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Two Types of Intuitionistic Fuzzy Covering Rough Sets and an Application to Multiple Criteria Group Decision Making

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Abstract: Intuitionistic fuzzy rough sets are constructed by combining intuitionistic fuzzy sets with rough sets. Recently, Huang et al. proposed the definition of an intuitionistic fuzzy (IF) β-covering and an IF covering rough set model. In this paper, some properties of IF β-covering approximation spaces and the IF covering rough set model are investigated further. Moreover, we present a novel methodology to the problem of multiple criteria group decision making. Firstly, some new notions and properties of IF β-covering approximation spaces are proposed. Secondly, we study the characterizations of Huang et al.'s IF covering rough set model and present a new IF covering rough set model for crisp sets in an IF environment. The relationships between these two IF covering rough set models and some other rough set models are investigated. Finally, based on the IF covering rough set model, Huang et al. also defined an optimistic multi-granulation IF rough set model. We present a novel method to multiple criteria group decision making problems under the optimistic multi-granulation IF rough set model.

Keywords: intuitionistic fuzzy; covering; neighborhood system; decision making

1. Introduction

Rough set theory was proposed by Pawlak [1,2] in 1982 as a tool to conceptualize, organize and analyze various types of data in data mining. There are other generalizations of his original concepts, for example by general binary relations [3], multi-granulations [4] and coverings [5]. Aiming at covering-based rough sets [6,7], they have been applied to decision rule synthesis [8,9], knowledge reduction [10,11] and other fields [12,13]. In theory, covering-based rough set theory has been connected with other theories. For example, it has been connected with lattice theory [14,15], matroid theory [16,17] and fuzzy set theory [18–23].

Zadeh's fuzzy set theory [24] addresses the problem of how to understand and manipulate imperfect knowledge. Recent investigations have shown that rough set and fuzzy set theories can be combined into various models, which are used for incomplete information in information systems. Dübois and Prade [25] first presented a fuzzy rough set model. Based on their work, some extended models and corresponding applications have been investigated in [26,27]. As far as fuzzy covering rough sets, D'eer et al. [28] discussed some fuzzy covering-based rough set models. Ma [18] proposed two new types of fuzzy covering rough set models. Inspired by Ma's work, Yang and Hu [29] investigated some types of fuzzy covering-based rough sets. Then, Yang and Hu studied other problems in fuzzy covering-based rough sets [30].

Intuitionistic fuzzy set (IFS) theory, as a straightforward extension of fuzzy set theory, was proposed by Atanassov [31]. The combination of IFS and rough set theories has attracted more interesting studies. For example, Zhang et al. [32] studied a general frame of IF rough sets. Huang et al. [33] presented an IF rough set model by combining β-neighborhoods induced by an IF β-covering. There are also other new notions and properties in Huang et al.'s IF rough set model, so it is necessary to investigate the IF rough set model further in this paper.

Recently, many researchers have studied decision making problems by rough set models [19,34], especially multiple criteria group decision making [20,23]. Multiple criteria group decision making (MCGDM) involves ranking from all feasible alternatives in conflicting and interactive criteria. After presenting the IF covering rough set model, Huang et al. also defined an optimistic multi-granulation IF rough set model. By investigation, multi-granulation IF rough set models have not been used for MCGDM problems. According to the characterizations of MCGDM problems, we construct the multi-granulation IF decision information systems and present a novel approach to MCGDM problems based on the optimistic multi-granulation IF rough set model in this paper.

The rest of this paper is organized as follows. Section 2 recalls some notions about covering-based rough sets and intuitionistic fuzzy sets. In Section 3, some properties of IF β-covering approximation space are investigated further. In Section 4, we investigate two IF covering rough set models. Among these two models, one is presented by Huang et al. [33], which concerns the IF sets; and the other is proposed by us, which concerns the crisp sets in an IF environment. In Section 5, we present a novel approach to MCGDM problems based on Huang et al.'s optimistic multi-granulation IF rough set model. This paper is concluded and further work is indicated in Section 6.

2. Basic Definitions

This section reviews some fundamental notions related to covering-based rough sets and intuitionistic fuzzy sets. Suppose U is a nonempty and finite set called a universe.

Definition 1. *(Covering [35,36]) Let U be a universe and $\mathbf{C}$ a family of subsets of U. If none of the subsets in $\mathbf{C}$ are empty and $\bigcup \mathbf{C} = U$, then $\mathbf{C}$ is called a covering of U.*

Definition 2. *(Neighborhood [35]) Let $\mathbf{C}$ be a covering of U. For any $x \in U$,*

$$N_{\mathbf{C}}(x) = \bigcap\{K \in \mathbf{C} : x \in K\}$$

is called the neighborhood of x with respect to $\mathbf{C}$. When the covering is clear, we omit the lowercase $\mathbf{C}$ in the neighborhood.

Definition 3. *(Approximation operators [37]) Let $\mathbf{C}$ be a covering of U. For any $X \subseteq U$,*

$$SH_{\mathbf{C}}(X) = \{x \in U : N(x) \cap X \neq \emptyset\},$$
$$SL_{\mathbf{C}}(X) = \{x \in U : N(x) \subseteq X\}.$$

$SH_{\mathbf{C}}(X)$ and $SL_{\mathbf{C}}(X)$ are called the upper and lower approximation operators with respect to $\mathbf{C}$.

Definition 4. *(Intuitionistic fuzzy set [31]) Let U be a fixed set. An intuitionistic fuzzy set (IFS) A in U is defined as:*

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in U\},$$

where $\mu_A : U \to [0,1]$ is called the degree of membership of the element $x \in U$ to A and $\nu_A : U \to [0,1]$ is called the degree of non-membership. They satisfy $\mu_A(x) + \nu_A(x) \leq 1$ for all $x \in U$. The family of all intuitionistic fuzzy sets in U is denoted by $IF(U)$.

We call $\langle a, b\rangle$ with $0 \leq a, b \leq 1$ and $a + b \leq 1$ an IF value. As is well known, for two IF values $\alpha = \langle a, b\rangle$ and $\beta = \langle c, d\rangle$, $\alpha \leq \beta \Leftrightarrow a \leq c$ and $b \geq d$.

For any family $\gamma_i \in [0,1], i \in I, I \subseteq \mathbb{N}^+$ ($\mathbb{N}^+$ is the set of all positive integers), we write $\vee_{i\in I}\gamma_i$ for the supremum of $\{\gamma_i : i \in I\}$ and $\wedge_{i\in I}\gamma_i$ for the infimum of $\{\gamma_i : i \in I\}$. Some basic operations on $IF(U)$ are shown as follows [31]: for any $A, B \in IF(U)$,

1. $A \subseteq B$ iff $\mu_A(x) \leq \mu_B(x)$ and $\nu_B(x) \leq \nu_A(x)$ for all $x \in U$;
2. $A = B$ iff $A \subseteq B$ and $B \subseteq A$;
3. $A \cup B = \{\langle x, \mu_A(x) \vee \mu_B(x), \nu_A(x) \wedge \nu_B(x)\rangle : x \in U\}$;
4. $A \cap B = \{\langle x, \mu_A(x) \wedge \mu_B(x), \nu_A(x) \vee \nu_B(x)\rangle : x \in U\}$;
5. $A' = \{\langle x, \nu_A(x), \mu_A(x)\rangle : x \in U\}$.

3. Some Properties of IF β-Covering Approximation Space

In this section, we introduce the notions of intuitionistic fuzzy (IF) β-covering approximation space. There are two concepts presented by Huang et al. in [33], which are IF β-covering and IF β-neighborhood in this approximation space. We mainly investigate some of their properties, and some new notions are presented.

3.1. IF β-Neighborhood and IF β-Neighborhood System

Definition 5. *([33]) Let U be a universe and $\beta = \langle a, b\rangle$ be an IF value. Then, we call $\widehat{\mathbf{C}} = \{C_1, C_2, ..., C_m\}$, with $C_i \in IF(U)(i = 1, 2, ..., m)$, an IF β-covering of U, if for any $x \in U$, there exists $C_i \in \widehat{\mathbf{C}}$, such that $C_i(x) \geq \beta$. We also call $(U, \widehat{\mathbf{C}})$ an IF β-covering approximation space.*

Let $\Gamma_\beta = \{\widehat{\mathbf{C}_1}, \widehat{\mathbf{C}_2}, \cdots, \widehat{\mathbf{C}_n}\}$. If any $\widehat{\mathbf{C}}_i$ $(i = 1, 2, ..., n)$ is an IF β-covering of U, then we call (U, Γ_β) a n-IF β-covering approximation space.

Definition 6. *([33]) Let $\widehat{\mathbf{C}}$ be an IF β-covering of U and $\widehat{\mathbf{C}} = \{C_1, C_2, \ldots, C_m\}$. For each $x \in U$, the IF β-neighborhood $\widetilde{N}_x^\beta$ of x induced by $\widehat{\mathbf{C}}$ can be defined as:*

$$\widetilde{N}_x^\beta = \cap\{C_i \in \widehat{\mathbf{C}} : C_i(x) \geq \beta\}.$$

Note that $C_i(x)$ is an IF value $\langle \mu_{C_i}(x), \nu_{C_i}(x)\rangle$ in Definitions 5 and 6. Hence, $C_i(x) \geq \beta$ means $\mu_{C_i}(x) \geq a$ and $\nu_{C_i}(x) \leq b$ where IF value $\beta = \langle a, b\rangle$.

Remark 1. *Let $\widehat{\mathbf{C}}$ be an IF β-covering of U, $\beta = \langle a, b\rangle$ and $\widehat{\mathbf{C}} = \{C_1, C_2, \ldots, C_m\}$. For each $x \in U$:*

$$\widetilde{N}_x^\beta = \cap\{C_i \in \widehat{\mathbf{C}} : \mu_{C_i}(x) \geq a, \nu_{C_i}(x) \leq b\}.$$

Example 1. *Let $U = \{x_1, x_2, x_3, x_4, x_5\}$ and $\widehat{\mathbf{C}} = \{C_1, C_2, C_3, C_4\}$, where:*

$$C_1 = \frac{(0.7,0.2)}{x_1} + \frac{(0.5,0.3)}{x_2} + \frac{(0.4,0.5)}{x_3} + \frac{(0.6,0.1)}{x_4} + \frac{(0.3,0.2)}{x_5},$$
$$C_2 = \frac{(0.6,0.2)}{x_1} + \frac{(0.3,0.2)}{x_2} + \frac{(0.2,0.3)}{x_3} + \frac{(0.4,0.5)}{x_4} + \frac{(0.7,0.3)}{x_5},$$
$$C_3 = \frac{(0.4,0.1)}{x_1} + \frac{(0.4,0.5)}{x_2} + \frac{(0.5,0.2)}{x_3} + \frac{(0.3,0.6)}{x_4} + \frac{(0.6,0.3)}{x_5},$$
$$C_4 = \frac{(0.1,0.5)}{x_1} + \frac{(0.6,0.1)}{x_2} + \frac{(0.6,0.3)}{x_3} + \frac{(0.5,0.3)}{x_4} + \frac{(0.8,0.1)}{x_5}.$$

According to Definition 5, we know $\widehat{\mathbf{C}}$ is an IF β-covering of U ($\beta = \langle a, b\rangle$ with $0 \le a \le 0.6, 0.3 \le b \le 1$). Let $\beta = \langle 0.5, 0.3\rangle$. Then, the IF β-neighborhoods are shown as follows:

$$\begin{aligned}
\widetilde{N}^{\beta}_{x_1} &= C_1 \cap C_2 = \frac{(0.6,0.2)}{x_1} + \frac{(0.3,0.3)}{x_2} + \frac{(0.2,0.5)}{x_3} + \frac{(0.4,0.5)}{x_4} + \frac{(0.3,0.3)}{x_5},\\
\widetilde{N}^{\beta}_{x_2} &= C_1 \cap C_4 = \frac{(0.1,0.5)}{x_1} + \frac{(0.5,0.3)}{x_2} + \frac{(0.4,0.5)}{x_3} + \frac{(0.5,0.3)}{x_4} + \frac{(0.3,0.2)}{x_5},\\
\widetilde{N}^{\beta}_{x_3} &= C_3 \cap C_4 = \frac{(0.1,0.5)}{x_1} + \frac{(0.4,0.5)}{x_2} + \frac{(0.5,0.3)}{x_3} + \frac{(0.3,0.6)}{x_4} + \frac{(0.6,0.3)}{x_5},\\
\widetilde{N}^{\beta}_{x_4} &= C_1 \cap C_4 = \frac{(0.1,0.5)}{x_1} + \frac{(0.5,0.3)}{x_2} + \frac{(0.4,0.5)}{x_3} + \frac{(0.5,0.3)}{x_4} + \frac{(0.3,0.2)}{x_5},\\
\widetilde{N}^{\beta}_{x_5} &= C_2 \cap C_3 \cap C_4 = \frac{(0.1,0.5)}{x_1} + \frac{(0.3,0.5)}{x_2} + \frac{(0.2,0.3)}{x_3} + \frac{(0.3,0.6)}{x_4} + \frac{(0.6,0.3)}{x_5}.
\end{aligned}$$

Theorem 1. *([33]) Let $\widehat{\mathbf{C}}$ be an intuitionistic fuzzy β-covering of U and $\widehat{\mathbf{C}} = \{C_1, C_2, \ldots, C_m\}$. Then, the following statements hold:*

1. *$\widetilde{N}^{\beta}_x(x) \ge \beta$ for any $x \in U$;*
2. *For $x, y, z \in U$, if $\widetilde{N}^{\beta}_x(y) \ge \beta$, $\widetilde{N}^{\beta}_y(z) \ge \beta$, then $\widetilde{N}^{\beta}_x(z) \ge \beta$;*
3. *For two IF values β_1, β_2, if $\beta_1 \le \beta_2 \le \beta$, then $\widetilde{N}^{\beta_1}_x \subseteq \widetilde{N}^{\beta_2}_x$ for all $x \in U$.*

For two different IF β-neighborhoods, a relationship between them is presented.

Proposition 1. *Let $\widehat{\mathbf{C}}$ be an IF β-covering of U. For any $x, y \in U$, $\widetilde{N}^{\beta}_x(y) \ge \beta$ if and only if $\widetilde{N}^{\beta}_y \subseteq \widetilde{N}^{\beta}_x$.*

Proof. Suppose the IF value $\beta = \langle a, b\rangle$.
($\Rightarrow$): Since $\widetilde{N}^{\beta}_x(y) \ge \beta$, so

$$\mu_{\widetilde{N}^{\beta}_x}(y) = \mu_{\bigcap\limits_{\substack{\mu_{C_i}(x)\ge a\\ \nu_{C_i}(x)\le b}} C_i}(y) = \bigwedge_{\substack{\mu_{C_i}(x)\ge a\\ \nu_{C_i}(x)\le b}} \mu_{C_i}(y) \ge a, \nu_{\widetilde{N}^{\beta}_x}(y) = \nu_{\bigcap\limits_{\substack{\mu_{C_i}(x)\ge a\\ \nu_{C_i}(x)\le b}} C_i}(y) = \bigvee_{\substack{\mu_{C_i}(x)\ge a\\ \nu_{C_i}(x)\le b}} \nu_{C_i}(y) \le b.$$

Hence,

$$\{C_i \in \widehat{\mathbf{C}} : \mu_{C_i}(x) \ge a, \nu_{C_i}(x) \le b\} \subseteq \{C_i \in \widehat{\mathbf{C}} : \mu_{C_i}(y) \ge a, \nu_{C_i}(y) \le b\}.$$

Therefore, for each $z \in U$,

$$\begin{aligned}
\mu_{\widetilde{N}^{\beta}_x}(z) &= \bigwedge_{\substack{\mu_{C_i}(x)\ge a\\ \nu_{C_i}(x)\le b}} \mu_{C_i}(z) \ge \bigwedge_{\substack{\mu_{C_i}(y)\ge a\\ \nu_{C_i}(y)\le b}} \mu_{C_i}(z) = \mu_{\widetilde{N}^{\beta}_y}(z),\\
\nu_{\widetilde{N}^{\beta}_x}(z) &= \bigvee_{\substack{\mu_{C_i}(x)\ge a\\ \nu_{C_i}(x)\le b}} \nu_{C_i}(z) \le \bigvee_{\substack{\mu_{C_i}(y)\ge a\\ \nu_{C_i}(y)\le b}} \nu_{C_i}(z) = \nu_{\widetilde{N}^{\beta}_y}(z).
\end{aligned}$$

Hence, $\widetilde{N}^{\beta}_y \subseteq \widetilde{N}^{\beta}_x$.
($\Leftarrow$): For any $x, y \in U$, since $\widetilde{N}^{\beta}_y \subseteq \widetilde{N}^{\beta}_x$, so $\mu_{\widetilde{N}^{\beta}_x}(y) \ge \mu_{\widetilde{N}^{\beta}_y}(y) \ge a$, $\nu_{\widetilde{N}^{\beta}_x}(y) \le \nu_{\widetilde{N}^{\beta}_y}(y) \le b$. Therefore $\widetilde{N}^{\beta}_x(y) \ge \beta$. $\square$

By the notion of IF β-neighborhood, we propose the following definition of the IF β-neighborhood system of $x \in U$.

Definition 7. *Let $\widehat{\mathbf{C}}$ be an IF β-covering of U and $\widehat{\mathbf{C}} = \{C_1, C_2, \ldots, C_m\}$. For each $x \in U$, the IF β-neighborhood system $\widetilde{\mathcal{N}}^{\beta}_{\widehat{\mathbf{C}}}(x)$ of x induced by $\widehat{\mathbf{C}}$ is defined as:*

$$\widetilde{\mathcal{N}}^{\beta}_{\widehat{\mathbf{C}}}(x) = \{C_i \in \widehat{\mathbf{C}} : C_i(x) \ge \beta\}.$$

According to Definition 6, we know $\widetilde{N}_x^\beta = \cap \widetilde{\mathcal{N}}_{\widehat{\mathbf{C}}}^\beta(x)$ for each $x \in U$. Let $\widehat{\mathbf{C}_1}$, $\widehat{\mathbf{C}_2}$ be two IF β-coverings of U. The statement does not hold: if $\widetilde{\mathcal{N}}_{\widehat{\mathbf{C}_1}}^\beta(x) = \widetilde{\mathcal{N}}_{\widehat{\mathbf{C}_2}}^\beta(x)$ for each $x \in U$, then $\widehat{\mathbf{C}_1} = \widehat{\mathbf{C}_2}$. The following example can illustrate it.

Example 2. *(Continued from Example 1) Let $\beta = \langle 0.5, 0.3 \rangle$, $\widehat{\mathbf{C}_1} = \widehat{\mathbf{C}} \cup \{C_5\}$, where $C_5 = \frac{\langle 0.2,0.5 \rangle}{x_1} + \frac{\langle 0.5,0.4 \rangle}{x_2} + \frac{\langle 0.4,0.6 \rangle}{x_3} + \frac{\langle 0.3,0.4 \rangle}{x_4} + \frac{\langle 0.3,0.5 \rangle}{x_5}$. Then, $\widetilde{\mathcal{N}}_{\widehat{\mathbf{C}}}^\beta(x_i)$ and $\widetilde{\mathcal{N}}_{\widehat{\mathbf{C}_1}}^\beta(x_i)$ $(i = 1,2,3,4,5)$ are listed in Table 1.*

Table 1. $\widetilde{\mathcal{N}}_{\widehat{\mathbf{C}}}^\beta(x_i)$ and $\widetilde{\mathcal{N}}_{\widehat{\mathbf{C}_1}}^\beta(x_i)$ $(i = 1,2,3,4,5)$.

U	x_1	x_2	x_3	x_4	x_5
$\widetilde{\mathcal{N}}_{\widehat{\mathbf{C}}}^\beta$	$\{C_1, C_2\}$	$\{C_1, C_4\}$	$\{C_3, C_4\}$	$\{C_1, C_4\}$	$\{C_2, C_3, C_4\}$
$\widetilde{\mathcal{N}}_{\widehat{\mathbf{C}_1}}^\beta$	$\{C_1, C_2\}$	$\{C_1, C_4\}$	$\{C_3, C_4\}$	$\{C_1, C_4\}$	$\{C_2, C_3, C_4\}$

Hence, $\widetilde{\mathcal{N}}_{\widehat{\mathbf{C}}}^\beta(x_i) = \widetilde{\mathcal{N}}_{\widehat{\mathbf{C}_1}}^\beta(x_i)$ for any $i = 1,2,3,4,5$, but $\widehat{\mathbf{C}} \neq \widehat{\mathbf{C}_1}$.

Inspired by this statement, we consider in which conditions two IF β-coverings generate the same β-neighborhood system for any element of the universe. In order to find the conditions, we introduce two new concepts in IF β-covering approximation space firstly.

Definition 8. *Let $\widehat{\mathbf{C}}$ be an IF β-covering of U and $C \in \widehat{\mathbf{C}}$. If there exists $x \in U$ such that $C(x) \geq \beta$, then C is called an IF β-dependent element of $\widehat{\mathbf{C}}$; otherwise, C is called an IF β-independent element of $\widehat{\mathbf{C}}$. If every element in $\widehat{\mathbf{C}}$ is an IF β-dependent element, then $\widehat{\mathbf{C}}$ is IF β-dependent; otherwise, $\widehat{\mathbf{C}}$ is IF β-independent.*

Example 3. *(Continued from Example 2) Let $\beta = \langle 0.5, 0.3 \rangle$. Then, C_5 is an IF β-independent element of $\widehat{\mathbf{C}_1}$ and C_i $(i = 1,2,3,4)$ are IF β-dependent elements of $\widehat{\mathbf{C}_1}$. Hence, $\widehat{\mathbf{C}_1}$ is IF β-independent and $\widehat{\mathbf{C}}$ is IF β-dependent.*

Proposition 2. *Let $\widehat{\mathbf{C}}$ be an IF β-covering of U and $C \in \widehat{\mathbf{C}}$. If C is an IF β-independent element of $\widehat{\mathbf{C}}$, then $\widehat{\mathbf{C}} - \{C\}$ is still an IF β-covering of U.*

Proof. According to Definitions 5 and 8, it is straightforward. □

Proposition 3. *Let $\widehat{\mathbf{C}}$ be an IF β-covering of U, C be an IF β-independent element of $\widehat{\mathbf{C}}$ and $C_1 \in \widehat{\mathbf{C}} - \{C\}$. Then, C_1 is an IF β-independent element of $\widehat{\mathbf{C}}$ iff C_1 is an IF β-independent element of $\widehat{\mathbf{C}} - \{C\}$.*

Proof. According to Definitions 5 and 8, it is straightforward. □

According to Propositions 2 and 3, it is still an IF β-covering after deleting all IF β-independent elements of an IF β-covering.

Definition 9. *Let $\widehat{\mathbf{C}}$ be an IF β-covering of U and $\widehat{\mathbf{B}} \subseteq \widehat{\mathbf{C}}$. If $\widehat{\mathbf{B}}$ is the set of all IF β-dependent elements of $\widehat{\mathbf{C}}$, then $\widehat{\mathbf{B}}$ is called the IF β-base of $\widehat{\mathbf{C}}$ and is denoted as $\Delta^\beta(\widehat{\mathbf{C}})$.*

Proposition 4. *Let $\widehat{\mathbf{C}}$ be an IF β-covering of U. For any $x \in U$,*

$$\widetilde{\mathcal{N}}_{\widehat{\mathbf{C}}}^\beta(x) = \widetilde{\mathcal{N}}_{\Delta^\beta(\widehat{\mathbf{C}})}^\beta(x).$$

Proof. According to Definitions 8, 9 and Proposition 2, it is straightforward. □

Theorem 2. *Let $\widehat{\mathbf{C}_1}$, $\widehat{\mathbf{C}_2}$ be two IF β-coverings of U. For any $x \in U$,*

$$\widetilde{\mathcal{N}}^{\beta}_{\widehat{C_1}}(x) = \widetilde{\mathcal{N}}^{\beta}_{\widehat{C_2}}(x) \text{ iff } \Delta^{\beta}(\widehat{C_1}) = \Delta^{\beta}(\widehat{C_2}).$$

Proof. By Proposition 4 and Definition 7, it is straightforward. □

Corollary 1. *Let $\widehat{C_1}$, $\widehat{C_2}$ be two IF β-coverings of U. For any $x \in U$, if $\Delta^{\beta}(\widehat{C_1}) = \Delta^{\beta}(\widehat{C_2})$, then $\widetilde{N}^{\beta}_{x} = \widetilde{N'}^{\beta}_{x}$, where $\widetilde{N}^{\beta}_{x}$ and $\widetilde{N'}^{\beta}_{x}$ are the β-neighborhoods induced by $\widehat{C_1}$ and $\widehat{C_2}$, respectively.*

Proof. According to Theorem 2, $\Delta^{\beta}(\widehat{C_1}) = \Delta^{\beta}(\widehat{C_2}) \Rightarrow \widetilde{\mathcal{N}}^{\beta}_{\widehat{C_1}}(x) = \widetilde{\mathcal{N}}^{\beta}_{\widehat{C_2}}(x) \Rightarrow \cap\widetilde{\mathcal{N}}^{\beta}_{\widehat{C_1}}(x) = \cap\widetilde{\mathcal{N}}^{\beta}_{\widehat{C_2}}(x) \Rightarrow \widetilde{N}^{\beta}_{x} = \widetilde{N'}^{\beta}_{x}$. □

3.2. β-Neighborhood

In this subsection, the definition of β-neighborhood is presented by the IF β-neighborhood.

Definition 10. *Let $(U, \widehat{C})$ be an IF β-covering approximation space and $\widehat{C} = \{C_1, C_2, \ldots, C_m\}$. For each $x \in U$, we define the β-neighborhood $\overline{N}^{\beta}_{x}$ of x as:*

$$\overline{N}^{\beta}_{x} = \{y \in U : \widetilde{N}^{\beta}_{x}(y) \geq \beta\}.$$

Note that $\widetilde{N}^{\beta}_{x}(y)$ is an IF value $\langle \mu_{\widetilde{N}^{\beta}_{x}}(y), \nu_{\widetilde{N}^{\beta}_{x}}(y) \rangle$ in Definition 10. Based on this, the β-neighborhood can be represented by the following remark.

Remark 2. *Let $\widehat{C}$ be an IF β-covering of U, $\beta = \langle a, b \rangle$ and $\widehat{C} = \{C_1, C_2, \ldots, C_m\}$. For each $x \in U$,*

$$\overline{N}^{\beta}_{x} = \{y \in U : \mu_{\widetilde{N}^{\beta}_{x}}(y) \geq a, \nu_{\widetilde{N}^{\beta}_{x}}(y) \leq b\}.$$

Example 4. *(Continued from Example 1) Let $\beta = \langle 0.5, 0.3 \rangle$, then we have:*

$$\overline{N}^{\beta}_{x_1} = \{x_1\},\ \overline{N}^{\beta}_{x_2} = \{x_2, x_4\},\ \overline{N}^{\beta}_{x_3} = \{x_3, x_5\},$$
$$\overline{N}^{\beta}_{x_4} = \{x_2, x_4\},\ \overline{N}^{\beta}_{x_5} = \{x_5\}.$$

The following theorem shows the basic properties of β-neighborhoods.

Theorem 3. *Let $\widehat{C}$ be an IF β-covering of U and $\widehat{C} = \{C_1, C_2, \ldots, C_m\}$. Then:*

1. *$x \in \overline{N}^{\beta}_{x}$ for any $x \in U$;*
2. *For any $x, y, z \in U$, if $x \in \overline{N}^{\beta}_{y}$, $y \in \overline{N}^{\beta}_{z}$, then $x \in \overline{N}^{\beta}_{z}$.*

Proof. (1) According to the first statement in Theorem 1, we know $\widetilde{N}^{\beta}_{x}(x) \geq \beta$ for each $x \in U$. Hence, $x \in \{y \in U : \widetilde{N}^{\beta}_{x}(y) \geq \beta\} = \overline{N}^{\beta}_{x}$ for each $x \in U$.
(2) For any $x, y, z \in U$, $x \in \overline{N}^{\beta}_{y} \Leftrightarrow \widetilde{N}^{\beta}_{y}(x) \geq \beta \Leftrightarrow \widetilde{N}^{\beta}_{x} \subseteq \widetilde{N}^{\beta}_{y}$, and $y \in \overline{N}^{\beta}_{z} \Leftrightarrow \widetilde{N}^{\beta}_{z}(y) \geq \beta \Leftrightarrow \widetilde{N}^{\beta}_{y} \subseteq \widetilde{N}^{\beta}_{z}$. Hence, $\widetilde{N}^{\beta}_{x} \subseteq \widetilde{N}^{\beta}_{z}$. By Proposition 1, we have $\widetilde{N}^{\beta}_{z}(x) \geq \beta$, i.e., $x \in \overline{N}^{\beta}_{z}$. □

The following proposition shows a relationship between $\overline{N}^{\beta}_{x}$ and $\overline{N}^{\beta}_{y}$.

Proposition 5. *Let $\widehat{C}$ be an IF β-covering of U and $\widehat{C} = \{C_1, C_2, \ldots, C_m\}$. Then, for any $x \in U$, $x \in \overline{N}^{\beta}_{y}$ iff $\overline{N}^{\beta}_{x} \subseteq \overline{N}^{\beta}_{y}$.*

Proof. ($\Rightarrow$): For any $z \in \overline{N}_x^\beta$, we know $\widetilde{N}_x^\beta(z) \geq \beta$. Since $x \in \overline{N}_y^\beta$, so $\widetilde{N}_y^\beta(x) \geq \beta$. According to (2) in Theorem 1, we have $\widetilde{N}_y^\beta(z) \geq \beta$. Hence, $z \in \overline{N}_y^\beta$. Therefore, $\overline{N}_x^\beta \subseteq \overline{N}_y^\beta$.
($\Leftarrow$): According to (1) in Theorem 3, $x \in \overline{N}_x^\beta$ for all $x \in U$. Since $\overline{N}_x^\beta \subseteq \overline{N}_y^\beta$, so $x \in \overline{N}_y^\beta$. $\square$

A relationship between IF β-neighborhoods and β-neighborhoods is proposed in the following proposition.

Proposition 6. *Let $\widehat{\mathbf{C}}$ be an IF β-covering of U. For any $x, y \in U$, $\widetilde{N}_x^\beta \subseteq \widetilde{N}_y^\beta$ iff $\overline{N}_x^\beta \subseteq \overline{N}_y^\beta$.*

Proof. For any $x, y \in U$, $\widetilde{N}_x^\beta \subseteq \widetilde{N}_y^\beta \Leftrightarrow \widetilde{N}_y^\beta(x) \geq \beta \Leftrightarrow x \in \overline{N}_y^\beta \Leftrightarrow \overline{N}_x^\beta \subseteq \overline{N}_y^\beta$. $\square$

4. Two Intuitionistic Fuzzy Covering Rough Set Models

In this section, we investigate two IF covering rough set models on the basis of the IF β-neighborhoods and the β-neighborhoods, respectively. Firstly, one model is presented by Huang et al., and we study the properties of the IF lower and upper approximations of each IF set further. Secondly, we propose another new model, which concerns crisp sets in an IF environment. Moreover, some properties of the approximations are investigated. Finally, the relationships between these two IF covering rough set models with other rough set models are revealed.

4.1. Characterizations of Huang et al.'s Intuitionistic Fuzzy Covering Rough Set Model

Huang et al. [33] presented an IF covering rough set model.

Definition 11. *([33]) Let $(U, \widehat{\mathbf{C}})$ be an IF β-covering approximation space. For each $A \in IF(U)$ where $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in U\}$, we define the intuitionistic fuzzy (IF) covering upper approximation $\widetilde{\mathbf{C}}(A)$ and lower approximation $\underset{\sim}{\mathbf{C}}(A)$ of A as:*

$$\widetilde{\mathbf{C}}(A) = \{\langle x, \vee_{y\in U}[\mu_{\widetilde{N}_x^\beta}(y) \wedge \mu_A(y)], \wedge_{y\in U}[\nu_{\widetilde{N}_x^\beta}(y) \vee \nu_A(y)]\rangle : x \in U\},$$
$$\underset{\sim}{\mathbf{C}}(A) = \{\langle x, \wedge_{y\in U}[\nu_{\widetilde{N}_x^\beta}(y) \vee \mu_A(y)], \vee_{y\in U}[\mu_{\widetilde{N}_x^\beta}(y) \wedge \nu_A(y)]\rangle : x \in U\}.$$

If $\widetilde{\mathbf{C}}(A) \neq \underset{\sim}{\mathbf{C}}(A)$, then A is called the first type of IF covering rough set.

Example 5. *(Continued from Example 1) Let $\beta = \langle 0.5, 0.3\rangle$, $A = \frac{(0.6,0.3)}{x_1} + \frac{(0.4,0.5)}{x_2} + \frac{(0.3,0.2)}{x_3} + \frac{(0.5,0.3)}{x_4} + \frac{(0.7,0.2)}{x_5}$.*

$$\widetilde{\mathbf{C}}(A) = \{\langle x_1, 0.6, 0.3\rangle, \langle x_2, 0.5, 0.2\rangle, \langle x_3, 0.6, 0.3\rangle, \langle x_4, 0.5, 0.2\rangle, \langle x_5, 0.6, 0.3\rangle\},$$
$$\underset{\sim}{\mathbf{C}}(A) = \{\langle x_1, 0.4, 0.3\rangle, \langle x_2, 0.4, 0.5\rangle, \langle x_3, 0.3, 0.4\rangle, \langle x_4, 0.4, 0.5\rangle, \langle x_5, 0.3, 0.3\rangle\}.$$

Some characterizations of Huang et al.'s IF covering rough set model are shown in the following proposition.

Proposition 7. *([33]) Let $(U, \widehat{\mathbf{C}})$ be an IF β-covering approximation space. Then, for all $A, B \in IF(U)$,*

1. $\underset{\sim}{\mathbf{C}}(U) = U$, $\widetilde{\mathbf{C}}(\emptyset) = \emptyset$;
2. $\widetilde{\mathbf{C}}(A') = (\underset{\sim}{\mathbf{C}}(A))'$, $\underset{\sim}{\mathbf{C}}(A') = (\widetilde{\mathbf{C}}(A))'$;
3. *If* $A \subseteq B$, *then* $\underset{\sim}{\mathbf{C}}(A) \subseteq \underset{\sim}{\mathbf{C}}(B)$, $\widetilde{\mathbf{C}}(A) \subseteq \widetilde{\mathbf{C}}(B)$;
4. $\underset{\sim}{\mathbf{C}}(A \bigcap B) = \underset{\sim}{\mathbf{C}}(A) \bigcap \underset{\sim}{\mathbf{C}}(B)$, $\widetilde{\mathbf{C}}(A \bigcup B) = \widetilde{\mathbf{C}}(A) \bigcup \widetilde{\mathbf{C}}(B)$;

5. $\underset{\sim}{C}(A \bigcup B) \supseteq \underset{\sim}{C}(A) \bigcup \underset{\sim}{C}(B)$, $\widetilde{C}(A \bigcap B) \subseteq \widetilde{C}(A) \bigcap \widetilde{C}(B)$.

Besides these characterizations shown in Proposition 7, there are other characterizations that should be investigated.

Example 6. *(Continued from Example 1) Let $\beta = \langle 0.5, 0.3 \rangle$.*

$$\widetilde{C}(U) = \{\langle x_1, 0.6, 0.2\rangle, \langle x_2, 0.5, 0.2\rangle, \langle x_3, 0.6, 0.3\rangle, \langle x_4, 0.5, 0.2\rangle, \langle x_5, 0.6, 0.3\rangle\},$$
$$\underset{\sim}{C}(\emptyset) = \{\langle x_1, 0.2, 0.6\rangle, \langle x_2, 0.2, 0.5\rangle, \langle x_3, 0.3, 0.6\rangle, \langle x_4, 0.2, 0.5\rangle, \langle x_5, 0.3, 0.6\rangle\}.$$

According to Example 6, we know $\widetilde{\mathbf{C}}(U) \neq U$ and $\underset{\sim}{\mathbf{C}}(\emptyset) \neq \emptyset$ in an IF β-covering approximation space. However, there are $\widetilde{\mathbf{C}}(U) = U$ and $\underset{\sim}{\mathbf{C}}(\emptyset) = \emptyset$ based on some conditions.

Proposition 8. *Let $(U, \widehat{C})$ be an IF β-covering approximation space with $\beta = \langle a, b\rangle$ and $\widehat{C} = \{C_1, C_2, \ldots, C_m\}$. $\underset{\sim}{C}(\emptyset) = \emptyset$ iff $\vee_{y \in U} \mu_{\widetilde{N}_x^\beta}(y) = 1, \wedge_{y \in U} \nu_{\widetilde{N}_x^\beta}(y) = 0$ for any $x \in U$.*

Proof.

$$\begin{aligned} \underset{\sim}{C}(\emptyset) = \emptyset \quad &\Leftrightarrow \widetilde{C}(U) = U \\ &\Leftrightarrow \vee_{y \in U}[\mu_{\widetilde{N}_x^\beta}(y) \wedge \mu_U(y)] = 1, \wedge_{y \in U}[\nu_{\widetilde{N}_x^\beta}(y) \vee \nu_U(y)] = 0, (\forall x \in U) \\ &\Leftrightarrow \vee_{y \in U} \mu_{\widetilde{N}_x^\beta}(y) = 1, \wedge_{y \in U} \nu_{\widetilde{N}_x^\beta}(y) = 0, (\forall x \in U). \end{aligned}$$

□

Remark 3. *Let $(U, \widehat{C})$ be an IF β-covering approximation space. The IF covering approximation operators $\underset{\sim}{C}$ and $\widetilde{C}$ in Definition 11 do not satisfy the following statements: for all $A, B \in IF(U)$,*

1. $\underset{\sim}{C}(\underset{\sim}{C}(A)) = \underset{\sim}{C}(A)$, $\widetilde{C}(\widetilde{C}(A)) = \widetilde{C}(A)$;
2. $\underset{\sim}{C}((\underset{\sim}{C}(A))') = (\underset{\sim}{C}(A))'$, $\widetilde{C}((\widetilde{C}(A))') = (\widetilde{C}(A))'$;
3. *For any $C \in \widehat{C}$, $\underset{\sim}{C}(C) = C$ and $\widetilde{C}(C) = C$;*

In order to illustrate this remark, the following example is introduced.

Example 7. *(Continued from Example 1) Let $\beta = \langle 0.5, 0.3\rangle$, and:*

$$A = \frac{(0.6,0.3)}{x_1} + \frac{(0.4,0.5)}{x_2} + \frac{(0.3,0.2)}{x_3} + \frac{(0.5,0.3)}{x_4} + \frac{(0.7,0.2)}{x_5},$$
$$C_1 = \frac{(0.7,0.2)}{x_1} + \frac{(0.5,0.3)}{x_2} + \frac{(0.4,0.5)}{x_3} + \frac{(0.6,0.1)}{x_4} + \frac{(0.3,0.2)}{x_5},$$

then:

$$\begin{aligned}
&\widetilde{C}(A) = \{\langle x_1, 0.6, 0.3\rangle, \langle x_2, 0.5, 0.2\rangle, \langle x_3, 0.6, 0.3\rangle, \langle x_4, 0.5, 0.2\rangle, \langle x_5, 0.6, 0.3\rangle\},\\
&\underset{\sim}{C}(A) = \{\langle x_1, 0.4, 0.3\rangle, \langle x_2, 0.4, 0.5\rangle, \langle x_3, 0.3, 0.4\rangle, \langle x_4, 0.4, 0.5\rangle, \langle x_5, 0.3, 0.3\rangle\},\\
&\widetilde{C}(\widetilde{C}(A)) = \{\langle x_1, 0.6, 0.3\rangle, \langle x_2, 0.5, 0.3\rangle, \langle x_3, 0.6, 0.3\rangle, \langle x_4, 0.5, 0.3\rangle, \langle x_5, 0.6, 0.3\rangle\},\\
&\underset{\sim}{C}(\underset{\sim}{C}(A)) = \{\langle x_1, 0.3, 0.4\rangle, \langle x_2, 0.3, 0.5\rangle, \langle x_3, 0.3, 0.4\rangle, \langle x_4, 0.3, 0.5\rangle, \langle x_5, 0.3, 0.3\rangle\},\\
&\widetilde{C}(C_1) = \{\langle x_1, 0.6, 0.2\rangle, \langle x_2, 0.5, 0.2\rangle, \langle x_3, 0.4, 0.3\rangle, \langle x_4, 0.5, 0.2\rangle, \langle x_5, 0.3, 0.3\rangle\},\\
&\underset{\sim}{C}(C_1) = \{\langle x_1, 0.3, 0.3\rangle, \langle x_2, 0.3, 0.4\rangle, \langle x_3, 0.3, 0.5\rangle, \langle x_4, 0.3, 0.4\rangle, \langle x_5, 0.3, 0.3\rangle\},\\
&(\widetilde{C}(A))' = \{\langle x_1, 0.3, 0.6\rangle, \langle x_2, 0.2, 0.5\rangle, \langle x_3, 0.3, 0.6\rangle, \langle x_4, 0.2, 0.5\rangle, \langle x_5, 0.3, 0.6\rangle\},\\
&(\underset{\sim}{C}(A))' = \{\langle x_1, 0.3, 0.4\rangle, \langle x_2, 0.5, 0.4\rangle, \langle x_3, 0.4, 0.3\rangle, \langle x_4, 0.5, 0.4\rangle, \langle x_5, 0.3, 0.3\rangle\},\\
&\widetilde{C}((\widetilde{C}(A))') = \{\langle x_1, 0.3, 0.5\rangle, \langle x_2, 0.3, 0.5\rangle, \langle x_3, 0.3, 0.5\rangle, \langle x_4, 0.3, 0.5\rangle, \langle x_5, 0.3, 0.5\rangle\},\\
&\underset{\sim}{C}((\underset{\sim}{C}(A))') = \{\langle x_1, 0.3, 0.4\rangle, \langle x_2, 0.3, 0.4\rangle, \langle x_3, 0.3, 0.4\rangle, \langle x_4, 0.3, 0.4\rangle, \langle x_5, 0.3, 0.3\rangle\}.
\end{aligned}$$

Hence, $\underset{\sim}{C}(\underset{\sim}{C}(A)) \neq \underset{\sim}{C}(A)$, $\widetilde{C}(\widetilde{C}(A)) \neq \widetilde{C}(A)$, $\underset{\sim}{C}((\underset{\sim}{C}(A))') \neq (\underset{\sim}{C}(A))'$, $\widetilde{C}((\widetilde{C}(A))') \neq (\widetilde{C}(A))'$, $\underset{\sim}{C}(C_1) \neq C_1$ *and* $\widetilde{C}(C_1) \neq C_1$.

A condition for $\underset{\sim}{\mathbf{C}}(A) \subseteq A \subseteq \widetilde{\mathbf{C}}(A)$ is proposed in the following proposition.

Proposition 9. *Let* $(U, \widehat{C})$ *be an IF β-covering approximation space with* $\beta = \langle a, b\rangle$ *and* $\widehat{C} = \{C_1, C_2, \ldots, C_m\}$*. Then, the following statements are equivalent:*

1. $\mu_{C_i}(x) \geq a, \nu_{C_i}(x) \leq b \Rightarrow \mu_{C_i}(x) = 1, \nu_{C_i}(x) = 0$ *for any* $x \in U$, $i \in \{1, 2, \ldots, m\}$;
2. $\underset{\sim}{C}(A) \subseteq A$ *for any* $A \in IF(U)$;
3. $A \subseteq \widetilde{C}(A)$ *for any* $A \in IF(U)$.

Proof. (1) ⇒ (2): According to (1), we know $\mu_{\widetilde{N}_x^\beta}(x) = 1$ and $\nu_{\widetilde{N}_x^\beta}(x) = 0$ for each $x \in U$. For each $A \in IF(U)$,

$$\begin{aligned}
\mu_{\underset{\sim}{\mathbf{C}}(A)}(x) &= \wedge_{y\in U}[\nu_{\widetilde{N}_x^\beta}(y) \vee \mu_A(y)]\\
&= (\wedge_{y\in U-\{x\}}[\nu_{\widetilde{N}_x^\beta}(y) \vee \mu_A(y)]) \wedge [\nu_{\widetilde{N}_x^\beta}(x) \vee \mu_A(x)]\\
&\leq \nu_{\widetilde{N}_x^\beta}(x) \vee \mu_A(x)\\
&= \mu_A(x).
\end{aligned}$$

$$\begin{aligned}
\nu_{\underset{\sim}{\mathbf{C}}(A)}(x) &= \vee_{y\in U}[\mu_{\widetilde{N}_x^\beta}(y) \wedge \nu_A(y)]\\
&= (\vee_{y\in U-\{x\}}[\mu_{\widetilde{N}_x^\beta}(y) \wedge \nu_A(y)]) \vee [\mu_{\widetilde{N}_x^\beta}(x) \wedge \nu_A(x)]\\
&\geq \mu_{\widetilde{N}_x^\beta}(x) \wedge \nu_A(x)\\
&= \nu_A(x).
\end{aligned}$$

Hence, $\underset{\sim}{\mathbf{C}}(A) \subseteq A$.

(2) ⇒ (3): According to (2) and the duality, it is immediate.

(3) ⇒ (1): For any $x \in U$, let $\mu_{1_x}(x) = 1$, $\nu_{1_x}(x) = 0$, elsewhere $\mu_{1_x}(y) = 0$, $\nu_{1_x}(y) = 1$ for all $y \in U - \{x\}$. Hence, $1_x \in IF(U)$. Since $A \subseteq \widetilde{\mathbf{C}}(A)$ for any $A \in IF(U)$, so:

$$\mu_{\widetilde{N}_x^\beta}(x) = \mu_{\widetilde{\mathbf{C}}(1_x)}(x) \geq \mu_{1_x}(x) = 1,\ \nu_{\widetilde{N}_x^\beta}(x) = \nu_{\widetilde{\mathbf{C}}(1_x)}(x) \leq \nu_{1_x}(x) = 0.$$

Therefore, $\mu_{C_i}(x) \geq a, \nu_{C_i}(x) \leq b \Rightarrow \mu_{C_i}(x) = 1, \nu_{C_i}(x) = 0$ for each $x \in U$, $i \in \{1, 2, \ldots, m\}$. □

4.2. An Intuitionistic Fuzzy Covering Rough Set Model for Crisp Subsets

In [18], Ma presented a fuzzy covering rough set model for crisp subsets. Inspired by his work, we propose an IF covering rough set model for crisp subsets.

Definition 12. *Let $(U, \widehat{C})$ be an IF β-covering approximation space. For each crisp subset $X \in P(U)$ ($P(U)$ is the power set of U), the IF covering upper approximation $\overline{\mathbf{C}}(X)$ and lower approximation $\underline{\mathbf{C}}(X)$ of X are defined as:*

$$\overline{\mathbf{C}}(X) = \{x \in U : \overline{N}_x^{\beta} \cap X \neq \emptyset\},$$
$$\underline{\mathbf{C}}(X) = \{x \in U : \overline{N}_x^{\beta} \subseteq X\}.$$

If $\overline{\mathbf{C}}(X) \neq \underline{\mathbf{C}}(X)$, then X is called the second type of IF covering rough set.

Example 8. *(Continued from Example 4) Let $\beta = \langle 0.5, 0.3 \rangle$, $X = \{x_1, x_2\}$, $Y = \{x_2, x_4, x_5\}$. Then:*

$$\overline{\mathbf{C}}(X) = \{x_1, x_2, x_4\}, \underline{\mathbf{C}}(X) = \{x_1\},$$
$$\overline{\mathbf{C}}(Y) = \{x_2, x_3, x_4, x_5\}, \underline{\mathbf{C}}(Y) = \{x_2, x_4, x_5\},$$
$$\overline{\mathbf{C}}(U) = U, \underline{\mathbf{C}}(U) = U, \overline{\mathbf{C}}(\emptyset) = \emptyset, \underline{\mathbf{C}}(\emptyset) = \emptyset.$$

The basic characterizations of the IF covering rough set model for crisp subsets are investigated in the following proposition.

Proposition 10. *Let $(U, \widehat{C})$ be an IF β-covering approximation space. Then, for all $X, Y \in P(U)$,*

1. $\underline{\mathbf{C}}(U) = U$, $\overline{\mathbf{C}}(\emptyset) = \emptyset$;
2. $\underline{\mathbf{C}}(\emptyset) = \emptyset$, $\overline{\mathbf{C}}(U) = U$;
3. $\underline{\mathbf{C}}(X') = (\overline{\mathbf{C}}(X))'$, $\overline{\mathbf{C}}(X') = (\underline{\mathbf{C}}(X))'$;
4. *If* $X \subseteq Y$, *then* $\underline{\mathbf{C}}(X) \subseteq \underline{\mathbf{C}}(Y)$, $\overline{\mathbf{C}}(X) \subseteq \overline{\mathbf{C}}(Y)$;
5. $\underline{\mathbf{C}}(X \bigcap Y) = \underline{\mathbf{C}}(X) \bigcap \underline{\mathbf{C}}(Y)$, $\overline{\mathbf{C}}(X \bigcup Y) = \overline{\mathbf{C}}(X) \bigcup \overline{\mathbf{C}}(Y)$;
6. $\underline{\mathbf{C}}(X \bigcup Y) \supseteq \underline{\mathbf{C}}(X) \bigcup \underline{\mathbf{C}}(Y)$, $\overline{\mathbf{C}}(X \bigcap Y) \subseteq \overline{\mathbf{C}}(X) \bigcap \overline{\mathbf{C}}(Y)$;
7. $\underline{\mathbf{C}}(X) \subseteq X \subseteq \overline{\mathbf{C}}(X)$;
8. $\underline{\mathbf{C}}(\underline{\mathbf{C}}(X)) \subseteq \underline{\mathbf{C}}(X), \overline{\mathbf{C}}(\overline{\mathbf{C}}(X)) \supseteq \overline{\mathbf{C}}(X)$;
9. $X \subseteq Y$ *or* $Y \subseteq X \Leftrightarrow \underline{\mathbf{C}}(X \cap Y) = \underline{\mathbf{C}}(X) \cap \underline{\mathbf{C}}(Y), \overline{\mathbf{C}}(X \cup Y) = \overline{\mathbf{C}}(X) \cup \overline{\mathbf{C}}(Y)$.

Proof. According to Definitions 10 and 12, it is immediate. □

4.3. Relationships between These Two Models and Some Other Rough Set Models

These two types of IF covering rough set models introduced above can be viewed as a bridge linking intuitionistic fuzzy sets and covering-based rough sets. In these models, $\widetilde{\mathbf{C}}$, $\underset{\sim}{\mathbf{C}}$ are IF approximation operators, and $\overline{\mathbf{C}}$, $\underline{\mathbf{C}}$ are crisp approximation operators in the IF environment. Firstly, we present the relationship between the IF covering rough set model defined in Section 4.1 and the generalized IF rough set model proposed by Zhou et al. in [38].

Definition 13. *([38]) Let U be a universe of discourse and $R \in IFR(U \times U)$. For any $A \in IF(U)$, the upper approximation $\widetilde{R}(A)$ and lower approximation $\underset{\sim}{R}(A)$ of A are defined as:*

$$\widetilde{R}(A) = \{\langle x, \vee_{y \in U}[\mu_R(x,y) \wedge \mu_A(y)], \wedge_{y \in U}[\nu_R(x,y) \vee \nu_A(y)]\rangle : x \in U\},$$
$$\underset{\sim}{R}(A) = \{\langle x, \wedge_{y \in U}[\nu_R(x,y) \vee \mu_A(y)], \vee_{y \in U}[\mu_R(x,y) \wedge \nu_A(y)]\rangle : x \in U\}.$$

For an IF β-covering $\widehat{\mathbf{C}}$ of U, one can define an IF relation R on the universe U as:

$$\mu_R(x,y) = \mu_{\widetilde{N}_x^\beta(y)}, \nu_R(x,y) = \nu_{\widetilde{N}_x^\beta(y)} \text{ for any } x, y \in U.$$

The induced IF relation R is related to all $C \in \widehat{\mathbf{C}}$. Hence, the IF covering rough set model defined in Section 4.1 can be viewed as a generalized IF rough set model presented by Zhou and Wu in Definition 13: for each $A \in IF(U)$,

$$\widetilde{\mathbf{C}}(A) = \{\langle x, \vee_{y\in U}[\mu_R(x,y) \wedge \mu_A(y)], \wedge_{y\in U}[\nu_R(x,y) \vee \nu_A(y)]\rangle : x \in U\} = \widetilde{R}(A),$$
$$\underset{\sim}{\mathbf{C}}(A) = \{\langle x, \wedge_{y\in U}[\nu_R(x,y) \vee \mu_A(y)], \vee_{y\in U}[\mu_R(x,y) \wedge \nu_A(y)]\rangle : x \in U\} = \underset{\sim}{R}(A).$$

Then, we present the relationship between the IF covering rough set model defined in Section 4.2 and a covering-based rough set model in Definition 3 proposed by Samanta and Chakraborty in [37].

Proposition 11. *Let $(U, \widehat{\mathbf{C}})$ be an IF β-covering approximation space and $\widehat{\mathbf{C}} = \{C_1, C_2, \ldots, C_m\}$.*

1. *If $K_i = \{x \in U : C_i(x) \geq \beta\}$ $(i = 1, 2, \ldots, m)$, then $\mathbf{C} = \{K_1, K_2, \ldots, K_m\}$ is a covering of U;*
2. *If (1) holds, then $N_{\mathbf{C}}(x) = \overline{N}_x^\beta$.*

Proof. (1) Since $\widehat{\mathbf{C}} = \{C_1, C_2, \ldots, C_m\}$ is an IF β-covering on U, so for any $x \in U$, there exists $C_i \in \widehat{\mathbf{C}}$ such that $C_i(x) \geq \beta$. Thus, $x \in K_i$. Hence, $\mathbf{C} = \{K_1, K_2, \ldots, K_m\}$ is a covering of U.

(2) Since $\mathbf{C} = \{K_1, K_2, \ldots, K_m\}$ is a covering of U with $K_i = \{x \in U : C_i(x) \geq \beta\}$, so:

$$\begin{aligned}
\overline{N}_x^\beta &= \{y \in U : \widetilde{N}_x^\beta(y) \geq \beta\} \\
&= \{y \in U : (\bigcap_{C_i(x)\geq\beta} C_i)(y) \geq \beta\} \\
&= \{y \in U : (\bigcap_{x\in K_i} C_i)(y) \geq \beta\} \\
&= \{y \in U : x \in K_i \Rightarrow y \in K_i, i = 1, 2, \ldots, m\} \\
&= \cap\{K_i \in \mathbf{C} : x \in K_i\} \\
&= N_{\mathbf{C}}(x).
\end{aligned}$$

□

According to Proposition 11, for any fixed $\beta = \langle a, b\rangle$ $(0 \leq a, b \leq 1$ and $a + b \leq 1)$, an IF β-covering $\widehat{\mathbf{C}}$ of U induces a covering $\mathbf{C}$ of U. Then, the second type of intuitionistic fuzzy covering rough set model defined in Subsection 4.2 can be viewed as a covering-based rough set model in Definition 3: for each $X \in P(U)$,

$$\overline{\mathbf{C}}(X) = \{x \in U : \overline{N}_x^\beta \cap X \neq \emptyset\} = \{x \in U : N_{\mathbf{C}}(x) \cap X \neq \emptyset\} = SH_{\mathbf{C}}(X),$$
$$\underline{\mathbf{C}}(X) = \{x \in U : \overline{N}_x^\beta \subseteq X\} = \{x \in U : N_{\mathbf{C}}(x) \subseteq X\} = SL_{\mathbf{C}}(X).$$

5. An Application to Multiple Criteria Group Decision Making

In [33], Huang et al. also defined an optimistic multi-granulation IF rough set model. In this section, we present a novel approach to MCGDM based on the optimistic multi-granulation IF rough set model. We investigate the basic description of an MCGDM problem under the framework of multi-granulation spaces. Then, we put forth a general decision making methodology for MCGDM problems by means of the optimistic multi-granulation IF rough set model in the case of patient ranking.

5.1. An Optimistic Multi-Granulation IF Rough Set Model

Let $\widehat{\mathbf{C}}$ be an IF β-covering of U. For each $x \in U$,

$$\widetilde{N}_x^{\beta} = \cap\{C_i \in \widehat{\mathbf{C}} : C_i(x) \geq \beta\}.$$

$\widetilde{N}_{\widehat{\mathbf{C}}}^{\beta} = \{\widetilde{N}_x^{\beta} : x \in U\}$, where $\mu_{\widetilde{N}_{\widehat{\mathbf{C}}}^{\beta}}(x,y) = \mu_{\widetilde{N}_x^{\beta}}(y)$ and $\nu_{\widetilde{N}_{\widehat{\mathbf{C}}}^{\beta}}(x,y) = \nu_{\widetilde{N}_x^{\beta}}(y)$ for any $y \in U$.

Definition 14. *([33]) Let $U = \{x_1, x_2, \cdots, x_m\}$ be a universe, (U, Γ_β) be a n-IF β-coverings approximation space with $\Gamma_\beta = \{\widehat{C_1}, \widehat{C_2}, \cdots, \widehat{C_n}\}$ and $(U, \mathfrak{N}_{\Gamma_\beta})$ be the IF β-neighborhood (graded neighborhood) approximation space induced by the n-IF β-covering approximation space (U, Γ_β). For each $A \in IF(U)$ where $A = \{\langle x_j, \mu_A(x_j), \nu_A(x_j)\rangle : 1 \leq j \leq m\}$, the optimistic upper approximation $\overline{\mathfrak{N}_{\Gamma_\beta}}^{(o)}(A)$ and lower approximation $\underline{\mathfrak{N}_{\Gamma_\beta}}^{(o)}(A)$ of A are defined as:*

$$\overline{\mathfrak{N}_{\Gamma_\beta}}^{(o)}(A) = \{\langle x_i, \mu_{\overline{\mathfrak{N}_{\Gamma_\beta}}^{(o)}(A)}(x_i), \nu_{\overline{\mathfrak{N}_{\Gamma_\beta}}^{(o)}(A)}(x_i)\rangle : 1 \leq i \leq m\},$$
$$\underline{\mathfrak{N}_{\Gamma_\beta}}^{(o)}(A) = \{\langle x_i, \mu_{\underline{\mathfrak{N}_{\Gamma_\beta}}^{(o)}(A)}(x_i), \nu_{\underline{\mathfrak{N}_{\Gamma_\beta}}^{(o)}(A)}(x_i)\rangle : 1 \leq i \leq m\},$$

where:

$$\mu_{\overline{\mathfrak{N}_{\Gamma_\beta}}^{(o)}(A)}(x_i) = \bigwedge_{k=1}^{n}\bigvee_{j=1}^{m}[\mu_{\widetilde{N}_{\widehat{C_k}}^{\beta}}(x_i,x_j) \wedge \mu_A(x_j)](1 \leq i \leq m),$$
$$\nu_{\overline{\mathfrak{N}_{\Gamma_\beta}}^{(o)}(A)}(x_i) = \bigvee_{k=1}^{n}\bigwedge_{j=1}^{m}[\nu_{\widetilde{N}_{\widehat{C_k}}^{\beta}}(x_i,x_j) \vee \nu_A(x_j)](1 \leq i \leq m),$$
$$\mu_{\underline{\mathfrak{N}_{\Gamma_\beta}}^{(o)}(A)}(x_i) = \bigvee_{k=1}^{n}\bigwedge_{j=1}^{m}[\nu_{\widetilde{N}_{\widehat{C_k}}^{\beta}}(x_i,x_j) \vee \mu_A(x_j)](1 \leq i \leq m),$$
$$\nu_{\underline{\mathfrak{N}_{\Gamma_\beta}}^{(o)}(A)}(x_i) = \bigwedge_{k=1}^{n}\bigvee_{j=1}^{m}[\mu_{\widetilde{N}_{\widehat{C_k}}^{\beta}}(x_i,x_j) \wedge \nu_A(x_j)](1 \leq i \leq m).$$

5.2. The Problem of Multiple Criteria Group Decision Making

Let $U = \{x_k : k = 1, 2, \cdots, l\}$ be the set of patients and $V = \{y_j | j = 1, 2, \cdots, m\}$ be the m main symptoms (for example, fever, cough, and so on) for a disease B. Assume that the duty doctor X invites n experts R_i $(i = 1, 2, \cdots, n)$ to evaluate every patient x_k $(k = 1, 2, \cdots, l)$.

Assume that every expert R_i $(i = 1, 2, \cdots, n)$ believes each patient $x_k \in U$ $(k = 1, 2, \cdots, l)$ has a symptom value C_{ij} $(j = 1, 2, \cdots, m)$, denoted by $C_{ij}(x_k) = \langle \mu_{C_{ij}}(x_k), \nu_{C_{ij}}(x_k)\rangle$, where $\mu_{C_{ij}}(x_k) \in [0, 1]$ is the degree that expert R_i confirms patient x_k has symptom y_j, $\nu_{C_{ij}}(x_k) \in [0, 1]$ is the degree that expert R_i confirms patient x_k does not have symptom y_j and $\mu_{C_{ij}}(x_k) + \nu_{C_{ij}}(x_k) \leq 1$.

Let $\beta = \langle a, b\rangle$ be the critical value. If any patient $x_k \in U$, there is at least one symptom $y_j \in V$ such that the symptom value C_{ij} for the patient x_k, which is diagnosed by the expert R_i, is not less than β, respectively, then $\Gamma_\beta = \{\widehat{\mathbf{C}_1}, \widehat{\mathbf{C}_2}, \cdots, \widehat{\mathbf{C}_n}\}$, where $\widehat{\mathbf{C}}_i = \{C_{i1}, C_{i2}, \cdots, C_{im}\}$, for all $1 \leq i \leq n$, is a n-IF β-coverings of U for some IF value β.

The IF β-neighborhood $\widetilde{N}_x^{\beta}$ of x induced by $\widehat{\mathbf{C}}_i$ $(1 \leq i \leq n)$ is an IFS:

$$\widetilde{N}_x^{\beta} = \cap\{C_{ij} \in \widehat{\mathbf{C}}_i : C_{ij}(x) \geq \beta, j = 1, 2, \cdots, m\}.$$

$\widetilde{N}_{\widehat{\mathbf{C}}_i}^{\beta} = \{\widetilde{N}_x^{\beta} : x \in U\}$, where $\mu_{\widetilde{N}_{\widehat{\mathbf{C}}_i}^{\beta}}(x, x_t) = \mu_{\widetilde{N}_x^{\beta}}(x_t)$ and $\nu_{\widetilde{N}_{\widehat{\mathbf{C}}_i}^{\beta}}(x, x_t) = \nu_{\widetilde{N}_x^{\beta}}(x_t)$ for any $t = 1, 2, \cdots, l$.

$\mu_{\widetilde{N}_{\widehat{\mathbf{C}}_i}^{\beta}}(x, x_t)$ denotes the minimum value among the degree of sickness of every patient x_t $(t = 1, 2, \cdots, l)$ according to the diagnoses of the expert R_i $(i = 1, 2, \cdots, n)$, respectively. $\nu_{\widetilde{N}_{\widehat{\mathbf{C}}_i}^{\beta}}(x, x_t)$ denotes the maximum value among the degree of non-sickness of every patient x_t $(t = 1, 2, \cdots, l)$ according to the diagnoses of the expert R_i $(i = 1, 2, \cdots, n)$, respectively.

If c is a possible degree and d is an impossible degree of the disease B of every patient $x_k \in U$ that is diagnosed by the duty doctor X, denoted by $A(x_k) = \langle c, d\rangle$, then the decision maker (the duty doctor X) for the MCGDM problem needs to know how to evaluate whether or not the patients $x_k \in U$ have the disease B.

5.3. Decision Making Methodology and Process

In this subsection, we give an approach to decision making for the problem of MCGDM with the above characterizations by means of the optimistic multi-granulation IF rough set model. According to the characterizations of the MCGDM problem in Subsection 5.2, we construct the multi-granulation intuitionistic fuzzy decision information systems and present the process of decision making under the framework of optimistic multi-granulation IF rough set model.

- **Input**: Multi-granulation fuzzy decision information systems $(U, \beta, \Gamma_\beta, A)$.
- **Output**: The score ordering for all alternatives.
- **Step 1**: Computing the IF β-neighborhood $\widetilde{N}_x^\beta$ of x induced by $\widehat{\mathbf{C}}_i \in \Gamma_\beta$, for all $x \in U$ and $i = 1, 2, \cdots, n$.
- **Step 2**: Computing the optimistic upper approximation $\overline{\mathfrak{N}_{\Gamma_\beta}}^{(o)}(A)$ and the optimistic lower approximation $\underline{\mathfrak{N}_{\Gamma_\beta}}^{(o)}(A)$.
- **Step 3**: Giving the right weight value of ζ, where $\zeta \in [0, 1]$.
- **Step 4**: Computing:

$$\sum_{i=1}^{n} \widetilde{R}_i(A) \quad = \zeta \underline{\mathfrak{N}_{\Gamma_\beta}}^{(o)}(A) + (1 - \zeta)\overline{\mathfrak{N}_{\Gamma_\beta}}^{(o)}(A).$$

- **Step 5**: Computing:

$$s(x) = \mu_{\sum\limits_{i=1}^{n} \widetilde{R}_i(A)}(x) - \nu_{\sum\limits_{i=1}^{n} \widetilde{R}_i(A)}(x) \text{ for any } x \in U.$$

- **Step 6**: Obtain the ranking for all $s(x)$ by using the principle of numerical size.

According to the above process, we can get the decision making according to the ranking. In Step 3, ζ reflects the preference of the decision maker for the risk of decision making problems. The decision maker can adjust ζ according to the goal in the real world. In Step 4, $\sum\limits_{i=1}^{n} \widetilde{R}_i(A)$ can be regarded as the compromise rule with the right weight ζ from the view of risk decision making with uncertainty. We define $\sum\limits_{i=1}^{n} \widetilde{R}_i(A) = \{\langle x_k, \zeta\mu_{\underline{\mathfrak{N}_{\Gamma_\beta}}^{(o)}(A)}(x_k) + (1-\zeta)\mu_{\overline{\mathfrak{N}_{\Gamma_\beta}}^{(o)}(A)}(x_k), \zeta\nu_{\underline{\mathfrak{N}_{\Gamma_\beta}}^{(o)}(A)}(x_k) + (1-\zeta)\nu_{\overline{\mathfrak{N}_{\Gamma_\beta}}^{(o)}(A)}(x_k)\rangle : 1 \le k \le l\}$.

5.4. An Applied Example

Assume that $U = \{x_1, x_2, x_3, x_4, x_5\}$ is a set of patients. According to the patients' symptoms, we write $V = \{y_1, y_2, y_3, y_4\}$ to be four main symptoms (fever, cough, sore and headache) for a disease B. Assume that the duty doctor X invites two experts R_i $(i = 1, 2)$ to evaluate every patient x_k $(k = 1, 2, \cdots, 5)$ as in Tables 2 and 3.

Table 2. Symptom values of expert R_1 for every patient x_k $(k = 1, 2, \cdots, 5)$.

U	C_{11}	C_{12}	C_{13}	C_{14}
x_1	$\langle 0.7, 0.2\rangle$	$\langle 0.6, 0.2\rangle$	$\langle 0.4, 0.1\rangle$	$\langle 0.1, 0.5\rangle$
x_2	$\langle 0.5, 0.3\rangle$	$\langle 0.3, 0.2\rangle$	$\langle 0.4, 0.5\rangle$	$\langle 0.6, 0.1\rangle$
x_3	$\langle 0.4, 0.5\rangle$	$\langle 0.2, 0.3\rangle$	$\langle 0.5, 0.2\rangle$	$\langle 0.6, 0.3\rangle$
x_4	$\langle 0.6, 0.1\rangle$	$\langle 0.4, 0.5\rangle$	$\langle 0.3, 0.6\rangle$	$\langle 0.5, 0.3\rangle$
x_5	$\langle 0.3, 0.2\rangle$	$\langle 0.7, 0.3\rangle$	$\langle 0.6, 0.3\rangle$	$\langle 0.8, 0.1\rangle$

Table 3. Symptom values of expert R_2 for every patient x_k ($k = 1, 2, \cdots, 5$).

U	C_{21}	C_{22}	C_{23}	C_{24}
x_1	$\langle 0.6, 0.2\rangle$	$\langle 0.7, 0.2\rangle$	$\langle 0.4, 0.2\rangle$	$\langle 0.2, 0.6\rangle$
x_2	$\langle 0.4, 0.2\rangle$	$\langle 0.5, 0.3\rangle$	$\langle 0.5, 0.4\rangle$	$\langle 0.6, 0.3\rangle$
x_3	$\langle 0.5, 0.4\rangle$	$\langle 0.3, 0.4\rangle$	$\langle 0.5, 0.2\rangle$	$\langle 0.7, 0.2\rangle$
x_4	$\langle 0.6, 0.1\rangle$	$\langle 0.4, 0.5\rangle$	$\langle 0.4, 0.6\rangle$	$\langle 0.5, 0.4\rangle$
x_5	$\langle 0.4, 0.2\rangle$	$\langle 0.6, 0.3\rangle$	$\langle 0.5, 0.3\rangle$	$\langle 0.7, 0.2\rangle$

Let $\beta = \langle 0.5, 0.3\rangle$ be the critical value. Then, $\Gamma_\beta = \{\widehat{\mathbf{C}_1}, \widehat{\mathbf{C}_2}\}$, where $\widehat{\mathbf{C}}_i = \{C_{i1}, C_{i2}, C_{i3}, C_{i4}\}$, for all $i = 1, 2$, is a 2-IF β-coverings of U. $\widetilde{N}^{\beta}_{\widehat{\mathbf{C}_i}}$ ($i = 1, 2$) are shown in Tables 4 and 5, respectively.

Table 4. $\widetilde{N}^{\beta}_{\widehat{\mathbf{C}_1}}$.

$\widetilde{N}^{\beta}_{\widehat{\mathbf{C}_1}}$	x_1	x_2	x_3	x_4	x_5
$\widetilde{N}^{\beta}_{x_1}$	$\langle 0.6, 0.2\rangle$	$\langle 0.3, 0.3\rangle$	$\langle 0.2, 0.5\rangle$	$\langle 0.4, 0.5\rangle$	$\langle 0.3, 0.3\rangle$
$\widetilde{N}^{\beta}_{x_2}$	$\langle 0.1, 0.5\rangle$	$\langle 0.5, 0.3\rangle$	$\langle 0.4, 0.5\rangle$	$\langle 0.5, 0.3\rangle$	$\langle 0.3, 0.2\rangle$
$\widetilde{N}^{\beta}_{x_3}$	$\langle 0.1, 0.5\rangle$	$\langle 0.4, 0.5\rangle$	$\langle 0.5, 0.3\rangle$	$\langle 0.3, 0.6\rangle$	$\langle 0.6, 0.3\rangle$
$\widetilde{N}^{\beta}_{x_4}$	$\langle 0.1, 0.5\rangle$	$\langle 0.5, 0.3\rangle$	$\langle 0.4, 0.5\rangle$	$\langle 0.5, 0.3\rangle$	$\langle 0.3, 0.2\rangle$
$\widetilde{N}^{\beta}_{x_5}$	$\langle 0.1, 0.5\rangle$	$\langle 0.3, 0.5\rangle$	$\langle 0.2, 0.3\rangle$	$\langle 0.3, 0.6\rangle$	$\langle 0.6, 0.3\rangle$

Table 5. $\widetilde{N}^{\beta}_{\widehat{\mathbf{C}_2}}$.

$\widetilde{N}^{\beta}_{\widehat{\mathbf{C}_2}}$	x_1	x_2	x_3	x_4	x_5
$\widetilde{N}^{\beta}_{x_1}$	$\langle 0.6, 0.2\rangle$	$\langle 0.4, 0.3\rangle$	$\langle 0.3, 0.4\rangle$	$\langle 0.4, 0.5\rangle$	$\langle 0.4, 0.3\rangle$
$\widetilde{N}^{\beta}_{x_2}$	$\langle 0.2, 0.6\rangle$	$\langle 0.5, 0.3\rangle$	$\langle 0.3, 0.4\rangle$	$\langle 0.4, 0.5\rangle$	$\langle 0.6, 0.3\rangle$
$\widetilde{N}^{\beta}_{x_3}$	$\langle 0.2, 0.6\rangle$	$\langle 0.5, 0.4\rangle$	$\langle 0.5, 0.2\rangle$	$\langle 0.4, 0.6\rangle$	$\langle 0.5, 0.3\rangle$
$\widetilde{N}^{\beta}_{x_4}$	$\langle 0.6, 0.2\rangle$	$\langle 0.4, 0.2\rangle$	$\langle 0.5, 0.4\rangle$	$\langle 0.6, 0.1\rangle$	$\langle 0.4, 0.2\rangle$
$\widetilde{N}^{\beta}_{x_5}$	$\langle 0.2, 0.6\rangle$	$\langle 0.5, 0.4\rangle$	$\langle 0.3, 0.4\rangle$	$\langle 0.4, 0.6\rangle$	$\langle 0.5, 0.3\rangle$

Assume that the duty doctor X diagnosed the value $A = \frac{(0.6,0.3)}{x_1} + \frac{(0.4,0.5)}{x_2} + \frac{(0.3,0.2)}{x_3} + \frac{(0.5,0.3)}{x_4} + \frac{(0.7,0.2)}{x_5}$ of the disease B of every patient. Then:

$$\overline{\mathfrak{N}_{\Gamma_\beta}}^{(o)}(A) = \frac{(0.6,0.3)}{x_1} + \frac{(0.5,0.3)}{x_2} + \frac{(0.5,0.3)}{x_3} + \frac{(0.5,0.2)}{x_4} + \frac{(0.5,0.3)}{x_5},$$
$$\underline{\mathfrak{N}_{\Gamma_\beta}}^{(o)}(A) = \frac{(0.4,0.3)}{x_1} + \frac{(0.4,0.5)}{x_2} + \frac{(0.3,0.4)}{x_3} + \frac{(0.4,0.4)}{x_4} + \frac{(0.4,0.3)}{x_5}.$$

Let $\zeta = 0.7$. Then:

$$\begin{aligned}\sum_{i=1}^{2} \widetilde{R}_i(A) &= 0.7\underline{\mathfrak{N}_{\Gamma_\beta}}^{(o)}(A) + (1-0.7)\overline{\mathfrak{N}_{\Gamma_\beta}}^{(o)}(A) \\ &= \frac{(0.46,0.3)}{x_1} + \frac{(0.43,0.44)}{x_2} + \frac{(0.36,0.37)}{x_3} + \frac{(0.43,0.34)}{x_4} + \frac{(0.43,0.3)}{x_5}.\end{aligned}$$

Hence, we can obtain $s(x_k)$ ($k = 1, 2, \cdots, 5$) in Table 6.

Table 6. $s(x_k)$ ($k = 1, 2, \cdots, 5$).

U	x_1	x_2	x_3	x_4	x_5
$s(x_k)$	0.16	−0.01	−0.01	0.09	0.13

According to the principle of numerical size, we have:

$$s(x_2) = s(x_3) < s(x_4) < s(x_5) < s(x_1).$$

Therefore, the doctor X diagnoses the patient x_1 as more likely to be sick with the disease B.

6. Conclusions

Covering rough set models are important research topics, which investigate data mining in a more general manner. Huang et al. [33] presented an IF rough set model and an optimistic multi-granulation IF rough set model. By investigation, we have found that no one has applied multi-granulation IF rough set models to MCGDM problems. In this paper, by showing some new notions and properties of IF β-covering approximation spaces, we mainly study Huang et al.'s models and propose a novel approach to MCGDM problems. The main conclusions in this paper and the further work are listed as follows.

1. Some new notions and properties of IF β-covering approximation spaces are proposed. Aiming at the new notion of β-neighborhood systems, we present a necessary and sufficient condition for two IF β-coverings to induce the same IF β-neighborhood systems.
2. By introducing Huang et al.'s IF rough set model, some new characterizations of it are investigated. We present a new IF covering rough set model for crisp subsets, and the relationships between these two IF covering rough set models and some other rough set models are investigated. Neutrosophic sets and related algebraic structures [39–43] will be connected with the research content of this paper in further research.
3. We construct the multi-granulation intuitionistic fuzzy decision information systems and present a novel approach to MCGDM problems based on the optimistic multi-granulation IF rough set model. There are many MCGDM technologies by rough set models [20,23]. However, among these models, the multi-granulation IF rough set models are not used. We first use the optimistic multi-granulation IF rough set model to solve MCGDM problems.

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