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Approximation Operator Based on Neighborhood Systems

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Abstract: In this paper, we propose a new covering-based set in which the lower and the upper approximation operations are defined by neighborhood systems. We systematically discuss this new type of covering-based set in two steps. First, we study the basic properties of this covering-based set, such as normality, contraction, and monotone properties. Second, we discuss the relationship between the new type of covering-based set and the other ten proposed sets.

Keywords: rough sets; covering approximation space; neighborhood system; approximation operation; partition

1. Introduction

Pawlak proposed the rough-set concept in 1982, and wrote many works on the subject [1,2]. It is a powerful mathematical tool for handing uncertainty. It has been widely used in many fields, such as medical diagnosis, process control, biology, economics, biochemistry, chemistry, psychology, environmental science, and conflict analysis. It can also be combined with fuzzy sets. Comparing with other methods, classical rough-set theory has its merits. For instance, it does not add information in the process of processing information data. Since then, many scholars have made many significant contributions to developing rough theory [3–17]. However, classical rough-set theory is based on partition or equivalence relation. The definition of the lower and upper approximations, which is based on these relationships, is limited. Partition or equivalence relation is hard to be satisfied since it has its limitations, and it was only used for dealing with complete information systems. In order to solve this issue, one approach was to extend equivalence relations to tolerance or general relations [18]. Another important approach was to relax the equivalence relation to a covering and receive covering-based rough sets [19–21]. In 1983, Zakowski first proposed the concept of covering rough sets, which generalized classical rough-set theory by using a covering relation instead of a partition or equivalence relation [22]. This generalization is very useful because it disposes of classical rough-set limitations. Subsequently, many scholars defined approximation operators that are based on coverings. These covering approximation operators play an important role in theoretical and practical fields [23–25]. The relationships among covering-based approximation operators have attracted intensive research. There are also many scholars investigating various coverings. How to obtain this useful information and deal with uncertain data has become a widely studied problem. In the process of solving the problem, many scholars proposed certain methods, for example, rough-set theory, fuzzy-set theory [26,27], statistical methods, and computing words [28].

In the following, Bonikowski et al. studied covering-based rough sets from the viewpoint of formal concepts. At this time, covering-based rough sets, as a more powerful tool, can be used to deal with problems that cannot be solved by classical rough-set theory, such as granularity problems in information systems. Up to now, about ten types of covering-based rough sets have been proposed and studied [8,10,29], and there are many scholarly studies on the subject. T. Yang et al. discuss covering reduction [30], while J. Zhan et al. investigated two types of covering-based multigranulation rough fuzzy sets, and certain types of soft covering-based rough sets [31–33]. L. Zhang et al. investigated the class of fuzzy soft β-covering-based fuzzy rough sets and their applications; they also did work on multicriteria fuzzy group decision making [34]. $D'eer$ et al. studied neighborhood operators for covering-based rough sets [35,36]. José et al. considered fuzzy techniques for decision making and formal relationships among soft sets, fuzzy sets, and their extensions [37,38]. Przemyslaw Grzegorzewski discussed the separability of fuzzy relations [39] and Alcantud, J.C used fuzzy techniques for Decision making [40].

In this paper, a new type of covering-based rough set is proposed. This paper is arranged as follows: In Section 2, properties such as normality, contraction, and monotone are studied. If a property does not hold, the necessary and sufficient conditions of a neighborhood system in which this property holds are researched. In Section 3, the condition that the type of covering-based rough set equals the other ten sets proposed by other scholars is evaluated and discussed.

2. Definition and Properties of Covering-Based Approximation Operators

Let U be a finite and nonempty set, called a universe; R be an equivalence relation on U, then the partition induced by R is denoted by $U/R = \{X_1, X_2, \cdots X_n\}$. For any $X \subseteq U$, two subsets of U are given as follows:

$$\underline{R}(X) = \cup\{X_i \in U/R : X_i \subseteq X\}$$
$$\overline{R}(X) = \cup\{X_i \in U/R : X_i \cap X \neq \emptyset\}$$

The first subset $\underline{R}(X)$ and the second $\overline{R}(X)$ are the lower and upper approximation of X, respectively.

Obviously, a partition of U is a covering of U, but a covering of U is not necessarily a partition of U, so the definition of covering approximation space has been introduced. Before defining the new type of covering-based rough set, it is necessary for us to give some basic definitions about covering approximation space.

Definition 1 (**Covering approximation space** [8])**.** *Let U be a universe, $\mathcal{C}$ a covering of U, then we call U with covering $\mathcal{C}$ a covering approximation space, denoted by $(U, \mathcal{C})$.*

Definition 2 (**Membership of a point** x [12])**.** *Let $(U, \mathcal{C}, N)$ be a covering approximation space. For a point $x \in U$, $FM(x) = \{K \in \mathcal{C} : x \in K\}$, called the membership of x.*

Definition 3 (**Minimal description of a point** x [12])**.** *Let $(U, \mathcal{C}, N)$ be a covering approximation space. The minimal description of a point x is defined as*

$$Md(x) = \{K \in \mathcal{C} : x \in K \in C \wedge (\forall S \in C \wedge x \in S \subseteq K \Rightarrow K = S)\}$$

Definition 4 (**Neighborhood of point** x [13])**.** *Let $(U, \mathcal{C})$ be a covering approximation space. For any $x \in U$, we call $N(x) = \cap\{K \in \mathcal{C} : x \in K\}$ the neighborhood of point x.*

Definition 5 (**Neighborhood system** [13])**.** *Let $(U, \mathcal{C})$ be a covering approximation space. We call $N = \{N(x) : x \in U\}$ the neighborhood system induced by $(U, \mathcal{C})$.*

Definition 6 (**Covering** [14])**.** *Let U be a universe, a set of nonempty subsets $\mathcal{C} = \{K_i \subseteq U : i \in I\}$ is called a covering of U if it satisfies $\cup\mathcal{C} = U$, and $K_i \neq \emptyset$ for each $i \in I$.*

From now on, the symbol $(U, \mathcal{C}, N)$ is used to represent covering approximation space $(U, \mathcal{C})$, and N is the neighborhood system induced by $(U, \mathcal{C})$.

Lemma 1. *[13] Let $(U, \mathcal{C}, N)$ be a covering approximation space. If $x, y \in U$, such that $x \in N(y)$, then $N(x) \subseteq N(y)$.*

Proposition 1. *Let $(U, \mathcal{C}, N)$ be a covering approximation space. N forms a partition of $U \Leftrightarrow$, there does not exist a pair $x, y \in U$, such that $x \in N(y)$ and $y \notin N(x)$.*

Proof. Necessity is simple, we only need to prove sufficiency. Suppose there does not exist a pair $x, y \in U$, such that $x \in N(y)$ and $y \notin N(x)$, but N is not a partition of U. We take two conditions into consideration: (1) $\exists x_0, y_0 \in U$, such that $x_0 \in N(y_0)$ and $y_0 \notin N(x_0)$. This is a contradiction to the assumption. (2) $\exists x_1, y_1 \in U$ such that $N(x_1) \cap N(y_1) \neq \emptyset$, $x_1 \notin N(y_1)$ and $y_1 \notin N(x_1)$. Select $z_1 \in N(x_1) \cap N(y_1)$, since $y_1 \notin N(x_1)$, by Lemma 1, we obtain a pair $y_1, z_1 \in U$, such that $z_1 \in N(y_1)$ and $y_1 \notin N(z_1)$. It is also a contradiction to the assumption. From Conditions (1) and (2), the proof of the sufficiency is completed. □

Definition 7. *Let $(U, \mathcal{C}, N)$ be a covering approximation space. For $X \subseteq U$, the covering-based lower approximation operation $\underline{N} : 2^U \rightarrow 2^U$ is defined as*

$$\underline{N}(X) = \cup\{N(x) : N(x) \subseteq X\}$$

Tthe covering-based upper approximation operation $\overline{N} : 2^U \rightarrow 2^U$ is defined as

$$\overline{N}(X) = \underline{N}(X) \cup \{x \in U : N(x) \cap (X - \underline{N}(X)) \neq \emptyset\}$$

Definition 8. *Let $(U, \mathcal{C}, N)$ be a covering approximation space. For $X \subseteq U$,*
(1) If $\underline{N}(X) = X$, then X is called an inner definable subset.
(2) If $\overline{N}(X) = X$, then X is called an outer definable subset.
(3) If $\overline{N}(X) = X = \underline{N}(X)$, then X is called a definable subset.

The following theorem describes what the essence of an inner definable, outer definable, and definable subset is.

Theorem 1. *Let $(U, \mathcal{C}, N)$ be a covering approximation space. For $X \subseteq U$,*
(1) X is an inner definable subset $\Leftrightarrow \exists A \subseteq U$ such that $X = \cup\{N(x) : x \in A\}$.
(2) X is an outer definable subset $\Leftrightarrow \forall x \notin X \Rightarrow (N(x) \cap X) \subseteq \underline{N}(X)$.
(3) X is a definable subset $\Leftrightarrow X$ is an inner definable subset.

Proof. The proof is simple. □

Remark 1. *X is a definable subset $\Leftrightarrow X$ is an inner definable subset $\Rightarrow X$ is an outer definable subset, but X is an outer definable subset $\nRightarrow X$ is an inner definable subset.*

Example 1. *If let $U = \{1, 2, 3, 4, 5\}$, $C = \{\{1, 2\}, \{3, 4\}, \{4\}, \{5\}\}$, $X_0 = \{1, 2, 3\}$, then $\underline{N}(X_0) = \{1, 2\} \neq X_0$, but $\overline{N}(X_0) = \{1, 2, 3\} = X_0$.*

Proposition 2. *Let $(U, \mathcal{C}, N)$ be a covering approximation space. $\forall X, Y \subseteq U$, we have:*

$$(1)\ \underline{N}(U) = U(Conormality)$$
$$(2)\ \overline{N}(U) = U(Conormality)$$
$$(3)\ \underline{N}(\emptyset) = \emptyset(Normality)$$
$$(4)\ \overline{N}(\emptyset) = \emptyset(Normality)$$
$$(5)\ \underline{N}(X) \subseteq X \subseteq \overline{N}(X)(Contraction - Extension)$$
$$(6)\ \underline{N}(X \cap Y) = \underline{N}(X) \cap \underline{N}(Y)(Multiplication)$$
$$(7)\ X \subseteq Y \Rightarrow \underline{N}(X) \subseteq \underline{N}(Y)(Monotone)$$
$$(8)\ \underline{N}(\underline{N}(X)) = \underline{N}(X)(Idempotency)$$
$$(9)\ \overline{N}(\overline{N}(X)) = \overline{N}(X)(Idempotency)$$
$$(10)\ \underline{N}(X) \cup \underline{N}(Y) \subseteq \underline{N}(X \cup Y)$$
$$(11)\ \overline{N}(X \cup Y) \subseteq \overline{N}(X) \cup \overline{N}(Y).$$

Proof. The proofs of (1)–(7), (10), and (11) are obvious. We only prove (8) and (9).

Firstly, we prove (8). From Proposition 2, Property (5), $\underline{N}(\underline{N}(X)) \subseteq \underline{N}(X)$ holds. $\forall y \in \underline{N}(X)$, since $\underline{N}(X) = \cup\{N(x) : N(x) \subseteq X\}$, so $N(y) \subseteq \underline{N}(X)$. By the definition of $\underline{N}(\underline{N}(X))$, we have $y \in \underline{N}(\underline{N}(X))$. This means $\underline{N}(X) \subseteq \underline{N}(\underline{N}(X))$, combining $\underline{N}(\underline{N}(X)) \subseteq \underline{N}(X)$, the proof of Property (8) is completed.

Secondly, we prove Property (9). From Proposition 2 Property (5), $\overline{N}(X) \subseteq \overline{N}(\overline{N}(X))$ holds. $\forall x \in \overline{N}(\overline{N}(X))$, we take two conditions into consideration: (a) $x \in \underline{N}(\overline{N}(X))$, we have $x \in \overline{N}(X))$. (b) $N(x) \cap (\overline{N}(X) - \underline{N}(\overline{N}(X))) \neq \emptyset$, select $x_0 \in N(x) \cap (\overline{N}(X) - \underline{N}(\overline{N}(X)))$. Since $x_0 \in \overline{N}(X) - \underline{N}(\overline{N}(X))$, so $x_0 \notin \underline{N}(X)$ and $N(x_0) \cap (X - \underline{N}(X)) \neq \emptyset$. On the other hand, from the condition that $x_0 \in N(x)$ and Lemma 1, we have $N(x) \cap (X - \underline{N}(X)) \neq \emptyset$. This means that $x \in \overline{N}(X)$. According to $(a)(b)$, the proof of (9) is completed. □

Generally speaking, suppose $(U, \mathcal{C}, N)$ is a covering approximation space. $(\natural) X \subseteq Y \subseteq U \nRightarrow \overline{N}(X) \subseteq \overline{N}(Y)$, $(\sharp)$ $\overline{N}(X \cup Y) = \overline{N}(X) \cup \overline{N}(Y)$ does not always hold.

Example 2. *Let* $U = \{1,2,3,4,5,6\}, \mathcal{C} = \{\{1\},\{2\},\{3\},\{1,2,3,4,5\},\{1,2,3,4,5,6\}\}, X_0 = \{1,2,3,4\}, Y_0 = \{1,2,3,4,5\}$, *then* $X_0 \subseteq Y_0, \overline{N}(X_0) = \{1,2,3,4,5,6\} \nsubseteq \overline{N}(Y_0) = \{1,2,3,4,5\}$ *and* $\overline{N}(X_0 \cup Y_0) = \{1,2,3,4,5\} \neq \overline{N}(X_0) \cup \overline{N}(Y_0) = \{1,2,3,4,5,6\}$.

Theorem 2. *Let $(U, \mathcal{C}, N)$ be a covering approximation space.* $\forall X \forall Y[(X \subseteq Y \subseteq U) \Rightarrow (\overline{N}(X) \subseteq \overline{N}(Y))] \Leftrightarrow \forall X \forall Y(\overline{N}(X \cup Y) = \overline{N}(X) \cup \overline{N}(Y))$

Proof. "$\Rightarrow$". $\forall X, Y \subseteq U$, since $X, Y \subseteq (X \cup Y)$, so $\overline{N}(X) \cup \overline{N}(Y) \subseteq \overline{N}(X \cup Y)$. By Proposition 2 (11), we have $\overline{N}(X \cup Y) = \overline{N}(X) \cup \overline{N}(Y)$.

"$\Leftarrow$". $\forall X \subseteq Y \subseteq U$, since $X \cup Y = Y$, so $\overline{N}(Y) = \overline{N}(X \cup Y) = \overline{N}(X) \cup \overline{N}(Y)$. This means that $\overline{N}(X) \subseteq \overline{N}(Y))$. □

Theorem 3. *Let $(U, \mathcal{C}, N)$ be a covering approximation space.* $\forall X \forall Y[(X \subseteq Y \subseteq U) \Rightarrow (\overline{N}(X) \subseteq \overline{N}(Y))]$ $\Leftrightarrow$ *There does not exist a pair* $x, y[(|N(x)| > 1) \wedge (|N(y)| > 1) \wedge (x \in N(y)) \wedge (y \notin N(x))]$.

Proof. "$\Rightarrow$", proof by contradiction. Suppose $\exists x_0 \exists y_0[(|N(x_0)| > 1) \wedge (|N(y_0)| > 1) \wedge (x_0 \in N(y_0)) \wedge (y_0 \notin N(x_0))]$. Select $z_0 \in N(x_0)$ and $x_0 \neq z_0$, let $X_0 = N(x_0) - \{z_0\}$ and $Y_0 = N(x_0)$. We can learn that $X_0 \subseteq Y_0, y_0 \in \overline{N}(X_0)$ and $y_0 \notin \overline{N}(Y_0)$. This means $\overline{N}(X_0) \nsubseteq \overline{N}(Y_0)$, contradicts the necessity assumption.

"$\Leftarrow$", proof by contradiction. Suppose $\exists X_0 \exists Y_0 \exists p_0[(X_0 \subseteq Y_0) \wedge (p_0 \in \overline{N}(X_0)) \wedge (p_0 \notin \overline{N}(Y_0))]$. Since $p_0 \notin \overline{N}(Y_0)$, $p_0 \notin Y_0$ and $p_0 \notin X_0$. From the fact that $p_0 \in \overline{N}(X_0)$, we have $N(p_0) \cap (X_0 - \underline{N}(X_0)) \neq \emptyset$. Select $q_0 \in N(p_0) \cap (X_0 - \underline{N}(X_0))$, take the conditions $p_0 \notin X_0$ and $p_0 \notin \overline{N}(Y_0)$ into consideration, and we have $p_0 \neq q_0, q_0 \in N(p_0), |N(q_0)| > 1$ and $p_0 \notin N(q_0)$. This means that $p_0, q_0[(|N(p_0)| > 1) \wedge (|N(q_0)| > 1) \wedge (q_0 \in N(p_0)) \wedge (p_0 \notin N(q_0))]$, contradicting the assumption of sufficiency. $\square$

Corollary 1. *By using Theorems 2 and 3, we obtain the fact that* $\forall X \forall Y(\overline{N}(X \cup Y) = \overline{N}(X) \cup \overline{N}(Y)) \Leftrightarrow$ *There does not exist a pair* $x, y[(|N(x)| > 1) \wedge (|N(y)| > 1) \wedge (x \in N(y)) \wedge (y \notin N(x))]$.

Proposition 3. *Let* $(U, \mathcal{C}, N)$ *be a covering approximation space. The properties below hold:*

$$(1)\underline{N}(U - \underline{N}(X)) \subseteq U - \underline{N}(X)$$
$$(2)\underline{N}(U - X) \subseteq U - \overline{N}(X)$$

Generally speaking, equality $\underline{N}(U - \underline{N}(X)) = U - \underline{N}(X)$ and $\underline{N}(U - X) = U - \overline{N}(X)$ does not always hold.

Example 3. *Let* $U = \{1,2,3,4\}, \mathcal{C} = \{\{1\},\{2\},\{3\},\{3,4\}\}, X_0 = \{1,2,3\}$. *We have* $\underline{N}(U - \underline{N}(X_0)) = \emptyset \neq U - \underline{N}(X_0) = \{4\}$

Example 4. *Let* $U = \{1,2,3,4,5\}, \mathcal{C} = \{\{1\},\{2\},\{3\},\{1,2,3,4\},\{5\}\}, X_0 = \{1,2,3\}$. *We have* $\underline{N}(U - X_0) = \{5\} \neq U - \overline{N}(X_0) = \{4,5\}$.

Theorem 4. *Let* $(U, \mathcal{C}, N)$ *be a covering approximation space.* $\forall X[\underline{N}(U - \underline{N}(X)) \subseteq (U - \underline{N}(X))] \Leftrightarrow$ *N forms a partition of U.*

Proof. "$\Rightarrow$", proof by contradiction. Suppose N does not form a partition of U. From Proposition 1, we can obtain $x_0, y_0 \in U$, such that $x_0 \in N(y_0)$ and $y_0 \notin N(x_0)$. If we choose $X_0 = N(x_0)$, then $y_0 \in U - \underline{N}(X_0)$ and $y_0 \notin \underline{N}(U - \underline{N}(X_0))$. This means $\underline{N}(U - \underline{N}(X_0)) \neq U - \underline{N}(X_0)$, contradicts the assumption of necessity.

"$\Leftarrow$" is simple. $\square$

Theorem 5. *Let* $(U, \mathcal{C}, N)$ *be a covering approximation space.*

(1) $\forall X[\underline{N}(U - \underline{N}(X)) = U - \underline{N}(X)] \Leftrightarrow$ *N forms a partition of U.*

(2) $\forall X[\underline{N}(U - X) = (U - \overline{N}(X))] \Leftrightarrow$ *N forms a partition of U.*

Proof. (1)"$\Rightarrow$", proof by contradiction. Suppose N does not form a partition of U. From Proposition 1, we can obtain $x_0, y_0 \in U$, such that $x_0 \in N(y_0)$ and $y_0 \notin N(x_0)$. If we choose $X_0 = N(x_0)$, then $y_0 \in U - \underline{N}(X_0)$ and $y_0 \notin \underline{N}(U - \underline{N}(X_0))$. This means $\underline{N}(U - X_0) \neq \underline{N}(U - \underline{N}(X_0))$, contradicting the assumption of necessity.

"$\Leftarrow$" is simple.

(2)"$\Rightarrow$", proof by contradiction. Suppose N does not form a partition of U. From Proposition 1, we can obtain $x_0, y_0 \in U$, such that $x_0 \in N(y_0)$ and $y_0 \notin N(x_0)$. If we choose $X_0 = N(x_0)$, then $y_0 \in U - \overline{N}(X_0)$ and $y_0 \notin \underline{N}(U - X_0)$. This means $\underline{N}(U - X_0) \neq (U - \overline{N}(X_0))$, contradicting the assumption of necessity.

"$\Leftarrow$" is simple. $\square$

3. Relationships between the New Lower and Upper Approximation Type Operations and Other Types

For a covering of U, there are about ten types of lower and upper approximation operations. A common question is what the relationship among them is. To answer this question, we need to outline the definitions of the ten types of lower and upper approximation operations.

Definition 9. *Let $(U, \mathcal{C}, N)$ be a covering approximation space. For each $n \in \{1,2,3,4,5,6,7,8,9,10\}$, $\underline{C}_n$ and $\overline{C}_n$ are called the n-th lower approximation operation and upper approximation operation, respectively, defined as follows:*

$$
\begin{aligned}
&(1) && \underline{C}_1(X) = \cup\{K : K \in \mathcal{C} \wedge K \subseteq X\}, \\
& && \overline{C}_1(X) = \underline{C}_1(X) \cup (\cup\{\cup Md(x) : x \in X - \underline{C}_1(X)\}). \\
&(2) && \underline{C}_2(X) = \cup\{K : K \in \mathcal{C} \wedge K \subseteq X\}, \\
& && \overline{C}_2(X) = \cup\{K : K \in \mathcal{C} \wedge K \cap X \neq \emptyset\}. \\
&(3) && \underline{C}_3(X) = \cup\{K : K \in \mathcal{C} \wedge K \subseteq X\}, \\
& && \overline{C}_3(X) = \cup\{\cup Md(x) : x \in X\}. \\
&(4) && \underline{C}_4(X) = \cup\{K : K \in \mathcal{C} \wedge K \subseteq X\}, \\
& && \overline{C}_4(X) = \underline{C}_4(X) \cup (\cup\{K : K \in \mathcal{C} \wedge K \cap (X - \underline{C}_4(X) \neq \emptyset\}). \\
&(5) && \underline{C}_5(X) = \cup\{K : K \in \mathcal{C} \wedge K \subseteq X\}, \\
& && \overline{C}_5(X) = \underline{C}_5(X) \cup (\cup\{N(x) : x \in X - \underline{C}_5(X)\}). \\
&(6) && \underline{C}_6(X) = \{x \in U : N(x) \subseteq X\}, \\
& && \overline{C}_6(X) = \{x \in U : N(x) \cap X \neq \emptyset\}. \\
&(7) && \underline{C}_7(X) = \{x \in U : \forall K \in \mathcal{C}(x \in K \Rightarrow K \subseteq X)\}, \\
& && \overline{C}_7(X) = \cup\{K : K \in \mathcal{C} \wedge K \cap X \neq \emptyset\}. \\
&(8) && \underline{C}_8(X) = \cup\{K : K \in \mathcal{C} \wedge K \subseteq X\}, \\
& && \overline{C}_8(X) = U - \underline{C}_8(U - X). \\
&(9) && \underline{C}_9(X) = \{x \in U : \forall u(x \in N(u) \Rightarrow N(u) \subseteq X)\}, \\
& && \overline{C}_9(X) = \cup\{N(x) : x \in U \wedge N(x) \cap X \neq \emptyset\}. \\
&(10) && \underline{C}_{10}(X) = \{x \in U : \forall u(x \in N(u) \Rightarrow u \in X)\}, \\
& && \overline{C}_{10}(X) = \cup\{N(x) : x \in X\}.
\end{aligned}
$$

Remark 2. *$\underline{C}_n$ and $\overline{C}_n (n = 1,2,3)$ can be found from Reference [19], $\underline{C}_4$ and $\overline{C}_4$ can be found from Reference [21], $\underline{C}_5$ and $\overline{C}_5$ can be found from Reference [18], $\underline{C}_6$ and $\overline{C}_6$ can be found from Reference [21], $\underline{C}_7$ and $\overline{C}_7$ can be found from Reference [25], and $\underline{C}_n$ and $\overline{C}_n(n = 8,9,10)$ can be found from Reference [10].*

Proposition 4. *Let $(U, \mathcal{C}, N)$ be a covering approximation space. The properties below hold, but all the "$\subseteq$" symbols cannot be replaced by the "$=$" symbol.*

$$
\begin{aligned}
&(1) && \forall X(\underline{C}_1(X) \subseteq \underline{N}(X)), \\
&(2) && \forall X(\overline{N}(X)) \subseteq \overline{C}_2(X)), \\
&(3) && \forall X(\overline{N}(X)) \subseteq \overline{C}_4(X)).
\end{aligned}
$$

Example 5. *Let $U = \{1,2,3,4,5\}, \mathcal{C} = \{\{1,2\},\{3,4\},\{5\},\{3,5\}\}, X_0 = \{1,2,3\}$. We have $\underline{N}(X_0) = \{1,2,3\} \neq \underline{C}_1(X_0) = \{1,2\}$.*

Example 6. *Let* $U = \{1,2,3,4,5\}, \mathcal{C} = \{\{1,2,3\},\{3,4,5\}\}, X_0 = \{3\}$. *We have* $\overline{N}(X_0) = \{3\} \neq \overline{C}_2(X_0) = \{1,2,3,4,5\} = \overline{C}_4(X_0)$.

Theorem 6. *Let* $(U, \mathcal{C}, N)$ *be a covering approximation space.*
(1) $\forall X(\underline{C}_1(X) = \underline{N}(X)) \Leftrightarrow \forall x(|Md(x)| = 1)$,
(2) $\forall X(\overline{C}_1(X) = \overline{N}(X)) \Leftrightarrow [\forall x(|Md(x)| = 1) \wedge N$ *forms a partition of* $U]$,
(3) $\forall X(\overline{C}_2(X) = \overline{N}(X)) \Leftrightarrow C$ *forms a partition of* U,
(4) $\forall X(\overline{C}_3(X) = \overline{N}(X)) \Leftrightarrow [\forall x(|Md(x)| = 1) \wedge N$ *forms a partition of* $U]$,
(5) $\forall X(\overline{C}_4(X) = \overline{N}(X)) \Leftrightarrow \{[\forall x(|Md(x)| = 1)] \wedge \forall K \in \mathcal{C}[\forall y \in K(\{y\} \in \mathcal{C}) \vee \forall z \in K(Md(z) = \{K\})]\}$.

Proof. (1)"$\Rightarrow$", proof by contradiction. Suppose $\exists x_0(|Md(x_0)| > 1)$, we find $K_1, K_2 \in Md(x_0)$, such that $x_0 \in K_1 \cap K_2, K_1 \cap K_2 \subsetneq K_1$, and $K_1 \cap K_2 \subsetneq K_2$. If we choose $X_0 = K_1 \cap K_2$, then $\underline{N}(X_0) = K_1 \cap K_2 \neq \underline{C}_1(X_0) = \emptyset$. This contradicts the assumption of necessity.

"$\Leftarrow$" is simple.

(2)"$\Rightarrow$", proof by contradiction. Firstly, we prove $\forall x(|Md(x)| = 1)$. Suppose $\exists x_0(|Md(x_0)| > 1)$, select $K_1, K_2 \in Md(x_0)$, such that $x_0 \in K_1 \cap K_2, K_1 \cap K_2 \subsetneq K_1$, and $K_1 \cap K_2 \subsetneq K_2$. Without loss of generality, if we choose $y_0 \in K_2$ and $y_0 \notin K_1$, then $y_0 \notin \overline{N}(K_1 \cap K_2)$. Since $K \in C$ does not exist, such that $x_0 \in K \subseteq K_1 \cap K_2$, $x_0 \in K_1 \cap K_2 - \underline{C}_1(K_1 \cap K_2)$ and $y_0 \in \overline{C}_1(K_1 \cap K_2)$. This means $\overline{N}(K_1 \cap K_2) \neq \overline{C}_1(K_1 \cap K_2)$, contradicting the assumption of necessity.

Secondly, we prove that N forms a partition of U. Suppose N is not a partition of U, by Proposition 1, $\exists x_1, y_1 \in U$ such that $x_1 \in N(y_1)$ and $y_1 \notin N(x_1)$. If we choose $X_0 = N(y_1) - N(x_1)$, then $x_1 \notin \overline{N}(X_0)$. Since $x_1 \in N(y_1)$, $K \in C$ does not exist, such that $y_1 \in K \subseteq X_0$. Thus, $y_1 \in X_0 - \underline{C}_1(X_0)$ and $x_1 \in \cup Md(y_1) \subseteq \overline{C}_1(X_0)$. This means $\overline{C}_1(X_0) \neq \overline{N}(X_0)$, contradicting the assumption of necessity.

"$\Leftarrow$", $\forall X \subseteq U$, by Theorem 6 (1) and $\forall x(|Md(x)| = 1)$, we have $\underline{C}_1(X) = \underline{N}(X)$ and $\forall y(N(y) = \cup Md(y))$. For $\forall z \in \overline{C}_1(X)$, we take two conditions into consideration, ($\sharp$) $z \in \underline{C}_1(X) \subseteq X$, and we have $z \in \overline{N}(X)$. ($\natural$) $\exists z_0 \in X - \underline{C}_1(X)$ such that $z \in \cup Md(z_0) = N(z_0)$. Since N is a partition of U, so $N(z) = N(z_0)$. This means $z_0 \in N(z) \cap (X - \underline{C}_1(X)) = N(z) \cap (X - \underline{N}(X) \neq \emptyset$. By the definition of $\overline{N}(X)$, we have $z \in \overline{N}(X)$. Coming ($\sharp$) with ($\natural$), $\overline{C}_1(X) \subseteq \overline{N}(X)$. On the other hand, for $\forall p \in \overline{N}(X)$, we also take two conditions into consideration, ($\sharp\sharp$)$p \in \underline{N}(X) \subseteq X$,and we have $p \in \overline{C}_1(X)$. ($\natural\natural$)$N(p) \cap (X - \underline{N}(X)) \neq \emptyset$, We can choose $p_0 \in N(p) \cap (X - \underline{N}(X)) = N(p) \cap (X - \underline{C}_1(X))$, consider that N is a partition of U, thus $p \in N(p_0) = \cup Md(p_0)$. By the definition of $\overline{C}_1(X)$, we have $p \in \overline{C}_1(X)$. Combining ($\sharp$)($\natural$) with ($\sharp\sharp$)($\natural\natural$), $\overline{N}(X) = \overline{C}_1(X)$ holds.

(3) the proof of (3) is simple.

(4) the proof of (4) is similar to (2).

(5) "$\Rightarrow$", proof by contradiction. Firstly, we prove $\forall x(|Md(x)| = 1)$. Suppose $\exists x_0(|Md(x_0)| > 1)$, select $K_1, K_2 \in Md(x_0)$, so that $x_0 \in K_1 \cap K_2, K_1 \cap K_2 \subsetneq K_1$ and $K_1 \cap K_2 \subsetneq K_2$. Without loss of generality, if we choose $y_0 \in K_2$ and $y_0 \notin K_1$, then $y_0 \notin \overline{N}(K_1 \cap K_2)$. Since $K \in C$ does not exist, then $x_0 \in K \subseteq K_1 \cap K_2$, so $x_0 \in K_1 \cap K_2 - \underline{C}_4(K_1 \cap K_2)$, and $y_0 \in \overline{C}_4(K_1 \cap K_2)$. This means $\overline{N}(K_1 \cap K_2) \neq \overline{C}_4(K_1 \cap K_2)$, contradicting the assumption of necessity.

Secondly, we prove $\forall K \in \mathcal{C}[\forall y \in K(\{y\} \in \mathcal{C}) \vee \forall z \in K(Md(z) = \{K\})]$. For $\forall K \in \mathcal{C}$, we take two conditions into consideration: ($\sharp$) $\exists p_0 \in K(\{p_0\} \in \mathcal{C}) \Rightarrow \forall y \in K(\{y\} \in \mathcal{C})$. Otherwise, $\exists q_0 \in K(Md(q_0) = \{K_3\} \wedge |K_3| > 1)$. If we select $q' \in K_3, q' \neq q_0$ and let $Y_0 = K_3 - \{q', p_0\}$, then $p_0 \notin \overline{N}(Y_0)$ and $p_0 \in \overline{C}_4(Y_0)$. This means $\overline{N}(Y_0) \neq \overline{C}_4(Y_0)$, contradicting the assumption of necessity. ($\natural$) $\forall m \in K(\{m\} \notin C) \Rightarrow \forall n \in K(Md(n) = \{K\})$. Otherwise, $\exists m_0 \in K$ such that $Md(m_0) = \{K_4\}, |K_4| > 1$ and $K_4 \subsetneq K$. By selecting $n_0 \in K - K_4, m' \in K_4, m' \neq m_0$ and let $Z_0 = \{n_0\}$, we obtain that $m_0 \notin \overline{N}(Z_0)$ and $m_0 \in \overline{C}_4(Z_0)$. This means $\overline{C}_4(Z_0) \neq \overline{N}(Z_0)$, contradicting the assumption of necessity.

"$\Leftarrow$" is simple. □

Proposition 5. *Let $(U, \mathcal{C}, N)$ be a covering approximation space. The properties below hold, but all the "$\subseteq$" symbols cannot be replaced by the "$=$" symbol.*

$$\begin{array}{ll} (1) & \forall X(\underline{C}_7(X) \subseteq \underline{N}(X)), \\ (2) & \forall X(\overline{N}(X)) \subseteq \overline{C}_6(X)). \end{array}$$

Example 7. *Let $U = \{1,2,3,4\}, \mathcal{C} = \{\{1,2,3\},\{1,2,4\}\}, X_0 = \{1,2\}$. We have $\underline{N}(X_0) = \{1,2\} \neq \underline{C}_7(X_0) = \emptyset$.*

Example 8. *Let $U = \{1,2,3,4\}, \mathcal{C} = \{\{1\},\{2\},\{3,4\},\{4,5\}\}, X_0 = \{4\}$. We have $\overline{N}(X_0) = \{4\} \neq \overline{C}_6(X_0) = \{3,4,5\}$.*

Theorem 7. *Let $(U, \mathcal{C}, N)$ be a covering approximation space.*
(1) $\forall X(\underline{C}_7(X) = \underline{N}(X)) \Leftrightarrow C$ forms a partition of U,
(2) $\forall X(\overline{C}_5(X) = \overline{N}(X)) \Leftrightarrow N$ forms a partition of U,
(3) $\forall X(\overline{C}_6(X) = \overline{N}(X)) \Leftrightarrow N$ forms a partition of U.

Proof. (1) the proof of (1) is simple.

(2) "$\Rightarrow$", proof by contradiction. Suppose N is not a partition of U, by Proposition 1, $\exists x_0 \exists y_0 (x_0 \in N(y_0) \wedge y_0 \notin N(x_0))$. If we let $X_0 = N(y_0) - N(x_0)$, then $x_0 \notin \overline{N}(X_0)$ and $x_0 \in \overline{C}_5(X_0)$. This means $\overline{C}_5(X_0) \neq \overline{N}(X_0)$, contradicting the assumption of necessity.

"$\Leftarrow$", $\forall X \subseteq U$. Firstly, we prove $\overline{N}(X) \subseteq \overline{C}_5(X)$. For $\forall x \in \overline{N}(X)$, we take two conditions into consideration: $(\sharp), x \in \underline{N}(X)$, and we have $x \in X \subseteq \overline{C}_5(X)$. $(\natural) N(x) \cap (X - \underline{N}(X)) \neq \emptyset$, take $x_0 \in N(x) \cap (X - \underline{N}(X))$, from Proposition 4 (1), and $x_0 \in X - \underline{N}(X) \subseteq X - \underline{C}_5(X)$ holds. By the assumption that N is a partition of U, we have $N(x_0) = N(x)$. According to the definition of $\overline{C}_5(X)$, we have $x \in \overline{C}_5(X)$, which means $\overline{N}(X) \subseteq \overline{C}_5(X)$. Secondly, we prove $\overline{C}_5(X) \subseteq \overline{N}(X)$. For $\forall y \in \overline{C}_5(X)$, we also take two conditions into consideration: $(\sharp\sharp)$ $y \in X$, and we have $y \in \overline{N}(X)$. $(\natural\natural)$ $y \in \overline{C}_5(X) - X$, $\exists y_0 \in X - \underline{C}_5(X)$ such that $y \in N(y_0)$. By the assumption that N is a partition of U, we have $N(y) = N(y_0)$. That is to say, $y_0 \in X - \underline{N}(X)$ and $y_0 \in N(y) \cap (X - \underline{N}(X)) \neq \emptyset$. By the definition of $\overline{N}(X)$, we have $y \in \overline{N}(X)$. This means $\overline{C}_5(X) \subseteq \overline{N}(X)$. Therefore, $\overline{C}_5(X) = \overline{N}(X)$ holds.

(3) "$\Rightarrow$", proof by contradiction. Suppose N is not a partition of U, by Proposition 1, $\exists x_0 \exists y_0 (x_0 \in N(y_0) \wedge y_0 \notin N(x_0))$. If we let $X_0 = N(x_0)$, then $y_0 \in \overline{C}_6(X_0)$ and $y_0 \notin \overline{N}(X_0)$, which means $\overline{C}_6(X_0) \neq \overline{N}(X_0)$, contradicting the assumption of necessity.

"$\Leftarrow$" is simple. □

Proposition 6. *Let $(U, \mathcal{C}, N)$ be a covering approximation space. The properties below hold, but all the "$\subseteq$" symbols cannot be replaced by the "$=$" symbol.*

$$\begin{array}{ll} (1) & \forall X(\underline{C}_9(X) \subseteq \underline{N}(X)), \\ (2) & \forall X(\overline{N}(X)) \subseteq \overline{C}_8(X)), \\ (3) & \forall X(\overline{N}(X)) \subseteq \overline{C}_9(X)). \end{array}$$

Example 9. *Let $U = \{1,2,3,4,5\}, \mathcal{C} = \{\{1,2,3\},\{1,2,4\},\{1,2,3,4,5\}\}, X_0 = \{1,2,3\}$. We have $\underline{N}(X_0) = \{1,2,3\} \neq \underline{C}_9(X_0) = \emptyset$.*

Example 10. *Let $U = \{1,2,3,4,5\}, \mathcal{C} = \{\{1,2\},\{1,2,3,4,5\}\}, X_0 = \{1,2\}$. We have $\overline{N}(X_0) = \{1,2\} \neq \overline{C}_8(X_0) = \{1,2,3,4,5\}$.*

Example 11. *Let $U = \{1,2,3,4\}, \mathcal{C} = \{\{1\},\{2\},\{3,4\},\{4\}\}, X_0 = \{3\}$. We have $\overline{N}(X_0) = \{3\} \neq \overline{C}_9(X_0) = \{3,4\}$.*

Theorem 8. *Let $(U, \mathcal{C}, N)$ be a covering approximation space.*
(1) $\forall X(\underline{C}_9(X) = \underline{N}(X)) \Leftrightarrow N$ forms a partition of U,
(2) $\forall X(\overline{C}_8(X) = \overline{N}(X)) \Leftrightarrow [\forall x(|Md(x)| = 1) \wedge N$ forms a partition of $U]$,
(3) $\forall X(\overline{C}_9(X) = \overline{N}(X)) \Leftrightarrow N$ forms a partition of U,
(4) $\forall X(\overline{C}_{10}(X) = \overline{N}(X)) \Leftrightarrow N$ forms a partition of U.

Proof. (1) "$\Rightarrow$", proof by contradiction. Suppose N is not a partition of U, by Proposition 1, $\exists x_0 \exists y_0 (x_0 \in N(y_0) \wedge y_0 \notin N(x_0))$. If we let $X_0 = N(x_0)$, then $x_0 \in \underline{N}(X_0)$ and $x_0 \notin \underline{C}_9(X_0)$. This means $\underline{N}(X_0) \neq \underline{C}_9(X_0)$, contradicting the assumption of necessity.

"$\Leftarrow$" is simple.

(2) "$\Rightarrow$", proof by contradiction. Firstly, we prove $\forall x(|Md(x)| = 1)$. Suppose $\exists x_0(|Md(x_0)| > 1)$, we can find $K_1, K_2 \in Md(x_0)$, such that $x_0 \in K_1 \cap K_2, K_1 \cap K_2 \subsetneq K_1$, and $K_1 \cap K_2 \subsetneq K_2$. By the assumption that $\forall X(\overline{C}_8(X) = \overline{N}(X))$, and the fact that $\overline{N}(K_1 \cap K_2) = K_1 \cap K_2, \exists L_1, L_2, \cdots L_n \in C$, such that $U - (K_1 \cap K_2) = L_1 \cup L_2 \cup \cdots \cup L_n$. Since $\overline{N}(L_1 \cup L_2 \cup \cdots \cup L_n) = L_1 \cup L_2 \cup \cdots \cup L_n = U - (K_1 \cap K_2)$, so $\exists L^1, L^2, \cdots L^m \in C$, such that $K_1 \cap K_2 = L^1 \cup L^2 \cup \cdots \cup L^m$. This means $\exists i_0 \in \{1, 2, \cdots, m\}$, such that $x_0 \in L^{i_0} \subseteq K_1 \cap K_2 \subsetneq K_2$, contradicting the fact that $K_2 \in Md(x_0)$. Secondly, we prove that N is a partition of U. Otherwise, by Proposition 1, $\exists y_0 \exists z_0 (y_0 \in N(z_0) \wedge z_0 \notin N(y_0))$. If we let $X_0 = N(y_0)$, then $z_0 \in \overline{C}_8(X_0)$ and $z_0 \notin \overline{N}(X_0)$. This means $\overline{C}_8(X_0) = \overline{N}(X_0)$, contradicting the assumption of necessity.

"$\Leftarrow$". $\forall X \subseteq U$, by Proposition 3.3(2), we only need to prove $\overline{C}_8(X) \subseteq \overline{N}(X)$. For $\forall x \in \overline{C}_8(X)$, we take two conditions into consideration, ($\sharp$) $x \in X$, and we have $x \in \overline{N}(X)$. ($\natural$) $x \in \overline{C}_8(X) - X$, since $\forall y(|Md(y)| = 1)$, so $\cup Md(x) = N(x)$ and $N(x) \cap X \neq \emptyset$. We can select $x_0 \in N(x) \cap X$, by the condition that N is a partition of U, and we have $N(x) = N(x_0)$ and $N(x) = N(x_0) \nsubseteq X$. This means $x_0 \notin \underline{N}(X)$ and $x_0 \in N(x) \cap (X - \underline{N}(X)) \neq \emptyset$. From the definition of $\overline{N}(X)$, we have $x \in \overline{N}(X)$. According to ($\sharp$)($\natural$), we finally have $\overline{C}_8(X) \subseteq \overline{N}(X)$.

(3) "$\Rightarrow$", proof by contradiction. Suppose N is not a partition of U, by Proposition 1, $\exists x_0 \exists y_0 (x_0 \in N(y_0) \wedge y_0 \notin N(x_0))$. If we let $X_0 = N(x_0)$, then $y_0 \in \overline{C}_9(X_0)$ and $y_0 \notin \overline{N}(X_0)$. This means $\overline{C}_9(X_0) \neq \overline{N}(X_0)$, contradicting the assumption of necessity.

"$\Leftarrow$" is simple.

(4) "$\Rightarrow$", proof by contradiction. Suppose N is not a partition of U, by Proposition 1, $\exists x_0 \exists y_0 (x_0 \in N(y_0) \wedge y_0 \notin N(x_0))$. If we let $X_0 = N(y_0) - N(x_0)$, then $x_0 \in \overline{C}_{10}(X_0)$ and $x_0 \notin \overline{N}(X_0)$. This means $\overline{C}_{10}(X_0) \neq \overline{N}(X_0)$, contradicting the assumption of necessity.

"$\Leftarrow$" is simple. □

In order to more clearly show the structures of $\underline{N}(X)$ and $\overline{N}(X)$, we introduce the conception of an Alexander topological space. Let (U, C, N) be a covering approximation space. As a topological base, N can induce a topology T on U. Topological space (U, T) is called an Alexander topological space.

For $\forall X \subseteq U$, let symbol $int(X)$ represent the interior of X, and $cl(X)$ represent the closure of X, then

$$\underline{N}(X) = int(X),$$
$$\overline{N}(X) = int(X) \cup cl(X - int(X))$$

As the end, we introduce definitions of n-th inner and outer accuracy to show the reason why we introduce this type of covering-based generalized rough set.

Definition 10. *Let $(U, \mathcal{C}, N)$ be a covering approximation space. For a subset X of U, denote $\rho_i(X) = \frac{|\underline{C}_i(X)|}{|X|}(i \in \{1, 2, \cdots, 10\})$, the n-th inner accuracy of X, and $\rho^i(X) = \frac{|\overline{C}^i(X)|}{|X|}(i \in \{1, 2, \cdots, 10\})$, the n-th outer accuracy of X, where symbol $|.|$ represents the cardinality of a set. For $i = 0$, denote $\rho_0(X) = \frac{|\underline{N}_(X)|}{|X|}$ and $\rho^0(X) = \frac{|\overline{N}_(X)|}{|X|}$.*

From Definition 10, we easily see that $\rho_i(X) \leqslant 1$ for each i and X, and $\rho^i(X) \geqslant 1$ for each i and X. For a fixed subset X of U, if $\rho_i(X) \geqslant \rho_j(X)$,we say that the i-th inner accuracy of X is higher than the j-th inner accuracy of X; similarly, if $\rho^i(X) \leqslant \rho^j(X)$, we say that the i-th outer accuracy of X is higher than the j-th outer accuracy of X.

Theorem 9. *Let $(U, \mathcal{C}, N)$ be a covering approximation space.*

$$
\begin{array}{ll}
(1) & \forall X(\rho_0(X) \geqslant \rho_7(X)), \\
(3) & \forall X(\rho_0(X) \geqslant \rho_9(X)), \\
(4) & \forall X(\rho^0(X) \leqslant \rho^2(X)), \\
(5) & \forall X(\rho^0(X) \leqslant \rho^4(X)), \\
(6) & \forall X(\rho^0(X) \leqslant \rho^6(X)), \\
(7) & \forall X(\rho^0(X) \leqslant \rho^8(X)), \\
(8) & \forall X(\rho^0(X) \leqslant \rho^9(X)).
\end{array}
$$

Proof. Straightforwardly by Propositions 4–6. □

Definition 10 and Theorem 9 indicate that the type of covering-based rough set possesses good inner and outer accuracy; this is the meaning we propose for this kind of covering-based rough set.

4. Conclusions

In this paper, we have presented a new type of covering-based generalized rough set, and proved some properties of $\underline{N}(X)$ and $\overline{N}(X)$. Here, we could not obtain the sufficient and necessary condition for $\forall X(\overline{N}(U - \overline{N}(X)) = U - \overline{N}(X))$. We mainly discussed the sufficient and necessary conditions for $\forall X(\underline{C}_i(X) = \underline{N}(X))$ and $\forall X(\overline{C}_i(X) = \overline{N}(X))(i = \{1, 2, \cdots, 10\})$. The most important sufficient and necessary condition is that N forms a partition of U. This article introduces two interesting questions: (1) Which conditions of C should be satisfied to infer that N is a partition of U, and (2) which conditions, $\underline{N}(X)$ or $\overline{N}(X)$, should be satisfied to infer that N is a partition of U? Solving Problems (1) and (2) will be our future work.

Author Contributions: The authors discuss the results. P.W. proofs the propositions and Q.W. proofs the theorems and J.H. provides counterexamples and X.S. writes the paper and improve the language.

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