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Sehgal Type Contractions on *b*-Metric Space

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Abstract: In this paper, we analyze two discontinuous self-mappings that satisfy Sehgal-type inequalities in the setup of complete *b*-metric space. The main results of the paper cover and extend a few existing results in the corresponding literature. Furthermore, we give some illustrative examples to verify the effectiveness and strength of our derived results. Thereafter, as an application, we consider the obtained result to aggregate the existence and uniqueness of the solution for nonlinear Fredholm integral equations.

Keywords: b-metric; Sehgal-type contraction; common fixed point; discontinuous mapping

1. Introduction and Preliminaries

In 1969, Sehgal [1] formulated an inequality that can be considered an extension of the renowned Banach contractions mapping principle in the setting of a complete metric space. Indeed, Sehgal [1] investigated the contractive iteration of each point of continuous self-mappings in the circumstance of complete metric spaces.

Theorem 1. *[1] Suppose that O is a continuous self-mapping on a complete metric space $(\mathcal{M}, d)$. If there exists a positive real number c with $c < 1$ such that for each $p \in \mathcal{M}$ there exists a positive integer $n(p)$ such that:*

$$d(O^{n(p)}p, O^{n(p)}q) \leq cd(p,q), \text{ for each } q \in \mathcal{M}, \tag{1}$$

then O possesses an unique fixed point in $\mathcal{M}$.

Sehgal [1] gave also an example of a mapping O that does not form a contraction, but it satisfies (1) and possesses a fixed point. This result has been refined by Guseman [2] by relaxing the continuity condition on the mapping. Our purpose in this study is to extend the existing common fixed point results in a more general abstract structure. The idea of the extension of a metric notion, in particular the concept of b-metric, is quite natural, and it has appeared in several papers, such as Bourbaki [3], Bakhtin [4], Czerwik [5], Heinonen [6] and many others. In brief, the b-metric was obtained by substituting the triangle inequality of the metric

$$(T) \quad d(p,q) \leq d(p,\mu) + d(\mu,q), \text{for every } p,q,\mu \in \mathcal{M},$$

with the inequality:

$$(S) \quad d(p,q) \leq s[d(p,\mu) + d(\mu,q)], \text{for every } p,q,\mu \in \mathcal{M},$$

for a fixed $s \geq 1$. In this case, the triplet $(\mathcal{M}, d, s)$ is termed a b-metric space. It is clear that $(\mathcal{M}, d, s)$ forms a standard metric space in the case of $s = 1$.

A typical example of b-metric is the following:

Example 1. *[7] For any metric space $(\mathcal{M}, d)$, it is possible to define a function $d_\alpha : \mathcal{M} \times \mathcal{M} \to \mathbb{R}_0^+$ as $d_\alpha(p, q) = (d(p, q))^\alpha$ with $\alpha > 1$, where α is a real number and $\mathbb{R}_0^+$ is the collection of all nonnegative real numbers. I this case, $(\mathcal{M}, d_\alpha, s)$ forms a b-metric with $s = 2^{p-1}$. Indeed, for $1 < \alpha < \infty$, the function $J : (0, \infty) \to \mathbb{R}$ defined by $J(t) = t^\alpha$ is convex, and therefore, it verifies Jensen's inequality $J\left(\frac{a+b}{2}\right) \leq \frac{J(a)+J(b)}{2}$. Hence, $\left(\frac{a+b}{2}\right)^\alpha \leq \frac{a^\alpha+b^\alpha}{2}$, meaning $(a + b)^\alpha \leq 2^{\alpha-1}(a^\alpha + b^\alpha)$.*

For more interesting examples and fundamental results on the b-metric, we refer to, e.g., [8–17] and the related references therein. With respect to the analogy with the standard metric space, the topology on b-metric space is easily setup. On the other hand, in general, b-metrics are not necessary continuous. We say that a sequence $\{p_n\}$ in a b-metric space $(\mathcal{M}, d, s)$ converges to p if $\lim_{n\to\infty} d(p_n, p) = 0$. A sequence $\{p_n\}$ is Cauchy if $d(p_n, p_m) \to 0$ as $n, m \to \infty$. We notice also that each convergent sequence in a b-metric space is Cauchy. As usual, if each Cauchy sequence is convergent, then we say that a b-metric space $(\mathcal{M}, d, s)$ is complete. We mention also that in a b-metric space $(\mathcal{M}, d, s)$, a convergent sequence has a unique limit.

2. Main Results

First of all, inspired by the ideas from [18,19], we consider a new type of contractive condition.

Definition 1. *Let O, R be self-mappings on a b-metric space $(\mathcal{M}, d, s)$. We say that O, R forms Δ-contraction if there exists $c \in (0, \frac{1}{2s-1})$ such that for each $p, q \in \mathcal{M}$, there exist positive integers $n(p), m(q)$ such that:*

$$d(O^{n(p)}p, R^{m(q)}q) \leq cE(p, q), \tag{2}$$

where:

$$E(p, q) = \left[d(p, q) + \left|d(p, O^{n(p)}p) - d(q, R^{m(q)}q)\right|\right].$$

Theorem 2. *If O, R form Δ-contraction on a complete b-metric space $(\mathcal{M}, d, s)$, then O and R possess exactly one common fixed point.*

Proof. Firstly, we notice that:

$$E(p, q) = 0 \text{ if and only if } p = q \text{ and } O^{n(p)}p = R^{m(p)}p.$$

Indeed, if $E(p, q) = d(p, q) + \left|d(p, O^{n(p)}p) - d(q, R^{m(q)}q)\right| = 0$, then:

$$0 = d(p, q) + \left|d(p, O^{n(p)}p) - d(q, R^{m(q)}q)\right| \geq d(p, q) \geq 0,$$

and hence,

$$d(p, q) = 0 \Rightarrow p = q. \tag{3}$$

On the other hand, for $p = q$, we have by (2):

$$0 \leq d(O^{n(p)}p, R^{m(p)}p) \leq cE(p, p) = 0,$$

which yields that:

$$d(O^{n(p)}p, R^{m(p)}p) = 0 \Rightarrow O^{n(p)}p = R^{m(p)}p. \tag{4}$$

Therefore, by combining (3) and (4), we get that $O^{n(p)}p = p = q = R^{m(q)}q$.
Conversely, if $p = q$ and $O^{n(p)}p = R^{m(p)}p$, then:

$$d(p,p) + \left|d(p, O^{n(p)}p) - d(p, R^{m(p)}p)\right| = 0.$$

Let $p_0 \in \mathcal{M}$ be an arbitrary point. Starting from p_0, we will inductively construct a sequence $\{p_k\}$, by:

$$p_1 = R^{m(p_0)}p_0,\ p_2 = O^{n(p_1)}p_1, ...\ p_{2k+1} = R^{m(p_{2k})}p_{2k},\ p_{2k+2} = O^{n(p_{2k+1})}p_{2k+1}, ... \tag{5}$$

or if we use the notation $m_k = m(p_{2k})$, $n_k = n(p_{2k-1})$, we can write $p_{2k+1} = R^{m_k}p_{2k}$, respectively $p_{2k+2} = O^{n_{k+1}}p_{2k+1}$.

If we suppose that for some $k_0 \in \mathbb{N}_0$, $p_{k_0} = p_{k_0+1}$, then the proof is completed, since $t = p_{k_0}$ is a common fixed point for O and R. Hence, without loss of generality, we presume that $p_k \neq p_{k+1}$ for each $k \in \mathbb{N}$. We examine the following cases:

(a) For $p = p_{2k-1}$ and $q = p_{2k}$, the inequality (2) becomes:

$$\begin{aligned} 0 < d(p_{2k}, p_{2k+1}) &= d(O^{n_k}p_{2k-1}, R^{m_k}p_{2k}) \leq c\left[d(p_{2k-1}, p_{2k}) + |d(p_{2k-1}, O^{n_k}p_{2k-1}) - d(p_{2k}, R^{m_k}p_{2k})|\right] \\ &= c\cdot\left[d(p_{2k-1}, p_{2k}) + |d(p_{2k-1}, p_{2k}) - d(p_{2k}, p_{2k+1})|\right]. \end{aligned} \tag{6}$$

If $\max\{d(p_{2k-1}, p_{2k}), d(p_{2k}, p_{2k+1})\} = d(p_{2k}, p_{2k+1})$, then from (6), we obtain a contradiction:

$$0 < d(p_{2k}, p_{2k+1}) = d(O^{n_k}p_{2k-1}, R^{m_k}p_{2k}) \leq c\cdot d(p_{2k}, p_{2k+1}) < d(p_{2k}, p_{2k+1}), \tag{7}$$

since $c \in (0, \frac{1}{2s-1})$. Therefore, $\max\{d(p_{2k-1}, p_{2k}), d(p_{2k}, p_{2k+1})\} = d(p_{2k-1}, p_{2k})$, and (6) becomes:

$$d(p_{2k}, p_{2k+1}) \leq c\cdot(2d(p_{2k-1}, p_{2k}) - d(p_{2k}, p_{2k+1})), \tag{8}$$

or equivalently:

$$d(p_{2k}, p_{2k+1}) \leq \frac{2c}{1+c}d(p_{2k-1}, p_{2k}),\ \text{for all } k \geq 1. \tag{9}$$

(b) For $p = p_{2k+1}$, $q = p_{2k}$, we have:

$$\begin{aligned} 0 < d(p_{2k+2}, p_{2k+1}) &= d(O^{n_{k+1}}p_{2k+1}, R^{m_k}p_{2k}) \\ &\leq c\cdot\left[d(p_{2k+1}, p_{2k}) + |d(p_{2k+1}, O^{n_{k+1}}p_{2k+1}) - d(p_{2k}, R^{m_k}p_{2k})|\right] \\ &= c\cdot(d(p_{2k+1}, p_{2k}) + |d(p_{2k+1}, p_{2k+2}) - d(p_{2k}, p_{2k+1})|). \end{aligned} \tag{10}$$

If $\max\{d(p_{2k+1}, p_{2k+2}), d(p_{2k}, p_{2k+1})\} = d(p_{2k+1}, p_{2k+2})$, then (10) turns into:

$$d(p_{2k+1}, p_{2k+2}) \leq cd(p_{2k+1}, p_{2k+2}) < d(p_{2k+1}, p_{2k+2}),$$

which is a contradiction. Hence, for all $k \geq 1$, $\max\{d(p_{2k+1}, p_{2k+2}), d(p_{2k}, p_{2k+1})\} = d(p_{2k}, p_{2k+1})$. Thus,

$$0 < d(p_{2k+2}, p_{2k+1}) \leq c\left[2d(p_{2k}, p_{2k+1}) - d(p_{2k+1}, p_{2k+2})\right]$$

or:

$$d(p_{2k+1}, p_{2k+2}) \leq \frac{2c}{1+c}d(p_{2k}, p_{2k+1}),\ \text{for all } k \geq 1. \tag{11}$$

By routine calculation and based on (9) and (11), we get:

$$0 < d(p_{2k+1}, p_{2k+2}) \leq \frac{2c}{1+c}d(p_{2k}, p_{2k+1}) \leq ... \leq \left(\frac{2c}{1+c}\right)^{2k} d(p_1, p_2), \tag{12}$$

and:

$$0 < d(p_{2k}, p_{2k+1}) \leq \frac{2c}{1+c} d(p_{2k-1}, p_{2k}) \leq ... \leq \left(\frac{2c}{1+c}\right)^{2k} d(p_0, p_1). \tag{13}$$

Therefore, combining (12) and (13), we can conclude that:

$$d(p_n, p_{n+1}) \leq \left(\frac{2c}{1+c}\right)^n r(p_0), \tag{14}$$

for all $n \in \mathbb{N}$, where we denoted by $r(p_0) = \max\{d(p_0, p_1), d(p_1, p_2))\}$. Letting $n \to \infty$ in (14), we obtain:

$$\lim_{n\to\infty} d(p_n, p_{n+1}) = 0. \tag{15}$$

We will establish that $\{p_n\}$ is a Cauchy sequence in $(\mathcal{M}, d, s)$. For $p \in \mathbb{N}$, using the triangle inequality and taking (14) into account, we have:

$$\begin{aligned} d(p_k, p_{k+p}) &\leq s \cdot \left[d(p_k, p_{k+1}) + d(p_{k+1}, p_{k+p})\right] \\ &\leq s \cdot d(p_k, p_{k+1}) + s^2 d(p_{k+1}, p_{k+2}) + ... + s^p \cdot d(p_{k+p-1}, p_{k+p}) \\ &\leq s \cdot \left(\tfrac{2c}{1+c}\right)^k r(p_0) + s^2 \cdot \left(\tfrac{2c}{1+c}\right)^{k+1} r(p_0) + ... + s^p \cdot \left(\tfrac{2c}{1+c}\right)^{k+p-1} r(p_0) \\ &= s \cdot \left(\tfrac{2c}{1+c}\right)^k r(p_0) \left[1 + s \cdot \left(\tfrac{2c}{1+c}\right) + s^2 \cdot \left(\tfrac{2c}{1+c}\right)^2 + ... + s^{p-1} \cdot \left(\tfrac{2c}{1+c}\right)^{p-1}\right] \\ &= s \cdot \left(\tfrac{2c}{1+c}\right)^k r(p_0) \cdot \frac{1 - \left(s \cdot \frac{2c}{1+c}\right)^p}{1 - s \cdot \frac{2c}{1+c}} \to 0, \end{aligned}$$

when $k \to \infty$. Consequently, $\{p_n\}$ is a Cauchy sequence. By completeness, $p_n \to t$ as $n \to \infty$ for some point $t \in \mathcal{M}$, that is:

$$\lim_{n\to\infty} d(p_n, t) = 0. \tag{16}$$

Letting $p = t$ and $q = p_{2k}$ in (2), we have:

$$\begin{aligned} d(O^{n(t)}t, p_{2k+1}) &= d(O^{n(t)}t, R^{m_k} p_{2k}) \leq c \cdot \left(d(t, p_{2k}) + \left|d(t, O^{n(t)}t) - d(p_{2k}, R^{m_k} p_{2k})\right|\right) \\ &= c\left(d(t, p_{2k}) + \left|d(t, O^{n(t)}t) - d(p_{2k}, p_{2k+1})\right|\right). \end{aligned} \tag{17}$$

Taking the limit of $k \to \infty$ in (17) and using (15), (16), we obtain:

$$d(O^{n(t)}t, t) \leq c \cdot d(t, O^{n(t)}t) < d(t, O^{n(t)}t),$$

which means that $O^{n(t)}t = t$. Using the same reasoning, we observe that for $p = p_{2k-1}$ and $q = t$,

$$\begin{aligned} d(p_{2k}, R^{m(t)}t) &= d(O^{n_k} p_{2k-1}, R^{m(t)}t) \leq c \cdot \left(d(p_{2k-1}, t) + \left|d(p_{2k-1}, O^{n_k} p_{2k-1}) - d(t, R^{m(t)}t)\right|\right) \\ &= c \cdot \left(d(p_{2k-1}, t) + \left|d(p_{2k-1}, p_{2k}) - d(t, R^{m(t)}t)\right|\right), \end{aligned} \tag{18}$$

and letting $k \to \infty$ in the inequality above, we derive that:

$$d(t, R^{m(t)}t) \leq c \cdot d(t, R^{m(t)}t) < d(t, R^{m(t)}t). \tag{19}$$

Hence, we get that $R^{m(t)}t = t$. We suppose now that there exists another point $v \in \mathcal{M}$, with $t \neq v$ such that:

$$O^{n(v)}v = v \text{ and } R^{m(v)}v = v.$$

We get from (2) that:

$$0 < d(t, v) = d(O^{n(t)}t, R^{m(v)}v) \leq c\left[d(t, v) + \left|d(t, O^{n(t)}t) - d(v, R^{m(v)}v)\right|\right] = c \cdot d(t, v) < d(t, v),$$

which is a contradiction. Hence, $t = v$. On the other hand, $Ot = O(O^{n(t)}t) = O^{n(t)}(Ot)$ and from the uniqueness of t, we can conclude that $Ot = t$. Similarly, we get $Rt = t$. In conclusion, O and R have exactly one common fixed point t. □

Example 2. *Let $\mathcal{M} = \{0,1,2,3\}$, and define $d : \mathcal{M} \times \mathcal{M} \to \mathbb{R}_0^+$ by $d(p,q) = d(q,p)$, for all $p,q \in \mathcal{M}$, $d(p,p) = 0$, $p \in \mathcal{M}$, $d(0,1) = d(1,2) = 3$, $d(1,3) = d(2,3) = 1$, $d(0,2) = 4$, $d(0,3) = 2$. Therefore, $(\mathcal{M}, d, s)$ is a b-metric space with $s = \frac{4}{3}$, but not a metric space since for $p = 0$ and $q = 2$:*

$$d(0,2) = 4 > 2 + 1 = d(0,3) + d(3,2).$$

Therefore, the triangle inequality is not satisfied. Let $O, R : \mathcal{M} \to \mathcal{M}$ be two mappings defined as:

$$O(1) = O(3) = O(2) = 1, \; O(0) = 2$$

and:

$$R(1) = 1, \; R(3) = 2, \; R(2) = 0, \; R(0) = 1.$$

It is easy to see that for any $p \in \mathcal{M}$, there is $n = n(p) \in \{3,4,...\}$ such that $O^{n(p)}p = 1$ and for any $q \in \mathcal{M}$, there is $m = m(q) \in \{4,5,...\}$ such that $R^{m(q)}q = 1$. Therefore, there exists $c \in (0, \frac{1}{2s-1})$ such that for each $p,q \in \mathcal{M}$, there exist positive integers $n(p), m(q)$ such that

$$0 = d(1,1) = d(O^{n(p)}p, R^{m(q)}q) \leq c \left[d(p,q) + \left| d(p, O^{n(p)}p) - d(q, R^{m(q)}q) \right| \right].$$

Hence, all the conditions of Theorem 2 are fulfilled, and O and R have exactly one fixed point, $p = 1$. In addition, we can observe that for $p = 1, q = 3$,

$$d(O(1), R(3)) \quad = d(1,2) = 3 > 2 = d(1,3) + |d(1, O(1)) - d(3, R(3))|.$$

Example 3. *Let $\mathcal{M} = [0,1]$ and $d : \mathcal{M} \times \mathcal{M} \to \mathbb{R}_0^+$ be defined as $d(p,q) = |p-q|^2$. It is clear, due to Example (1), that $(\mathcal{M}, d, s)$ is a b-metric space with the constant $s = 2$. Let $O, R : \mathcal{M} \to \mathcal{M}$ be defined by:*

$$O(p) = \begin{cases} \frac{p}{2} & \text{for } p \in [0,1) \\ \\ \frac{1}{8} & \text{for } p = 1. \end{cases}$$

and $R(p) = \frac{p}{4}$.

Due to this definition of mappings O and R, we consider two cases:

(a) For fixed $p \in [0,1)$, $q \in [0,1]$, if we denote $n(p) = n$ and $m(q) = m$, we have:

$$O^n(p) = \frac{p}{2^n}, \; R^m(q) = \frac{q}{2^{2m}}.$$

Considering $n = 4$, $m = 2$, we get:

$$d(O^4(p), R^2(q)) = d\left(\frac{p}{16}, \frac{q}{16}\right) = \frac{|p-q|^2}{256}$$

and:

$$\begin{aligned} d(p,q) + \left| d(p, O^4(p)) - d(q, R^2(q)) \right| &= |p-q|^2 + \left| \left|p - \tfrac{p}{16}\right|^2 - \left|q - \tfrac{q}{16}\right|^2 \right| \\ &= |p-q|^2 + 225 \cdot \tfrac{|p^2 - q^2|}{256}. \end{aligned}$$

In this case, (2) becomes:

$$\frac{|p-q|^2}{256} \le c\left(|p-q|^2 + 225 \cdot \frac{|p^2-q^2|}{256}\right).$$

Therefore, for $p \neq q$,

$$|p-q| \le c\,(256|p-q| + 225(p+q))$$

or, equivalently,

$$(1-256q)|p-q| \le 225(p+q)$$

which is true, for any $\frac{1}{256} < c < \frac{1}{2}$.

(b) For $p = 1$ and $q \in [0,1]$, we choose $n(1) = 2$ and $m(q) = 2$. Since $O^2(1) = \frac{1}{16}$ and $R^2(q) = \frac{q}{16}$, we find that:

$$\begin{aligned} d(O^2(1), R^2(q)) &= d(\tfrac{1}{16}, \tfrac{q}{16}) = \tfrac{|1-q|^2}{256}, \\ d(1, O^2(1)) &= d\left(1, \tfrac{1}{16}\right) = \tfrac{225}{256}, \\ d(q, R^2(q)) &= d(q, \tfrac{q}{16}) = \tfrac{225q^2}{256}, \\ d(1,q) &= |1-q|^2, \end{aligned}$$

and for any $\frac{1}{256} < c < \frac{1}{2}$, we get:

$$\frac{|1-q|^2}{256} \le c\left[|1-q|^2 + \left|\frac{225}{256} - \frac{225q^2}{256}\right|\right],$$

which shows that (2) is satisfied.

In conclusion, for any $p, q \in \mathcal{M}$, all the presumptions of Theorem 2 are satisfied. It follows that O and R have exactly one common fixed point in $\mathcal{M}$, $t = 0$.

Corollary 1. *Let $(\mathcal{M}, d, s)$ be a complete b-metric space with $s \ge 1$ and $O : \mathcal{M} \to \mathcal{M}$ be a mapping for which there exists a real number c, $0 < c < \frac{1}{2s-1}$ such that, for each $p, q \in \mathcal{M}$ there exists a positive integer $n(p)$ with:*

$$d(O^{n(p)}p, O^{n(q)}q) \le c\left[d(p,q) + \left|d(p, O^{n(p)}p) - d(q, O^{n(q)}q)\right|\right]. \tag{20}$$

Then, O has exactly one fixed point.

Theorem 3. *Let O, R be two self-mappings on a complete b-metric space $(\mathcal{M}, d, s)$ such that for all $p, q \in \mathcal{M}$, there exist positive integers $n(p), m(q)$ such that:*

$$d(O^{n(p)}p, R^{m(q)}q) \le a_1 d(p,q) + a_2 d(p, O^{n(p)}p) + a_3 d(q, R^{m(q)}q) + a_4\left(d(q, O^{n(p)}p) + d(p, O^{m(q)}q)\right), \tag{21}$$

where $a_i \ge 0$, $i \in \{1,2,3,4\}$ with $a_1 + a_2 + a_3 + 2sa_4 < \frac{1}{s}$. Then, the pair of mappings O, R possesses exactly one common fixed point t.

Proof. Starting with an arbitrary point $p_0 \in \mathcal{M}$, we construct a sequence $\{p_k\}$ in $\mathcal{M}$ as follows:

$$p_1 = R^{m(p_0)}p_0,\ p_2 = O^{n(p_1)}p_1, \ldots\ p_{2k+1} = R^{m(p_{2k})}p_{2k},\ p_{2k+2} = O^{n(p_{2k+1})}p_{2k+1}, \ldots \tag{22}$$

Let $m_k = m(p_{2k})$ and $n_k = n(p_{2k-1})$. For $p = p_{2k-1}$ and $q = p_{2k}$, the inequality (21) becomes:

$$\begin{aligned} d(p_{2k}, p_{2k+1}) &= d(O^{n_k} p_{2k-1}, R^{m_k} p_{2k}) \\ &\leq a_1 d(p_{2k-1}, p_{2k}) + a_2 d(p_{2k-1}, O^{n_k} p_{2k-1}) + a_3 d(p_{2k}, R^{m_k} p_2 k) \\ &\quad + a_4 \left(d(p_{2k}, O^{n_k} p_{2k-1}) + d(p_{2k-1}, R^{m_k} p_2 k) \right. \\ &= a_1 d(p_{2k-1}, p_{2k}) + a_2 d(p_{2k-1}, p_{2k}) + a_3 d(p_{2k}, p_{2k+1}) \\ &\quad + a_4 \left(d(p_{2k}, p_{2k}) + d(p_{2k-1}, p_{2k+1}) \right). \end{aligned} \tag{23}$$

By using the triangle inequality, we find that:

$$\begin{aligned} d(p_{2k}, p_{2k+1}) &\leq a_1 d(p_{2k-1}, p_{2k}) + a_2 d(p_{2k-1}, p_{2k}) + a_3 d(p_{2k}, p_{2k+1}) + \\ &\quad a_4 \cdot s \left[d(p_{2k-1}, p_{2k}) + d(p_{2k}, p_{2k+1}) \right] \\ &= (a_1 + a_2 + s a_4) d(p_{2k-1}, p_{2k}) + (a_3 + s a_4) d(p_{2k}, p_{2k+1}). \end{aligned} \tag{24}$$

Hence, from the above inequality, it follows that:

$$d(p_{2k}, p_{2k+1}) \leq \frac{a_1 + a_2 + s a_4}{1 - a_3 - s a_4} d(p_{2k-1}, p_{2k}) < d(p_{2k-1}, p_{2k}), \tag{25}$$

where $\dfrac{a_1 + a_2 + s a_4}{1 - a_3 - s a_4} < \dfrac{1}{s}$. Repeating the above process, we obtain for $p = p_{2k+1}$ and $q = p_{2k}$:

$$\begin{aligned} d(p_{2k+2}, p_{2k+1}) &\leq a_1 d(p_{2k+1}, p_{2k}) + a_2 d(p_{2k+1}, O^{n_{k+1}} p_{2k+1}) + a_3 d(p_{2k}, R^{m_k} p_2 k) \\ &\quad + a_4 \left(d(p_{2k}, O^{n_{k+1}} p_{2k+1}) + d(p_{2k+1}, R^{m_k)} p_{2k}) \right) \\ &= a_1 d(p_{2k+1}, p_{2k}) + a_2 d(p_{2k+1}, p_{2k+2}) + a_3 d(p_{2k}, p_{2k+1}) \\ &\quad + a_4 \left(d(p_{2k}, p_{2k+2}) + d(p_{2k+1}, p_{2k+1}) \right) \\ &< (a_1 + a_3 + s a_4) d(p_{2k+1}, p_{2k}) + (a_2 + s a_4) d(p_{2k+1}, p_{2k+2}). \end{aligned} \tag{26}$$

Therefore,

$$d(p_{2k+1}, p_{2k+2}) \leq \frac{a_1 + a_3 + s a_4}{1 - a_2 - s a_4} (d(p_{2k}, p_{2k+1}). \tag{27}$$

Putting together (25) and (27), we find that for any $k \in \mathbb{N}$:

$$\begin{aligned} d(p_{2k}, p_{2k+1}) &\leq \left(\frac{a_1 + a_2 + s a_4}{1 - a_3 - s a_4} \right)^{2k} d(p_0, p_1), \\ d(p_{2k+1}, p_{2k+2}) &\leq \left(\frac{a_1 + a_3 + s a_4}{1 - a_2 - s a_4} \right)^{2k} d(p_1, p_2). \end{aligned} \tag{28}$$

From here, considering $r(p_0) = \max \{ d(p_0, p_1), d(p_1, p_2) \}$ and $q = \max \left\{ \frac{a_1 + a_3 + s a_4}{1 - a_2 - s a_4}, \frac{a_1 + a_2 + s a_4}{1 - a_3 - s a_4} \right\} < \frac{1}{s}$, we can conclude that:

$$d(p_m, p_{m+1}) \leq q^m r(p_0), \text{ for any } m \in \mathbb{N}. \tag{29}$$

In order to prove that the sequence $\{p_k\}$ is Cauchy, we will estimate $d(p_k, p_{k+p})$. For $p \in \mathbb{N}$, we have:

$$\begin{aligned} d(p_k, p_{k+p}) &\leq s \cdot \Big[d(p_k, p_{k+1}) + d(p_{k+1}, p_{k+p}) \Big] \\ &\leq s \cdot d(p_k, p_{k+1}) + s^2 d(p_{k+1}, p_{k+2}) + ... + s^p \cdot d(p_{k+p-1}, p_{k+p}) \\ &\leq s \cdot q^k r(x_0) + s^2 \cdot q^{k+1} r(x_0) + ... + s^p \cdot q^{k+p-1} r(x_0) \\ &= r(p_0) \textstyle\sum_{i=k}^{k+p-1} s^{i-k+1} \cdot q^i \\ &\leq r(x_0) \textstyle\sum_{i=k}^{\infty} (s \cdot q)^i \to 0 \end{aligned}$$

as $n \to \infty$. Therefore, $\{p_n\}$ is a Cauchy sequence. When the b-metric space $(\mathcal{M}, d, s)$ is complete, there is a point $t \in \mathcal{M}$ such that:

$$\lim_{n \to \infty} d(p_n, t) = 0. \tag{30}$$

We prove that $O^{n(t)} t = t = R^{m(t)} t$, meaning that t is a common fixed point of $O^{n(t)}$, respectively $R^{m(t)}$. If we take $p = p_{2k-1}$ and $y = u$ in (21), we get:

$$\begin{aligned} d(O^{n_k} p_{2k-1}, R^{m(t)} t) &\leq a_1 d(p_{2k-1}, t) + a_2 d(p_{2k-1}, O^{n_k} p_{2k-1}) + a_3 d(t, R^{m(t)} t) \\ &+ a_4 \left(d(p_{2k-1}, R^{m(t)} t) + d(t, O^{n_k} p_{2k-1}) \right) \\ &= a_1 d(p_{2k-1}, t) + a_2 d(p_{2k-1}, p_{2k}) + a_3 d(t, R^{m(t)} t) \\ &+ a_4 \left(d(p_{2k-1}, R^{m(t)} t) + d(t, p_{2k}) \right), \end{aligned} \tag{31}$$

and taking the limit $k \to \infty$ in the previous inequality, we obtain:

$$d(t, R^{m(t)} t) \leq \lim_{n \to \infty} d(p_{2k}, R^{m(t)} t) \leq (a_3 + a_4)(d(t, R^{m(t)} t) < d(t, R^{m(t)} t), \tag{32}$$

which implies that $d(t, R^{m(t)} t) = 0$. Hence, $R^{m(t)} t = t$. Supposing that $O^{n(t)} t \neq t$, from (21) and (32), we have:

$$0 < d(O^{n(t)} t, t) = d(O^{n(t)} t, R^{m(t)} t) \leq a_2 d(t, O^{n(t)} t) + a_4 d(t, O^{n(t)} t)) < d(t, O^{n(t)} t) \tag{33}$$

which is a contradiction, and hence, $O^{n(t)} t = t$.

Finally, we will demonstrate the uniqueness of the fixed point. For this, we presume that on the contrary, there exists another point $v \in \mathcal{M}$ such that $O^{n(v)} v = v = R^{m(v)} v$ and $t \neq v$. Therefore,

$$\begin{aligned} 0 < d(t, v) &= d(O^{n(t)} t, R^{m(v)} v)) \\ &\leq a_1 d(t, v) + a_2 d(t, O^{n(t)} t) + a_3 d(v, R^{m(v)} v) + a_4 d(v, O^{n(t)} t) + d(t, R^{m(v)} v) \\ &= a_1 d(t, v) < d(t, v). \end{aligned} \tag{34}$$

This is a contradiction, hence $t = v$. Since the fixed point is unique, we can conclude that t is a common fixed point for O and R. Indeed,

$$Ot = O(O^{n(t)} t) = O^{n(t)}(Ot) \tag{35}$$

shows that Ot is also a fixed point of $O^{n(t)}$. However, $O^{n(t)}$ has a unique fixed point t; hence, $Ot = t$. Similarly $Rt = t$. $\square$

Corollary 2. *Let O, R be two self-mappings on a complete b-metric space $(\mathcal{M}, d, s)$. Suppose that there exists a positive constant c, $0 < c < \frac{1}{s}$ and for all $p, q \in \mathcal{M}$ there exist positive integers $n(p), m(q)$ such that:*

$$d(O^{n(p)}p, R^{m(q)}q) \leq cd(p, q). \tag{36}$$

Then, the pair of mappings O, R has exactly one common fixed point t.

Proof. The proof follows from Theorem 3 by taking $a_2 = a_3 = a_4 = 0$ and $a_1 = c < 1/s$. □

Example 4. *Let $\mathcal{M} = \left\{ A(p) = \begin{pmatrix} 2p & 4p \\ p & 0 \end{pmatrix} : p \in \mathbb{R} \right\}$ and $d : \mathcal{M} \times \mathcal{M} \to \mathbb{R}_0^+$ be defined as $d(A(p), A(q)) = |tr(A(p) - A(q))|^2$. The triplet $(\mathcal{M}, d, s)$ forms a complete two-metric space. Let $O, R : \mathcal{M} \to \mathcal{M}$ be defined by $O(A(p)) = BA(p)$, $S(A(p)) = CA(p)$, where $B = \begin{pmatrix} 1/2 & 1/2 \\ 0 & 1/2 \end{pmatrix}$, respectively $C = \begin{pmatrix} 1/4 & 2/3 \\ 0 & 1/2 \end{pmatrix}$. By regular calculation, we have $O^3(A(p)) = B^3A(p) = \begin{pmatrix} 5p/8 & p/2 \\ p/8 & 0 \end{pmatrix}$ and $R^3(A(q)) = C^2A(q) = \begin{pmatrix} 5q/8 & q/4 \\ q/4 & 0 \end{pmatrix}$. For $c = \frac{1}{4} < \frac{1}{2}$, we have:*

$$d(O^3(A(p)), R^3(A(q))) = \frac{25}{64}|p - q|^2 < \frac{1}{4} \cdot 4|p - q|^2 = d(A(p), A(q))$$

so, for all $A(p), A(q) \in \mathcal{M}$ we can find $n(p) = 3$ and $m(q) = 2$ such that the assumptions of Corollary 2 are satisfied, which means that $A(0)$ is the unique common fixed point for O and R. We can remark that, in fact, choosing for example $a_1 = \frac{1}{64}$, the presumptions of Theorem 3 are satisfied for any $a_2, a_3, a_4 \geq 0$ such that $a_2 + a_3 + 4a_4 < \frac{31}{64} = \frac{1}{2} - \frac{1}{64}$. The system $BA(p) = CA(p) = A(p)$ has exactly one solution.

Letting $O = R$ and $m(q) = n(q)$ in Theorem 3, we obtain the next result:

Corollary 3. *Let O be a self-mapping on a complete b-metric space $(\mathcal{M}, d, s)$. Suppose that for all $p, q \in \mathcal{M}$, there exist positive integers $n(p), n(q)$ such that:*

$$d(O^{n(p)}p, O^{n(q)}q) \leq a_1 d(p, q) + a_2 d(p, O^{n(p)}p) + a_3 d(q, O^{n(q)}q) + a_4 \left(d(q, O^{n(p)}p) + d(p, O^{n(q)}q) \right), \tag{37}$$

where $a_i \geq 0$, $i \in \{1, 2, 3, 4\}$ and $a_1 + a_2 + a_3 + 2sa_4 < \frac{1}{s}$. Then, the mapping O has exactly one fixed point t.

Corollary 4. *Let O be a self-mapping on a complete b-metric space $(\mathcal{M}, d, s)$. Suppose that there exists a positive constant c, $0 < c < \frac{1}{s}$ such that for each $p, q \in \mathcal{M}$, there exist positive integers $n(p), n(q)$ such that:*

$$d(O^{n(p)}p, O^{n(q)}q) \leq cd(p, q). \tag{38}$$

Then, the mapping O has exactly one fixed point.

Proof. In Corollary 3, set $a_1 = c < \frac{1}{s}$ and $a_2 = a_3 = a_4 = 0$. □

3. Application to Nonlinear Fredholm Integral Equation

In this section, as an application, we use Corollary 12 to study the existence and uniqueness of the common solution of nonlinear Fredholm integral equations. Let $\mathcal{M} = \mathbb{C}([0,1], \mathbb{R}_0^+)$ be the space of all continuous real valued functions defined on [0,1], where: $\mathbb{R}_0^+ = [0, \infty)$. Define $d_\tau : \mathcal{M} \times \mathcal{M} \to \mathbb{R}_0^+$ by:

$$\begin{aligned} d_\tau(p,q) &= \sup_{t\in[0,1]} \{|p(t) - q(t)|^2 e^{-\tau t}\} \\ &= ||p-q||_\tau \end{aligned}$$

for all $p, q \in \mathbb{C}([0,1], \mathbb{R}_0^+)$.

For $p \in \mathbb{C}([0,1], \mathbb{R}_0^+)$, we define norm as $||p||_\tau = \sup_{t\in[0,1]} \{|p(t)|e^{-\tau t}\}$ where $\tau > 0$ is taken arbitrarily. Then, $(\mathcal{M}, d_\tau)$ becomes a complete b-metric space. Let us study the Fredholm integral equations as:

$$p(t) = \int_0^t \Upsilon_1(t, s, p(s)ds) + \varrho(t) \tag{39}$$

$$q(t) = \int_0^t \Upsilon_2(t, s, q(s)ds) + \varrho(t) \tag{40}$$

for all $s, t \in [0,1]$ and $n(p), n(q)$ are positive integers; where $\varrho : [0,1] \to \mathbb{R}_0^+$ and $\Upsilon_1, \Upsilon_2 : [0,1] \times [0,1] \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ are continuous functions.

Now, we shall state and prove the following theorem to ensure the existence and uniqueness of the common solution of nonlinear Fredholm integral Equations (39) and (40).

Theorem 4. *Let $(\mathcal{M}, d_\tau)$ be a complete b-metric space defined above. Further, we presume that the following conditions are fulfilled.*

1. *Define*

$$O^{n(p)} p(t) = \int_0^t \Upsilon_1(t, s, p(s))ds + \varrho(t), \tag{41}$$

$$O^{n(q)} q(t) = \int_0^t \Upsilon_2(t, s, q(s))ds + \varrho(t). \tag{42}$$

2. *Suppose there exists $\tau > 0$ and a non-negative constant z, where $0 < z < \frac{1}{s}$ such that:*

$$|\Upsilon_1(t, s, p(s)) - \Upsilon_2(t, s, q(s))|^2 \leq z\tau |p(s) - q(s)|^2.$$

Then, the Fredholm integral Equations (39) and (40) have exactly one common solution.

Proof. For any $p, q \in \mathbb{C}([0,1]), t \in [0,1]$ and $n(p), n(q)$ positive integers, we consider:

$$
\begin{aligned}
|O^{n(p)}p(t) - O^{n(q)}q(t)|^2 &= \left| \int_0^t \Upsilon_1(t,s,p(s))ds + \varrho(t) - \int_0^t \Upsilon_2(t,s,q(s))ds + \varrho(t) \right|^2 \\
&= \left| \int_0^t \Upsilon_1(t,s,p(s))ds - \int_0^t \Upsilon_2(t,s,q(s))ds \right|^2 \\
&\leq \int_0^t |\Upsilon_1(t,s,p(s)) - \Upsilon_2(t,s,q(s))|^2 \, ds \\
&\leq \int_0^t z\tau |p(s) - q(s)|^2 ds \\
&= \int_0^t z\tau e^{-s\tau}.e^{s\tau} |p(s) - q(s)|^2 ds \\
&= \int_0^t z\tau e^{s\tau} ||p - q|| ds \\
&= \int_0^t z\tau e^{s\tau} d_\tau(p,q) ds \\
&= z\tau d_\tau(p,q) \int_0^t e^{s\tau} ds \\
&= z\tau d_\tau(p,q) \left[\frac{e^{s\tau}}{\tau} \right]_0^t \\
&< z d_\tau(p,q) e^{t\tau},
\end{aligned} \tag{43}
$$

which bring us:

$$
\begin{aligned}
& e^{-t\tau} |O^{n(p)}p(t) - O^{n(q)}q(t)|^2 \leq z d_\tau(p,q) \\
& \Rightarrow ||O^{n(p)}p - O^{n(q)}q||^2 \leq z d_\tau(p,q) \\
& \Rightarrow d_\tau(O^{n(p)}p, O^{n(q)}q) \leq z d_\tau(p,q).
\end{aligned}
$$

Thus, all the conditions of Corollary 12 are satisfied. Hence, given the fact that the nonlinear Fredholm integral Equations (39) and (40) have a common solution, this yields the existence and uniqueness of the common solution of nonlinear Fredholm integral equations. □

4. Conclusions

In this paper, we have extended several existing results two-fold. Firstly, we proved our results in the most generalized setting, the b-metric space. Secondly, we considered distinct contraction conditions that are not commonly studied in the metric fixed point theory. It is obvious that the given results cover the existing results of Sehgal [1], Guseman [2], Ray and Rhoades [19], and so on. It is also clear that by taking $n(p) = 1$ in the setting of the b-metric space, we get some more corollaries. On the other hand, if $n(p) \geq 2$, the continuity of the mappings cannot be derived from the contraction conditions. Hence, our results can be considered as fixed point results in the frame of discontinuous functions [20]. Moreover, as an application, we used the obtained results to aggregate the existence and uniqueness of the solution for nonlinear Fredholm integral equations.

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