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On the Statistical Convergence of Order α in Paranormed Space

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Abstract: The aim of the present work is to introduce notions of statistical convergence, strongly p-Cesàro summability and the statistically Cauchy sequence of order α in paranormed spaces. Some certain topological properties of these new concepts are examined. Furthermore, we introduce the some inclusion relations among them.

Keywords: sequences; statistical convergence; Cesàro summability; paranormed space

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1. Introduction

Zygmund introduced the idea of statistical convergence in [1]. Fast and Steinhaus independently in the same year introduced statistical convergence to assign a limit to sequences that are not convergent in the usual sense (see [2,3]).

The notion of the asymptotic (or natural) density of a set $A \subset \mathbb{N}$ is defined such that:

$$\delta(A) = \lim_{n \to \infty} \frac{1}{n} \left| \{ k \le n : k \in A \} \right|,$$

whenever the limit exists. $|\{.\}|$ indicates the cardinality of the enclosed set. A sequence (x_k) of numbers is called statistically convergent to a number ξ provided that for every $\varepsilon > 0$,

$$\lim_{n \to \infty} \frac{1}{n} \left| \{ k \le n : |x_k - \xi| \ge \varepsilon \} \right| = 0.$$

In this case, the set of all statistically convergent sequences is denoted by S, and a sequence that is statistically convergent to ξ is denoted by $S - \lim\limits_{k \to \infty} x_k = \xi$.

This notion has been used as an effective tool to resolve many problems in ergodic theory, fuzzy set theory, trigonometric series and Banach spaces in the past few years. Furthermore, many researchers studied related topics with summability theory. (see [4,5]).

A paranorm $g : X \to \mathbb{R}$ is defined on a linear space X provided that for all $x, y, z \in X$:

(i) $g(x) = 0$ if $x = \theta$,
(ii) $g(-x) = g(x)$,
(iii) $g(x + y) \le g(x) + g(y)$,
(iv) if (α_n) is a sequence of scalars with $\alpha_n \to \alpha_0$ $(n \to \infty)$ and $x_n, a \in X$ with $x_n \to a$ $(n \to \infty)$ in the sense that $g(x_n - a) \to 0$ $(n \to \infty)$, then $\alpha_n x_n \to \alpha_0 a$ $(n \to \infty)$, in the sense that $g(\alpha_n x_n - \alpha_0 a) \to 0$ $(n \to \infty)$.

A paranorm g for which $g(x) = 0$ implies that $x = \theta$ is said to be a total paranorm on X. (X, g) is said to be a total paranormed space. We recall that each seminorm (norm) g on X is a paranorm (total). The opposite is not true.

Recently, Alotaibi and Alroqi [6] studied strong Cesàro summability, statistical convergence and the statistical Cauchy sequencein paranormed space. Then, Alghamdi and Mursaleen [7] introduced λ-statistical convergence in paranormed space.

Definition 1. *A sequence (x_k) in (X, g) is said to be convergent (or g-convergent) to the number ξ if for every $\varepsilon > 0$, there exists a positive integer k_0 such that $g(x_k - \xi) < \varepsilon$ whenever $k \geq k_0$. In this case, we write $g - \lim_{k\to\infty} x_k = \xi$ and ξ is called the g-limit of (x_k).*

The statistical convergence with degree $0 < \beta < 1$ was introduced by Gadjiev and Orhan in [8]. Later the statistical convergence of order α and strong p-Cesàro summability of order α were studied by Çolak in [9]. Çolak defined statistical convergence of order α as follows:

Let (x_k) be a sequence and $\alpha \in (0, 1]$. (x_k) be called statistically convergent of order α if there is a number ξ provided that:

$$\lim_{n\to\infty} \frac{1}{n^\alpha} |\{k \leq n : |x_k - \xi| \geq \varepsilon\}| = 0$$

for every $\varepsilon > 0$.

As a continuation of this study, many authors studied related topics and the generalized concept of this notion (see [10–14]).

2. Main Results

We begin by recalling the α-density of a set $A \subseteq \mathbb{N}$ where α is any real number such that $0 < \alpha \leq 1$. The α-density of A is defined by:

$$\delta_\alpha(A) = \lim_{n\to\infty} \frac{1}{n^\alpha} |\{k \leq n : k \in A\}|$$

provided the limit exists, where $|\{k \leq n : k \in A\}|$ denotes the number of elements of A not exceeding n. We note that the α-density notion reduces to the natural density notion in case $\alpha = 1$.

Definition 2. *A sequence (x_k) in (X, g) is said to be statistically convergent of order α (or $S^\alpha(g)$-convergent) to the number ξ if for every $\varepsilon > 0$,*

$$\lim_{n\to\infty} \frac{1}{n^\alpha} |\{k \leq n : g(x_k - \xi) \geq \varepsilon\}| = 0$$

where $0 < \alpha \leq 1$. In this case, we write $S^\alpha(g) - \lim_{k\to\infty} x_k = \xi$. $S^\alpha(g)$ indicates the set of these sequences in (X, g).

This notion reduces to the statistical convergence in paranormed space, which is introduced in [6] for $\alpha = 1$.

The statistical convergence of order α in paranormed space is well defined for $0 < \alpha \leq 1$. However, it is not well defined for $\alpha > 1$. For this, let X be a paranormed space with the paranorm $g(x) = |x|$. Consider a sequence (x_k) where $x_k = 1$ for $k = 2n$ and $x_k = 0$ for $k \neq 2n$. Then, we have:

$$g(x_k) = \begin{cases} 1, & k = 2n \\ 0, & k \neq 2n \end{cases}$$

for $n \in \mathbb{N}$. Then, both:

$$\lim_{n\to\infty} \frac{1}{n^\alpha} |\{k \leq n : g(x_k - 1) \geq \varepsilon\}| \leq \lim_{n\to\infty} \frac{n}{2n^\alpha} = 0$$

and:

$$\lim_{n\to\infty}\frac{1}{n^{\alpha}}\left|\{k\leq n:g\left(x_k\right)\geq\varepsilon\}\right|\leq\lim_{n\to\infty}\frac{n}{2n^{\alpha}}=0$$

for $\alpha > 1$. (x_k) in (X, g) is a statistically convergent sequence of order α both to one and zero. This is not possible.

Now, let us give a simple example to demonstrate the significance of this new type of convergence and to investigate the relationship between this new type of convergence and other approaches. For instance, $c(g)$, which is the set of of all convergent sequences, is a paranormed space by $g(x) = \sup_{k\in\mathbb{N}} |x_k|$. If we take a sequence (x_k) defined by:

$$x_k=\begin{cases}1, & k\neq m^2\\ \sqrt{k}, & k=m^2\end{cases}$$

for $m = 1, 2, 3, ...$ For $\varepsilon > 0$, we have:

$$\left|\{k\leq n:g\left(x_k-1\right)\geq\varepsilon\}\right|\leq\left|\{k\leq n:g\left(x_k\right)\neq 1\}\right|\leq\sqrt{n}$$

and:

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n^{\alpha}}\left|\{k\leq n:g\left(x_k-1\right)\geq\varepsilon\}\right| &\leq \lim_{n\to\infty}\frac{1}{n^{\alpha}}\left|\{k\leq n:g\left(x_k\right)\neq 1\}\right|\\ &\leq \lim_{n\to\infty}\frac{\sqrt{n}}{n^{\alpha}}=0\end{aligned}$$

for $\alpha > \frac{1}{2}$. From this inequality, it seems that the sequence (x_k) is statistically convergent of order α to one, and it belongs to the set $S^{\alpha}(g)$ where $\alpha > \frac{1}{2}$. We state in advance that from the example that is given above, we obtain the inclusion $c(g) \subset S^{\alpha}(g)$ that strictly holds where $\alpha = 1$. This means that a sequence that is not ordinary convergent in paranormed space can be statistically convergent of order α in this space. Furthermore, from this example, it seems that some sequences that are unbounded divergent can be statistically summable of order α in paranormed spaces.

Definition 3. *A sequence (x_k) in (X, g) is said to be a statistical Cauchy sequence of order α to the number ξ, if for every $\varepsilon > 0$, there exists a number $N = N(\varepsilon)$ such that:*

$$\lim_{n\to\infty}\frac{1}{n^{\alpha}}\left|\left\{k\leq n:g\left(x_j-x_N\right)\geq\varepsilon\right\}\right|=0.$$

Definition 4. *A sequence (x_k) in (X, g) is said to be strongly p-Cesàro summable of order α to the number ξ provided that:*

$$\lim_{n\to\infty}\frac{1}{n^{\alpha}}\sum_{k=1}^{n}g\left(x_k-\xi\right)^p=0$$

where α is any real number such that $0 < \alpha \leq 1$. We write it as $x_k \to \xi \left(w_p^{\alpha}(g)\right)$. In this case, ξ is called the $w_p^{\alpha}(g)$-limit of (x_k). The set of all these sequences is denoted by $w_p^{\alpha}(g)$.

We note that the strongly p-Cesàro summable of order α in (X, g) reduces to the strongly p-Cesàro summable in (X, g), which was introduced in [6] for $\alpha = 1$.

Theorem 1. *If a sequence (x_k) in total paranormed space (X, g) is statistically convergent of order α, then the $S^{\alpha}(g)$-limit is unique.*

Proof. Assume that $S^{\alpha}(g) - \lim_{k\to\infty} x = \xi_1$ and $S^{\alpha}(g) - \lim_{k\to\infty} x = \xi_2$. Define the following sets as:

$$\begin{aligned} K_1(\varepsilon) &= \{k \leq n : g(x_k - \xi_1) \geq \varepsilon/2\}, \\ K_2(\varepsilon) &= \{k < n : g(x_k - \xi_2) \geq \varepsilon/2\} \end{aligned}$$

for $\varepsilon > 0$. Since $S^{\alpha}(g) - \lim_{k\to\infty} x_k = \xi_1$ and $S^{\alpha}(g) - \lim_{k\to\infty} x_k = \xi_2$, we have $\delta_{\alpha}(K_1(\varepsilon)) = 0$ and $\delta_{\alpha}(K_2(\varepsilon)) = 0$. Let $K(\varepsilon) = K_1(\varepsilon) \cup K_2(\varepsilon)$. Then, $\delta_{\alpha}(K(\varepsilon)) = 0$ and $\delta_{\alpha}(K^c(\varepsilon)) = 1$. Now, if $k \in \mathbb{N}\backslash K(\varepsilon)$, then we have $g(\xi_1 - \xi_2) \leq g(x_k - \xi_1) + g(x_k - \xi_2) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. Consequently, we get $g(\xi_1 - \xi_2) = 0$, and hence, $\xi_1 = \xi_2$ for arbitrary $\varepsilon > 0$. □

Remark 1. *$S^{\alpha}(g)$ is different from $S_{\lambda}(g)$ defined in [7], in general. If we take $\lambda_n = n^{\alpha}$ for $0 < \alpha < 1$, then $S^{\alpha}(g) \subseteq S_{\lambda}(g)$. If $\lambda_n = n^{\alpha}$ with $\alpha = 1$, that is $\lambda_n = n$, then $S^{\alpha}(g) = S_{\lambda}(g) = S(g)$.*

Theorem 2. *Every g-convergent sequence is a statistically convergent sequence of order α in (X, g) for $\alpha = 1$. However, the converse case is not true.*

Proof. Let (x_k) be a g-convergent sequence. Then, there is a positive integer N such that:

$$g(x_n - \xi) < \varepsilon$$

for $\varepsilon > 0$ and for all $n \geq N$. Since the set $A(\varepsilon) = \{k \in \mathbb{N} : g(x_k - \xi) \geq \varepsilon\} \subset \{1, 2, 3, ...\}$, $\delta_{\alpha}(A(\varepsilon)) = 0$. Hence, (x_k) is a statistically convergent sequence of order α where $\alpha = 1$. □

To show the converse case is not true, let X be a paranormed space with the paranorm $g(x) = |x|$ and a sequence (x_k) in (X, g) defined by $x_k = 1$ for $k = n^3$, $x_k = 0$ for $k \neq n^3$. Then, we have:

$$g(x_k) = \begin{cases} 1, & k = n^3 \\ 0, & k \neq n^3 \end{cases}. \tag{1}$$

(x_k) in (X, g) is statistically convergent of order α to zero for $\alpha > \frac{1}{3}$. However, it is not g-convergent.

Now, we give another example to see the meaning of Definition 2. Let us consider the space $L_2(0, 2\pi)$ as X with the paranorm $g(x) = |f(x)|$ where f is a linear functional on $L_2(0, 2\pi)$. Choose a sequence (x_k) belonging to $L_2(0, 2\pi)$ given by:

$$x_k(t) = \sin kt$$

for $k = 1, 2, 3, ...$ It is well known that (x_k) is weakly convergent to zero in $L_2(0, 2\pi)$ (see [15]); hence, we have that $\lim_{n\to\infty} g(x_k) = 0$. By Theorem 2, $(x_k) \in S^{\alpha}(g)$ where $\alpha = 1$. On the other hand, we have that:

$$\|x_k - 0\|^2 = \int_0^{2\pi} \sin^2 kt dt = \pi$$

for all $k = 1, 2, 3, ...$ If we take $\alpha = 1$,

$$\lim_{n\to\infty} \frac{1}{n^{\alpha}} |\{k \leq n : \|x_k - 0\| \geq \varepsilon\}| = \lim_{n\to\infty} \frac{1}{n^{\alpha}} \left|\{k \leq n : \sqrt{\pi} \geq \varepsilon\}\right| = 1 \neq 0$$

for every $0 < \varepsilon < 1$. Hence, (x_k) is not statistically convergent of order α to zero in this space with the metric induced by the norm.

As seen from this example, this new concept provides us an opportunity for the statistical summability of these sequences in paranormed spaces.

Theorem 3. *Let $0 < \alpha \leq \beta \leq 1$. Then, $S^{\alpha}(g) \subseteq S^{\beta}(g)$ holds, and this inclusion is strict for some α and β such that $\alpha < \beta$.*

Proof. If $0 < \alpha \leq \beta \leq 1$, then:

$$\frac{1}{n^{\beta}}\left|\{k \leq n : g(x_k - \xi) \geq \varepsilon\}\right| \leq \frac{1}{n^{\alpha}}\left|\{k \leq n : g(x_k - \xi) \geq \varepsilon\}\right|$$

for every $\varepsilon > 0$, and this means that $S^{\alpha}(g) \subseteq S^{\beta}(g)$. To prove that the inclusion is strict, consider a paranormed space X with the paranorm $g(x) = |x|$, and also, the sequence (x_k) is defined by $x_k = 1$ for $k = n^2$, $x_k = 0$ for $k \neq n^2$. Then, we have:

$$g(x_k) = \begin{cases} 1 & k = n^2 \\ 0 & k \neq n^2 \end{cases}. \tag{2}$$

Hence, $S^{\beta}(g) - \lim\limits_{k\to\infty} x_k = 0$, i.e., $x \in S^{\beta}(g)$ for $1/2 < \beta \leq 1$, but $x \notin S^{\alpha}(g)$ for $0 < \alpha \leq 1/2$. □

Remark 2. *If we take $\beta = 1$, then $S^{\alpha}(g) \subseteq S(g)$ strictly holds.*

Remark 3. *(i) If $\alpha = \beta$, then $S^{\alpha}(g) = S^{\beta}(g)$.*
(ii) If $\alpha = 1$, then $S^{\alpha}(g) = S(g)$.

Theorem 4. *If a sequence (x_k) in (X, g) is statistically convergent of order α to ξ, then there exists a set $K = \{k_n : n \in \mathbb{N}$ and $k_1 < k_2 < ... < k_n < ...\} \subseteq \mathbb{N}$ with $\delta_{\alpha}(K) = 1$ such that $g(x_{k_n} - \xi) \to 0$ $(n \to \infty)$.*

Proof. Assume that (x_k) in (X, g) is statistically convergent of order α to ξ, that is $S^{\alpha}(g) - \lim\limits_{k\to\infty} x_k = \xi$. (x_k) in (X, g) is also statistically convergent to ξ from Remark 2. Now, write $K_r = \left\{n \in \mathbb{N} : g(x_{k_n} - \xi) \geq \frac{1}{r}\right\}$, $M_r = \left\{n \in \mathbb{N} : g(x_{k_n} - \xi) < \frac{1}{r}\right\}$ for $r = 1, 2, ...$ Then, $\delta_{\alpha}(K_r) = 0$,

$$M_1 \supset M_2 \supset ... \supset M_i \supset M_{i+1} \supset ..., \tag{3}$$

$$\delta_{\alpha}(M_r) = 1, \quad r = 1, 2, ... \tag{4}$$

We have to show that, for $n \in M_r$, (x_{k_n}) is g-convergent to ξ. On the contrary, suppose that (x_{k_n}) is not g-convergent to ξ. Therefore, there is $\varepsilon > 0$ such that $g(x_{k_n} - \xi) \geq \varepsilon$ for infinitely many terms. Let $M_{\varepsilon} = \{n \in \mathbb{N} : g(x_{k_n} - \xi) < \varepsilon\}$ and $\varepsilon > 1/r$, $r \in \mathbb{N}$. Then, $\delta_{\alpha}(M_{\varepsilon}) = 0$, and by (3), $M_r \subset M_{\varepsilon}$. Hence, $\delta_{\alpha}(M_r) = 0$. This contradicts (4), and we have that (x_{k_n}) is g-convergent to ξ. □

Theorem 5. *A sequence (x_k) in a complete paranormed space (X, g) is statistically Cauchy of order α if and only if it is statistically convergent of order α.*

Proof. Assume that (x_k) is $S^{\alpha}(g)$-Cauchy, but not statistically convergent of order α in (X, g). Then, we have $m \in \mathbb{N}$ such that $\delta_{\alpha}(G(\varepsilon)) = 0$, where $G(\varepsilon) = \{n \in \mathbb{N} : g(x_n - x_m) \geq \varepsilon\}$, and $\delta_{\alpha}(D(\varepsilon)) = 0$, where $D(\varepsilon) = \{n \in \mathbb{N} : g(x_n - \xi) < \varepsilon/2\}$, i.e., $\delta_{\alpha}\left(D^{C}(\varepsilon)\right) = 1$. If $g(x_n - \xi) < \frac{\varepsilon}{2}$, then $g(x_n - x_m) \leq 2g(x_n - \xi) < \varepsilon$. Moreover, $\delta_{\alpha}\left(G^{C}(\varepsilon)\right) = 0$, i.e., $\delta_{\alpha}(G(\varepsilon)) = 1$, which leads to a contradiction, since (x_k) was $S^{\alpha}(g)$-Cauchy. Hence, (x_k) must be statistically convergent of order α in (X, g). □

Conversely, let assume that $S^{\alpha}(g) - \lim\limits_{k\to\infty} x_k = \xi$. Then, we have $\delta_{\alpha}(K(\varepsilon)) = 0$ where:

$$K(\varepsilon) = \left\{n \in \mathbb{N} : g(x_n - \xi) \geq \frac{\varepsilon}{2}\right\}.$$

This implies that:

$$\delta\left(\mathbb{N}\backslash K\left(\varepsilon\right)\right)=\delta\left(\left\{n\in\mathbb{N}:g\left(x_{n}-\xi\right)<\frac{\varepsilon}{2}\right\}\right)=1.$$

Let $m,n\notin K\left(\varepsilon\right)$, then $g\left(x_{m}-x_{n}\right)<\varepsilon$. Let:

$$M\left(\varepsilon\right)=\left\{n\in\mathbb{N}:g\left(x_{m}-x_{n}\right)<\varepsilon\right\}$$

for a fix $m\notin K\left(\varepsilon\right)$. Then, $\mathbb{N}\backslash K\left(\varepsilon\right)\subset M\left(\varepsilon\right)$. Hence,

$$1=\delta\left(\mathbb{N}\backslash K\left(\varepsilon\right)\right)\leq\delta\left(M\left(\varepsilon\right)\right)\leq 1.$$

This will imply $\delta\left(\mathbb{N}\backslash M\left(\varepsilon\right)\right)=0$, where $\mathbb{N}\backslash M\left(\varepsilon\right)=\left\{n\in\mathbb{N}:g\left(x_{m}-x_{n}\right)\geq\varepsilon\right\}$. This implies that $\left(x_{n}\right)$ is statistically Cauchy of order α in $\left(X,g\right)$.

Remark 4. *Let $0<\alpha\leq\beta\leq 1$. Then, if $\left(x_{k}\right)$ in $\left(X,g\right)$ is a statistically Cauchy sequence of order α, then it is also a statistically Cauchy sequence of order β for some α and β such that $\alpha<\beta$.*

Theorem 6. *Let $0<\alpha\leq\beta\leq 1$ and p be a positive real number. Then, $w_{p}^{\alpha}\left(g\right)\subseteq w_{p}^{\beta}\left(g\right)$ strictly holds for some α and β such that $\alpha<\beta$.*

Proof. Suppose that $\left(x_{k}\right)\in w_{p}^{\alpha}\left(g\right)$. Then, we have:

$$\frac{1}{n^{\beta}}\sum_{k=1}^{n}g\left(x_{k}-\xi\right)^{p}\leq\frac{1}{n^{\alpha}}\sum_{k=1}^{n}g\left(x_{k}-\xi\right)^{p}$$

for a positive real number p and given α and β such that $0<\alpha\leq\beta\leq 1$. This means $w_{p}^{\alpha}\left(g\right)\subseteq w_{p}^{\beta}\left(g\right)$. To prove the strictness of this inclusion, let us consider the sequence defined in (2). We have that:

$$\frac{1}{n^{\beta}}\sum_{k=1}^{n}g\left(x_{k}-0\right)^{p}\leq\frac{\sqrt{n}}{n^{\beta}}=\frac{1}{n^{\beta-\frac{1}{2}}}.$$

Since $\frac{1}{n^{\beta-\frac{1}{2}}}\to 0$ as $n\to\infty$, then $w_{p}^{\beta}\left(g\right)-\lim_{k\to\infty}x_{k}=0$, i.e., $x\in w_{p}^{\beta}$ for $1/2<\beta\leq 1$, but since:

$$\frac{\sqrt{n}-1}{n^{\alpha}}\leq\frac{1}{n^{\alpha}}\sum_{k=1}^{n}g\left(x_{k}-0\right)^{p}$$

and $\frac{\sqrt{n}-1}{n^{\alpha}}\to\infty$ as $n\to\infty$, then $x\notin w_{p}^{\alpha}\left(g\right)$ for $0<\alpha<1/2$. This completes the proof. □

Remark 5. *Let $0<\alpha\leq\beta\leq 1$ and $p\in\mathbb{R}^{+}$. Then,*

(i) If $\alpha=\beta$, then $w_{p}^{\alpha}\left(g\right)=w_{p}^{\beta}\left(g\right)$.
(ii) $w_{p}^{\alpha}\left(g\right)\subseteq w_{p}^{\beta}\left(g\right)$ for each $\alpha\in\left(0,1\right]$ and $0<p<\infty$.

Theorem 7. *Let $0<\alpha\leq 1$ and $0<q<p<\infty$. Then, $w_{p}^{\alpha}\left(g\right)\subset w_{q}^{\alpha}\left(g\right)$.*

Proof. It is seen easily by Hölder's inequality. This is an extension of a result of Maddox [16,17]. □

Remark 6. *If $\alpha=1$ in Theorem 7, then we have $w_{p}\left(g\right)\subset w_{q}\left(g\right)$.*

Theorem 8. *Let α and β be fixed real numbers such that $0<\alpha\leq\beta\leq 1$ and $0<p<\infty$. If a sequence in $\left(X,g\right)$ is strongly p-Cesàro summable of order α to ξ, then it is statistically convergent of order β to ξ.*

Proof. For any sequence (x_k) in (X, g) and $\varepsilon > 0$, we have:

$$\sum_{k=1}^{n} g\left(x_k - \xi\right)^p \geq \left|\left\{k \leq n : g\left(x_k - \xi\right)^p \geq \varepsilon\right\}\right| \varepsilon^p$$

and so that:

$$\begin{aligned}\frac{1}{n^{\alpha}} \sum_{k=1}^{n} g\left(x_k - \xi\right)^p &\geq \frac{1}{n^{\alpha}} \left|\left\{k \leq n : g\left(x_k - \xi\right)^p \geq \varepsilon\right\}\right| \varepsilon^p \\ &\geq \frac{1}{n^{\beta}} \left|\left\{k \leq n : g\left(x_k - \xi\right)^p \geq \varepsilon\right\}\right| \varepsilon^p.\end{aligned}$$

This means that (x_k) in (X, g) is strongly p-Cesàro summable of order α to ξ, then it is statistically convergent of order β to ξ. $\square$

Remark 7. *Let $\alpha = \beta$. If a sequence in (X, g) is strongly p-Cesàro summable of order α to ξ in (X, g), then it is statistically convergent of order α to ξ where α is a fixed real number such that $0 < \alpha \leq 1$ and $0 < p < \infty$.*

Corollary 1. *If $g\left(x_k - \xi\right) \leq M$ and (x_k) in (X, g) is a statistically convergent sequence of order α, then (x_k) need not be strongly p-Cesàro summable of order α for $0 < \alpha < 1$.*

Proof. Let consider a paranormed space X with the paranorm $g(x) = |x|$ and a sequence $x_k = 1/\sqrt{k}$ for $k \neq m^3$ and $x_k = 1$ for $k = m^3$. Then, we have:

$$g\left(x_k\right) = \begin{cases} \frac{1}{\sqrt{k}} & k \neq m^3 \\ 1 & k = m^3 \end{cases}.$$

It is clear that (x_k) in (X, g) is bounded and $x \in S^{\alpha}(g)$ for each α such that $1/3 < \alpha \leq 1$. If we remind about the inequality:

$$\sum_{k=1}^{n} \frac{1}{\sqrt{k}} > \sqrt{n}$$

holds for every positive integer $n \geq 2$. Define $A_n = \left\{k \leq n : k \neq m^3, m = 1, 2, 3, ...\right\}$ and $p = 1$. Since:

$$\begin{aligned}\sum_{k=1}^{n} g\left(x_k\right)^p &= \sum_{k=1}^{n} g\left(x_k\right) = \sum_{k \in A_n, 1 \leq k \leq n} g\left(x_k\right) + \sum_{k \notin A_n, 1 \leq k \leq n} g\left(x_k\right) \\ &= \sum_{k \in A_n, 1 \leq k \leq n} \frac{1}{\sqrt{n}} + \sum_{k \notin A_n, 1 \leq k \leq n} 1 > \sum_{k=1}^{n} \frac{1}{\sqrt{n}} > \sqrt{n}\end{aligned}$$

we have:

$$\begin{aligned}\frac{1}{n^{\alpha}} \sum_{k=1}^{n} g\left(x_k\right)^p &= \frac{1}{n^{\alpha}} \sum_{k=1}^{n} g\left(x_k\right) \\ &> \frac{1}{n^{\alpha}} \sum_{k=1}^{n} \frac{1}{\sqrt{k}} \\ &> \frac{1}{n^{\alpha}} \sqrt{n} \\ &= \frac{1}{n^{\alpha - \frac{1}{2}}} \to \infty\end{aligned}$$

as $n \to \infty$. Hence, $x \notin w_p^{\alpha}$ for $0 < \alpha < 1/2$ if $p = 1$. Consequently, $x \in S^{\alpha}(g)$, but $x \notin w_p^{\alpha}(g)$ for $1/3 < \alpha < 1/2$ if $p = 1$. $\square$

Theorem 9. *Let $0 < \alpha \leq 1$ and $p \in \mathbb{R}^+$. Then, $w_p^\alpha(g) \subset S(g)$ holds. This inclusion strictly holds for $0 < \alpha < 1$.*

Proof. From Remarks 2 and 7, we have $w_p^\alpha(g) \subset S(g)$. To show that this inclusion is strict, let us consider the sequence and paranorm, which are defined in (1). Then, we have:

$$\begin{aligned} \frac{1}{n^\alpha} \sum_{k=1}^{n} g(x_k - 0)^p &= \frac{1}{n^\alpha} \sum_{k=1}^{n} g(x_k) \\ &\geq \frac{\sqrt[3]{n} - 1}{n^\alpha}. \end{aligned}$$

Since $\frac{\sqrt[3]{n}-1}{n^\alpha} \to \infty$ as $n \to \infty$, then $x \notin w_p^\alpha(g)$ for $0 < \alpha < 1/3$ and $p = 1$. Clearly, it is seen (x_k) in (X, g) is a statistically convergent sequence of order α. Consequently, $(x_k) \in S(g)$, but $(x_k) \notin w_p^\alpha(g)$ for $0 < \alpha < 1/3$. □

Remark 8. *If a sequence $(x_k) \in S^\alpha(g)$ and $g(x_k - \xi) \leq M$, then $(x_k) \in w_p(g) \subset S(g)$ for each α, where $0 < \alpha \leq 1$ and $0 < p < \infty$.*

3. Conclusions

The concept of statistical convergence has applications in different fields of mathematics such as number theory, statistics and probability theory, approximation theory, optimization, probability theory and fuzzy set theory. In this paper, the concepts of statistical convergence, strongly p-Cesàro summability and statistically Cauchy sequence of order α in paranormed spaces are introduced. Some topological properties of these concepts in paranormed spaces are investigated. Relations between statistical convergence of order α and strongly p-Cesàro summable of order α in paranormed spaces are considered in Theorem 8, Corollary 1, Theorem 9, and examples are given for clear understanding. These definitions and results provide new tools to deal with the convergence problems of sequences occurring in many branches of science which are given above. We state that the concept of paranorm is a generalization of absolute value. Hence, the introduced constructions and obtained results in this paper open new directions for further research. It would be interesting to develop connections between statistical convergence of order α in paranormed spaces and many branches of science.

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