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A Class of Nonlinear Boundary Value Problems for an Arbitrary Fractional-Order Differential Equation with the Riemann-Stieltjes Functional Integral and Infinite-Point Boundary Conditions

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Abstract: In this paper, we investigate the existence of an absolute continuous solution to a class of first-order nonlinear differential equation with integral boundary conditions (BCs) or with infinite-point BCs. The Liouville-Caputo fractional derivative is involved in the nonlinear function. We first consider the existence of a solution for the first-order nonlinear differential equation with m-point nonlocal BCs. The existence of solutions of our problems is investigated by applying the properties of the Riemann sum for continuous functions. Several examples are given in order to illustrate our results.

Keywords: nonlinear boundary value problems; fractional-order differential equations; Riemann-Stieltjes functional integral; Liouville-Caputo fractional derivative; infinite-point boundary conditions; advanced and deviated arguments; existence of at least one solution

MSC: primary 26A33, 34B18, 34K37; secondary 34A08, 34B10

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1. Introduction

Our objective in this article is to investigate the existence of absolute continuous solutions of the nonlocal first-order boundary value problem (BVP) with the nonlinear function involving the Liouville-Caputo fractional derivative:

$$\frac{dx}{dt} = f\big(t, D^{\alpha}x(t)\big) \quad \text{a.e.} \qquad (0 < t < 1;\ 0 < \alpha \leqq 1), \tag{1}$$

together with either the Riemann-Stieltjes functional integral boundary condition (with the advanced or deviated argument ϕ) given by

$$\int_0^1 x\big(\phi(s)\big)\, dg(s) = x_0 \qquad \big(g : [0,1] \to [0,1];\ g(s) \geqq 0\big) \tag{2}$$

or the infinite-point boundary conditions given by

$$\sum_{k=1}^{\infty} a_k\, x\big(\phi(\tau_k)\big) = x_0 \qquad \big(a_k > 0;\ \tau_k \in (0,1);\ \phi(\tau_k) \leqq \tau_k\big), \tag{3}$$

where $g : [0,1] \to [0,1]$ is an increasing function, $\alpha \in (0,1]$ and D^α denotes the Liouville-Caputo fractional derivative of order α. The integral in (2) is the Riemann-Stieltjes type with respect to $g(s)$. In the case when $g(s) = s$, the Riemann-Stieltjes integral in the boundary condition given by (2) reduces to the relatively more familiar Riemann integral.

In the case when $\alpha = 1$, the BVP (1) becomes the implicit differential problem given by

$$\frac{dx}{dt} = f\big(t, \frac{dx}{dt}\big) \quad \text{a.e.} \qquad (0 < t < 1)$$

under the Riemann-Stieltjes functional integral BC (2) or infinite-point BCs (3).

Our results in this article are based upon *Kolmogorov's Compactness Criterion* (see [1]) and upon Schauder's Fixed Point Theorem (see [2]).

Nonlinear BVPs with nonlocal multi-point BCs have received a lot of attention in recent years. In fact, various conditions are obtained for the existence of solutions by (for example) Alvan et al. [3], Benchohra et al. [4], Boucherif [5], El-Sayed and Bin-Taher [6], Gao and Han [7], Hamani et al. [8] and Nieto et al. [9] (see also the references to the related earlier works which are cited in each of these investigations).

BVPs with integral BCs arise naturally in semiconductor problems [10], thermal conduction problems [11], hydrodynamic problems [12], population dynamics model [13], and so on (see also [14]). Recently, these BVPs were extensively studied by (among others) Akcan and Çetin [15], Boucherif [16], Benchohra et al. [17], Chalishajar and Kumar [18], Dou et al. [19], Li and Zhang [20], Liu et al. [21], Song et al. [22], Tokmagambetov and Torebek [23], Wang et al. [24] and Yang and Qin [25] (see also the references to the related earlier works which are cited in each of these investigations).

The study of BVPs involving infinite-point BCs has become attractive recently. In the year 2011, Gao and Han [7] firstly studied the solutions to thefractional-order differential equation problem with infinite-point BCs. Ever since then, many significant and interesting cases of BVPs of fractional order were considered with infinite-point BCs by (for example) Ge et al. [26], Guo et al. [27], Hu and Zhang [28], Li et al. [29], Liu et al. [30], Zhang and Zhong [31] and Zhang [32] (see also to the references cited therein). In the year 2016, Xu and Yang [33] proposed a generalization of the PID controller and studied two kinds of fractional-order differential equations arising in control theory together with the infinite point boundary conditions. Their results can describe the corresponding control system accurately and also provide a platform for the understanding of our environment. However, investigations on the infinite-point BVPs for differential equations of fractional or integer order have gradually aroused people's attentions and interests, but such investigations are still not too many.

Motivated by the above-mentioned developments and results, we consider the BVP given by (1) and (2) or by (1) and (3). In each case, we determine sufficient conditions on f guaranteeing that the problem (1) under the Riemann-Stieltjes functional integral BC (2) or the problem (1) under infinite-point BC (3) has a solution. We first find the solutions of the problem (1) with the m-point BCs given by

$$\sum_{k=1}^{m} a_k\, x\big(\phi(\tau_k)\big) = x_0 \qquad \big(a_k \neq 0;\ 0 < \tau_k < 1\big), \tag{4}$$

and then, by using the properties of the Riemann sum for continuous functions, we investigate the solutions of the BVP given by (1) and (2) as well as the BVP given by Equations (1) and (3). The solutions of our problems in the Carathéodory sense are given under some weak conditions on f, which are sufficiently general and easy to check.

Our work has the following salient features. Firstly, a unified investigation involving both the Riemann-Stieltjes integral as well as infinite points is presented here in the BCs of the BVP (1). Secondly, to the best of our knowledge, most (if not all) of the earlier works dealt with the Riemann-Stieltjes integral BCs or infinite-point BCs as separate cases. Here, if we have a way of getting the continuous solution of the m-point BVP, we can (in a simple way) get a solution to the BVP with the Riemann-Stieltjes integral or infinite points in the BCs.

2. Preliminaries

Let $C(I)$ be the space of continuous functions defined on I with the norm given by

$$||x|| = \sup_{t\in I} |x(t)|,$$

and $AC[0,1]$ be the space of all absolutely continuous functions on $[0,1]$.

In addition, let $L_1(I)$ denote the class of the Lebesgue-integrable functions on the interval $I = [0,1]$ with the norm given by

$$||y||_{L_1} = \int_0^1 y(\xi)d\xi.$$

Definition 1. *The Riemann-Liouville fractional integral of the function $f \in L_1[0,T]$ of order $\beta > 0$ is defined by (see [34,35])*

$$I^\beta f(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s)\, ds.$$

Definition 2. *The Caputo (or, more precisely, the Liouville-Caputo) fractional derivative of $f(t)$ of order α $(0 < \alpha \leqq 1)$ is defined as follows (see [34,35])*

$$D^\alpha f(t) = I^{1-\alpha} \frac{d}{dt}\{f(t)\} = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \frac{d}{ds}\{f(s)\}\, ds.$$

3. Existence of Solutions to (1) with the m-Point BCs (4)

Definition 3. *A function x is called a solution of problem (1) with the m-point BCs (4) if $x \in AC[0,1]$ and satisfies (1) and (4).*

We make several assumptions as detailed below:

(i) The function $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, that is, it possesses the following properties:

 (a) For each $t \in [0,1]$, $f(t,\cdot)$ is continuous;

 (b) For each $u \in \mathbb{R}$, $f(\cdot,u)$ is measurable.

(ii) The function $\phi : [0,1] \to [0,1]$ is continuous and advanced, $\phi(t) \geqq t$, or continuous and deviated, $\phi(t) \leqq t$.

(iii) There exists an integrable function $a \in L_1[0,1]$ and a constant $b > 0$ such that

$$|f(t,u)| \leqq a(t) + b\,|u| \qquad \text{for each } t \in [0,1] \text{ and } u \in \mathbb{R}$$

Lemma 1. *The boundary value problem given by (1) and (4) is equivalent to the following integral equation:*

$$x(t) = A\left(x_0 - \sum_{k=1}^m a_k \int_0^{\phi(\tau_k)} y(\xi)\, d\xi\right) + \int_0^t y(\xi)\, d\xi, \tag{5}$$

where $y(t)$ is the solution of fractional-order integral equation given by

$$y(t) = f\big(t, I^{1-\alpha} y(t)\big) \qquad \big(t \in [0,1]\big) \tag{6}$$

and

$$A = \left(\sum_{k=1}^{m} a_k \right)^{-1}.$$

Proof. We begin by considering the problem (1) with the m-point BCs in (4). If we put $y(t) = x'(t)$ in (1), then Definition 2 implies that

$$y(t) = f\big(t, I^{1-\alpha} y(t)\big).$$

We also have

$$x(t) = x(0) + I^1 y(t). \tag{7}$$

We now use the nonlocal condition (4) in order to compute the constant $x(0)$. Indeed, upon setting $t = \phi(\tau_k) \in (0,1)$ in Equation (7), we get

$$x(\phi(\tau_k)) = \int_0^{\phi(\tau_k)} y(\xi)\, d\xi + x(0),$$

so that we have

$$\sum_{k=1}^{m} a_k\, x(\phi(\tau_k)) = \sum_{k=1}^{m} a_k \int_0^{\phi(\tau_k)} y(\xi)\, d\xi + x(0) \sum_{k=1}^{m} a_k.$$

From Equation (4), we find that

$$x_0 = \sum_{k=1}^{m} a_k \int_0^{\phi(\tau_k)} y(\xi)\, d\xi + x(0) \sum_{k=1}^{m} a_k,$$

which yields

$$x(0) = A \left(x_0 - \sum_{k=1}^{m} a_k \int_0^{\phi(\tau_k)} y(\xi)\, d\xi \right).$$

Substituting this last evaluation in Equation (7), we obtain formula (5).

Finally, in order to complete the proof of the above Lemma, we show that Equation (5) satisfies problem (1) together with the m-point BCs in (4). In fact, from (5), we obtain

$$D^\alpha x(t) = I^{1-\alpha} \frac{d}{dt}\{x(t)\} = I^{1-\alpha} y(t).$$

In addition, upon differentiating (5) with respect to t, we have

$$\frac{dx}{dt} = y(t) = f\big(t, I^{1-\alpha} y(t)\big) = f\big(t, D^\alpha x(t)\big).$$

Again, from (5), we have

$$\sum_{k=1}^{m} a_k\, x\big(\phi(\tau_k)\big) = x_0 - \sum_{k=1}^{m} a_k \int_0^{\phi(\tau_k)} y(\xi)\, d\xi + \sum_{k=1}^{m} a_k \int_0^{\phi(\tau_k)} y(\xi)\, d\xi = x_0.$$

This proves the equivalence between the nonlocal problem given by (1) and (2) and the integral Equation (5). □

For the problem (1) with the m-point BCs (4), we prove Theorem 1 below.

Theorem 1. *Suppose that the assumptions* (i) *to* (iii) *are satisfied. If*

$$\frac{b}{\Gamma(2-\alpha)} < 1,$$

then the fractional-order integral equation (6) *has a solution* $y \in L_1[0,1]$.

Suppose also that the coefficients a_k *satisfy the following inequality*:

$$\sum_{k=1}^{m} a_k \neq 0.$$

Then the problem (1) *together with the m-point BCs in* (4) *has at least one solution* $x \in AC[0,1]$ *given by* (5).

Proof. Let us define the operator T associated with Equation (6) by

$$(Ty)(t) = f\big(t, I^{1-\alpha}y(t)\big).$$

In addition, for a positive number r, let

$$B_r = \{y : y \in \ L_1(I) \quad \text{and} \quad ||y||_{L_1} \leqq r\} \subset L_1[0,1],$$

where

$$r \geqq \frac{||a||_{L_1}}{1 - \frac{b}{\Gamma(2-\alpha)}}.$$

Clearly, B_r is nonempty, closed, convex and bounded.

From the assumption (i), we can deduce that the operator T is continuous.

Suppose that y is an arbitrary element in B_r. We will show that $TB_r \subset r$. Indeed, from (6) and the assumptions (i) and (iii), we get

$$\begin{aligned} ||Ty||_{L_1} &= \int_0^1 \ |Ty(t)| \ dt \\ &\leqq \int_0^1 \ |a(t)|dt + b \ \int_0^1 \int_0^t \ \frac{(t-\xi)^{-\alpha}}{\Gamma(1-\alpha)} \ |y(\xi)| \ d\xi \ dt \\ &\leqq ||a||_{L_1} + b \ \int_0^1 \ \int_\xi^1 \ \frac{(t-\xi)^{-\alpha}}{\Gamma(1-\alpha)} \ dt \ |y(\xi)| \ d\xi \\ &\leqq ||a||_{L_1} + \frac{b}{\Gamma(2-\alpha)} \ ||y||_{L_1} \leqq ||a||_{L_1} + \frac{b}{\Gamma(2-\alpha)} \ r \leqq r, \end{aligned}$$

which implies that $TB_r \subset B_r$.

We will now show that T is a compact operator. In fact, if we let Ω be a bounded subset of B_r, then $T(\Omega)$ is clearly seen to be bounded in $L_1[0,1]$, that is, the first condition of Kolmogorov's Compactness Criterion (see [1]) is satisfied.

We next prove that

$$(Ty)_h \ \to \ Ty \ \text{ uniformly in } \ L_1[0,1] \qquad (h \ \to \ 0),$$

where

$$(Ty)_h(t) = \frac{1}{h} \int_t^{t+h} \ (Ty)(\xi) \ d\xi.$$

For each $y \in \Omega$, we thus find that

$$\begin{aligned}||(Ty)_h - Ty||_{L_1} &= \int_0^1 |(Ty)_h(t) - (Ty)(t)|\, dt \\ &= \int_0^1 |\frac{1}{h}\int_t^{t+h} (Ty)(\xi)\, d\xi - (Ty)(t)|\, dt \\ &\leqq \int_0^1 \left(\frac{1}{h}\int_t^{t+h} |(Ty)(\xi) - (Ty)(t)|\, d\xi\right) dt \\ &\leqq \int_0^1 \frac{1}{h}\int_t^{t+h} \left|f\left(\xi, I^{1-\alpha}y(\xi)\right) - f\left(t, I^{1-\alpha}y(t)\right)\right| d\xi\, dt.\end{aligned}$$

By the assumptions (i) and (iii), $y \in \Omega$ implies that $f \in L_1[0,1]$, so it follows that (see [36])

$$\frac{1}{h}\int_t^{t+h} \left|f\left(\xi, I^{1-\alpha}y(\xi)\right) - f\left(t, I^{1-\alpha}y(t)\right)\right| d\xi \to 0 \qquad (h \to 0) \text{ a.e.} \qquad \big(t \in [0,1]\big).$$

Then, by Kolmogorov's Compactness Criterion (see [1]), we find that $T(\Omega)$ is relatively compact, that is, T is a compact operator.

As a consequence of Schauder's Fixed Point Theorem (see [2]), the operator T has a fixed point in B_r. This proves the existence of the solution $y \in L_1[0,1]$ of Equation (6). Consequently, based on the above Lemma, problem (1) together with the m-point BCs (4) possess a solution $x \in AC(0,1)$.

Now, from Equation (5), we have

$$x(0) = \lim_{t \to 0+} x(t) = A\, x_0 - A \sum_{k=1}^m a_k \int_0^{\phi(\tau_k)} y(\xi)\, d\xi$$

and

$$x(1) = \lim_{t \to 1-} x(t) = A\, x_0 - A \sum_{k=1}^m a_k \int_0^{\phi(\tau_k)} y(\xi)\, d\xi + \int_0^1 y(\xi)\, d\xi,$$

from which we deduce that Equation (5) has a solution $x \in AC[0,1]$.

Consequently, the nonlocal problem given by (1) and (4) has a solution $x \in AC[0,1]$ given by (5). $\square$

4. Riemann-Stieltjes Functional Integral BCs

Let $x \in AC[0,1]$ be a solution of the problem (1) with the m-point BCs in (4). Then, we have the following theorem.

Theorem 2. *Suppose that the assumptions (i) to (iii) are satisfied. If*

$$\frac{b}{\Gamma(2-\alpha)} < 1$$

and $g : [0,1] \to [0,1]$ is an increasing function, then there exists a solution $x \in AC[0,1]$ of the following problem:

$$x'(t) = f\left(t, D^\alpha x(t)\right) \text{ a.e.} \qquad \big(t \in (0,1);\ \alpha \in (0,1]\big),$$

together with the Riemann-Stieltjes functional integral condition:

$$\int_0^1 x\big(\phi(s)\big)\, dg(s) = x_0,$$

which is represented by

$$x(t) = [g(1) - g(0)]^{-1}\, x_0 - [g(1) - g(0)]^{-1} \cdot \int_0^1 \int_0^{\phi(s)} y(\xi)\, d\xi\, dg(s) + \int_0^t y(\xi)\, d\xi. \tag{8}$$

Proof. Let

$$a_k = g(t_k) - g(t_{k-1}) \qquad \left(\tau_k \in (t_{k-1}, t_k);\ 0 \leqq t_0 < t_1 < t_2, \cdots < t_n \leqq 1\right).$$

Then, the multi-point nonlocal condition (4) becomes

$$\sum_{k=1}^{m} [g(t_k) - g(t_{k-1})]\, x\big(\phi(\tau_k)\big) = x_0.$$

From the continuity of the solution x of the multi-point nonlocal problem given by (1) and (4), we can get

$$\lim_{m\to\infty} \sum_{k=1}^{m} [g(t_k) - g(t_{k-1})]\, x\big(\phi(\tau_k)\big) = \int_0^1 x\big(\phi(s)\big)\, dg(s).$$

Furthermore, the multi-point nonlocal boundary condition (4) can be transformed into the following Riemann-Stieltjes functional integral form:

$$\int_0^1 x\big(\phi(s)\big)\, dg(s) = x_0.$$

In addition, from the functional integral Equation (5), we have

$$\begin{aligned}
\lim_{m\to\infty} x(t) &= [g(1) - g(0)]^{-1}\, x_0 - [g(1) - g(0)]^{-1} \\
&\quad \cdot \lim_{m\to\infty} \sum_{k=1}^{m} [g(t_k) - g(t_{k-1})] \int_0^{\phi(\tau_k)} y(\xi) d\xi + \int_0^t y(\xi) d\xi \\
&= [g(1) - g(0)]^{-1}\, x_0 - [g(1) - g(0)]^{-1} \\
&\quad \cdot \int_0^1 \int_0^{\phi(s)} y(\xi)\, d\xi\, dg(s) + \int_0^t y(\xi) d\xi.
\end{aligned}$$

Hence, the continuous solution of the first-order nonlinear differential Equation (1) with the Riemann-Stieltjes functional integral condition (2) is given by (8). □

We would like to provide two examples of the first order BVP (1) with the Riemann-Stieltjes functional integral boundary condition (2) (with the advanced or deviated argument ϕ) whose solutions are ensured by Theorem 2.

Example 1. *Let the nonlinear function $f(t,u)$ in (1) be given by*

$$f(t,u) = \cos\big(3(t+1)\big) + \frac{1}{5}\left(t^3 \sin u + e^{-t}\, u\right).$$

It is clear that the assumptions (i) and (iii) of Theorem 2 are fulfilled with

$$a(t) = \cos\big(3(t+1)\big) \in L_1[0,1] \qquad \text{and} \qquad b = \frac{2}{5}.$$

Let the fractional order in (1) be $\alpha = \frac{1}{2}$. Then

$$\frac{b}{\Gamma(2-\alpha)} \approx 0.4503338 < 1.$$

In this case, the first-order BVP (1) has the following form:

$$\frac{dx}{dt} = \cos\big(3(t+1)\big) + \frac{1}{5}\big[t^3\,\sin D^{1/2}x(t) + e^{-t}\,D^{1/2}x(t)\big]. \tag{9}$$

Let the function $g : [0,1] \to [0,1]$ be defined by the formula

$$g(t) = t\,\ln(1+t)\,.$$

If $\beta \in (0,1)$, we consider the advanced function $\phi(t) = t^\beta$. Then, the integral condition (2) assumes the following form:

$$\int_0^1 x(t^\beta)\,d\big(t\,\ln(1+t)\big) = x_0. \tag{10}$$

Thus, clearly, one can obtain the existence of a solution of (9) and (10).

Example 2. *Let $f(t,u)$, $g(t)$, α and β be as in Example (1) and consider the deviated function $\phi(t) = \beta t$. Then the functional integral condition (2) becomes*

$$\int_0^1 x(\beta t)\,d\big(t\ln(1+t)\big) = x_0. \tag{11}$$

Therefore, we can obtain the existence of a solution of (9) and (11).

We now consider another Riemann-Stieltjes nonlocal integral boundary condition.

Corollary 1. *Let the assumptions of Theorem 2 be satisfied. Then there exists a solution $x \in AC[0,1]$ of the following problem*:

$$x'(t) = f\big(t, D^\alpha x(t)\big) \;\; a.e. \qquad \big(t \in (0,\,1);\; \alpha \in (0,1]\big),$$

together with the Riemann-Stieltjes nonlocal integral condition given by

$$\int_c^d x\big(\phi(s)\big)\,dg(s) = x_0 \qquad (0 < c < d < 1),$$

which is represented by

$$x(t) = [g(d) - g(c)]^{-1}\,x_0 - [g(d) - g(c)]^{-1} \cdot \int_c^d \int_0^{\phi(s)} y(\xi)\,d\xi\,dg(s) + \int_0^t y(\xi)\,d\xi.$$

Proof. The proof of the above corollary is similar to that of Theorem 2. Here, in this case, we let

$$a_k = g(t_k) - g(t_{k-1}) \qquad \big(\tau_k \in (t_{k-1}, t_k);\; 0 < c \leqq t_0 < t_1 < t_2, \cdots < t_n \leqq d < 1\big).$$

□

5. Infinite-Point Boundary Conditions

Let $x \in AC[0,1]$ be the solution of the nonlocal problem given by (1) and (4). Then, we have the following theorem.

Theorem 3. *Let the assumptions (i) and (iii) be satisfied and let*

$$\phi(\tau_k) \leqq \tau_k \qquad \text{and} \qquad \frac{b}{\Gamma(2-\alpha)} < 1.$$

Suppose also that the following series:

$$\sum_{k=1}^{\infty} a_k = B^{-1}$$

is convergent. Then there exists a solution $x \in AC[0,1]$ of the nonlocal problem (1) and (3) given by the following integral equation:

$$x(t) = B\,x_0 - B \sum_{k=1}^{\infty} a_k \int_0^{\phi(\tau_k)} y(\xi)\,d\xi + \int_0^t y(\xi)\,d\xi \tag{12}$$

for every solution y of the functional equation (6).

Proof. Let $x \in AC[0,1]$ be a solution of the infinite point BVP (1) and (4) given by (5). Since

$$\left|a_k\,x(\phi(\tau_k))\right| \leqq a_k\,||x|| \qquad \text{and} \qquad \left|a_k \int_0^{\phi(\tau_k)} y(\xi)\,d\xi\right| \leqq a_k\,||y||_{L_1},$$

by the comparison test, the series in (3) and

$$\sum_{k=1}^{\infty} a_k \int_0^{\phi(\tau_k)} y(\xi)\,d\xi$$

are convergent. Thus, by taking the limit as $m \to \infty$ in (5), we obtain

$$x(t) = B\,x_0 - B \sum_{k=1}^{\infty} a_k \int_0^{\phi(\tau_k)} y(\xi)\,d\xi + \int_0^t y(\xi)\,d\xi,$$

which, for every solution y of the functional Equation (6), satisfies the differential Equation (1). Furthermore, from (12), we have

$$\begin{aligned}\sum_{k=1}^{\infty} a_k\,x(\phi(\tau_k)) &= B^{-1}\,B\,x_0 - B^{-1}\,B \sum_{k=1}^{\infty} a_k \int_0^{\phi(\tau_k)} y(\xi)\,d\xi \\ &\quad + \sum_{k=1}^{\infty} a_k \int_0^{\phi(\tau_k)} y(\xi)\,d\xi = x_0.\end{aligned} \tag{13}$$

This proves that the solution of the integral equation (12) satisfies the problem given by (1) under infinite-point BCs (3). □

6. Further Illustrative Examples

In this section, we consider the following examples with a view to illustrating some of our main results.

Example 3. *Consider the following infinite-point BVP:*

$$\frac{dx}{dt} = \frac{\ln\left(1 + |D^{2/3}x(t)|\right)}{2+t^2} + t^3 e^{-t^2} \quad a.e. \qquad (0 < t < 1)$$

together with

$$\sum_{k=1}^{\infty} \frac{1}{k^3}\, x\left(\frac{k^2-1}{k^2} - \sigma \sin^2\left(\sqrt{\frac{k^2-1}{k^2}}\right)\right) \qquad (0 \leqq \sigma \leqq 1).$$

If we set

$$f(t,u) = \frac{\ln\left(1+|u(t)|\right)}{2+t^2} + t^3 e^{-t^2},$$

then

$$|f(t,u)| \leqq t^3 e^{-t^2} + \frac{1}{3}\,|u|.$$

We also set

$$a(t) = t^3 e^{-t^2} \in L_1[0,1] \qquad \textit{and} \qquad b = \frac{1}{3}.$$

Thus, clearly, assumptions (i) and (iii) are satisfied.

On the other hand, we have

$$\alpha = \frac{2}{3} \quad \textit{so that} \quad \frac{b}{\Gamma(2-\alpha)} \approx 0.3731148 < 1.$$

Now, if we let

$$\phi(\tau_k) = \tau_k - \lambda \sin^2\left(\sqrt{\tau_k}\right) \qquad \textit{and} \qquad \tau_k = \frac{k^2-1}{k^2} \in (0,1),$$

then

$$\phi(\tau_k) \leqq \tau_k.$$

In addition, the following series:

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{k^3}$$

is convergent. Therefore, by appealing to Theorem 3, the given infinite-point BVP has an absolute continuous solution.

Example 4. *Consider the following infinite-point implicit BVP:*

$$\frac{dx}{dt} = \frac{[x'(t)]^3}{2(1+|x'(t)|^2)} + \frac{1}{4\pi}\,\sin\left(\pi\, x'(t)\right) + \cos t^3 + 3 \quad \textit{a.e.} \qquad \left(0 < t < 1;\;\, x'(t) := \frac{dx}{dt}\right)$$

together with

$$\sum_{k=1}^{\infty} 10\,\left(\frac{3}{4}\right)^k x\left(\frac{1}{k^3} - \lambda \exp\left(-\frac{1}{k^3}\right)\right) \qquad (0 \leqq \lambda \leqq 1).$$

If we set

$$f(t,u) = \frac{u^3}{2(1+|u|^2)} + \frac{1}{4\pi}\,\sin\left(\pi\, u\right) + \cos t^3 + 3,$$

then

$$|f(t,u)| \leqq \cos t^3 + 3 + \frac{3}{4}\,|u|,$$

Now, putting

$$a(t) = \cos t^3 + 3 \in L_1[0,1] \;\; \textit{and} \;\; b = \frac{3}{4},$$

the assumptions (i) and (iii) hold true.

We have

$$\alpha = 1 \quad \textit{so that} \quad \frac{b}{\Gamma(2-\alpha)} = \frac{3}{4} < 1.$$

On the other hand, if we let

$$\phi(\tau_k) = \tau_k - \lambda \exp\left(-\tau_k\right) \qquad \text{and} \qquad \tau_k = \frac{1}{k^3} \in (0,1),$$

then

$$\phi(\tau_k) \leqq \tau_k.$$

We also see that the following series:

$$\sum_{k=1}^{\infty} a_k = 10 \sum_{k=1}^{\infty} \left(\frac{2}{3}\right)^{k-1}$$

is convergent. Therefore, by applying Theorem 3, the given infinite-point implicit BVP has an absolute continuous solution in $[0,1]$.

7. Conclusions

In our present investigation, we have considered the existence of an absolute continuous solution to a class of first-order nonlinear differential equation with integral boundary conditions (BCs) or with infinite-point BCs (see Theorems 2 and 3 and the above Corollary). We have demonstrated that, if we can get the continuous solutions to BVPs with m-point BCs, we can easily get the solutions to these problems with integral BCs or with infinite-point BCs. Several examples have also been given in order to illustrate some of our main results. We note that the fractional differential Equation (1) involves the ordinary derivative $\frac{dx}{dt}$ of order 1 on its left-hand side. In the foreseeable future, we propose to investigate the possibility of extending our results to such other higher-order derivatives as

$$\frac{d^2x}{dt^2}, \quad \frac{d^3x}{dt^3}, \quad \frac{d^4x}{dt^4}, \quad \cdots,$$

occurring on the left-hand side of the fractional differential Equation (1), involving the Liouville-Caputo fractional derivatives together with integral BCs and/or the infinite-point BCs.

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