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Modified Kudryashov Method to Solve Generalized Kuramoto-Sivashinsky Equation

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Abstract: The generalized Kuramoto–Sivashinsky equation is investigated using the modified Kudryashov method for the new exact solutions. The modified Kudryashov method converts the given nonlinear partial differential equation to algebraic equations, as a result of various steps, which upon solving the so-obtained equation systems yields the analytical solution. By this way, various exact solutions including complex structures are found, and their behavior is drawn in the 2D plane by Maple to compare the uniqueness and wave traveling of the solutions.

Keywords: generalized Kuramoto–Sivashinsky equation; modified Kudryashov method; exact solutions; Maple graphs

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1. Introduction

In engineering and science, the problems arising from the wave propagation of communication between two (or) more systems such as electromagnetic waves in wireless sensor networks, water flow in dams during an earthquake, stability of the output in electricity current, viscous flows in fluid dynamics, magneto hydro dynamics, turbulence in microtides and other physical phenomena are described by the non-linear evolution equations (NLEE). In modeling such aforesaid media continuously described by the generalized Kuramoto–Sivashinsky equation (GKSE) [1] given by the nonlinear partial differential equation for $u = u(x,t)$ and non-zero constants α, β and γ:

$$u_t + uu_x + \alpha u_{xx} + \beta u_{xxx} + \gamma u_{xxxx} = 0. \tag{1}$$

The GKSE and its solutions play huge roles in flowing in viscous fluids, feedback in the output of self-loop controllers, trajectory systems and gas dynamics. The process of solving NLEE analytically and numerically uses symbolic computation procedures such as exact solution techniques and cardinal function methods such as wavelet transforms, respectively. When $\alpha = \gamma = 1$ and $\beta = 0$, Equation (1) leads to the Kuramoto–Sivashinsky equation (KSE). N. A. Kudryashov solved Equation (1) by the method of Weiss–Tabor–Carnevale and obtained exact solutions in [1]. E. J. Parkes et al. applied the tanh method for Equation (1) by taking $\alpha = \beta = 1$ and solving using the Mathematica package; they also solved Equation (1) by taking $\alpha = -1$ and $\beta = 1$ in [2]. B. Abdel-Hamid in [3] assumed the initial solution as the PDE for u and solved exactly for $\alpha = 1$ and $\beta = 0$ in Equation (1). D. Baldwin et al. [4] applied the tanh and sech methods to Equation (1) with $\alpha = \gamma = 1$ and solved using the Mathematica package. C. Li et al. [5] solved Equation (1) of the form $u_t + \beta u^\alpha u_x + \gamma u^\tau u_{xx} + \delta u_{xxxx} = 0$ using the Bernoulli equation as the auxiliary differential equation. By the simplest equation method, again, N. A. Kudryashov solved Equation (1) by considering $u_x = u^m u_x$ and obtained the solution for general m with some restrictions in [6].

A. H. Khater et al. in [7] used Chebyshev polynomials and applied the collocation points to solve approximations of Equation (1). M. G. Porshokouhi et al. in [8] solved Equation (1) for different values of constants and approximately solved by the variational iteration method. In [9], C.M. Khalique reduced Equation (1) by Lie symmetry and solved exactly by the simplest equation method with Riccati and Bernoulli equations separately. D. Feng in [10] by taking $\beta = 0$ and $uu_x = \gamma uu_x$ in Equation (1) solved using the Riccati equation as the auxiliary differential equation. M. Lakestani et al. used the B-spline approximation function and solved Equation (1) numerically in [11], where they used tanh exact solutions for error estimations. J. Yang et al. in [12] used the sine-cosine method and dynamic bifurcation method to solve the more generalized GKSE and its related equations to Equation (1). In [13], J. Rashidinia et al. solved Equation (1) by Chebyshev wavelets. O. Acan et al. applied the reduced differential transform method to solve Equation (1) by taking $\beta = 0$ in [14].

For solving the nonlinear partial differential equations, there have been many schemes applied such as the Kudryashov method by M. Foroutan et al. in [15] and K. K. Ali et al. in [16]; the modified Kudryashov method by K. Hosseini et al. in [17,18], D. Kumar et al. in [19], A. K. Joardar et al. in [20] and A.R. Seadawy et al. in [21]; the generalized Kudryashov method by F. Mahmud et al. in [22], S. T. Demiray et al. in [23] and S. Bibi et al. in [24]; the sine-cosine method by K. R. Raslan et al. in [25]; the sine-Gordon method by H. Bulut et al. in [26]; the sinh-Gordon equation expansion method by H. M. Baskonus et al. in [27], Y. Xian-Lin et al. in [28] and A. Esen et al. in [29]; the extended trial equation method by K. A. Gepreel in [30],Y. Pandir et al. in [31] and Y. Gurefe et al. in [32]; the Exp-function method by L.K. Ravi et al. in [33], A. R. Seadawy et al. in [34] and M. Nur Alam et al. in [35]; the Jacobi elliptic function method by S. Liu et al. in [36]; the F-expansion method by A. Ebaid et al. in [37]; and the extended $\left(\frac{G'}{G}\right)$ method by E. M. E. Zayed and S. Al-Joudi et al. in [38].

The GKSE Equation (1) does not have the solution for general α and β; however, for the different values of α and β, the solution exists for (1), which can be found in [1–14]. In this work, we apply the modified Kudryashov method (MKM) to solve the GKSE in which we compute the constants α and β by the MKM. Then, for the each solution, a two-dimensional graph is drawn to show the wave traveling.

2. Analysis of the Modified Kudryashov Method

The modified Kudryashov method involves the following steps in solving the nonlinear partial differential equations (NLPDE) [17–21]:

Step 1. Consider the given NLPDE of the following form $u = u(x,t)$.

$$P\left(u, u_t, u_x, u_{tt}, u_{xx}, u_{xt}, \cdots\right) = 0. \tag{2}$$

Step 2. Apply the wave transformation $u(x,t) = u(\eta)$ in Equation (2), where:

$$\eta = \mu(x - \lambda t). \tag{3}$$

Here, μ is the wave variable and λ is the velocity; both are non-zero constants. Hence, Equation (2) transforms to the following ODE:

$$O\left(u, u', u'', uu', \cdots\right) = 0, \tag{4}$$

where the prime represents the derivative with respect to η.

Step 3. Let the initial solution guess of Equation (4) be,

$$u(\eta) = A_0 + \sum_{i=1}^{N} A_i \left[Q(\eta)\right]^i, \tag{5}$$

where N is a non-zero and positive constant calculated by the principle of homogeneous balancing of Equation (4), $A_i\ ;\ i = 0, 1, 2, \cdots$ are unknowns to be calculated and $Q(\eta)$ is the solution of the following auxiliary ODE:

$$\frac{dQ(\eta)}{d\eta} = Q(\eta)\left[Q(\eta) - 1\right]\ln(a)\ ;\ a \neq 1, \tag{6}$$

given by,

$$Q(\eta) = \frac{1}{1 + Da^{\eta}}, \tag{7}$$

where D is the integral constant and we assume $D = 1$.

Step 4. Substituting Equations (5) and (6) in Equation (4) leads to the polynomial in $Q(\eta)^i\ ;\ i = 0, 1, 2, \cdots$. As $Q(\eta)^i \neq 0$, so collecting its coefficients and then equating to zero give the systems of overdetermined algebraic equations, which upon solving give the unknowns of Equations (3) and (5).

Step 5. Finally, substituting the values of Step 4 in Equation (5) and then in Equation (3) gives the solution $u(x, t)$ of Equation (2).

3. MKM Application to Solve the Generalized Kuramoto–Sivashinsky Equation

Applying the wave transformation with Equation (3) to Equation (1) leads to the ODE, and then, integrating once the ODE by taking integration constant to zero transforms to the following ODE:

$$-\lambda u + \frac{u^2}{2} + \alpha\mu u^{(1)} + \beta\mu^2 u^{(2)} + \gamma\mu^3 u^{(3)} = 0, \tag{8}$$

where $u = u(\eta)$ and the superscripts (.) represent the derivatives w. r. t. η. By the homogeneous balancing of Equation (8), $N = 3$, and hence, the initial guess solution of Equation (8) from Equation (5) is given by,

$$u(\eta) = A_0 + A_1 Q(\eta) + A_2 \left(Q(\eta)\right)^2 + A_3 \left(Q(\eta)\right)^3. \tag{9}$$

Substituting Equations (6) and (9) in Equation (8) results in the sixth order polynomial of $Q(\eta)$. Collecting the coefficients of $\left(Q(\eta)\right)^i\ ;\ i = 0, 1, \cdots, 6$ and equating each coefficient to zero gives the systems of algebraic equations, which upon solving by Maple give the unknowns in Equations (9), (3) and (α, β) in Equation (8). The resulting values are substituted in Equation (9) along with Equations (3) and (7), which give the exact solution of Equation (1) for the specific values of constants α and β. Substituting the α and β values in Equation (1) and the unknowns $A_i\ ;\ i = 0, 1, 2, 3$ in Equation (9) where $Q(\eta)$ is given by Equation (7) yields the following exact solutions. Let $\delta_1 = \gamma\mu\ln(a)$, $\delta_2 = \gamma\mu^2\ln(a)^2$ and $\delta_3 = \gamma\mu^3\ln(a)^3$ in the following cases.

Case 1. For $\alpha = \delta_2$ and $\beta = 4\delta_1$ in Equation (1), the unknown coefficients are given by,

$$A_0 = A_1 = 0\ ,\ A_2 = 120\delta_3\ ,\ A_3 = -120\delta_3\ ,\ \lambda = 6\delta_3.$$

Therefore, the exact solution of Equation (1) is given by (Figure 1),

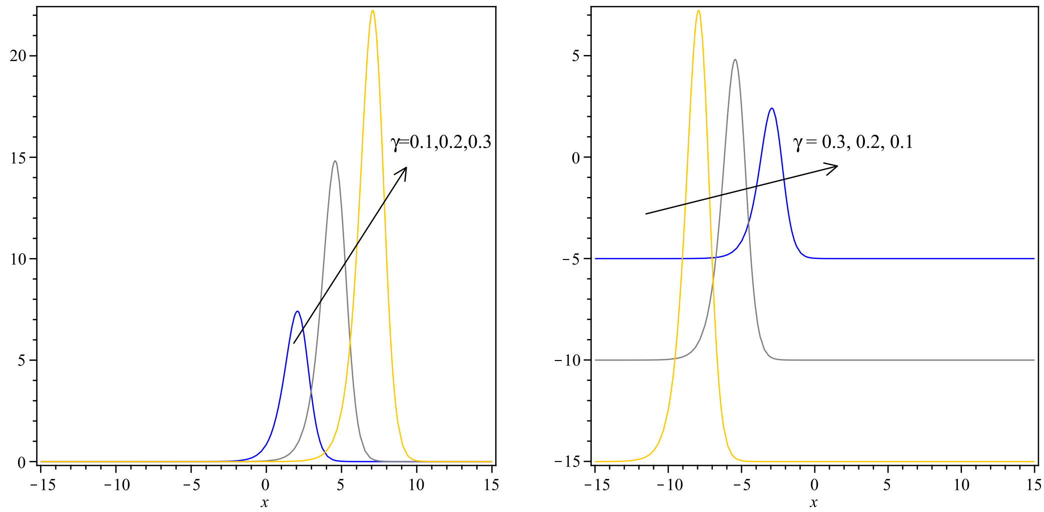
$$u_1(x,t) := \frac{120\delta_3 a^{\mu x - 6\delta_3 \mu t}}{\left(1 + a^{\mu x - 6\delta_3 \mu t}\right)^3}. \tag{10}$$

Further, for the same α and β value, the second set of unknown coefficients are given by,

$$A_0 = -12\delta_3\,,\ A_1 = 0\,,\ A_2 = 120\delta_3\,,\ A_3 = -120\delta_3\,,\ \lambda = -6\delta_3.$$

Therefore, the exact solution of Equation (1) is given by (Figure 1),

$$u_2(x,t) := -\frac{12\delta_3\left(1 + a^{3\mu x} e^{3(6\delta_3 \mu \ln(a) t)} + 3a^{2\mu x} e^{2(6\delta_3 \mu \ln(a) t)} - 7a^{\mu x} e^{6\delta_3 \mu \ln(a) t}\right)}{\left(1 + a^{\mu x} e^{6\delta_3 \mu \ln(a) t}\right)^3}. \tag{11}$$



(a) $u_1(x,t)$ **(b)** $u_2(x,t)$

Figure 1. Solutions in Case 1, Equations (10) and (11), respectively from left to right for $a = 5$, $\mu = 1$ and $t = 1$ in $x \in [-15, 15]$ for different values of γ.

Case 2. For $\alpha = \delta_2$ and $\beta = -4\delta_1$ in Equation (1), the unknown coefficients are given by,

$$A_0 = 0\,,\ A_1 = -120\delta_3\,,\ A_2 = 240\delta_3\,,\ A_3 = -120\delta_3\,,\ \lambda = -6\delta_3.$$

Therefore, the exact solution of Equation (1) is given by (Figure 2),

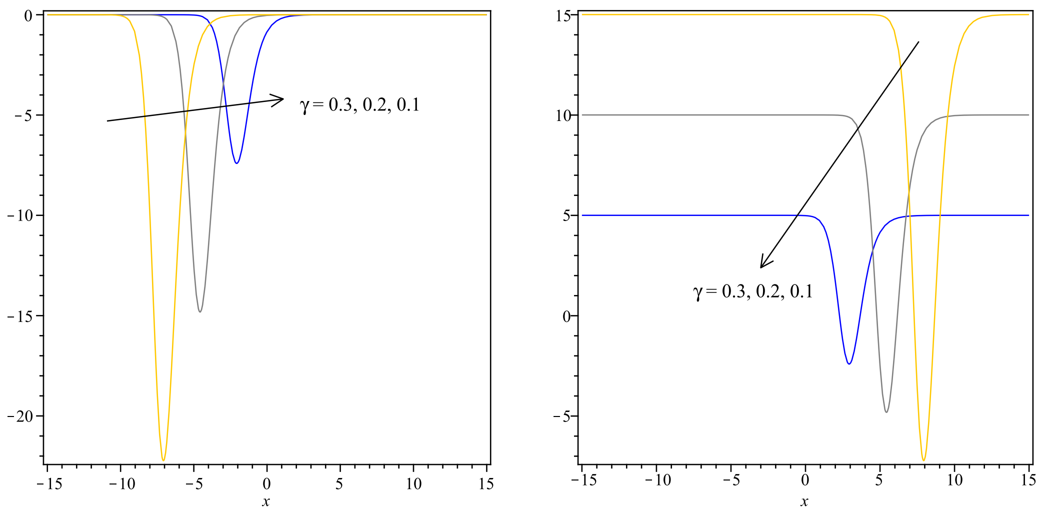
$$u_3(x,t) := -\frac{120\delta_3 a^{2(\mu x + 6\delta_3 \mu t)}}{\left(1 + a^{\mu x + 6\delta_3 \mu t}\right)^3}. \tag{12}$$

Further, for the same α and β value, the second set of unknown coefficients are given by,

$$A_0 = 12\delta_3\,,\ A_1 = -120\delta_3\,,\ A_2 = 240\delta_3\,,\ A_3 = -120\delta_3\,,\ \lambda = 6\delta_3.$$

Therefore, the exact solution of Equation (1) is given by (Figure 2),

$$u_4(x,t) := \frac{12\delta_3\left(a^{3(\mu x - 6\delta_3 \mu t)} - 7a^{2(\mu x - 6\delta_3 \mu t)} + 3a^{\mu x - 6\delta_3 \mu t} + 1\right)}{\left(1 + a^{\mu x - 6\delta_3 \mu t}\right)^3}. \tag{13}$$



(a) $u_3(x,t)$ **(b)** $u_4(x,t)$

Figure 2. Solutions in Case 2, Equations (12) and (13), respectively from left to right for $a = 5$, $\mu = 1$ and $t = 1$ in $x \in [-15, 15]$ for different values of γ.

Case 3. For $\alpha = -19\delta_2$ and $\beta = 0$ in Equation (1), the unknown coefficients are given by,

$$A_0 = -60\delta_3\,,\ A_1 = 0\,,\ A_2 = 180\delta_3\,,\ A_3 = -120\delta_3\,,\ \lambda = -30\delta_3.$$

Therefore, the exact solution of Equation (1) is given by (Figure 3),

$$u_5(x,t) := -\frac{60\delta_3 e^{2(30\delta_3\mu\ln(a)t)}\left(a^{3\mu x}e^{30\delta_3\mu\ln(a)t} + 3a^{2\mu x}\right)}{\left(1 + a^{\mu x}e^{30\delta_3\mu\ln(a)t}\right)^3}. \tag{14}$$

Further, for the same α and β value, the second set of unknown coefficients are given by,

$$A_0 = A_1 = 0\,,\ A_2 = 180\delta_3\,,\ A_3 = -120\delta_3\,,\ \lambda = 30\delta_3.$$

Therefore, the exact solution of Equation (1) is given by (Figure 3),

$$u_6(x,t) := \frac{60\delta_3\left(1 + 3a^{\mu x - 30\delta_3\mu t}\right)}{\left(1 + a^{\mu x - 30\delta_3\mu t}\right)^3}. \tag{15}$$

Case 4. For $\alpha = 47\delta_2$ and $\beta = 12\delta_1$ in Equation (1), the unknown coefficients are given by,

$$A_0 = A_1 = A_2 = 0\,,\ A_3 = -120\delta_3\,,\ \lambda = -60\delta_3.$$

Therefore, the exact solution of Equation (1) is given by (Figure 4),

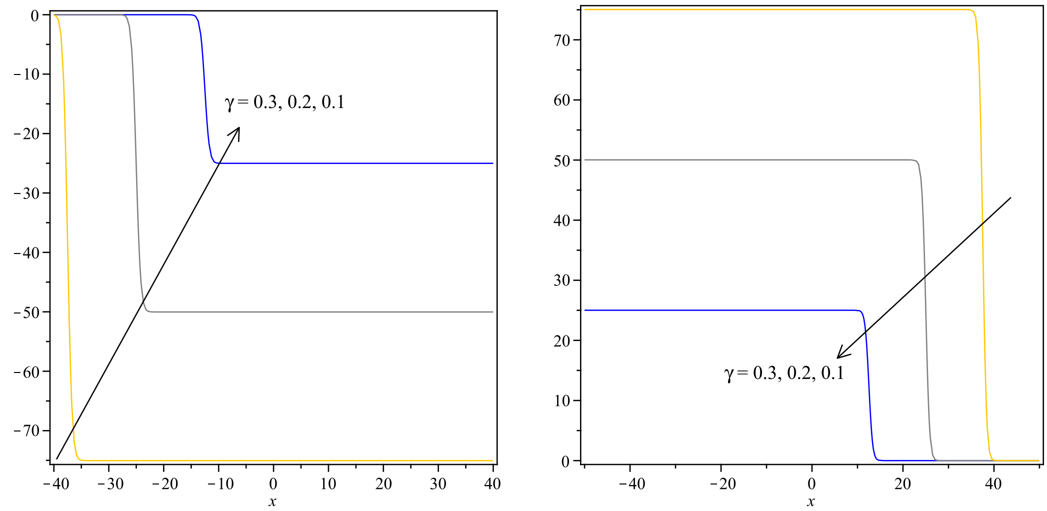
$$u_7(x,t) := -\frac{120\delta_3}{\left(1 + a^{\mu x + 60\delta_3\mu t}\right)^3}. \tag{16}$$

Further, for the same α and β, the second set of unknown coefficients are given by,

$$A_0 = 120\delta_3\,,\ A_1 = A_2 = 0\,,\ A_3 = -120\delta_3\,,\ \lambda = 60\delta_3.$$

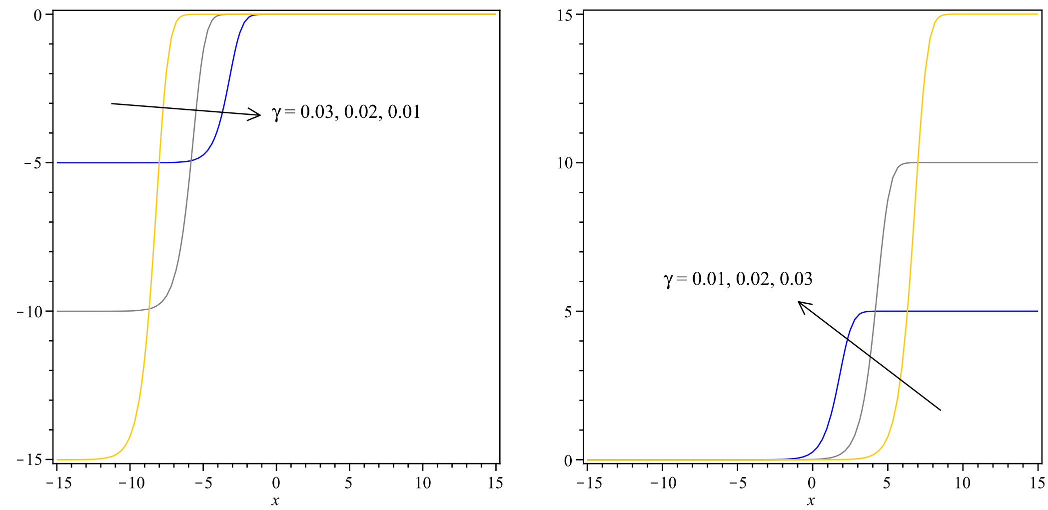
Therefore, the exact solution of Equation (1) is given by (Figure 4),

$$u_8(x,t) := \frac{120\delta_3\left(3a^{\mu x}e^{2(60\delta_3\mu\ln(a)t)} + 3a^{2\mu x}e^{60\delta_3\mu\ln(a)t} + a^{3\mu x}\right)}{\left(a^{\mu x} + e^{60\delta_3\mu\ln(a)t}\right)^3}. \tag{17}$$



(a) $u_5(x,t)$ **(b)** $u_6(x,t)$

Figure 3. Solutions in Case 3, Equations (14) and (15), respectively from left to right for $a = 5$, $\mu = 1$ and $t = 1$ in $x \in [-40, 40]$ for $u_5(x,t)$ and in $x \in [-50, 50]$ for $u_6(x,t)$ for different values of γ.



(a) $u_7(x,t)$ **(b)** $u_8(x,t)$

Figure 4. Solutions in Case 4, Equations (16) and (17), respectively from left to right for $a = 5$, $\mu = 1$ and $t = 1$ in $x \in [-15, 15]$ for different values of γ.

Case 5. For $\alpha = 47\delta_2$ and $\beta = -12\delta_1$ in Equation (1), the unknown coefficients are given by,

$$A_0 = 0\,,\ A_1 = -360\delta_3\,,\ A_2 = 360\delta_3\,,\ A_3 = -120\delta_3\,,\ \lambda = -60\delta_3.$$

Therefore, the exact solution of Equation (1) is given by (Figure 5),

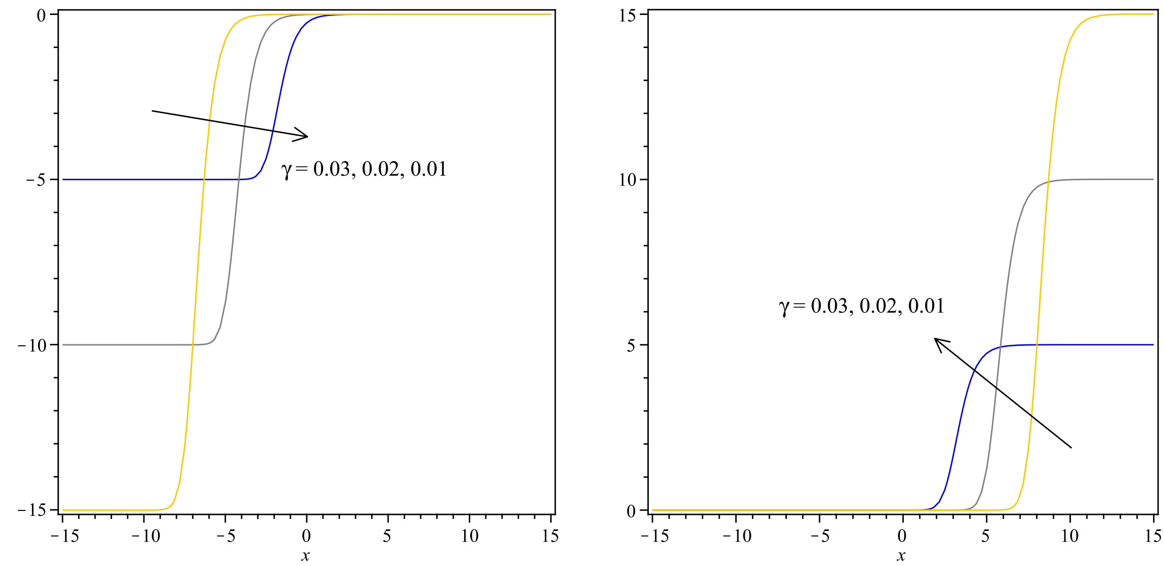
$$u_9(x,t) := -\frac{120\delta_3\left(3a^{2\mu x}e^{2(60\delta_3\mu\ln(a)t)} + 3a^{\mu x}e^{60\delta_3\mu\ln(a)t} + 1\right)}{\left(1 + a^{\mu x}e^{60\delta_3\mu\ln(a)t}\right)^3}. \tag{18}$$

Further, for the same α and β value, the second set of unknown coefficients are given by,

$$A_0 = 120\delta_3\,,\ A_1 = -360\delta_3\,,\ A_2 = 360\delta_3\,,\ A_3 = -120\delta_3\,,\ \lambda = 60\delta_3.$$

Therefore, the exact solution of Equation (1) is given by (Figure 5),

$$u_{10}(x,t) := \frac{120\delta_3 a^{3(\mu x - 60\delta_3\mu t)}}{\left(1 + a^{\mu x - 60\delta_3\mu t}\right)^3}. \tag{19}$$



(a) $u_9(x,t)$ **(b)** $u_{10}(x,t)$

Figure 5. Solutions in Case 5, Equations (18) and (19), respectively from left to right for $a = 5$, $\mu = 1$ and $t = 1$ in $x \in [-15, 15]$ for different values of γ.

Case 6. For $\alpha = 73\delta_2$ and $\beta = 16\delta_1$ in Equation (1), the unknown coefficients are given by,

$$A_0 = 180\delta_3\,,\ A_1 = 0\,,\ A_2 = -60\delta_3\,,\ A_3 = -120\delta_3\,,\ \lambda = 90\delta_3.$$

Therefore, the exact solution of Equation (1) is given by (Figure 6),

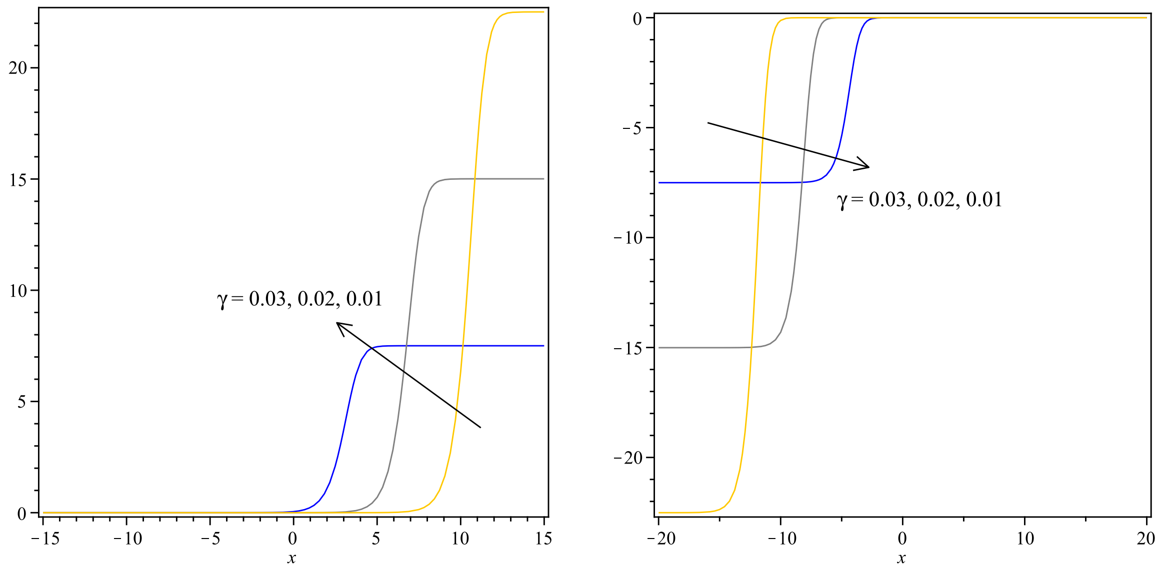
$$u_{11}(x,t) := \frac{60\delta_3\left(8a^{\mu x}e^{2(90\delta_3\mu\ln(a)t)} + 9a^{2\mu x}e^{90\delta_3\mu\ln(a)t} + 3a^{3\mu x}\right)}{\left(e^{90\delta_3\mu\ln(a)t} + a^{\mu x}\right)^3}. \tag{20}$$

Further, for the same α and β value, the second set of unknown coefficients are given by,

$$A_0 = A_1 = 0\,,\ A_2 = -60\delta_3\,,\ A_3 = -120\delta_3\,,\ \lambda = -90\delta_3.$$

Therefore, the exact solution of Equation (1) is given by (Figure 6),

$$u_{12}(x,t) := -\frac{60\delta_3\left(3 + a^{\mu x + 90\delta_3 \mu t}\right)}{\left(1 + a^{\mu x + 90\delta_3 \mu t}\right)^3}. \tag{21}$$



(a) $u_{11}(x,t)$ **(b)** $u_{12}(x,t)$

Figure 6. Solutions in Case 6, Equations (20) and (21), respectively from left to right for $a = 5$, $\mu = 1$ and $t = 1$ in $x \in [-15, 15]$ for $u_{11}(x,t)$ and $x \in [-20, 20]$ for $u_{12}(x,t)$ for different values of γ.

Case 7. For $\alpha = 73\delta_2$ and $\beta = -16\delta_1$ in Equation (1), the unknown coefficients are given by,

$$A_0 = 180\delta_3\,,\ A_1 = -480\delta_3\,,\ A_2 = 420\delta_3\,,\ A_3 = -120\delta_3\,,\ \lambda = 90\delta_3.$$

Therefore, the exact solution of Equation (1) is given by (Figure 7),

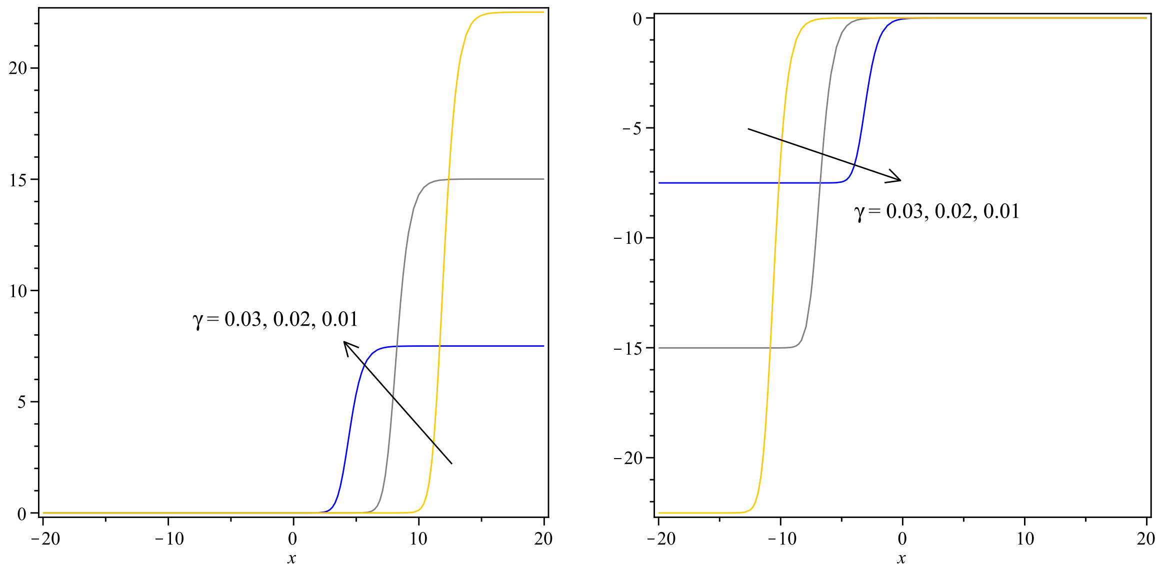
$$u_{13}(x,t) := \frac{60\delta_3\left(a^{2\mu x} e^{90\delta_3 \mu \ln(a) t} + 3a^{3\mu x}\right)}{\left(e^{90\delta_3 \mu \ln(a) t} + a^{\mu x}\right)^3}. \tag{22}$$

Further, for the same α and β value, the second set of unknown coefficients are given by,

$$A_0 = 0\,,\ A_1 = -480\delta_3\,,\ A_2 = 420\delta_3\,,\ A_3 = -120\delta_3\,,\ \lambda = -90\delta_3.$$

Therefore, the exact solution of Equation (1) is given by (Figure 7),

$$u_{14}(x,t) := -\frac{60\delta_3\left(8a^{2\mu x} e^{2(90\delta_3 \mu \ln(a) t)} + 9a^{\mu x} e^{90\delta_3 \mu \ln(a) t} + 3\right)}{\left(1 + a^{\mu x} e^{90\delta_3 \mu \ln(a) t}\right)^3}. \tag{23}$$



(a) $u_{13}(x,t)$ **(b)** $u_{14}(x,t)$

Figure 7. Solutions in Case 7, Equations (22) and (23), respectively from left to right for $a = 5$, $\mu = 1$ and $t = 1$ in $x \in [-20, 20]$ for different values of γ.

Case 8. For $\alpha = \frac{19}{11}\delta_2$ and $\beta = 0$ in Equation (1), the unknown coefficients are given by,

$$A_0 = \frac{60}{11}\delta_3\,,\ A_1 = -\frac{720}{11}\delta_3\,,\ A_2 = 180\delta_3\,,\ A_3 = -120\delta_3\,,\ \lambda = \frac{30}{11}\delta_3.$$

Therefore, the exact solution of Equation (1) is given by (Figure 8),

$$u_{15}(x,t) := \frac{60\delta_3 a^{\left(\mu x - \frac{30}{11}\delta_3 \mu t\right)}\left(a^{2\left(\mu x - \frac{30}{11}\delta_3 \mu t\right)} - 9a^{\left(\mu x - \frac{30}{11}\delta_3 \mu t\right)} + 12\right)}{11\left(1 + a^{\left(\mu x - \frac{30}{11}\delta_3 \mu t\right)}\right)^3}. \tag{24}$$

Further, for the same α and β value, the second set of unknown coefficients are given by,

$$A_0 = 0\,,\ A_1 = -\frac{720}{11}\delta_3\,,\ A_2 = 180\delta_3\,,\ A_3 = -120\delta_3\,,\ \lambda = -\frac{30}{11}\delta_3.$$

Therefore, the exact solution of Equation (1) is given by (Figure 8),

$$u_{16}(x,t) := -\frac{60\delta_3\left(1 - 9a^{\left(\mu x + \frac{30}{11}\delta_3 \mu t\right)} + 12a^{2\left(\mu x + \frac{30}{11}\delta_3 \mu t\right)}\right)}{11\left(1 + a^{\left(\mu x + \frac{30}{11}\delta_3 \mu t\right)}\right)^3}. \tag{25}$$

Case 9. For $\alpha = -\delta_2$ and $\beta = 4i\delta_1$ in Equation (1), the unknown coefficients are given by,

$$A_0 = 0\,,\ A_1 = -60\mu^3 \ln(a)^3\left(\gamma - i\gamma\right)\,,\ A_2 = 60(3 - i)\delta_3\,,\ A_3 = -120\delta_3\,,\ \lambda = 4i\delta_3.$$

Therefore, the exact complex solution of Equation (1) is given by,

$$u_{17}(x,t) := \frac{60\delta_3 a^{\mu x - 4i\delta_3\mu t}\left(i + 1 + (i - 1)\, a^{\mu x - 4i\delta_3 \mu t}\right)}{\left(1 + a^{\mu x - 4i\delta_3\mu t}\right)^3}. \tag{26}$$

The 2D graph of real and imaginary parts of $u_{17}(x,t)$ are drawn in Figure 9.

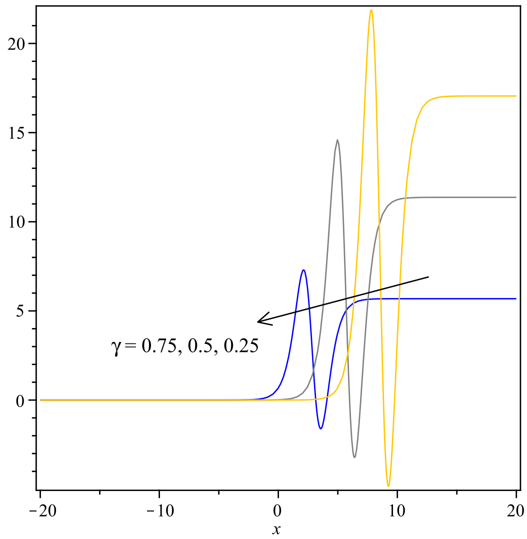
Further, for the same α and β value, the second set of unknown coefficients are given by,

$$A_0 = -8i\delta_3\,,\ A_1 = -60\mu^3\ln(a)^3\,(\gamma - i\gamma)\,,\ A_2 = 60(3-i)\delta_3\,,\ A_3 = -120\delta_3\,,\ \lambda = -4i\delta_3.$$

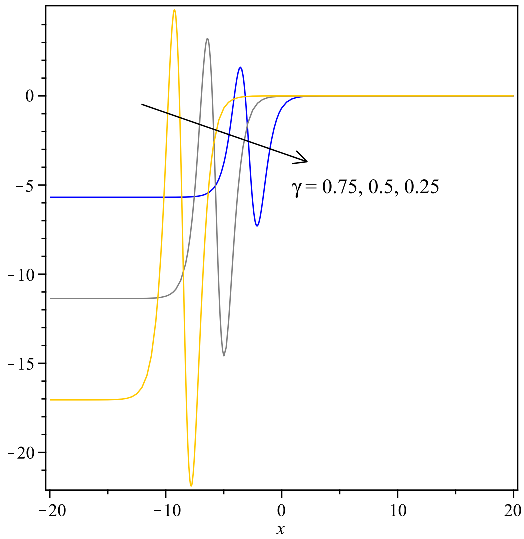
Therefore, the exact complex solution of Equation (1) is given by,

$$u_{18}(x,t) := -\frac{8\delta_3}{\left(1+a^{\mu x+4i\delta_3\mu t}\right)^3}\left[i(1+a^{3(\mu x+4i\delta_3\mu t)}) + \left(\frac{15-9i}{2}\right)a^{2(\mu x+4i\delta_3\mu t)} - \left(\frac{15+9i}{2}\right)a^{\mu x+4i\delta_3\mu t}\right]. \quad (27)$$

where $i = \sqrt{-1}$. The 2D graphs of the real and imaginary parts of $u_{18}(x,t)$ are drawn in Figure 10.

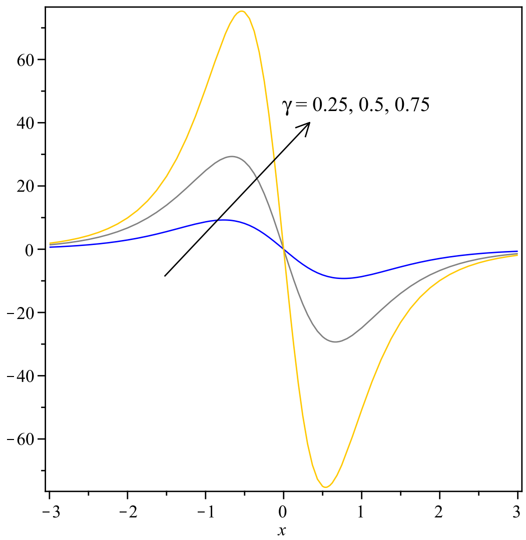


(a) $u_{15}(x,t)$

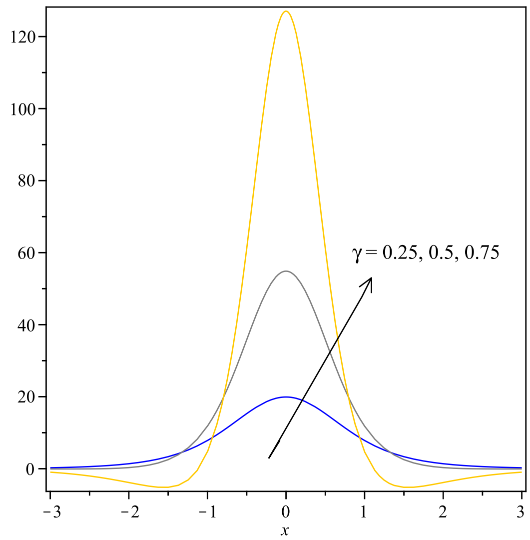


(b) $u_{16}(x,t)$

Figure 8. Solutions in Case 8, Equations (24) and (25), respectively from left to right for $a = 5$, $\mu = 1$ and $t = 1$ in $x \in [-20, 20]$ for different values of γ.

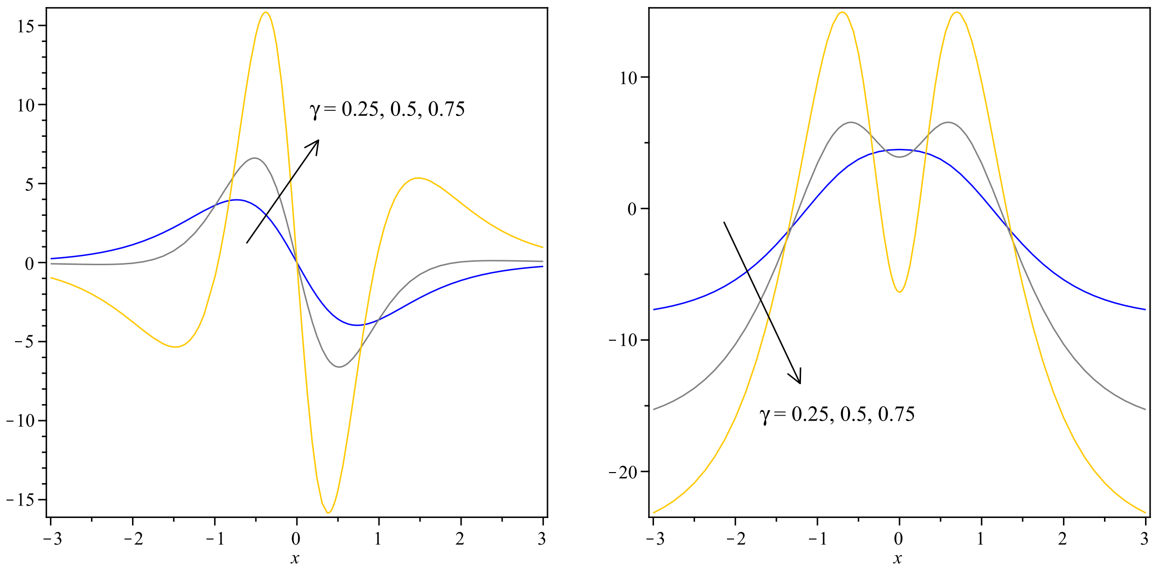


(a) Real part of $u_{17}(x,t)$



(b) Imaginary part of $u_{17}(x,t)$

Figure 9. Real and imaginary part of the solution in Case 9, Equation (26), respectively from left to right for $a = 5$, $\mu = 1$ and $t = 1$ in $x \in [-3, 3]$ for different values of γ.



(**a**) Real part of $u_{18}(x,t)$ (**b**) Imaginary part of $u_{18}(x,t)$

Figure 10. Real and imaginary part of the solution in Case 9, Equation (27), respectively from left to right for $a = 5$, $\mu = 1$ and $t = 1$ in $x \in [-3, 3]$ for different values of γ.

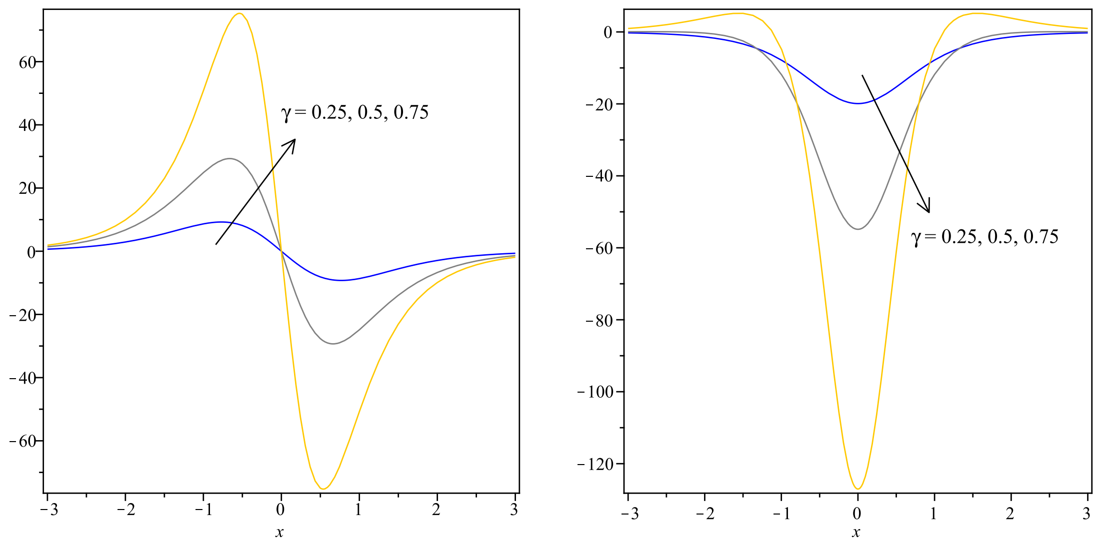
Case 10. For $\alpha = -\delta_2$ and $\beta = -4i\delta_1$ in Equation (1), the unknown coefficients are given by,

$$A_0 = 0\,,\ A_1 = -60\mu^3 \ln(a)^3(\gamma + i\gamma)\,,\ A_2 = 60(3+i)\delta_3\,,\ A_3 = -120\delta_3\,,\ \lambda = -4i\delta_3.$$

Therefore, the exact complex solution of Equation (1) is given by,

$$u_{19}(x,t) := -\frac{60\delta_3 a^{\mu x + 4i\delta_3\mu t}\left(i - 1 + (i+1)a^{\mu x + 4i\delta_3\mu t}\right)}{\left(1 + a^{\mu x + 4i\delta_3\mu t}\right)^3}. \tag{28}$$

The 2D graphs of real and imaginary parts of $u_{19}(x,t)$ are drawn in Figure 11.



(**a**) Real part of $u_{19}(x,t)$ (**b**) Imaginary part of $u_{19}(x,t)$

Figure 11. Real and imaginary part of the solution in Case 10, Equation (28), respectively from left to right for $a = 5$, $\mu = 1$ and $t = 1$ in $x \in [-3, 3]$ for different values of γ.

Further, for the same α and β value, the second set of unknown coefficients are given by,

$$A_0 = 8i\delta_3\,,\ A_1 = -60\mu^3\ln(a)^3(\gamma + i\gamma)\,,\ A_2 = 60(3+i)\delta_3\,,\ A_3 = -120\delta_3\,,\ \lambda = 4i\delta_3.$$

Therefore, the exact complex solution of Equation (1) is given by,

$$u_{20}(x,t) := \frac{8\delta_3}{(1+a^{\mu x - 4i\delta_3\mu t})^3}\left[i(1+a^{3(\mu x-4i\delta_3\mu t)}) - \left(\frac{15+9i}{2}\right)a^{2(\mu x-4i\delta_3\mu t)} + \left(\frac{15-9i}{2}\right)a^{\mu x-4i\delta_3\mu t}\right]. \tag{29}$$

where $i = \sqrt{-1}$. The 2D graphs of the real and imaginary parts of $u_{20}(x,t)$ are drawn in Figure 12.

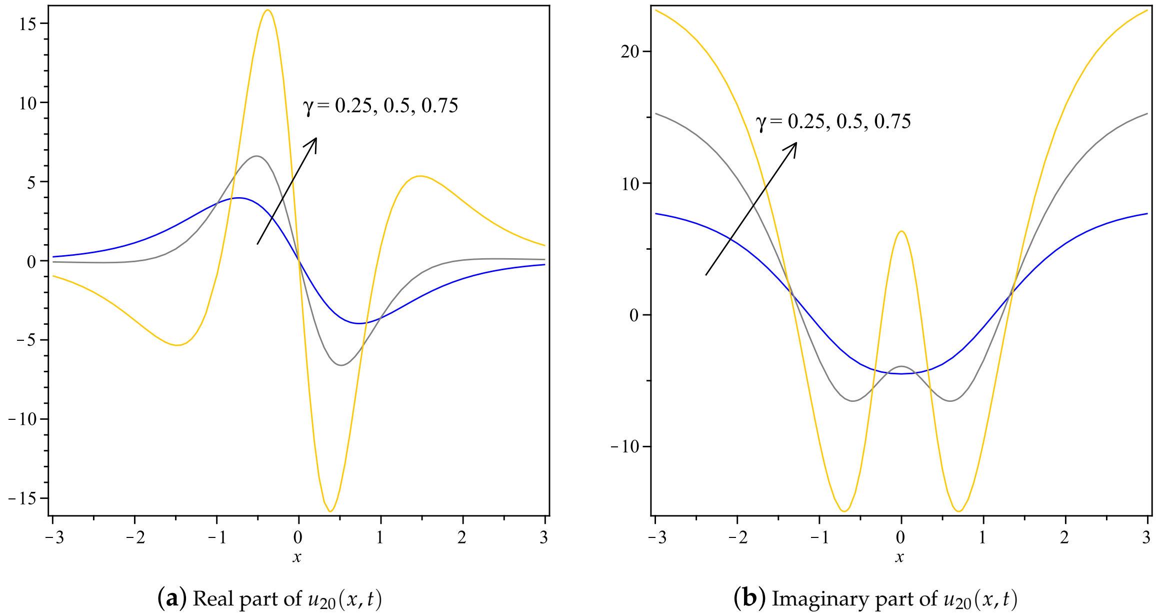


Figure 12. Real and imaginary part of the solution in Case 10, Equation (29), respectively from left to right for $a = 7$, $\mu = 1$ and $t = 1$ in $x \in [-3,3]$ for different values of γ.

4. Conclusions

In this work, the generalized Kuramoto–Sivashinsky equation is solved, and the exact solutions have been found. The aforesaid GKSE has solutions for the different values of α and β, which we obtained by the application of the modified Kudryashov method, and we found 10 classes of $(\alpha\,,\ \beta)$ pairs and their corresponding two distinct exact solutions for each pair of Equation (1) in Cases 1–10. The two-dimensional simulations of the solutions in Figures 1–12 show their behavioral pattern and wave train traveling for different values of γ. However, the wave structures vary when the values of a, μ, t and the domain changes in the 2D plane. The solutions found in this work will be useful in studying electromagnetic waves, fluid flows and the areas where GKSE plays a vital role. All the solutions are validated in the Maple computer algebra system by substituting them in the original equation. Our new solutions are compared with the previous solutions of GKSE in Appendices A and B.

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Appendix A. GKSE in the Previous Studies

N.A. Kudryashov in [6] solved for the exact solution of Equation (1). Based on the homogeneous balancing, he has taken the following initial solution.

$$u(\eta) = A_0 + A_1 g(\eta) + A_2 g(\eta)^2 + A_3 g(\eta)^3.$$

where $g(\eta)$ is the solution of $\frac{dg(\eta)}{d\eta} = b - g(\eta)^2$, and obtained the following values.

1. $$A_0 = -\frac{\beta^3}{576\gamma^2}\,,\ A_1 = \frac{5\beta^2}{4\gamma}\,,\ A_2 = -15\beta\,,\ A_3 = 120\gamma\,,\ \alpha = \frac{47\beta^2}{144\gamma}\,,\ b = \frac{\beta^2}{576\gamma^2}\,,\ C_0 = -\frac{5\beta^3}{144\gamma^2}.$$

2. $$A_0 = \frac{30\beta^3}{128\gamma^2}\,,\ A_1 = -\frac{30\beta^2}{16\gamma}\,,\ A_2 = -30\beta\,,\ A_3 = 120\gamma\,,\ \alpha = \frac{\beta^2}{16\gamma}\,,\ b = \frac{\beta^2}{64\gamma^2}\,,\ C_0 = \frac{3\beta^3}{32\gamma^2}.$$

In the same work, he solved Equation (1) with the auxiliary equations $\left(\frac{dg(z)}{dz}\right)^2 + 4g(z)^3 - ag(z)^2 - 2bg(z) + d = 0$ and $\frac{d^2g(z)}{dz^2} + 6g(z)^2 - ag(z) - b = 0$ and obtained other values for unknowns.

C.M. Khalique in [9] solved Equation (1) by taking the Bernoulli equation $\frac{dh(\eta)}{d\eta} = ah(\eta) + bh(\eta)^2$ and Riccati equation $\frac{dh(\eta)}{d\eta} = ah(\eta)^2 + bh(\eta) + c$ as the auxiliary ODE and obtained the following values respectively by using each ODE. For both the auxiliary equation the constant values are $a = 1$, $b = 3$ and $c = 1$:

1. $$A_0 = \nu - 6a^3\gamma\,,\ A_1 = -120a^2b\gamma\,,\ A_2 = 240ab^2\gamma\,,\ A_3 = -120b^3\gamma\,,\ \alpha = a^2\gamma\,,\ \beta = 4a\gamma.$$

2. $$A_0 = -990\gamma + 60\gamma k + \nu\,,\ A_1 = 60\gamma + 180\gamma k\,,\ A_2 = 60\gamma k\,,\ A_3 = -120\gamma\,,\ \alpha = 365\gamma\,,\ \beta = -36\gamma - 4\gamma k.$$

While comparing the above values, our solutions of Equation (1) in this work are new to the surveyed literature.

Appendix B. Studying GKSE by GKM and SGEEM

1. For solving Equation (1) by the generalized Kudryashov method [22–24], the homogeneous balancing of Equation (8) gives $N = M + 3$, which has infinite solutions. For the value $M = 1$, this gives $N = 4$. Therefore,

$$u(\eta) = \frac{A_0 + A_1 Q(\eta) + A_2 (Q(\eta))^2 + A_3 (Q(\eta))^3 + A_4 (Q(\eta))^4}{B_0 + B_1 Q(\eta)}.$$

   where $Q(\eta)$ is the solution of $\frac{dQ(\eta)}{d\eta} = Q(\eta)(Q(\eta) - 1)$, Applying these equations to Equation (8) leads to the polynomial in $Q(\eta)$ and its powers. Collecting the coefficients of $(Q(\eta))^i$; $i = 0, 1, 2, \cdots$ and attempting to solve the overdetermined equations results in the continuous execution of Maple. Hence, we conclude that Equation (1) cannot be solved by the generalized Kudryashov method.
2. Next, for solving Equation (1) by the sine-Gordon equation expansion method [26], the homogeneous balancing is the same as the MKM given by $N = 3$. Thus,

$$u(\eta) = A_0 + A_1 \tanh(\eta) + B_1 \text{sech}(\eta) + A_2 \tanh^2(\eta) + B_2 \tanh(\eta)\text{sech}(\eta) + A_3 \tanh^3(\eta) + B_3 \tanh^2(\eta)\text{sech}(\eta).$$

Substituting the above equation $u(\eta)$ in Equation (8) and following the steps in [26] lead to the polynomials in $\sin(w)$, $\cos(w)$, their products and powers. Collecting the coefficients, equating them to zero and solving in Maple result in the continuous execution. Thus, we conclude that Equation (1) cannot be solved by the sine-Gordon equation expansion method either.

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