such that $\{a, b\} \subseteq \underbrace{z_1 * \ldots * z_n}_{n}$. Thus, $\beta(a) \neq \beta(b)$, i.e., $|H/\beta| \geq 2$. □

In the following example, we have no minimal elements, yet H/β is again not trivial. Notice that in this way we can get cyclic hypergroups, which are not *s*-cyclic. Also worth mentioning is the fact that all infinite cyclic groups are isomorphic to the cyclic group $(\mathbb{Z}, +)$.

Example 8. *Regard the additive group of "selected complex numbers", i.e., $H = \{a + bi \mid a \in \mathbb{R}, b \in \mathbb{Z}\}$ with the usual component wise addition. Obviously, $(H, +)$ is a group with the neutral element $0 + 0i$ such that the opposite of $a + bi$ is $-a - bi$. Now, say that $a_1 + b_1 i \leq a_2 + b_2 i$ whenever $b_1 = b_2$ and $a_1 \leq a_2$. Obviously, "$\leq$" is compatible with the addition. Thus, we set $z_1 * z_2 = \{x \in H \mid z_1 + z_2 \leq x\}$ and know that $(H, *)$ is a hypergroup.*

Now, regard e.g., $z_1 = 2 + 3i$ and $z_2 = 4 + 5i$. In this case, $z_1 \in \{a + 3i \mid a \in \mathbb{R}\}$ while $z_2 \in \{a + 5i \mid a \in \mathbb{R}\}$ and we will never be able to find such $x, y \in H$ that both z_1 and z_2 belong to $x * y$ because the sum of imaginary parts is always unique. In other words, for an arbitrary $z = a + bi$ and $n \geq 2$, we have that $\beta_n(z) = \{x + bi \mid x \in \mathbb{R}\} = \mathbb{R} + bi$, i.e., $\beta(a + bi) = \mathbb{R} + bi$. Thus, it can be easily observed that $H/\beta = \{\mathbb{R} + bi \mid b \in \mathbb{Z}\}$, which is isomorphic to the cyclic group $(\mathbb{Z}, +)$. Therefore, by Definition 7, $(H, *)$ is a cyclic hypergroup with generators $a \pm i$, where $a \in \mathbb{R}$, i.e., infinitely many generators which could be grouped as $\mathbb{R} + i$, $\mathbb{R} - i$. In contrast to Example 5, the fundamental group is not trivial, though. Obviously, $(H, *)$ is not s-cyclic because we are not able to get the whole of H from any $a \in H$ by means of uniting powers of a because—in contrast to cyclicity in groups—only positive powers of generators are available.

Finally, what does it mean that elements of $\mathbb{R} + i$, $\mathbb{R} - i$ "generate" $(H, *)$? As has been mentioned in Section 1.4, H must be the intersection of all subhypergroups which are complete parts and contain the generator. For a generator h, we denote this $\langle h \rangle$. In [13], Osservazione 1.4, De Salvo and Freni show that all subhypergroups of a cyclic hypergroup are complete parts, i.e., it is enough to consider the intersection of subhypergroups. Now, what are the subhypergroups of $(H, *)$ which contain e.g., the generator belonging to $\mathbb{R} + i$? Since $(a + i) + (a + i) = 2a + 2i$ for an arbitrary $a \in \mathbb{R}$, we have that every subhypergroup G of H containing an element from $\mathbb{R} + i$ must (in order to provide for $G * G \subseteq G$) contain $K + \mathbb{Z}^+ i$, where K is a suitable subset of $\mathbb{R}$. However, this is not enough for the reproductive axiom, i.e., $a * G = G * a = G$ for all $a \in G$, to hold because the sum of non-negative imaginary parts is always greater than the parts themselves and $a + i \in G$. Therefore, we need also (all) negative imaginary parts. The same reasoning holds for real parts. Thus, $\langle a + i \rangle$, where $a \in \mathbb{R}$ is arbitrary, equals H, which is in accordance with Theorem 1 (the same holds also for generators from the set $\mathbb{R} - i$).

Remark 2. *Obviously, if $H = \{a + bi \mid a \in \mathbb{R}, b \in \mathbb{Z}/n\mathbb{Z}, n \text{ is prime}\}$, then H is cyclic because $H/\beta \cong \mathbb{Z}/n\mathbb{Z}$ and $\mathbb{Z}/n\mathbb{Z}$ is a cyclic group. See also Figure 2.*
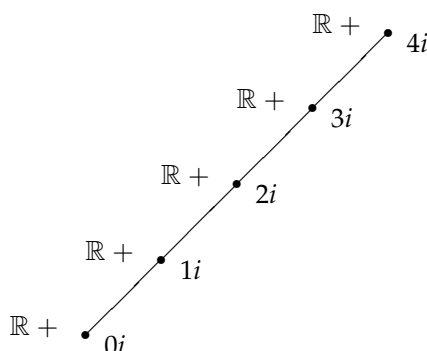

Figure 2. Related to Example 8 and Remark 2 when $H/\beta \cong \mathbb{Z}/5\mathbb{Z}$. Elements are in relation based on the imaginary part and within the same class based on the real part.

Given Remark 2, we know that we can construct cyclic *EL*–hypergroups such that their fundamental group has any cardinality we choose (given Example 8 including infinity). Thus, there arises an obvious question: "How can this be done?"

First of all, in a *finite* case, the notion of a *partially ordered cyclic group* is absurd because antisymmetry of the relation conflicts with the compatibility of the relation and the group operation. Therefore, if we insist on the fact that the relation "$\leq$" used in the construction of *EL*–hyperstructures is a partial order, we cannot start with a *finite* cyclic group (such as, e.g., groups of n elements, where n is prime). However, we can take any cyclic group and "expand" it by "attaching" copies of some (finite or infinite) set to each of its elements—as has been done in Example 8.

4. Conclusions and Future Work

The aim of this paper has been to clarify one of the fundamental properties of hypergroups, *cyclicity*. We have pointed out two approaches to this concept: via the fundamental relation β and the canonical projection, or by repeatedly applying the hyperoperation on the generator. Thus, we were able to link the three existing concepts of cyclicity: the class of *single power cyclic hypergroups*, which is included (not strictly) in the class of *s-cyclic hypergroups*, which is included (again, not strictly) in the class of *cyclic hypergroups*—see Figure 1. As a tool to show this, we have chosen the concept of *EL*–hyperstructures. It is to be noted that, even though numerous properties of *EL*–hyperstructures have already been studied, the important issue of *cyclicity* in this class of hyperstructures constructed from partially ordered (semi)groups has been neglected so far.

We intend to continue our study of cyclicity in hypergroups by linking it to the concept of *reducibility*. The notion of *reduced hypergroups* was introduced by Jantosciak [35], when he noticed that the elements of a non-empty set H can play interchangeable roles with respect to a hyperoperation defined on it, and explained this property by means of three equivalencies on H, called *fundamental relations*, that identify the elements with the same properties. To be more precise, two elements x, y of a hypergroup $(H, \circ)$ are called

- *operationally equivalent* if $x \circ a = y \circ a$ and $a \circ x = a \circ y$, for all $a \in H$,
- *inseparable* if for $a, b \in H$ we have $x \in a \circ b$ if and only if $y \in a \circ b$,
- *essentially indistinguishable* if they are operationally equivalent and inseparable.

Then, a hypergroup $(H, \circ)$ is called *reduced* if the equivalence class of each element of H with respect to the essentially indistinguishable relation is a singleton.

Based on this, we intend to find out based on which assumptions the various types of cyclic hypergroups presented in this paper are reduced hypergroups.

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