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Review

Supersymmetric Higher Spin Models in Three Dimensional Spaces

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Abstract: We review the component Lagrangian construction of the supersymmetric higher spin models in three-dimensional (3D) Minkowski and anti de Sitter (AdS) spaces. The approach is based on the frame-like gauge-invariant formulation, where massive higher spin fields are realized through a system of massless ones. We develop a supersymmetric generalization of this formulation to the Lagrangian construction of the on-shell $\mathcal{N} = 1$, 3D higher spin supermultiplets. In 3D Minkowski space, we show that the massive supermultiplets can be constructed from one extended massless supermultiplet by adding the mass terms to the Lagrangian and the corresponding corrections to the supertransformations of the fermionic fields. In 3D AdS space, we construct massive supermultiplets using a formulation of the massive fields in terms of the set of gauge-invariant objects (curvatures) in the process of their consistent supersymmetric deformation.

Keywords: supersymmetry; higher spins; three-dimensional space times; gauge invariance

1. Introduction

As soon as the supersymmetry (see, e.g., the books [1–3]) has been discovered, it aroused an immediate interest in finding the supersymmetric generalization of the known theories. In a short time, there appeared the supersymmetric extensions of such famous theories as the standard model, the Einstein's gravity and string theory. Construction of supersymmetric models and the study of their properties on classical and quantum levels became one of the most attractive trends in modern theoretical physics.

In the last few decades, there was an essential progress in the higher spin field theory (see, e.g., the reviews [4–6]). The purpose of this review is to describe a recent development of Lagrangian construction for massless and massive supersymmetric higher spin models in three-dimensional Minkowski and anti de Sitter spaces.

In the papers [7,8], the massless $\mathcal{N} = 1$ supersymmetric higher spin field theory was formulated in four-dimensional (4D) Minkowski space. The basic results of these papers were the global on-shell $\mathcal{N} = 1$ supertransformations leaving invariant the pair of (Fang)-Fronsdal Lagrangians for free massless higher spin-$(s, s + 1/2)$ fields [9,10]. The off-shell formulation of such a system was given in [11,12], where the $\mathcal{N} = 1$ superfield extension of (Fang)-Fronsdal Lagrangians in 4D Minkowski space was obtained. Later on, this result was generalized for 4D AdS space [13]. In both cases, the constructed superfield models, up to elimination of auxiliary fields, reduce to the sum of spin-s

and spin-($s + 1/2$) (Fang)-Fronsdal Lagrangians, thus describing $\mathcal{N} = 1$, 4D massless higher spin supermultiplets. Later, making use the same technique, the off-shell formulation of 4D, $\mathcal{N} = 2$ massless higher spin supermultiplets was found [14].

There exist much fewer results in the study of supersymmetric massive higher spin models. The reason is that shifting from massless component formulation to the massive one, we have to introduce the very complicated higher derivative corrections to the supertransformations. Moreover, the higher the spin of the fields entering the supermultiplet, the higher the number of derivatives one has to consider. The problem of the supersymmetric description of the 4D massive higher spin supermultiplet was resolved explicitly only in 2007 for the case $\mathcal{N} = 1$ on-shell Poincare superalgebra [15]. The solution was based on the generalization of the gauge-invariant formulation of the massive higher spin fields [16–18] to the case of massive supermultiplets. In such a formulation, the massive supermultiplets are described as a system of the appropriate massless ones coupled by local symmetries. On the other hand, this system of massless supermultiplets should be invariant under the initial massless supertransformations corrected in a certain way. In [15], it was shown that to obtain the massive deformation, it is enough to add the non-derivative corrections to the supertransformations for the fermions only. Complicated higher derivative corrections to the supertransformations reappear if one tries to fix all local symmetries breaking gauge invariance (attempts to develop the off-shell superfield formulation of the massive 4D higher spin supermultiplets were considered for some examples in [19–21]). Surprisingly, in 4D, the above results are still the main results in massive supersymmetric higher spin theory till now.

Taking into account the difficulties in constructing the Lagrangian formulation for 4D massive higher spin supermultiplets, it is natural to study the same problems in a simpler case, for example to consider the massive higher spin supermultiplets in three dimensions. Indeed, in the last few years, much attention in the supersymmetric higher spin theory has been focused on 3D spaces where higher spin theory is much more simple (see, e.g., [4,22]). Here, it is important to emphasize that in general, a supersymmetry in different dimensions is realized quite differently. The matter is that the supersymmetry operates with spinor fields, which are formulated separately for each space-time dimension. Therefore, the 3D supersymmetry is an independent type of symmetry and should be considered by itself.

It is known that in 3D, the massless higher spin fields ($k \geq 3/2$) do not propagate any physical degrees of freedom, and one of the reasons to study such models can be the possibility to consider their deformation to a massive theory. In turn, the massive higher spin fields in 3D do propagate two physical degrees of freedom [23]. It is important to note that massive higher spin fields can be realized in different ways. One of the possibilities is to generate the mass for 3D massless gauge fields by adding a Chern–Simons-like term [24] generalizing 3D topologically massive gravity [25]. Note that such a description is based on the higher derivative parity odd Lagrangians. The existence of such Lagrangians is a very specific feature of the three-dimensional theories, so they do not admit any straightforward generalization to higher dimensions. Recently for these models, the off-shell $\mathcal{N} = 1$ and $\mathcal{N} = 2$ superfield extensions have been constructed in 3D Minkowski space [26,27].

In our work, we use another possibility to describe massive higher spin fields similar to the one used in higher dimensions [28]. There are two main ingredients in this formalism. The first one is the gauge-invariant formulation for the massive bosonic and fermionic fields (for the non-gauge-invariant description, see [29]). The main idea is to begin with the appropriately chosen set of massless fields and then glue them together in such a way as to keep all (though modified) their gauge symmetries. This guarantees the correct number of the physical degrees of freedom and the absence of Ghosts. Moreover, such a formalism nicely works both in flat Minkowski space, as well as in anti de Sitter space. Let us stress once again that it is the usage of the gauge-invariant description for massive higher spin bosonic and fermionic fields that allowed one of us to construct massive higher spin supermultiplets in $D = 4$ [15]. The second ingredient is the frame-like formalism, which is just the higher spin generalization of the rather well-known frame-like formalism for gravity. It has a number of very convenient features. In particular, being completely antisymmetric on the world indices, it allows one to use the language of differential forms admitting, e.g., a coordinate-free description for the background Minkowski and anti de Sitter spaces (see below). For the three dimensions, this possibility has been realized in the papers [30,31]. The goal of this review is to present the general methods of supersymmetric Lagrangian construction for massive higher spin fields in 3D Minkowski and AdS spaces. These methods are based on the gauge-invariant description of the massive fields [30,31] and are realized in the component approach for the case of on-shell $\mathcal{N} = 1$ supersymmetry. The main content of this review is based on the papers [32,33].

The review is organized as follows. In the rest of the Introduction, we fix our notations and conventions on 3D field variables. In Sections 2 and 3, we present the Lagrangian formulation of 3D free bosonic and fermionic higher spin fields, respectively. In Section 4, we construct the Lagrangian formulation for higher spin supermultiplets in 3D Minkowski space. Here, we show that it is possible to construct one extended massless supermultiplet and then smoothly deform it into the massive one. Another approach is used in Section 5 for the construction of the massive supermultiplets in 3D AdS. It is based on the Lagrangian formulations in terms of the explicitly gauge-invariant objects and their consistent supersymmetric deformation. Such an approach is more elegant, but it requires the introduction of the so-called extra fields.

Notations and conventions: In this review, we use a language of differential forms where all the objects are some p-forms Ω (p = 0, 1, 2, 3). It is defined as:

$$\Omega = \theta^{\mu_1}...\theta^{\mu_p}\Omega_{\mu_1...\mu_p}, \qquad \theta^\mu\theta^\nu = -\theta^\nu\theta^\mu$$

In particular, the derivative is defined as one-form $d = \theta^\mu\partial_\mu$.

In 3D, it is more convenient to use a frame-like multispinor formalism where all the objects have totally symmetric local spinor indices. To simplify the expressions, we will use the condensed notations for the spinor indices such that, e.g.,

$$\Omega^{\alpha(2k)} = \Omega^{(\alpha_1\alpha_2...\alpha_{2k})}$$

Furthermore, we always assume that spinor indices denoted by the same letters and placed on the same level are symmetrized, e.g.,

$$\Omega^{\alpha(2k)}\zeta^\alpha = \Omega^{(\alpha_1...\alpha_{2k}}\zeta^{\alpha_{2k+1})}$$

In flat space, usual derivative d commutes $d \wedge d = 0$, while for AdS space, we use the following normalization of the covariant derivative:

$$D \wedge D\zeta^\alpha = -\lambda^2 E^\alpha{}_\beta\zeta^\beta$$

Basis elements of $1,2,3$-form spaces are $e^{\alpha(2)}$, $E^{\alpha(2)}$, E, respectively, where the last two are defined as the double and triple wedge product of the frame $e^{\alpha(2)}$:

$$e^{\alpha\alpha} \wedge e^{\beta\beta} = \varepsilon^{\alpha\beta}E^{\alpha\beta}, \qquad E^{\alpha\alpha} \wedge e^{\beta\beta} = \varepsilon^{\alpha\beta}\varepsilon^{\alpha\beta}E.$$

Furthermore, we write some useful relations for these basis elements:

$$E^\alpha{}_\gamma \wedge e^{\gamma\beta} = 3\varepsilon^{\alpha\beta}E, \qquad e^\alpha{}_\gamma \wedge e^{\gamma\beta} = 4E^{\alpha\beta}.$$

Further on, the sign of wedge product $\wedge$ will be omitted.

2. Free Higher Spin Bosonic Models

In this section, we review the Lagrangian description of the arbitrary spin bosonic fields in three-dimensional Minkowski and its cosmological extension AdS spaces [30]. Both for massless and massive fields, we present the gauge-invariant formulation using the frame-like field variables. These fields generalize the tetrad and Lorentz connection in the frame formulation of gravity. Such an approach allows us to construct gauge-invariant objects (we will call them curvatures) similar to the gravitational curvature and torsion and use them to simplify many constructions.

2.1. Massless Fields

It is well known that all massless fields with spin $k \geq 1$ are gauge ones; therefore, the gauge-invariant formulation for them is a natural form of description. As the gravity, which can be described in terms of the metric $g_{\mu\nu}$ field or in terms of the frame field $e_\mu{}^a$ and the Lorentz connection $\omega_\mu{}^{a,b}$, the massless higher spins can be described in two ways: metric-like or frame-like. In 4D Minkowski or AdS spaces, the metric-like approach leads to the Fronsdal formulation of massless integer spin-k in terms of totally symmetric tensors $\varphi_{\mu_1\mu_2\ldots\mu_k}$ subject to the double tracelessness condition $\varphi^\sigma{}_\sigma{}^\rho{}_{\rho\mu_5\ldots\mu_k}$, which becomes nontrivial for $k \geq 4$. In the frame-like approach, such a field is described by the generalized frame and Lorentz connection fields:

$$f_\mu{}^{a_1\ldots a_{k-1}}, \qquad \Omega_\mu{}^{a_1\ldots a_{k-1},b}$$

(Actually, for 4D massless fields with spin $k > 2$, one should consider extra gauge fields $\Omega_\mu{}^{a_1\ldots a_{k-1},b_1\ldots b_t}$ where $2 \leq t \leq (k-1)$. They do not enter the free Lagrangian, but do play a crucial role in the Vasiliev interacting theory. However, in 3D, these extra fields are absent in the massless case). Here, μ is the curved world index and a, b the flat tangent indices. World and flat indices are related by the background Minkowski or AdS frame $e_\mu{}^a$. The flat indices of these generalized fields correspond to the irreducible $so(3,1)$ Lorentz tensors. The Fronsdal formulation is recovered by eliminating the auxiliary field $\Omega_\mu{}^{a_1\ldots a_{k-1},b}$ and considering symmetric combination:

$$\varphi_{\mu_1\ldots\mu_k} = e_{(\mu_1}{}^{a_1} e_{\mu_2}{}^{a_2} \ldots e_{\mu_{k-1}}{}^{a_{k-1}} f_{\mu_k)a_1a_2\ldots a_{k-1}}$$

In 3D, it is convenient to use a dual higher spin connection:

$$\Omega_\mu{}^{a_1\ldots a_{k-1}} = \varepsilon_{bc}{}^{(a_1}\Omega_\mu{}^{a_2\ldots a_{k-1})b,c}$$

where ε^{abc} is a totally antisymmetric tensor. Therefore, in 3D, the frame-like and Lorentz-like higher spin gauge fields have the same index structure, and their local indices form irreducible $so(2,1)$ Lorentz tensors. Due to the isomorphism $so(2,1) \sim sp(2)$, it is very convenient to use multispinor formalism in which our gauge fields take the form:

$$f_\mu{}^{\alpha_1\alpha_2\ldots\alpha_{2k-2}}, \qquad \Omega_\mu{}^{\alpha_1\alpha_2\ldots\alpha_{2k-2}}$$

where $\alpha = 1, 2$ are spinor indices. Below, we make use of the language of differential forms, considering the above higher spin field variables as one-forms (we omit index μ) and condensed notations for the spinor indices given in the Introduction.

Spin-$(k+1)$ $(k \geq 1)$:

In the frame-like formalism, it is described by the physical one-form $f^{\alpha(2k)}$ and the auxiliary one-form $\Omega^{\alpha(2k)}$. The Lagrangian in 3D AdS looks like:

$$\begin{aligned}\mathcal{L} &= (-1)^{k+1}[k\Omega_{\alpha(2k-1)\beta}e^{\beta}{}_{\gamma}\Omega^{\alpha(2k-1)\gamma} + \Omega_{\alpha(2k)}Df^{\alpha(2k)} \\ &\qquad + \frac{k\lambda^2}{4}f_{\alpha(2k-1)\beta}e^{\beta}{}_{\gamma}f^{\alpha(2k-1)\gamma}]\end{aligned} \tag{1}$$

It is invariant under gauge transformations:

$$\begin{aligned}\delta\Omega^{\alpha(2k)} &= D\eta^{\alpha(2k)} + \tfrac{\lambda^2}{4}e^{\alpha}{}_{\beta}\xi^{\alpha(2k-1)\beta} \\ \delta f^{\alpha(2k)} &= D\xi^{\alpha(2k)} + e^{\alpha}{}_{\beta}\eta^{\alpha(2k-1)\beta}\end{aligned} \tag{2}$$

One can construct a pair of the gauge-invariant curvatures:

$$\begin{aligned}\mathcal{R}^{\alpha(2k)} &= D\Omega^{\alpha(2k)} + \tfrac{\lambda^2}{4}e^{\alpha}{}_{\beta}f^{\alpha(2k-1)\beta} \\ \mathcal{T}^{\alpha(2k)} &= Df^{\alpha(2k)} + e^{\alpha}{}_{\beta}\Omega^{\alpha(2k-1)\beta}\end{aligned} \tag{3}$$

Using these curvatures, the Lagrangian can be rewritten as follows:

$$\mathcal{L} = \frac{(-1)^{k+1}}{2}[\Omega_{\alpha(2k)}\mathcal{T}^{\alpha(2k)} + f_{\alpha(2k)}\mathcal{R}^{\alpha(2k)}] \tag{4}$$

In order to obtain the formulation in 3D Minkowski space, one should put $\lambda \to 0$. Thus, kinetic terms for higher spin fields are just the first line in (1) where $D \to d$. Let us also present kinetic terms for the lower spin fields. The frame-like formulation for them is just the first-order formalism.

Spin-1 is described by the physical one-form A and the auxiliary zero-form $B^{\alpha(2)}$. The Lagrangian has the form:

$$\mathcal{L} = EB_{\alpha\beta}B^{\alpha\beta} - B_{\alpha\beta}e^{\alpha\beta}dA$$

and it is invariant under the gauge transformations with zero-form parameter ξ:

$$\delta A = d\xi$$

Spin-0 is described by the physical zero-form φ and the auxiliary zero-form $\pi^{\alpha(2)}$. The expression for the Lagrangian looks like:

$$\mathcal{L} = -E\pi_{\alpha\beta}\pi^{\alpha\beta} + \pi_{\alpha\beta}E^{\alpha\beta}d\varphi$$

As will be seen below, all these constructions play a role in the gauge-invariant formulation of the massive bosonic fields.

2.2. Massive Fields

In the gauge-invariant form, the massive spin s field can be described as a system of the massless fields with spins $s, (s-1), ..., 0$. In the frame-like approach, the corresponding set of fields consists of:

$$(\Omega^{\alpha(2k)}, f^{\alpha(2k)}) \quad 1 \le k \le (s-1), \quad (B^{\alpha(2)}, A), \quad (\pi^{\alpha(2)}, \varphi)$$

The Lagrangian for the free fields with mass m in 3D AdS has the form:

$$
\begin{aligned}
\mathcal{L} \quad = \quad & \sum_{k=1}^{s-1}(-1)^{k+1}[k\Omega_{\alpha(2k-1)\beta}e^{\beta}{}_{\gamma}\Omega^{\alpha(2k-1)\gamma} + \Omega_{\alpha(2k)}Df^{\alpha(2k)}] \\
& + EB_{\alpha\beta}B^{\alpha\beta} - B_{\alpha\beta}e^{\alpha\beta}DA - E\pi_{\alpha\beta}\pi^{\alpha\beta} + \pi_{\alpha\beta}E^{\alpha\beta}D\varphi \\
& + \sum_{k=1}^{s-2}(-1)^{k+1}a_k[-\tfrac{(k+2)}{k}\Omega_{\alpha(2)\beta(2k)}e^{\alpha(2)}f^{\beta(2k)} + \Omega_{\alpha(2k)}e_{\beta(2)}f^{\alpha(2k)\beta(2)}] \\
& + 2a_0\Omega_{\alpha(2)}e^{\alpha(2)}A - a_0 f_{\alpha\beta}E^{\beta}{}_{\gamma}B^{\alpha\gamma} + 2sM\pi_{\alpha\beta}E^{\alpha\beta}A \\
& + \sum_{k=1}^{s-1}(-1)^{k+1}b_k f_{\alpha(2k-1)\beta}e^{\beta}{}_{\gamma}f^{\alpha(2k-1)\gamma} + b_0 f_{\alpha(2)}E^{\alpha(2)}\varphi + \tfrac{3a_0{}^2}{2}E\varphi^2
\end{aligned}
\tag{5}
$$

where:

$$
\begin{aligned}
a_k{}^2 &= \frac{k(s+k+1)(s-k-1)}{2(k+1)(k+2)(2k+3)}[M^2-(k+1)^2\lambda^2] \\
a_0{}^2 &= \frac{(s+1)(s-1)}{3}[M^2-\lambda^2] \\
b_k &= \frac{s^2M^2}{4k(k+1)^2}, \qquad b_0 = \frac{sMa_0}{2}, \qquad M^2 = m^2+(s-1)^2\lambda^2
\end{aligned}
\tag{6}
$$

Let us briefly discuss the structure of the Lagrangian (5). The first two lines are kinetic terms. It is just the sum of the massless Lagrangians for spins $s, (s-1), ..., 0$ where ordinary derivatives are replaced by the AdS covariant ones. The third and the fourth lines contain cross-terms for neighboring spins. These cross-terms couple the individual massless fields into the whole system describing the massive spin-s field. The last line in (5) contains the mass terms. The coefficients in (6) are determined by the invariance of the Lagrangian under the following gauge transformations:

$$
\begin{aligned}
\delta\Omega^{\alpha(2k)} &= D\eta^{\alpha(2k)} + \frac{(k+2)a_k}{k}e_{\beta(2)}\eta^{\alpha(2k)\beta(2)} \\
&\quad + \frac{a_{k-1}}{k(2k-1)}e^{\alpha(2)}\eta^{\alpha(2k-2)} + \frac{b_k}{k}e^{\alpha}{}_{\beta}\xi^{\alpha(2k-1)\beta} \\
\delta f^{\alpha(2k)} &= D\xi^{\alpha(2k)} + e^{\alpha}{}_{\beta}\eta^{\alpha(2k-1)\beta} + a_k e_{\beta(2)}\xi^{\alpha(2k)\beta(2)} \\
&\quad + \frac{(k+1)a_{k-1}}{k(k-1)(2k-1)}e^{\alpha(2)}\xi^{\alpha(2k-2)} \\
\delta\Omega^{\alpha(2)} &= D\eta^{\alpha(2)} + 3a_1 e_{\beta(2)}\eta^{\alpha(2)\beta(2)} + b_1 e^{\alpha}{}_{\gamma}\xi^{\alpha\gamma} \\
\delta f^{\alpha(2)} &= D\xi^{\alpha(2)} + e^{\alpha}{}_{\gamma}\eta^{\alpha\gamma} + a_1 e_{\beta(2)}\xi^{\alpha(2)\beta(2)} + 2a_0 e^{\alpha(2)}\xi \\
\delta B^{\alpha(2)} &= 2a_0\eta^{\alpha(2)}, \qquad \delta A = D\xi + \frac{a_0}{4}e_{\alpha(2)}\xi^{\alpha(2)} \\
\delta\pi^{\alpha(2)} &= \frac{Msa_0}{2}\xi^{\alpha(2)}, \qquad \delta\varphi = -2Ms\xi
\end{aligned}
\tag{7}
$$

Comparing with the massless case in the previous subsection, one can see that we still have all the gauge symmetries that our massless fields possessed modified so as to be consistent with the structure of the massive Lagrangian. This gauge-invariant formulation of the massive theory in 3D AdS space possesses some remarkable features. First, we can consider a flat limit $\lambda \to 0$ and immediately obtain the description of the massive fields in 3D Minkowski space. Second, there is a correct massless limit $m \to 0$ without the gap in the number of physical degrees of freedom. In such a limit, our system decomposes into two systems describing the massless spin-s and the massive spin-$(s-1)$ fields. At last, in dS space, when $\lambda^2 < 0$, one can consider the so-called partially massless limits $a_k \to 0$. In such a limit, the system decomposes into the two subsystems describing the partially massless spin-s field and the massive spin-k field.

Now, let us return to the general case of the 3D AdS massive spin-s field. Having at our disposal the explicit expressions for the gauge transformations (7), we can construct the gauge-invariant curvatures. After the change of the normalization:

$$
B^{\alpha(2)} \to 2a_0 B^{\alpha(2r)} \qquad \pi^{\alpha(2)} \to b_0\pi^{\alpha(2)}
\tag{8}
$$

the first part of the curvatures looks like:

$$
\begin{aligned}
\mathcal{R}^{\alpha(2k)} &= D\Omega^{\alpha(2k)} + \frac{(k+2)a_k}{k} e_{\beta(2)}\Omega^{\alpha(2k)\beta(2)} \\
&+ \frac{a_{k-1}}{k(2k-1)} e^{\alpha(2)}\Omega^{\alpha(2k-2)} + \frac{b_k}{k} e^{\alpha}{}_{\beta} f^{\alpha(2k-1)\beta} \\
\mathcal{T}^{\alpha(2k)} &= Df^{\alpha(2k)} + e^{\alpha}{}_{\beta}\Omega^{\alpha(2k-1)\beta} + a_k e_{\beta(2)} f^{\alpha(2k)\beta(2)} \\
&+ \frac{(k+1)a_{k-1}}{k(k-1)(2k-1)} e^{\alpha(2)} f^{\alpha(2k-2)} \\
\mathcal{R}^{\alpha(2)} &= D\Omega^{\alpha(2)} + 3a_1 e_{\beta(2)}\Omega^{\alpha(2)\beta(2)} + b_1 e^{\alpha}{}_{\gamma} f^{\alpha\gamma} - a_0{}^2 E^{\alpha}{}_{\beta} B^{\alpha\beta} + b_0 E^{\alpha(2)}\varphi \\
\mathcal{T}^{\alpha(2)} &= Df^{\alpha(2)} + e^{\alpha}{}_{\gamma}\Omega^{\alpha\gamma} + a_1 e_{\beta(2)} f^{\alpha(2)\beta(2)} + 2a_0 e^{\alpha(2)} A \\
\mathcal{A} &= DA + \frac{a_0}{4} e_{\alpha(2)} f^{\alpha(2)} - 2a_0 E_{\gamma(2)} B^{\gamma(2)} \\
\Phi &= D\varphi + 2MsA - b_0 e_{\alpha(2)}\pi^{\alpha(2)}
\end{aligned}
\tag{9}
$$

There is a peculiarity when we try to construct the curvatures for the $B^{\alpha(2)}$ and $\pi^{\alpha(2)}$ fields. Namely, in order to achieve gauge invariance for them, we should introduce the so-called extra fields $B^{\alpha(4)}, \pi^{\alpha(4)}$ with the following gauge transformations:

$$\delta B^{\alpha(4)} = \eta^{\alpha(4)} \qquad \delta\pi^{\alpha(4)} = \xi^{\alpha(4)}$$

Then, the corresponding gauge-invariant curvatures look like:

$$
\begin{aligned}
\mathcal{B}^{\alpha(2)} &= DB^{\alpha(2)} - \Omega^{\alpha(2)} + b_1 e^{\alpha}{}_{\beta}\pi^{\alpha\beta} + 3a_1 e_{\beta(2)} B^{\alpha(2)\beta(2)} \\
\Pi^{\alpha(2)} &= D\pi^{\alpha(2)} - f^{\alpha(2)} + e^{\alpha}{}_{\beta} B^{\alpha\beta} - \frac{a_0}{sM} e^{\alpha(2)}\varphi + a_1 e_{\beta(2)}\pi^{\alpha(2)\beta(2)}
\end{aligned}
\tag{10}
$$

In turn, to construct gauge-invariant curvatures for the $B^{\alpha(4)}, \pi^{\alpha(4)}$, we should introduce the extra fields $B^{\alpha(6)}, \pi^{\alpha(6)}$, and so on. The procedure ends when we construct curvatures for $B^{\alpha(2s-2)}, \pi^{\alpha(2s-2)}$. Thus, the full set of extra fields is $B^{\alpha(2k)}, \pi^{\alpha(2k)}$, $2 \le k \le s-1$ with the following gauge transformations:

$$\delta B^{\alpha(2k)} = \eta^{\alpha(2k)} \qquad \delta\pi^{\alpha(2k)} = \xi^{\alpha(2k)}$$

and the gauge-invariant curvatures:

$$
\begin{aligned}
\mathcal{B}^{\alpha(2k)} &= DB^{\alpha(2k)} - \Omega^{\alpha(2k)} + \tfrac{b_k}{k} e^{\alpha}{}_{\beta}\pi^{\alpha(2k-1)\beta} + \tfrac{a_{k-1}}{k(2k-1)} e^{\alpha(2)} B^{\alpha(2k-2)} \\
&+ \tfrac{(k+2)}{k} a_k e_{\beta(2)} B^{\alpha(2k)\beta(2)} \\
\Pi^{\alpha(2k)} &= D\pi^{\alpha(2k)} - f^{\alpha(2k)} + e^{\alpha}{}_{\beta} B^{\alpha(2k-1)\beta} + \tfrac{(k+1)a_{k-1}}{k(k-1)(2k-1)} e^{\alpha(2)}\pi^{\alpha(2k-2)} \\
&+ a_k e_{\beta(2)}\pi^{\alpha(2k)\beta(2)}
\end{aligned}
\tag{11}
$$

In three dimensions, it is possible to rewrite the Lagrangian in terms of the curvatures only [34]. In the case of arbitrary integer spin field, the corresponding Lagrangian (5) can be rewritten in the following simple form:

$$\mathcal{L} = -\frac{1}{2}\sum_{k=1}^{s-1}(-1)^{k+1}[\mathcal{R}_{\alpha(2k)}\Pi^{\alpha(2k)} + \mathcal{T}_{\alpha(2k)}\mathcal{B}^{\alpha(2k)}] + \frac{a_0}{2sM} e_{\alpha(2)}\mathcal{B}^{\alpha(2)}\Phi \tag{12}$$

Thus, there are two approaches to the construction of the supersymmetric (or interacting in general case) higher spin models. According to one of them, one can work with the explicit field variables and the Lagrangian in the form (5). It is straightforward, but a rather cumbersome way.

We will use it in Section 4 for the more simple case of 3D Minkowski space. According to the other way, one can work in terms of the gauge-invariant curvatures and Lagrangian in the form of (12). It is a more elegant way, and we use it in Section 5 to study the supersymmetric higher spin models in 3D AdS.

3. Free Higher Spin Fermionic Models

In this section, we review the Lagrangian description of arbitrary spin fermionic fields in three-dimensional Minkowski and its cosmological extension AdS spaces [31]. As in the bosonic case, we present the frame-like gauge-invariant formulation. It is a natural form of description for massless fields, and for massive fields, the gauge-invariant formulation is realized as a system of massless fields coupled by Stueckelberg symmetries.

3.1. Massless Fields

As in the integer spin case, all massless fields with half-integer spin $(k+1/2) \geq 3/2$ are gauge ones and can be described according to the metric-like or the frame-like approaches. In 4D Minkowski or AdS spaces, the metric-like approach leads to the Fang–Fronsdal formulation of the massless half-integer spin-$(k+1/2)$ fields in terms of totally symmetric spin-tensors $\psi_{\mu_1\mu_2\ldots\mu_k,\alpha}$ (here, $\alpha, \beta = 1,2$ is the spinor index) subject to the γ-tracelessness condition $\gamma^\sigma{}_\alpha{}^\beta \psi^\rho{}_{\rho\sigma\mu_3\ldots\mu_k,\beta}$, which becomes nontrivial for $k \geq 3$. In the frame-like approach, such a field is described by the generalized frame-like field

$$\Phi_\mu{}^{a_1\ldots a_{k-1},\alpha}$$

(As in the bosonic case in 4D for the massless fields with half-integer spins $(k+1/2) > 3/2$, one should consider extra gauge fields $\Phi_\mu{}^{a_1\ldots a_{k-1},b_1\ldots b_t,\alpha}$ where $2 \leq t \leq (k-1)$. They play a crucial role in Vasiliev interacting theory. In 3D, these extra fields vanish in the massless case). Here, μ is the curved world index and a_i are Lorentz flat tangent indices. World and flat indices are related by the background Minkowski or AdS frame $e_\mu{}^a$. Their flat indices correspond to the irreducible $so(3,1)$ Lorentz spin-tensors. The Fang–Fronsdal spin-tensors are recovered by considering symmetric combination:

$$\psi_{\mu_1\ldots\mu_k,\alpha} = e_{(\mu_1}{}^{a_1} e_{\mu_2}{}^{a_2} \ldots e_{\mu_{k-1}}{}^{a_{k-1}} \Phi_{\mu_k)a_1a_2\ldots a_{k-1},\alpha}$$

Again, due to the isomorphism $so(2,1) \sim sp(2)$, it is more convenient to use the multispinor formalism in which our gauge field takes the form:

$$\Phi_\mu{}^{\alpha_1\alpha_2\ldots\alpha_{2k-1}}$$

Below, we make use of the language of differential forms, considering the higher spin field variables as one-forms and using condensed notations for the spinor indices given in the Introduction.

Spin $k+3/2$ ($k \geq 0$):

In the frame-like formalism, it is described by physical one-form $\Phi^{\alpha(2k-1)}$. The Lagrangian in 3D AdS has the following form:

$$\mathcal{L} \;=\; i\frac{(-1)^{k+1}}{2}[\Phi_{\alpha(2k+1)} D\Phi^{\alpha(2k+1)} + \frac{(2k+1)\lambda}{2}\Phi_{\alpha(2k)\beta} e^\beta{}_\gamma \Phi^{\alpha(2k)\gamma}] \tag{13}$$

It is invariant under the following gauge transformations:

$$\delta\Phi^{\alpha(2k+1)} \;=\; D\xi^{\alpha(2k+1)} + \frac{\lambda}{2} e^\alpha{}_\beta \xi^{\alpha(2k)\beta} \tag{14}$$

The gauge-invariant curvature looks like:

$$\mathcal{F}^{\alpha(2k+1)} = D\Phi^{\alpha(2k+1)} + \frac{\lambda}{2} e^{\alpha}{}_{\beta}\Phi^{\alpha(2k)\beta} \tag{15}$$

Using this curvature, the Lagrangian can be rewritten as follows:

$$\mathcal{L} = i\frac{(-1)^{k+1}}{2}\Phi_{\alpha(2k+1)}\mathcal{F}^{\alpha(2k+1)} \tag{16}$$

The 3D Minkowski case corresponds to the flat limit $\lambda \to 0$. Let us also write out the kinetic term for the spin-1/2 field.

Spin 1/2 is described by the physical zero-form ϕ^{α}. It is not a gauge field, and the Lagrangian looks like:

$$\mathcal{L} = \frac{1}{2}\phi_{\alpha}E^{\alpha}{}_{\beta}d\phi^{\beta}$$

3.2. Massive Fields

To describe the massive spin-$(s+1/2)$ field in the gauge-invariant form, we have to consider a system of the massless fields with spins $(s+1/2), (s-1/2), ..., 1/2$. In the frame-like approach, the corresponding set of fields consists of:

$$\Phi^{\alpha(2k+1)} \quad 0 \le k \le (s-1), \quad \phi^{\alpha}$$

The Lagrangian for the free field with mass m_1 in 3D AdS space looks like:

$$\begin{aligned}
\frac{1}{i}\mathcal{L} = {} & \textstyle\sum_{k=0}^{s-1}(-1)^{k+1}[\frac{1}{2}\Phi_{\alpha(2k+1)}D\Phi^{\alpha(2k+1)}] + \frac{1}{2}\phi_{\alpha}E^{\alpha}{}_{\beta}D\phi^{\beta} \\
& \textstyle + \sum_{k=1}^{s-1}(-1)^{k+1}c_k\Phi_{\alpha(2k-1)\beta(2)}e^{\beta(2)}\Phi^{\alpha(2k-1)} + c_0\Phi_{\alpha}E^{\alpha}{}_{\beta}\phi^{\beta} \\
& \textstyle + \sum_{k=0}^{s-1}(-1)^{k+1}\frac{d_k}{2}\Phi_{\alpha(2k)\beta}e^{\beta}{}_{\gamma}\Phi^{\alpha(2k)\gamma} - \frac{3d_0}{2}E\phi_{\alpha}\phi^{\alpha}
\end{aligned} \tag{17}$$

where:

$$\begin{aligned}
c_k{}^2 &= \frac{(s+k+1)(s-k)}{2(k+1)(2k+1)}[M_1{}^2 - (2k+1)^2\frac{\lambda^2}{4}] \\
c_0{}^2 &= 2s(s+1)[M_1{}^2 - \frac{\lambda^2}{4}] \\
d_k &= \frac{(2s+1)}{(2k+3)}M_1, \qquad M_1{}^2 = m_1{}^2 + (s-\frac{1}{2})^2\lambda^2
\end{aligned} \tag{18}$$

The structure of Lagrangian (17) is the same as in the bosonic case. The first line is kinetic terms; the second line is cross-terms; and the third line is mass terms. The coefficients in (18) are determined by the requirement of the invariance of the Lagrangian under the following gauge transformations:

$$\begin{aligned}
\delta\Phi^{\alpha(2k+1)} &= D\xi^{\alpha(2k+1)} + \frac{d_k}{(2k+1)}e^{\alpha}{}_{\beta}\xi^{\alpha(2k)\beta} \\
& \quad + \frac{c_k}{k(2k+1)}e^{\alpha(2)}\xi^{\alpha(2k-1)} + c_{k+1}e_{\beta(2)}\xi^{\alpha(2k+1)\beta(2)} \\
\delta\phi^{\alpha} &= c_0\xi^{\alpha}
\end{aligned} \tag{19}$$

In such a formulation, we can take the correct massless limit $m_1 \to 0$ in AdS ($\lambda^2 > 0$) and correct partially massless limits $c_k \to 0$ in dS ($\lambda^2 < 0$). Taking flat limit $\lambda \to 0$, we obtain the description of the massive field in 3D Minkowski space. Note that the Lagrangian (17) describes the massive Majorana left fermion carrying one physical degree of freedom.

Let us return to the general massive fermion and reformulate the theory in terms of the gauge-invariant curvatures. Having at our disposal the explicit expressions for the gauge transformations (19), we can construct the gauge-invariant objects. After the change of normalization:

$$\phi^\alpha \to c_0\phi^\alpha \tag{20}$$

they take the form:

$$\begin{aligned}
\mathcal{F}^{\alpha(2k+1)} &= D\Phi^{\alpha(2k+1)} + \frac{d_k}{(2k+1)}e^\alpha{}_\beta\Phi^{\alpha(2k)\beta} \\
&\quad + \frac{c_k}{k(2k+1)}e^{\alpha(2)}\Phi^{\alpha(2k-1)} + c_{k+1}e_{\beta(2)}\Phi^{\alpha(2k+1)\beta(2)} \\
\mathcal{F}^\alpha &= D\Phi^\alpha + d_0e^\alpha{}_\beta\Phi^\beta + c_1e_{\beta(2)}\Phi^{\alpha\beta(2)} - c_0{}^2E^\alpha{}_\beta\phi^\beta \\
\mathcal{C}^\alpha &= D\phi^\alpha - \Phi^\alpha + d_0e^\alpha{}_\beta\phi^\beta + c_1e_{\beta(2)}\phi^{\alpha\beta(2)}
\end{aligned} \tag{21}$$

As in the case of integer spins, in order to achieve gauge invariance for $\mathcal{C}^\alpha$, we have introduced extra zero-form $\phi^{\alpha(3)}$ with the gauge transformations:

$$\delta\phi^{\alpha(3)} = \xi^{\alpha(3)}$$

In turn, to construct gauge-invariant curvatures for the $\phi^{\alpha(3)}$ field, we should introduce extra zero-form $\phi^{\alpha(5)}$, and so on. Iterations end at the case of $\phi^{\alpha(2s-1)}$, so that the full set of extra fields we should introduce is $\phi^{\alpha(2k+1)}$, $1 \le k \le (s-1)$ with the following gauge transformations:

$$\delta\phi^{\alpha(2k+1)} = \xi^{\alpha(2k+1)}$$

The gauge-invariant curvatures for them have the form:

$$\begin{aligned}
\mathcal{C}^{\alpha(2k+1)} &= D\phi^{\alpha(2k+1)} - \Phi^{\alpha(2k+1)} + \frac{d_k}{(2k+1)}e^\alpha{}_\beta\phi^{\alpha(2k)\beta} \\
&\quad + \frac{c_k}{k(2k+1)}e^{\alpha(2)}\phi^{\alpha(2k-1)} + c_{k+1}e_{\beta(2)}\phi^{\alpha(2k+1)\beta(2)}
\end{aligned} \tag{22}$$

Finally, the Lagrangian (17) can be rewritten in terms of these curvatures as follows:

$$\mathcal{L} = -\frac{i}{2}\sum_{k=0}^{s-2}(-1)^{k+1}\mathcal{F}_{\alpha(2k+1)}\mathcal{C}^{\alpha(2k+1)} \tag{23}$$

In the next sections, we study supersymmetric higher spin models. In 3D Minkowski space, we use the gauge-invariant formulation in terms of the explicit fields and the Lagrangian in the form (17). In 3D AdS space, we use the formulation in terms of the gauge-invariant curvatures and the Lagrangian in the form (23).

4. Lagrangian Construction of Higher Spin Supermultiplets in 3D Minkowski Space

In this section, we show how to combine the bosonic and the fermionic higher spin fields into one supermultiplet in 3D Minkowski space and restrict ourselves with $\mathcal{N} = 1$ on-shell supersymmetry. Our construction is based on the gauge-invariant formulation given above in terms of the field variables. As we have shown in such a formulation, the massive field in the massless limit decomposes into a system of massless fields. If we take such a decomposition for each field in the massive supermultiplet, we obtain in general its decomposition into supersymmetric system of the massless fields. Therefore, the main idea is to start with this supersymmetric system of the massless fields and construct smooth massive deformation. In other words, we generalize the gauge-invariant formulation to the supersymmetric case.

4.1. 3D vs. 4D Supermultiplets

For the first time, the idea to construct the massive higher spin supermultiplets from the supersymmetric system of massless fields was realized for 4D Minkowski space [15]. Moreover, it was shown that this supersymmetric system of the massless fields is perfectly combined into the system of the massless supermultiplets. Therefore, it is useful to consider how the familiar massive 4D supermultiplets decomposes into the massless ones and then compare it with decomposition of the more specific massive 3D supermultiplets. In both cases, it is used such that in the gauge invariant formulation, massive bosonic and fermionic fields decompose into a set of the massless ones:

$$\begin{array}{lll} s & \xrightarrow{m=0} & s \oplus (s-1) \oplus ... \oplus 0 = \sum_{k=0}^{s} k \\ (s+\frac{1}{2}) & \xrightarrow{m=0} & (s+\frac{1}{2}) \oplus (s-\frac{1}{2}) \oplus ... \oplus \frac{1}{2} = \sum_{k=0}^{s} (k+\frac{1}{2}) \end{array} \tag{24}$$

Massive 4D, $\mathcal{N}=1$ supermultiplets with the half-integer spin-$(s+1/2)$ as the highest one contain four massive fields $s+1/2, s, s'$ and $s-1/2$. Recall that in 4D, massive bosonic spin-s has $2s+1$ d.o.f., and massive fermionic spin-$(s+1/2)$ has $2s+2$ d.o.f. All 4D massless fields have two d.o.f., except for Spin 0, which has one. In the massless limit, the massive supermultiplet decomposes into massless ones in the same way as (24). Simple counting of d.o.f. gives:

$$\begin{pmatrix} & s+\frac{1}{2} & \\ s & & s' \\ & s-\frac{1}{2} & \end{pmatrix} \xrightarrow{m=0} \sum_{k=1}^{s} \begin{pmatrix} k+\frac{1}{2} \\ k \end{pmatrix} \oplus \sum_{k=1}^{s} \begin{pmatrix} k' \\ k-\frac{1}{2} \end{pmatrix} \oplus \begin{pmatrix} \frac{1}{2} \\ 0,0' \end{pmatrix} \tag{25}$$

Therefore, to construct massive supermultiplets, we have to start with $2s+1$ massless ones and find a massive deformation. However, in [15], it was shown that the crucial point in the whole construction is the possibility to make a dual mixing of the massless supermultiplets by rotating fields with spin k and spin k':

$$\begin{pmatrix} k+\frac{1}{2} \\ k \end{pmatrix} \oplus \begin{pmatrix} k' \\ k-\frac{1}{2} \end{pmatrix} \rightarrow \begin{pmatrix} & k+\frac{1}{2} & \\ k & & k' \\ & k-\frac{1}{2} & \end{pmatrix} \tag{26}$$

In such mixing, the massless bosonic fields with spin k and k' must have equal spins, but opposite parities. Thus, the structure of the decomposition of the 4D massive supermultiplets with higher half-integer spin into the massless ones looks like:

$$\begin{pmatrix} & s+\frac{1}{2} & \\ s & & s' \\ & s-\frac{1}{2} & \end{pmatrix} \xrightarrow{m=0} \sum_{k=1}^{s} \begin{pmatrix} & k+\frac{1}{2} & \\ k & & k' \\ & k-\frac{1}{2} & \end{pmatrix} \oplus \begin{pmatrix} \frac{1}{2} \\ 0,0' \end{pmatrix}$$

Analogously, we obtain that the 4D massive supermultiplets with higher integer spin in the massless limit has decomposition:

$$\begin{pmatrix} & s+1 & \\ s+\frac{1}{2} & & s'+\frac{1}{2} \\ & s & \end{pmatrix} \xrightarrow{m=0} \begin{pmatrix} s+1 \\ s+\frac{1}{2} \end{pmatrix} \oplus \sum_{k=1}^{s} \begin{pmatrix} & k'+\frac{1}{2} & \\ k & & k \\ & k-\frac{1}{2} & \end{pmatrix} \oplus \begin{pmatrix} \frac{1}{2}' \\ 0,0 \end{pmatrix}$$

Now, let us consider 3D massive supermultiplets decomposition in the massless limit and compare it with 4D case. First of all, recall that all 3D massless bosonic and fermionic higher spin fields do not propagate any degrees of freedom. Only massless fields with Spins $1, 1/2$ and 0 propagate one physical degree of freedom. In turn, 3D massive higher spin fields do propagate physical degrees of freedom, two and one d.o.f. for bosons and fermions, respectively. In the gauge-invariant formulation, this is clearly seen from (24).

As in 4D, the minimal massless supermultiplets in 3D contain one boson and one fermion. However, unlike 4D, massless higher spin fields in 3D do not have physical degrees of freedom, which is why one can extend the massless supermultiplet adding such fields. In some sense, it plays an analogous role as the mixing between two massless supermultiplets in the 4D case (26). As we will see in 3D, one can construct extended the massless supermultiplet, which will correspond to the massless decomposition of the massive supermultiplet.

Further in this section, we first of all construct minimal massless higher spin supermultiplets. They will play the role of initial blocks for the construction of the extended massless supermultiplet. Then, we fined a gauge-invariant massive deformation for them so that the resulting system describes massive supermultiplets.

4.2. Massless Higher Spin Supermultiplets

Supermultiplet $(k+\frac{3}{2}, k+1)$, $k \geq 1$:

It contains one fermionic field with spin $(k+\frac{3}{2})$ and one bosonic field with spin $(k+1)$. In the frame-like formulation, the corresponding field variables are the one one-form $\Phi^{\alpha(2k+1)}$ for the fermion and two one-forms $\Omega^{\alpha(2k)}$, $f^{\alpha(2k)}$ for the boson. The Lagrangian describing this supermultiplet is just the sum of their kinetic terms (see Sections 2.1 and 3.1 for details):

$$\mathcal{L}_0 = (-1)^{k+1}[l\Omega_{\alpha(2k-1)\beta}e^\beta{}_\gamma\Omega^{\alpha(2k-1)\gamma} + \Omega_{\alpha(2k)}df^{\alpha(2k)} + \frac{i}{2}\Phi_{\alpha(2k+1)}d\Phi^{\alpha(2k+1)}] \tag{27}$$

It is not hard to show that the Lagrangian is invariant under the following global supertransformations:

$$\delta f^{\alpha(2k)} = i(2k+1)\alpha_k\Phi^{\alpha(2k)\beta}\zeta_\beta, \qquad \delta\Phi^{\alpha(2k+1)} = \alpha_k\Omega^{\alpha(2k)}\zeta^\alpha \tag{28}$$

Let us calculate commutator of the supertransformations on the bosonic field:

$$[\delta_1, \delta_2]f^{\alpha(2l)} = i(2k+1){\alpha_k}^2\Omega^{\alpha(2l-1)\beta}(\zeta_{1\beta}{\zeta_2}^\alpha - \zeta_{2\beta}{\zeta_1}^\alpha) \tag{29}$$

In the frame-like formulation, the right side of the commutator corresponds to the translations, and it means that:

$$[Q_\alpha, Q_\beta] \sim P_{\alpha\beta}$$

In what follows, we will not fix the normalization of supertransformations.

Supermultiplet $(k+1, k+\frac{1}{2})$, $k \geq 1$:

It contains one fermionic field with spin $(k+\frac{1}{2})$ and one bosonic field with spin $(k+1)$. In the frame-like formulation, the corresponding field variables are one one-form $\Phi^{\alpha(2k-1)}$ for fermion and two one-forms $\Omega^{\alpha(2k)}$, $f^{\alpha(2k)}$ for boson. The Lagrangian describing this supermultiplets has the form:

$$\mathcal{L}_0 = (-1)^{k+1}[l\Omega_{\alpha(2k-1)\beta}e^\beta{}_\gamma\Omega^{\alpha(2k-1)\gamma} + \Omega_{\alpha(2k)}df^{\alpha(2k)} - \frac{i}{2}\Phi_{\alpha(2k-1)}d\Phi^{\alpha(2k-1)}] \tag{30}$$

It is invariant under the following supertransformations:

$$\delta f^{\alpha(2k)} = i\beta_k\Phi^{\alpha(2l-1)}\zeta^\alpha, \qquad \delta\Phi^{\alpha(2k-1)} = 2k\beta_k\Omega^{\alpha(2k-1)\beta}\zeta_\beta \tag{31}$$

Therefore, we have described the full set of the massless higher spin supermultiplets, but in the massive case, we will also need the massless lower spin supermultiplets $(1, \frac{1}{2})$ and $(\frac{1}{2}, 0)$.

Supermultiplet $(1, \frac{1}{2})$:

It contains the fermionic zero-form ψ^α, as well as the bosonic zero-form $B^{\alpha\beta}$ and the one-form A. The sum of their kinetic terms:

$$\mathcal{L}_0 = EB^{\alpha\beta}B_{\alpha\beta} - B_{\alpha\beta}e^{\alpha\beta}dA + \frac{i}{2}\phi_\alpha E^\alpha{}_\beta d\phi^\alpha \tag{32}$$

One can show that on the auxiliary field $B^{\alpha\beta}$ equations, we have:

$$E^\alpha{}_\beta dB^{\beta\gamma} = E^\gamma{}_\beta dB^{\beta\alpha} \tag{33}$$

Lagrangian (32) is invariant under the following supertransformations (Strictly speaking, this Lagrangian is invariant up to the terms proportional to the auxiliary field $B^{\alpha\beta}$ equation only (33). Thus, there are two possible approaches here. On the one hand, one can introduce non-trivial corrections to the supertransformations for this auxiliary field. Another possibility, which we will systematically follow here and further on, is to use equations for the auxiliary fields in calculating all variations.):

$$\delta A = i\beta_0\phi_\alpha e^{\alpha\beta}\zeta_\beta, \qquad \delta\phi^\alpha = 4\beta_0 B^{\alpha\beta}\zeta_\beta \tag{34}$$

Supermultiplet $(\frac{1}{2}, 0)$:

It contains the fermionic zero-form ϕ_α and two bosonic zero-forms $\pi^{\alpha\beta}$ and φ. The sum of their kinetic terms looks like:

$$\mathcal{L}_0 = \frac{i}{2}\phi_\alpha E^\alpha{}_\beta d\phi^\alpha - E\pi^{\alpha\beta}\pi_{\alpha\beta} + E^{\alpha\beta}\pi_{\alpha\beta}d\varphi \tag{35}$$

Using the fact that on the auxiliary field $\pi^{\alpha\beta}$ equations, we have:

$$E^\alpha{}_\gamma d\pi^{\beta\gamma} = \frac{1}{2}\varepsilon^{\alpha\beta}E_{\gamma\delta}d\pi^{\gamma\delta} \tag{36}$$

we can show that the Lagrangian is invariant under the following supertransformations:

$$\delta\varphi = \frac{i\tilde{\delta}_0}{2}\phi_\alpha\zeta^\alpha, \qquad \delta\phi^\alpha = \tilde{\delta}_0\pi^{\alpha\beta}\zeta_\beta. \tag{37}$$

4.3. Massive Higher Spin Supermultiplets

Supermultiplet $(s + \frac{1}{2}, s)$:

This supermultiplet contains spin-$s + \frac{1}{2}$ fermion with mass m_1 and spin-s boson with mass m. In the massless limit, it decomposes into the system of the massless fields with spins:

$$(s + \frac{1}{2}), s, (s - \frac{1}{2}), (s-1), ..., \frac{3}{2}, 1, \frac{1}{2}, 0$$

Correspondingly, we begin with the appropriate sum of the kinetic terms for all fields:

$$\begin{aligned}\mathcal{L} \;=\; & \textstyle\sum_{k=1}^{s-1}(-1)^{k+1}[k\Omega_{\alpha(2k-1)\beta}e^\beta{}_\gamma\Omega^{\alpha(2k-1)\gamma} + \Omega_{\alpha(2k)}df^{\alpha(2k)}] \\ & + EB_{\alpha\beta}B^{\alpha\beta} - B_{\alpha\beta}e^{\alpha\beta}dA - E\pi_{\alpha\beta}\pi^{\alpha\beta} + \pi_{\alpha\beta}E^{\alpha\beta}d\varphi \\ & + \tfrac{i}{2}\textstyle\sum_{k=0}^{s-1}(-1)^{k+1}\Phi_{\alpha(2k+1)}d\Phi^{\alpha(2k+1)} + \tfrac{i}{2}\phi_\alpha E^\alpha{}_\beta d\phi^\beta\end{aligned} \tag{38}$$

This Lagrangian possesses the following supersymmetry:

$$
\begin{aligned}
\delta f^{\alpha(2k)} &= i\beta_k \Phi^{\alpha(2k-1)}\zeta^\alpha + i\alpha_k \Phi^{\alpha(2k)\beta}\zeta_\beta \\
\delta f^{\alpha(2)} &= i\beta_1 \Phi^\alpha \zeta^\alpha + i\alpha_1 \Phi^{\alpha(2)\beta}\zeta_\beta \\
\delta A &= i\alpha_0 \Phi^\alpha \zeta_\alpha + i\beta_0 e_{\alpha\beta}\phi^\alpha \zeta^\beta, \qquad \delta\varphi = -\frac{i\tilde{\delta}_0}{2}\phi^\gamma \zeta_\gamma \\
\delta\Phi^{\alpha(2k+1)} &= \frac{\alpha_k}{(2k+1)}\Omega^{\alpha(2k)}\zeta^\alpha + 2(k+1)\beta_{k+1}\Omega^{\alpha(2k+1)\beta}\zeta_\beta \\
\delta\Phi^\alpha &= 2\beta_1 \Omega^{\alpha\beta}\zeta_\beta + \alpha_0 e_{\beta(2)} B^{\beta(2)}\zeta^\alpha \\
\delta\phi^\alpha &= 4\beta_0 B^{\alpha\beta}\zeta_\beta + \tilde{\delta}_0 \pi^{\alpha\beta}\zeta_\beta
\end{aligned}
\tag{39}
$$

These supertransformations are determined by the structure of the supertransformations for the massless supermultiplets given above. Therefore, in some sense, the Lagrangian (38) describes the "large" massless supermultiplet, which contains the full set of the massless fields.

To construct a massive deformation, we have to add the lower derivative terms. For the bosonic terms, we take the ones corresponding to the gauge-invariant description of massive spin-s boson with mass m (see Section 2.2):

$$
\begin{aligned}
\mathcal{L}_b &= \textstyle\sum_{k=1}^{s-2}(-1)^{k+1}a_k[-\frac{(k+2)}{k}\Omega_{\alpha(2)\beta(2k)}e^{\alpha(2)}f^{\beta(2k)} + \Omega_{\alpha(2k)}e_{\beta(2)}f^{\alpha(2k)\beta(2)}] \\
&\quad +2a_0\Omega_{\alpha(2)}e^{\alpha(2)}A - a_0 f_{\alpha\beta}E^\beta{}_\gamma B^{\alpha\gamma} + 2sM\pi_{\alpha\beta}E^{\alpha\beta}A \\
&\quad +\textstyle\sum_{k=1}^{s-1}(-1)^{k+1}b_k f_{\alpha(2k-1)\beta}e^\beta{}_\gamma f^{\alpha(2k-1)\gamma} + b_0 f_{\alpha(2)}E^{\alpha(2)}\varphi + \frac{3a_0{}^2}{2}E\varphi^2
\end{aligned}
\tag{40}
$$

Note that after such deformation, Equations (33) and (36) are also deformed. Now, they have the form:

$$
E^\alpha{}_\gamma dB^{\beta\gamma} = E^\beta{}_\gamma dB^{\alpha\gamma} - \frac{a_0}{4}\varepsilon^{\alpha\beta}e^{\gamma\delta}df_{\gamma\delta}
$$

$$
E^\alpha{}_\gamma d\pi^{\beta\gamma} = \frac{1}{2}\varepsilon^{\alpha\beta}E^{\gamma\delta}d\pi_{\gamma\delta} + \frac{sM}{2}e^{\alpha\beta}dA
$$

For the fermionic terms, we take the ones corresponding to the gauge invariant description of the massive spin-$s+1/2$ fermion with mass m_1 (see Section 3.2):

$$
\begin{aligned}
\tfrac{1}{i}\mathcal{L}_f &= \textstyle\sum_{k=1}^{s-1}(-1)^{k+1}c_k\Phi_{\alpha(2k-1)\beta(2)}e^{\beta(2)}\Phi^{\alpha(2k-1)} + c_0\Phi_\alpha E^\alpha{}_\beta\phi^\beta \\
&\quad +\textstyle\sum_{k=0}^{s-1}(-1)^{k+1}\frac{d_k}{2}\Phi_{\alpha(2k)\beta}e^\beta{}_\gamma\Phi^{\alpha(2k)\gamma} - \frac{3d_0}{2}E\phi_\alpha\phi^\alpha
\end{aligned}
\tag{41}
$$

Calculating the variations, we use auxiliary field equations:

$$
e^\alpha{}_\gamma\Omega^{\alpha(2k-1)\gamma} = -df^{\alpha(2k)} - \frac{a_{k-1}(k+1)}{k(k-1)(2k-1)}e^{\alpha(2)}f^{\alpha(2k-2)} - a_k e_{\beta(2)}f^{\alpha(2k)\beta(2)}
$$

$$
EB^{\alpha(2)} = \frac{1}{2}e^{\alpha(2)}dA - \frac{a_0}{4}E^\alpha{}_\gamma f^{\alpha\gamma}, \qquad 2E\pi^{\alpha(2)} = E^{\alpha(2)}d\varphi + 2sME^{\alpha(2)}A
$$

In this case, to find a massive deformation for the supertransformations, we have to add corrections for the fermionic fields only. From the variations with one derivative, we found that we must introduce the following set of the corrections to the supertransformations:

$$
\begin{aligned}
\delta\Phi^{\alpha(2k+1)} &= \gamma_k f^{\alpha(2k)}\zeta^\alpha + \delta_k f^{\alpha(2k+1)\beta}\zeta_\beta \\
\delta\Phi^\alpha &= \delta_0 f^{\alpha\beta}\zeta_\beta + \gamma_0 A\zeta^\alpha + \tilde{\gamma}_0\varphi e^\alpha{}_\beta\zeta^\beta \\
\delta\phi^\alpha &= \rho_0\varphi\zeta^\alpha
\end{aligned}
\tag{42}
$$

Taking into account that all the coefficients in the Lagrangian are fixed, it is straightforward to find solution for the parameters of the supertransformations, which appears to be valid when $m_1 = m$ only:

$$\begin{aligned} {\alpha_k}^2 &= k(s+k+1)\hat{\alpha}^2, \qquad {\beta_k}^2 = \frac{(k+1)(s-k)}{2k(2k+1)}\hat{\alpha}^2 \\ {\gamma_k}^2 &= \frac{s^2(s+k+1)}{4k(k+1)^2(2k+1)^2}m^2\hat{\alpha}^2 \\ {\delta_k}^2 &= \frac{s^2(s-k-1)}{2(k+1)(k+2)(2k+3)}m^2\hat{\alpha}^2 \end{aligned} \tag{43}$$

$$\tilde{\delta}_0 = 4\beta_0 = \sqrt{2s}\hat{\alpha}, \qquad \alpha_0 = \frac{1}{2}\sqrt{(s+1)}\hat{\alpha}, \qquad \hat{\alpha}^2 = \frac{{\alpha_{s-1}}^2}{2s(s-1)}$$

$$\gamma_0 = -2\tilde{\gamma}_0 = s\sqrt{(s+1)}m\hat{\alpha}, \quad \rho_0 = -\frac{(s+1)\sqrt{s}m}{\sqrt{2}}\hat{\alpha}$$

Supermultiplet $(s, s-\frac{1}{2})$:

We need the same set of fields as in the previous case, except the field $\Phi^{\alpha(2s-1)}$. Therefore, we take the same massless Lagrangian (38) with this field omitted and the same set of initial supertransformations (39) where now $\alpha_{s-1} = 0$. As far as the low derivative terms, the bosonic terms again have the same form (40), while the fermionic terms correspond to the gauge-invariant description of the massive spin-$s-1/2$ fermion (it coincides with (41), where $\Phi^{\alpha(2s-1)}$ is omitted). Since the structure of the bosonic and the fermionic terms is the same as in the previous case, the correction to the supertransformations also look the same, that is:

$$\begin{aligned} \delta\Phi^{\alpha(2k+1)} &= \gamma_k f^{\alpha(2k)}\zeta^\alpha + \delta_k f^{\alpha(2k+1)\beta}\zeta_\beta \\ \delta\Phi^\alpha &= \delta_0 f^{\alpha\beta}\zeta_\beta + \gamma_0 A\zeta^\alpha + \tilde{\gamma}_0\varphi e^\alpha{}_\beta\zeta^\beta \\ \delta\phi^\alpha &= \rho_0\varphi\zeta^\alpha \end{aligned} \tag{44}$$

Calculating all variations, we obtain the relation between masses $m = m_1$ and the following expressions for the parameters determining supertransformations:

$$\begin{aligned} {\alpha_k}^2 &= k(s-k-1)\hat{\alpha}^2, \qquad {\beta_k}^2 = \frac{(k+1)(s+k)}{2k(2k+1)}\hat{\alpha}^2 \\ {\gamma_k}^2 &= \frac{s^2(s-k-1)}{4k(k+1)^2(2k+1)^2}m^2\hat{\alpha}^2 \\ {\delta_k}^2 &= \frac{s^2(s+k+1)}{2(k+1)(k+2)(2k+3)}m^2\hat{\alpha}^2 \end{aligned} \tag{45}$$

$$\tilde{\delta}_0 = 4\beta_0 = \sqrt{2s}\hat{\alpha}, \qquad \alpha_0 = \frac{1}{2}\sqrt{(s-1)}\hat{\alpha}, \qquad \hat{\alpha}^2 = \frac{{\alpha_{s-2}}^2}{(s-2)}$$

$$\gamma_0 = -2\tilde{\gamma}_0 = s\sqrt{(s-1)}m\hat{\alpha}, \quad \rho_0 = -\frac{(s-1)\sqrt{s}}{\sqrt{2}}m\hat{\alpha}$$

Thus, we explicitly construct the Lagrangian description for the two massive supermultiplets $(s+1/2, s)$ and $(s, s-1/2)$. Both of these massive supermultiplets have two bosonic and one fermionic (left) degrees of freedom, i.e., they possess (1,0) supersymmetry. One can construct massive supermultiplets possessing (1,1) supersymmetry with an equal number of bosonic and fermionic degrees of freedom if one introduces one more massive fermion (right). In our previous work [32], we have constructed the particular cases of such massive supermultiplets when supertransformations take the form of diagonal superposition of two (1,1) supertransformations.

5. Higher Spin Supermultiplets in 3D *AdS* Space

In this section, we study $\mathcal{N} = 1$ supersymmetric construction for the higher spin models in 3D *AdS* space. As has been shown in [35], $\mathcal{N}$-extended supersymmetry in 3D *AdS* space exists in several incarnations. They are so-called (p, q) supersymmetries where p, q are integers and $\mathcal{N} = p + q$. The simplest $(1, 0)$, which we restrict ourselves to, is naturally associated with 3D *AdS* supergroup:

$$OSp(1,2) \otimes Sp(2)$$

so that we have supersymmetry in the left sector only. In practice, this means that the massive higher spin supermultiplets, as well as the massless ones contain only one bosonic and one fermionic degrees of freedom.

As in the previous section, our construction is based on the gauge-invariant description of the higher spin fields, but in contrast to Minkowski space, we work in terms of the gauge-invariant curvatures and the Lagrangians in the form (12) and (23). Below, we present a general scheme of our higher spin supermultiplet construction and apply it to the massless and massive cases. At the end, we demonstrate how 3D *AdS* superalgebra is realized in our construction. In particular, we show that the higher spin supertransformations satisfy (1,0) superalgebra for which the commutation relation of supercharges has the form:

$$\{Q_\alpha, Q_\beta\} \sim P_{\alpha\beta} + \frac{\lambda}{2} M_{\alpha\beta}$$

where $P_{\alpha(2)}$ and $M_{\alpha(2)}$ are the generators of 3D *AdS* algebra.

5.1. Procedure of Curvature Deformation

Here, we present the general scheme of our higher spin supermultiplet constructions in 3D based on the procedure of curvature deformation. As has been shown in Sections 2 and 3, the Lagrangian description of the bosonic and fermionic higher spin fields can be reformulated in terms of the gauge-invariant curvatures. To supersymmetrize the system of the bosonic and fermionic fields, we covariantly deform the corresponding curvatures by a background non-dynamical gravitino one-form Ψ^α, so such construction can be interpreted as a supersymmetric theory in terms of the background fields of the supergravity. Due to some difference in the structure of the gauge-invariant curvatures for the massless and massive case, we separately present the procedure of their deformation.

Massless fields:

Massless higher spin bosonic and fermionic fields are described by one-forms (see Sections 2.1 and 3.1), which we denote as Ω and Φ, respectively. Let us also denote corresponding gauge-invariant curvatures as $\mathcal{R}$ and $\mathcal{F}$, which are two-forms. They have the form (schematically):

$$\mathcal{R} = D\Omega + (e\Omega), \qquad \mathcal{F} = D\Phi + (e\Phi)$$

Here, $e \equiv e_\mu{}^{\alpha\beta}$ is a non-dynamical background 3D *AdS* frame. The Lagrangians then look as follows:

$$\mathcal{L}_\Omega = \Omega\mathcal{R}, \qquad \mathcal{L}_\Phi = \Phi\mathcal{F} \tag{46}$$

We note again that in the massless theory, all curvatures are two-form, and so, in 3D *AdS*, there is no possibility to rewrite the Lagrangian in terms of squares of them. It can be done in space-time dimensions greater than or equal to four.

At the first step in the supersymmetric construction, we deform the curvatures by the terms containing the background gravitino one-form Ψ^α:

$$\Delta\mathcal{R} = (\Psi\Phi), \qquad \Delta\mathcal{F} = (\Psi\Omega)$$

and require that the deformed curvatures transform covariantly, that is:

$$\delta\hat{\mathcal{R}} = \delta(\mathcal{R} + \Delta\mathcal{R}) \sim \mathcal{F}, \qquad \delta\hat{\mathcal{F}} = \delta(\mathcal{F} + \Delta\mathcal{F}) \sim \mathcal{R}$$

In turn, the covariant deformation of curvatures immediately defines the form of the supertransformations:

$$\delta_\zeta\Omega = \frac{\delta(\Psi\Phi)}{\delta\Psi^\alpha}\zeta^\alpha, \qquad \delta_\zeta\Phi = \frac{\delta(\Psi\Omega)}{\delta\Psi^\alpha}\zeta^\alpha$$

where ζ^α is the parameter of local supertransformations:

$$\delta\Psi^\alpha = D\zeta^\alpha + \frac{\lambda}{2}e^\alpha{}_\beta\zeta^\beta$$

Now, if we set $\Psi^\alpha = 0$, this leaves us with the global supertransformations, i.e., with the supertransformations, where the parameter satisfies the relation:

$$D\zeta^\alpha = -\frac{\lambda}{2}e^\alpha{}_\beta\zeta^\beta$$

Note that up to this point, the deformation procedures for the bosonic and fermionic curvatures (and as a result, the parameters of these deformations) are completely independent. Their relation appears when we consider a sum of the bosonic and fermionic Lagrangians:

$$\mathcal{L} = \Omega\mathcal{R} + \Phi\mathcal{F} \tag{47}$$

and require that the sum be invariant under the global supertransformations obtained.

Massive fields:

In the gauge-invariant description of the massive bosonic and fermionic higher spin fields, the set of field variables contain one-forms, as well as zero-forms (see Sections 2.2 and 3.2). We denote them respectively as Ω^A, B^A for bosons and Φ^A, ϕ^A for fermions. Let us also denote the corresponding gauge-invariant curvatures as $\mathcal{R}^A$, $\mathcal{B}^A$ and $\mathcal{F}^A$, $\mathcal{C}^A$ ($\mathcal{R}^A$ and $\mathcal{F}^A$ are two-forms, $\mathcal{B}^A$ and $\mathcal{C}^A$ are one-forms). Schematically, they have the following structure:

$$\mathcal{R}^A = D\Omega^A + (e\Omega)^A + (eeB)^A \qquad \mathcal{F}^A = D\Phi^A + (e\Phi)^A + (ee\phi)^A \tag{48}$$

$$\mathcal{B}^A = DB^A + \Omega + (eB)^A \qquad \mathcal{C}^A = D\phi^A + \Phi + (e\phi)^A \tag{49}$$

Note that in 3D terms $(eeB)^A$ and $(ee\phi)^A$ in the expressions for the two-form, curvatures disappear for higher spin components. This is typical for 3D higher spin models and is related to the fact that the massless higher spin fields do not propagate any degrees of freedom. Unlike the massless case, the Lagrangian for the massive higher spins can be presented as the sum of the quadratic expressions in two-form and one-form curvatures (12) and (23):

$$\mathcal{L}_\Omega = \sum \mathcal{R}^A\mathcal{B}^A, \qquad \mathcal{L}_\Phi = \sum \mathcal{F}^A\mathcal{C}^A \tag{50}$$

Let us turn to a realization of the supersymmetric construction. Deformation of the curvatures by the terms containing the background gravitino one-form Ψ^α schematically can be written as:

$$\Delta\mathcal{R}^A = (\Psi\Phi)^A + (e\Psi\phi)^A \qquad \Delta\mathcal{F}^A = (\Psi\Omega)^A + (e\Psi B)^A \tag{51}$$

$$\Delta\mathcal{B}^A = (\Psi\phi)^A \qquad \Delta\mathcal{C}^A = (\Psi B)^A \tag{52}$$

Note here that the presence of $(e\Psi\phi)^A$ and $(e\Psi B)^A$ terms in two-form curvatures is related to the presence of $(eeB)^A$ and $(ee\phi)^A$ ones. Hence, such terms in the deformations appear in the two-form curvatures for the low spin components only. The requirement of covariant deformation defines uniquely supertransformations:

$$\delta_\zeta \Omega^A = (\frac{\delta(\Psi\Phi)^A}{\delta\Psi^\alpha} + \frac{\delta(e\Psi\phi)^A}{\delta\Psi^\alpha})\zeta^\alpha \qquad \delta_\zeta \Phi^A = (\frac{\delta(\Psi\Omega)^A}{\delta\Psi^\alpha} + \frac{\delta(e\Psi B)^A}{\delta\Psi^\alpha})\zeta^\alpha \tag{53}$$

$$\delta_\zeta B^A = \frac{\delta(\Psi\phi)^A}{\Psi^\alpha}\zeta^\alpha \qquad \delta\phi^A = \frac{\delta(\Psi B)^A}{\Psi^\alpha}\zeta^\alpha \tag{54}$$

As in the massless case, we now set $\Psi^\alpha = 0$, so that our supersymmetric Lagrangian for the given supermultiplet is just the sum of the free bosonic and fermionic Lagrangians:

$$\mathcal{L} = \sum[\mathcal{R}^A\mathcal{B}^A + \mathcal{F}^A\mathcal{C}^A] \tag{55}$$

Remaining arbitrariness is fixed by the condition that the Lagrangian must be invariant under the global supertransformations. Below, we demonstrate how such a general scheme can actually be realized for the massless and massive supermultiplets in 3D *AdS*.

5.2. Massless Supermultiplets

Supermultiplet $(k+1, k+3/2)$:

The integer spin-$(k+1)$ field is described by two one-forms $\Omega^{\alpha(2k)}, f^{\alpha(2k)}$, and half-integer spin-$(k+3/2)$ is described by one one-form $\Phi^{\alpha(2k+1)}$. The initial curvatures for these system are defined by (3) and (15). Let us begin with the deformation of the curvatures for the integer spin. There is a unique possibility:

$$\begin{aligned}
\Delta\mathcal{R}^{\alpha(2k)} &= i\sigma_k \Phi^{\alpha(2k)\beta}\Psi_\beta \\
\Delta\mathcal{T}^{\alpha(2k)} &= i\alpha_k \Phi^{\alpha(2k)\beta}\Psi_\beta
\end{aligned}$$

where we have two arbitrary parameters σ_k and α_k. To construct covariant deformation, we write out the corresponding supertransformations:

$$\begin{aligned}
\delta\Omega^{\alpha(2k)} &= i\sigma_k \Phi^{\alpha(2k)\beta}\zeta_\beta \\
\delta f^{\alpha(2k)} &= i\alpha_k \Phi^{\alpha(2k)\beta}\zeta_\beta
\end{aligned} \tag{56}$$

Explicit calculations of the covariant transformations for the deformed curvatures give us, on the one hand:

$$\begin{aligned}
\delta\hat{\mathcal{R}}^{\alpha(2k)} &= i\sigma_k D\Phi^{\alpha(2k)\beta}\zeta_\beta + i\frac{\lambda^2}{4}\alpha_k e^\alpha{}_\gamma \Phi^{\alpha(2k-1)\gamma\beta}\zeta_\beta - i\frac{\lambda}{2}\sigma_k e_{\gamma\beta}\Phi^{\alpha(2k)\gamma}\zeta^\beta \\
\delta\hat{\mathcal{T}}^{\alpha(2k)} &= i\alpha_k D\Phi^{\alpha(2k)\beta}\zeta_\beta + i\sigma_k e^\alpha{}_\gamma \Phi^{\alpha(2k-1)\gamma\beta}\zeta_\beta - i\frac{\lambda}{2}\alpha_k e_{\gamma\beta}\Phi^{\alpha(2k)\gamma}\zeta^\beta
\end{aligned}$$

and on the other hand:

$$\begin{aligned}
\delta\hat{\mathcal{R}}^{\alpha(2k)} &= i\sigma_k \mathcal{F}^{\alpha(2k)\beta}\zeta_\beta \\
&= i\sigma_k[D\Phi^{\alpha(2k)\beta} + \frac{\lambda}{2}(e^\alpha{}_\gamma \Phi^{\alpha(2k-1)\beta\gamma} + e^\beta{}_\gamma \Phi^{\alpha(2k)\gamma})]\zeta_\beta \\
\delta\hat{\mathcal{T}}^{\alpha(2k)} &= i\alpha_k \mathcal{F}^{\alpha(2k)\beta}\zeta_\beta \\
&= i\alpha_k[D\Phi^{\alpha(2k)\beta} + \frac{\lambda}{2}(e^\alpha{}_\gamma \Phi^{\alpha(2k-1)\beta\gamma} + e^\beta{}_\gamma \Phi^{\alpha(2k)\gamma})]\zeta_\beta
\end{aligned}$$

Comparing these expressions, we obtain the relation on the parameters:

$$\sigma_k = \frac{\lambda}{2}\alpha_k$$

Now, let us turn to the curvature deformation for the half-integer spin fields. A unique possibility is:

$$\Delta\mathcal{F}^{\alpha(2k+1)} \quad = \quad \beta_k \Omega^{\alpha(2k)}\Psi^\alpha + \gamma_k f^{\alpha(2k)}\Psi^\alpha$$

where β_k and γ_k comprise another pair of arbitrary parameters. The structure of the deformed curvatures defines the form of the supertransformations:

$$\delta\Phi^{\alpha(2k+1)} \quad = \quad \beta_k \Omega^{\alpha(2k)}\zeta^\alpha + \gamma_k f^{\alpha(2k)}\zeta^\alpha \tag{57}$$

Explicit verification of the covariant transformation for the deformed curvatures gives, on the one hand:

$$\delta\hat{\mathcal{F}}^{\alpha(2k+1)} \quad = \quad \beta_k D\Omega^{\alpha(2k)}\zeta^\alpha + \gamma_k Df^{\alpha(2k)}\zeta^\alpha + \frac{\lambda}{2}e^\alpha{}_\gamma[\beta_k\Omega^{\alpha(2k-1)\gamma}\zeta^\alpha + \gamma_k f^{\alpha(2k-1)\gamma}\zeta^\alpha]$$

and on the other hand:

$$\begin{aligned}\delta\hat{\mathcal{F}}^{\alpha(2k+1)} \quad &= \quad \beta_k\mathcal{R}^{\alpha(2k)}\zeta^\alpha + \gamma_k\mathcal{T}^{\alpha(2k)}\zeta^\alpha \\ &= \quad \beta_k[D\Omega^{\alpha(2k)} + \frac{\lambda^2}{4}e^\alpha{}_\beta f^{\alpha(2k-1)\beta}]\zeta^\alpha + \gamma_k[Df^{\alpha(2k)} + e^\alpha{}_\beta\Omega^{\alpha(2k-1)\beta}]\zeta^\alpha\end{aligned}$$

Comparing these expressions, we conclude that:

$$\gamma_k = \frac{\lambda}{2}\beta_k$$

Therefore, we constructed the covariant supersymmetric deformation for the curvatures, but we still have two free parameters α_k and β_k. To relate them, we construct the supersymmetric Lagrangian. We choose the following form for it:

$$\mathcal{L} \quad = \quad \frac{(-1)^{k+1}}{2}[\Omega_{\alpha(2k)}\mathcal{T}^{\alpha(2k)} + f_{\alpha(2k)}\mathcal{R}^{\alpha(2k)} + i\Phi_{\alpha(2k+1)}\mathcal{F}^{\alpha(2k+1)}] \tag{58}$$

i.e., just the sum of the free Lagrangians for spin-$(k+1)$ and spin-$(k+3/2)$ fields. Using supertransformations for fields and curvatures, we obtain for the Lagrangian variations:

$$\begin{aligned}\delta\mathcal{L} \quad = \quad &\frac{(-1)^{k+1}}{2}[i(\sigma_k - (2k+1)\gamma_k)\Phi^{\alpha(2k)\beta}\zeta_\beta\mathcal{T}_{\alpha(2k)} + i(\alpha_k - (2k+1)\beta_k)\Omega_{\alpha(2k)}\mathcal{F}^{\alpha(2k)\beta}\zeta_\beta \\ &+ i(\alpha_k - (2k+1)\beta_k)\Phi^{\alpha(2k)\beta}\zeta_\beta\mathcal{R}_{\alpha(2k)} + i(\sigma_k - (2k+1)\gamma_k)f_{\alpha(2k)}\mathcal{F}^{\alpha(2k)\beta}\zeta_\beta]\end{aligned}$$

In order to achieve invariance, we must put:

$$\alpha_k = (2k+1)\beta_k$$

Thus, we showed in the explicit form how to use the curvature deformation procedure to construct the supersymmetric Lagrangian formulation of the multiplet $(k+1, k+3/2)$. The constructed model contains one free parameter β_k, which is related to the normalization of the superalgebra.

Supermultiplet $(k+1, k+1/2)$:

In this supermultiplet, the integer spin is the same as in the previous case and described by a pair of one-forms $\Omega^{\alpha(2k)}, f^{\alpha(2k)}$, and half-integer spin-$(k+1/2)$ is described by one one-form $\Phi^{\alpha(2k-1)}$.

Supersymmetric construction for the given supermultiplet is analogous, and so, let us present the final results only. The supersymmetric curvatures have the form:

$$\begin{aligned}
\hat{\mathcal{R}}^{\alpha(2k)} &= D\Omega^{\alpha(2k)} + \frac{\lambda^2}{4} e^\alpha{}_\beta f^{\alpha(2k-1)\beta} + i\frac{\lambda}{2}\beta_k \Phi^{\alpha(2k-1)}\Psi^\alpha \\
\hat{\mathcal{T}}^{\alpha(2k)} &= Df^{\alpha(2k)} + e^\alpha{}_\beta \Omega^{\alpha(2k-1)\beta} + i\beta_k \Phi^{\alpha(2k-1)}\Psi^\alpha \\
\hat{\mathcal{F}}^{\alpha(2k-1)} &= D\Phi^{\alpha(2k-1)} + \frac{\lambda}{2} e^\alpha{}_\beta \Phi^{\alpha(2k-2)\beta} + 2k\beta_k(\Omega^{\alpha(2k-1)\beta}\Psi_\beta + \frac{\lambda}{2} f^{\alpha(2k-1)\beta}\Psi_\beta)
\end{aligned}$$

They are covariant under the following supertransformations:

$$\begin{aligned}
\delta\Omega^{\alpha(2k)} &= i\frac{\lambda}{2}\beta_k \Phi^{\alpha(2k-1)}\zeta^\alpha \\
\delta f^{\alpha(2k)} &= i\beta_k \Phi^{\alpha(2k-1)}\zeta^\alpha \\
\delta\Phi^{\alpha(2k-1)} &= 2k\beta_k(\Omega^{\alpha(2k-1)\beta}\zeta_\beta + \frac{\lambda}{2} f^{\alpha(2k-1)\beta}\zeta_\beta)
\end{aligned}$$

The supersymmetric Lagrangian looks as follows:

$$\mathcal{L} = \frac{(-1)^{k+1}}{2}[\Omega_{\alpha(2k)}\mathcal{T}^{\alpha(2k)} + f_{\alpha(2k)}\mathcal{R}^{\alpha(2k)} - i\Phi_{\alpha(2k-1)}\mathcal{F}^{\alpha(2k-1)}] \tag{59}$$

where:

$$\begin{aligned}
\delta\mathcal{R}^{\alpha(2k)} &= i\frac{\lambda}{2}\beta_k \mathcal{F}^{\alpha(2k-1)}\zeta^\alpha \\
\delta\mathcal{T}^{\alpha(2k)} &= i\beta_k \mathcal{F}^{\alpha(2k-1)}\zeta^\alpha \\
\delta\mathcal{F}^{\alpha(2k-1)} &= 2k\beta_k(\mathcal{R}^{\alpha(2k-1)\beta}\zeta_\beta + \frac{\lambda}{2}\mathcal{T}^{\alpha(2k-1)\beta}\zeta_\beta)
\end{aligned}$$

5.3. Massive Supermultiplets

Supermultiplet $(s, s+\frac{1}{2})$:

For the realization of the given massive supermultiplets, let us first consider their structure at the massless flat limit $m, m_1, \lambda \to 0$. In this case, the Lagrangian is described by the system of massless fields with spins $(s+\frac{1}{2}), s, ..., \frac{1}{2}, 0$ in three-dimensional flat space:

$$\begin{aligned}
\mathcal{L} = &\textstyle\sum_{k=1}^{s-1}(-1)^{k+1}[k\Omega_{\alpha(2k-1)\beta}e^\beta{}_\gamma\Omega^{\alpha(2k-1)\gamma} + \Omega_{\alpha(2k)}Df^{\alpha(2k)}] \\
&+EB_{\alpha\beta}B^{\alpha\beta} - B_{\alpha\beta}e^{\alpha\beta}DA - E\pi_{\alpha\beta}\pi^{\alpha\beta} + \pi_{\alpha\beta}E^{\alpha\beta}D\varphi \\
&+\tfrac{i}{2}\textstyle\sum_{k=0}^{s-1}(-1)^{k+1}\Phi_{\alpha(2k+1)}D\Phi^{\alpha(2k+1)} + \tfrac{1}{2}\phi_\alpha E^\alpha{}_\beta D\phi^\beta
\end{aligned} \tag{60}$$

It is the same extended massless supermultiplet, with which we start the construction of the massive supermultiplets in Minkowski space. We have shown that this Lagrangian is supersymmetric. If the equations of motion (33) and (36) are fulfilled, the Lagrangian is invariant under the supertransformations (39):

$$
\begin{aligned}
\delta f^{\alpha(2k)} &= i\beta_k \Phi^{\alpha(2k-1)}\zeta^\alpha + i\alpha_k \Phi^{\alpha(2k)\beta}\zeta_\beta \\
\delta f^{\alpha(2)} &= i\beta_1 \Phi^\alpha \zeta^\alpha + i\alpha_1 \Phi^{\alpha(2)\beta}\zeta_\beta \\
\delta A &= i\alpha_0 \Phi^\alpha \zeta_\alpha + ic_0\beta_0 e_{\alpha\beta}\phi^\alpha\zeta^\beta, \qquad \delta\varphi = -\frac{ic_0\tilde{\delta}_0}{2}\phi^\gamma\zeta_\gamma \\
\delta\Phi^{\alpha(2k+1)} &= \frac{\alpha_k}{(2k+1)}\Omega^{\alpha(2k)}\zeta^\alpha + 2(k+1)\beta_{k+1}\Omega^{\alpha(2k+1)\beta}\zeta_\beta \\
\delta\Phi^\alpha &= 2\beta_1\Omega^{\alpha\beta}\zeta_\beta + 2a_0\alpha_0 e_{\beta(2)}B^{\beta(2)}\zeta^\alpha \\
\delta\phi^\alpha &= \frac{8a_0\beta_0}{c_0}B^{\alpha\beta}\zeta_\beta + \frac{b_0\tilde{\delta}_0}{c_0}\pi^{\alpha\beta}\zeta_\beta
\end{aligned}
$$

Here, we take into account the normalization (8) and (20). Thus, requiring that massive theory has a correct massless flat limit, we partially fix an arbitrariness in the choice of the supertransformations. Parameters $\alpha_k, \beta_k, \beta_0, \alpha_0, \tilde{\delta}_0$ at this step are still arbitrary.

As in the massless case, we will realize supersymmetry deforming the curvatures by the background gravitino field Ψ^α. We start with the construction of the deformations for the bosonic fields:

$$
\begin{aligned}
\Delta\mathcal{R}^{\alpha(2k)} &= i\rho_k \Phi^{\alpha(2k-1)}\Psi^\alpha + i\sigma_k \Phi^{\alpha(2k)\beta}\Psi_\beta \\
\Delta\mathcal{T}^{\alpha(2k)} &= i\beta_k \Phi^{\alpha(2k-1)}\Psi^\alpha + i\alpha_k \Phi^{\alpha(2k)\beta}\Psi_\beta \\
\Delta\mathcal{R}^{\alpha(2)} &= i\rho_1 \Phi^\alpha\Psi^\alpha + i\sigma_1 \Phi^{\alpha(2)\beta}\Psi_\beta + i\hat{\rho}_0 e^{\alpha(2)}\phi^\beta\Psi_\beta \\
\Delta\mathcal{T}^{\alpha(2)} &= i\beta_1 \Phi^\alpha\Psi^\alpha + i\alpha_1 \Phi^{\alpha(2)\beta}\Psi_\beta \\
\Delta\mathcal{A} &= i\alpha_0 \Phi^\alpha\Psi_\alpha + ic_0\beta_0 e_{\alpha(2)}\phi^\alpha\Psi^\beta, \qquad \Delta\Phi = \frac{ic_0\tilde{\delta}_0}{2}\phi^\alpha\Psi_\alpha \\
\Delta\mathcal{B}^{\alpha(2k)} &= -i\hat{\rho}_k \phi^{\alpha(2k-1)}\Psi^\alpha - i\hat{\sigma}_k \phi^{\alpha(2k)\beta}\Psi_\beta \\
\Delta\Pi^{\alpha(2k)} &= -i\hat{\beta}_k \phi^{\alpha(2k-1)}\Psi^\alpha - i\hat{\alpha}_k \phi^{\alpha(2k)\beta}\Psi_\beta
\end{aligned}
$$

The corresponding ansatz for the supertransformations has the form:

$$
\begin{aligned}
\delta\Omega^{\alpha(2k)} &= i\rho_k \Phi^{\alpha(2k-1)}\zeta^\alpha + i\sigma_k \Phi^{\alpha(2k)\beta}\zeta_\beta \\
\delta f^{\alpha(2k)} &= i\beta_k \Phi^{\alpha(2k-1)}\zeta^\alpha + i\alpha_k \Phi^{\alpha(2k)\beta}\zeta_\beta \\
\delta\Omega^{\alpha(2)} &= i\rho_1 \Phi^\alpha\zeta^\alpha + i\sigma_1 \Phi^{\alpha(2)\beta}\zeta_\beta + i\hat{\rho}_0 e^{\alpha(2)}\phi^\beta\zeta_\beta \\
\delta f^{\alpha(2)} &= i\beta_1 \Phi^\alpha\zeta^\alpha + i\alpha_1 \Phi^{\alpha(2)\beta}\zeta_\beta \\
\delta A &= i\alpha_0 \Phi^\alpha\zeta_\alpha + ic_0\beta_0 e_{\alpha(2)}\phi^\alpha\zeta^\beta, \qquad \delta\varphi = -\frac{ic_0\tilde{\delta}_0}{2}\phi^\gamma\zeta_\gamma \\
\delta B^{\alpha(2k)} &= i\hat{\rho}_k \phi^{\alpha(2k-1)}\zeta^\alpha + i\hat{\sigma}_k \phi^{\alpha(2k)\beta}\zeta_\beta \\
\delta\pi^{\alpha(2k)} &= i\hat{\beta}_k \phi^{\alpha(2k-1)}\zeta^\alpha + i\hat{\alpha}_k \phi^{\alpha(2k)\beta}\zeta_\beta
\end{aligned} \tag{61}
$$

All parameters are fixed by the requirement of covariant transformations of the curvatures. First of all, we consider:

$$
\begin{aligned}
\delta\hat{\mathcal{R}}^{\alpha(2k)} &= i\rho_k \mathcal{F}^{\alpha(2k-1)}\zeta^\alpha + i\sigma_k \mathcal{F}^{\alpha(2k)\beta}\zeta_\beta \\
\delta\hat{\mathcal{T}}^{\alpha(2k)} &= i\beta_k \mathcal{F}^{\alpha(2k-1)}\zeta^\alpha + i\alpha_k \mathcal{F}^{\alpha(2k)\beta}\zeta_\beta
\end{aligned} \tag{62}
$$

It leads to relation $M_1 = M + \frac{\lambda}{2}$ between mass parameters M_1 and M and defines the parameters:

$$
\begin{aligned}
{\alpha_k}^2 &= k(s+k+1)[M+(k+1)\lambda]\hat{\alpha}^2 \\
{\beta_k}^2 &= \frac{(k+1)(s-k)}{k(2k+1)}[M-k\lambda]\hat{\beta}^2 \\
{\sigma_k}^2 &= \frac{(s+k+1)}{k(k+1)^2}[M+(k+1)\lambda]\hat{\sigma}^2 \\
{\rho_k}^2 &= \frac{(s-k)}{k^3(k+1)(2k+1)}[M-k\lambda]\hat{\rho}^2
\end{aligned} \tag{63}
$$

where:

$$
\hat{\beta} = \frac{\hat{\alpha}}{\sqrt{2}}, \qquad \hat{\rho} = \frac{sM}{2\sqrt{2}}\hat{\alpha}, \qquad \hat{\sigma} = \frac{sM}{2}\hat{\alpha}, \qquad \hat{\alpha}^2 = \frac{{\alpha_{s-1}}^2}{2s(s-1)[M+s\lambda]}
$$

From the requirement that:

$$
\begin{aligned}
\delta\hat{\mathcal{B}}^{\alpha(2k)} &= i\hat{\rho}_k\mathcal{C}^{\alpha(2k-1)}\zeta^\alpha + i\hat{\sigma}_k\mathcal{C}^{\alpha(2k)\beta}\zeta_\beta \\
\delta\hat{\Pi}^{\alpha(2k)} &= i\hat{\beta}_k\mathcal{C}^{\alpha(2k-1)}\zeta^\alpha + i\hat{\alpha}_k\mathcal{C}^{\alpha(2k)\beta}\zeta_\beta
\end{aligned} \tag{64}
$$

we obtain:

$$
\hat{\rho}_k = \rho_k, \qquad \hat{\sigma}_k = \sigma_k, \qquad \hat{\beta}_k = \beta_k, \qquad \hat{\alpha}_k = \alpha_k
$$

The requirement of covariant transformations for the remaining curvatures:

$$
\begin{aligned}
\delta\hat{\mathcal{R}}^{\alpha(2)} &= i\rho_1\mathcal{F}^\alpha\zeta^\alpha + i\sigma_1\mathcal{F}^{\alpha(2)\beta}\zeta_\beta - i\hat{\rho}_0 e^{\alpha(2)}\mathcal{C}^\beta\zeta_\beta \\
\delta\hat{\mathcal{T}}^{\alpha(2)} &= i\beta_1\mathcal{F}^\alpha\zeta^\alpha + i\alpha_1\mathcal{F}^{\alpha(2)\beta}\zeta_\beta \\
\delta\hat{\mathcal{A}} &= i\alpha_0\mathcal{F}^\alpha\zeta_\alpha - ic_0\beta_0 e_{\alpha\beta}\mathcal{C}^\alpha\zeta^\beta, \qquad \delta\hat{\Phi} = -\tfrac{ic_0\tilde{\delta}_0}{2}\mathcal{C}^\gamma\zeta_\gamma
\end{aligned} \tag{65}
$$

gives the solution:

$$
\hat{\rho}_0 = -\frac{1}{8}{c_0}^2\beta_1, \qquad \tilde{\delta}_0 = 4\beta_0 = \frac{c_0}{a_0}\beta_1, \qquad \alpha_0 = \frac{{c_0}^2}{4sMa_0}\beta_1
$$

Now, let us consider the deformation of the curvatures for the fermions. We choose an ansatz in the form:

$$
\begin{aligned}
\Delta\mathcal{F}^{\alpha(2k+1)} &= \frac{\alpha_k}{(2k+1)}\Omega^{\alpha(2k)}\Psi^\alpha + 2(k+1)\beta_{k+1}\Omega^{\alpha(2k+1)\beta}\Psi_\beta \\
&\quad + \gamma_k f^{\alpha(2k)}\Psi^\alpha + \delta_k f^{\alpha(2k+1)\beta}\Psi_\beta \\
\Delta\mathcal{F}^\alpha &= 2\beta_1\Omega^{\alpha\beta}\Psi_\beta + 2a_0\alpha_0 e_{\beta(2)}B^{\beta(2)}\Psi^\alpha + \delta_0 f^{\alpha\beta}\Psi_\beta + \gamma_0 A\Psi^\alpha + \tilde{\gamma}_0\varphi e^\alpha{}_\beta\Psi^\beta \\
\Delta\mathcal{C}^\alpha &= -\frac{8a_0\beta_0}{c_0}B^{\alpha\beta}\Psi_\beta - \frac{b_0\tilde{\delta}_0}{c_0}\pi^{\alpha\beta}\Psi_\beta - \rho_0\varphi\Psi^\alpha \\
\Delta\mathcal{C}^{\alpha(2k+1)} &= -\tilde{\beta}_k B^{\alpha(2k+1)\beta}\Psi_\beta - \tilde{\alpha}_k B^{\alpha(2k)}\Psi^\alpha - \tilde{\delta}_k\pi^{\alpha(2k+1)\beta}\Psi_\beta - \tilde{\gamma}_k\pi^{\alpha(2k)}\Psi^\alpha
\end{aligned}
$$

and the ansatz for the supertransformations in the form:

$$\begin{aligned}
\delta\Phi^{\alpha(2k+1)} &= \frac{\alpha_k}{(2k+1)}\Omega^{\alpha(2k)}\zeta^\alpha + 2(k+1)\beta_{k+1}\Omega^{\alpha(2k+1)\beta}\zeta_\beta \\
&\quad +\gamma_k f^{\alpha(2k)}\zeta^\alpha + \delta_k f^{\alpha(2k+1)\beta}\zeta_\beta \\
\delta\Phi^\alpha &= 2\beta_1\Omega^{\alpha\beta}\zeta_\beta + 2a_0\alpha_0 e_{\beta(2)}B^{\beta(2)}\zeta^\alpha + \delta_0 f^{\alpha\beta}\zeta_\beta + \gamma_0 A\zeta^\alpha + \tilde{\gamma}_0\varphi e^\alpha{}_\beta\zeta^\beta \\
\delta\phi^\alpha &= \frac{8a_0\beta_0}{c_0}B^{\alpha\beta}\zeta_\beta + \frac{b_0\tilde{\delta}_0}{c_0}\pi^{\alpha\beta}\zeta_\beta + \rho_0\varphi\zeta^\alpha \\
\delta\phi^{\alpha(2k+1)} &= \tilde{\beta}_k B^{\alpha(2k+1)\beta}\zeta_\beta + \tilde{\alpha}_k B^{\alpha(2k)}\zeta^\alpha + \tilde{\delta}_k\pi^{\alpha(2k+1)\beta}\zeta_\beta + \tilde{\gamma}_k\pi^{\alpha(2k)}\zeta^\alpha
\end{aligned} \tag{66}$$

From the requirement that:

$$\begin{aligned}
\delta\hat{\mathcal{F}}^{\alpha(2k+1)} &= \frac{\alpha_k}{(2k+1)}\mathcal{R}^{\alpha(2k)}\zeta^\alpha + 2(k+1)\beta_{k+1}\mathcal{R}^{\alpha(2k+1)\beta}\zeta_\beta \\
&\quad +\gamma_k\mathcal{T}^{\alpha(2k)}\zeta^\alpha + \delta_k\mathcal{T}^{\alpha(2k+1)\beta}\zeta_\beta
\end{aligned} \tag{67}$$

we have the same relation between masses $M_1 = M + \frac{\lambda}{2}$. Besides, it leads to:

$$\begin{aligned}
\gamma_k{}^2 &= \frac{(s+k+1)}{k(k+1)^2(2k+1)^2}[M+(k+1)\lambda]\hat{\gamma}^2 \\
\delta_k{}^2 &= \frac{(s-k-1)}{(k+1)(k+2)(2k+3)}[M-(k+1)\lambda]\hat{\delta}^2
\end{aligned}$$

where:

$$\hat{\gamma} = \frac{sM}{2}\hat{\alpha}, \qquad \hat{\delta} = \frac{sM}{\sqrt{2}}\hat{\alpha}$$

In turn, the requirement that:

$$\delta\hat{\mathcal{C}}^{\alpha(2k+1)} = \tilde{\beta}_k\mathcal{B}^{\alpha(2k+1)\beta}\zeta_\beta + \tilde{\alpha}_k\mathcal{B}^{\alpha(2k)}\zeta^\alpha + \tilde{\delta}_k\Pi^{\alpha(2k+1)\beta}\zeta_\beta + \tilde{\gamma}_k\Pi^{\alpha(2k)}\zeta^\alpha \tag{68}$$

gives us:

$$\tilde{\gamma}_k = \gamma_k, \quad \tilde{\delta}_k = \delta_k, \quad \tilde{\alpha}_k = \frac{\alpha_k}{(2k+1)}, \quad \tilde{\beta}_k = 2(k+1)\beta_{k+1}$$

At last, the requirement for the other curvatures:

$$\begin{aligned}
\delta\hat{\mathcal{F}}^\alpha &= 2\beta_1\mathcal{R}^{\alpha\beta}\zeta_\beta - 2a_0\alpha_0 e_{\beta(2)}\mathcal{B}^{\beta(2)}\zeta^\alpha + \delta_0\mathcal{T}^{\alpha\beta}\zeta_\beta + \gamma_0\mathcal{A}\zeta^\alpha + \tilde{\gamma}_0\Phi e^\alpha{}_\beta\zeta^\beta \\
\delta\hat{\mathcal{C}}^\alpha &= \tfrac{8a_0\beta_0}{c_0}\mathcal{B}^{\alpha\beta}\zeta_\beta + \tfrac{b_0\tilde{\delta}_0}{c_0}\Pi^{\alpha\beta}\zeta_\beta + \rho_0\Phi\zeta^\alpha
\end{aligned} \tag{69}$$

yields solution:

$$\gamma_0 = -2\tilde{\gamma}_0 = \frac{c_0{}^2}{2a_0}\beta_1, \quad \rho_0 = -\frac{c_0{}^2}{4sMa_0}\beta_1$$

Now, all the arbitrary parameters are fixed.

The supersymmetric Lagrangian is the sum of the free Lagrangians:

$$\begin{aligned}
\hat{\mathcal{L}} &= -\tfrac{1}{2}\textstyle\sum_{k=1}^{s-1}(-1)^{k+1}[\mathcal{R}_{\alpha(2k)}\Pi^{\alpha(2k)} + \mathcal{T}_{\alpha(2k)}\mathcal{B}^{\alpha(2k)}] + \tfrac{a_0}{2sM}e_{\alpha(2)}\mathcal{B}^{\alpha(2)}\Phi \\
&\quad -\tfrac{i}{2}\textstyle\sum_{k=0}^{s-1}(-1)^{k+1}\mathcal{F}_{\alpha(2k+1)}\mathcal{C}^{\alpha(2k+1)}
\end{aligned} \tag{70}$$

The Lagrangian is invariant under the supertransformations for curvatures up to equations of motion for the fields $B^{\alpha(2)}$, $\pi^{\alpha(2)}$:

$$\Phi = 0, \quad \mathcal{A} = 0 \quad \Rightarrow \quad e_{\gamma(2)}\Pi^{\gamma(2)} = D\Phi - 2sM\mathcal{A} = 0 \tag{71}$$

The Lagrangian (70) is a final solution for the massive supermultiplet $(s, s+\frac{1}{2})$.

Supermultiplet $(s, s-\frac{1}{2})$:

In this section, we consider another massive higher spin supermultiplet when the highest spin is boson. The massive spin-s field was described in Section 3.1 in terms of massless fields. The massive spin-$(s-1/2)$ field can be obtained for the results in Section 3.2 if one makes the replacement $s \to (s-1)$. Therefore, the set of massless fields for the massive field with spin $s-1/2$ is $\Phi^{\alpha(2k+1)}$, $0 \le k \le s-2$ and ϕ^α. The gauge-invariant curvatures and the Lagrangian have the forms (19) and (23), where the parameters are:

$$\begin{aligned}
c_k{}^2 &= \frac{(s+k)(s-k-1)}{2(k+1)(2k+1)}[M_1{}^2 - (2k+1)^2\frac{\lambda^2}{4}] \\
c_0{}^2 &= 2s(s-1)[M_1{}^2 - \frac{\lambda^2}{4}] \\
d_k &= \frac{(2s-1)}{(2k+3)}M_1, \qquad M_1{}^2 = m_1{}^2 + (s-\frac{3}{2})^2\lambda^2
\end{aligned} \tag{72}$$

Following our procedure, we should construct the supersymmetric deformations for the curvatures. Actually, the structure of the deformed curvatures and supertransformations have the same form as in the previous subsection for the supermultiplets $(s, s+1/2)$. There is a difference in parameters (72) only. Therefore, we present here only the supertransformations for the curvatures. The requirement of covariant curvature transformations for the bosonic fields:

$$\begin{aligned}
\delta\hat{\mathcal{R}}^{\alpha(2k)} &= i\rho_k\mathcal{F}^{\alpha(2k-1)}\zeta^\alpha + i\sigma_k\mathcal{F}^{\alpha(2k)\beta}\zeta_\beta \\
\delta\hat{\mathcal{T}}^{\alpha(2k)} &= i\beta_k\mathcal{F}^{\alpha(2k-1)}\zeta^\alpha + i\alpha_k\mathcal{F}^{\alpha(2k)\beta}\zeta_\beta \\
\delta\hat{\mathcal{R}}^{\alpha(2)} &= i\rho_1\mathcal{F}^{\alpha}\zeta^\alpha + i\sigma_1\mathcal{F}^{\alpha(2)\beta}\zeta_\beta - i\hat{\rho}_0 e^{\alpha(2)}\mathcal{C}^\beta\zeta_\beta \\
\delta\hat{\mathcal{T}}^{\alpha(2)} &= i\beta_1\mathcal{F}^{\alpha}\zeta^\alpha + i\alpha_1\mathcal{F}^{\alpha(2)\beta}\zeta_\beta \\
\delta\hat{\mathcal{A}} &= i\alpha_0\mathcal{F}^\alpha\zeta_\alpha - ic_0\beta_0 e_{\alpha\beta}\mathcal{C}^\alpha\zeta^\beta, \qquad \delta\hat{\Phi} = -\frac{ic_0\tilde{\delta}_0}{2}\mathcal{C}^\gamma\zeta_\gamma \\
\delta\hat{\mathcal{B}}^{\alpha(2k)} &= i\hat{\rho}_k\mathcal{C}^{\alpha(2k-1)}\zeta^\alpha + i\hat{\sigma}_k\mathcal{C}^{\alpha(2k)\beta}\zeta_\beta \\
\delta\hat{\Gamma}^{\alpha(2k)} &= i\hat{\beta}_k\mathcal{C}^{\alpha(2k-1)}\zeta^\alpha + i\hat{\alpha}_k\mathcal{C}^{\alpha(2k)\beta}\zeta_\beta
\end{aligned}$$

gives us the relation $M_1 = M - \frac{\lambda}{2}$ between masses M_1 and M. Besides, it leads to:

$$\begin{aligned}
\sigma_k{}^2 &= \frac{(s-k-1)}{k(k+1)^2}[M-(k+1)\lambda]\hat{\sigma}^2 \\
\rho_k{}^2 &= \frac{(s+k)}{k^3(k+1)(2k+1)}[M+k\lambda]\hat{\rho}^2 \\
\alpha_k{}^2 &= k(s-k-1)[M-(k+1)\lambda]\hat{\alpha}^2 \\
\beta_k{}^2 &= \frac{(k+1)(s+k)}{k(2k+1)}[M+k\lambda]\hat{\beta}^2
\end{aligned} \tag{73}$$

$$\hat{\rho}_k = \rho_k, \qquad \hat{\sigma}_k = \sigma_k, \qquad \hat{\beta}_k = \beta_k, \qquad \hat{\alpha}_k = \alpha_k$$

$$\hat{\rho}_0 = -\frac{1}{8}c_0{}^2\beta_1, \qquad \tilde{\delta}_0 = 4\beta_0 = \frac{c_0}{a_0}\beta_1, \qquad \alpha_0 = \frac{c_0{}^2}{4sMa_0}\beta_1$$

where:

$$\hat{\beta} = \frac{\hat{\alpha}}{\sqrt{2}}, \qquad \hat{\rho} = \frac{sM}{2\sqrt{2}}\hat{\alpha}, \qquad \hat{\sigma} = \frac{sM}{2}\hat{\alpha}, \qquad \hat{\alpha}^2 = \frac{\alpha_{s-2}{}^2}{(s-2)[M-(s-1)\lambda]}$$

From the requirement of covariant supertransformations for the fermionic curvatures:

$$\begin{aligned}
\delta\hat{\mathcal{F}}^{\alpha(2k+1)} &= \frac{\alpha_k}{(2k+1)}\mathcal{R}^{\alpha(2k)}\zeta^\alpha + 2(k+1)\beta_{k+1}\mathcal{R}^{\alpha(2k+1)\beta}\zeta_\beta \\
&\quad +\gamma_k\mathcal{T}^{\alpha(2k)}\zeta^\alpha + \delta_k\mathcal{T}^{\alpha(2k+1)\beta}\zeta_\beta \\
\delta\hat{\mathcal{F}}^\alpha &= 2\beta_1\mathcal{R}^{\alpha\beta}\zeta_\beta - 2a_0\alpha_0 e_{\beta(2)}\mathcal{B}^{\beta(2)}\zeta^\alpha + \delta_0\mathcal{T}^{\alpha\beta}\zeta_\beta + \gamma_0\mathcal{A}\zeta^\alpha + \tilde{\gamma}_0\Phi e^\alpha{}_\beta\zeta^\beta \\
\delta\hat{\mathcal{C}}^\alpha &= \frac{8a_0\beta_0}{c_0}\mathcal{B}^{\alpha\beta}\zeta_\beta + \frac{b_0\tilde{\delta}_0}{c_0}\Pi^{\alpha\beta}\zeta_\beta + \rho_0\Phi\zeta^\alpha \\
\delta\hat{\mathcal{C}}^{\alpha(2k+1)} &= \tilde{\beta}_k\mathcal{B}^{\alpha(2k+1)\beta}\zeta_\beta + \tilde{\alpha}_k\mathcal{B}^{\alpha(2k)}\zeta^\alpha + \tilde{\delta}_k\Pi^{\alpha(2k+1)\beta}\zeta_\beta + \tilde{\gamma}_k\Pi^{\alpha(2k)}\zeta^\alpha
\end{aligned}$$

one gets:

$$\begin{aligned}
{\gamma_k}^2 &= \frac{(s-k-1)}{k(k+1)^2(2k+1)^2}[M-(k+1)\lambda]\hat{\gamma}^2 \\
{\delta_k}^2 &= \frac{(s+k+1)}{(k+1)(k+2)(2k+3)}[M+(k+1)\lambda]\hat{\delta}^2
\end{aligned}$$

$$\tilde{\gamma}_k = \gamma_k, \quad \tilde{\delta}_k = \delta_k, \quad \tilde{\alpha}_k = \frac{\alpha_k}{(2k+1)}, \quad \tilde{\beta}_k = 2(k+1)\beta_{k+1}$$

$$\gamma_0 = -2\tilde{\gamma}_0 = \frac{{c_0}^2}{2a_0}\beta_1, \quad \rho_0 = -\frac{{c_0}^2}{4sMa_0}\beta_1$$

where:

$$\hat{\gamma} = \frac{sM}{2}\hat{\alpha}, \qquad \hat{\delta} = \frac{sM}{\sqrt{2}}\hat{\alpha}$$

Supersymmetric Lagrangian have the form (70), and it is invariant under the supertransformations up to equations of motion for the auxiliary fields $B^{\alpha(2)}, \pi^{\alpha(2)}$ (71).

5.4. Realization of (Super)Algebra

In this section, we analyze the commutators of the (super)transformations and show how the (super)algebra is realized in our construction. All the considerations are valid both for the $(s, s+1/2)$ supermultiplets and the for $(s, s-1/2)$ one.

5.4.1. Description of the AdS Transformations

Before we turn to the supersymmetric theory, let us discuss the conventional AdS_3 algebra. In the frame formalism, AdS space is described by the background Lorentz connection field $\omega^{\alpha(2)}$ and the background frame field $e^{\alpha(2)}$. The first of them enters implicitly through the covariant derivative D, while the second one enters explicitly. Let $\eta^{\alpha(2)}$ and $\xi^{\alpha(2)}$ be the parameters of the Lorentz transformations and the pseudo-translations, respectively. The theory of the massive spin-s field has the following laws under these transformations:

$$\delta_\eta\Omega^{\alpha(2k)} = \eta^\alpha{}_\beta\Omega^{\alpha(2k-1)\beta} \qquad \delta_\eta f^{\alpha(2k)} = \eta^\alpha{}_\beta f^{\alpha(2k-1)\beta} \tag{74}$$

$$\begin{aligned}
\delta_\xi\Omega^{\alpha(2k)} &= \tfrac{(k+2)a_k}{k}\xi_{\beta(2)}\Omega^{\alpha(2k)\beta(2)} + \tfrac{a_{k-1}}{k(2k-1)}\xi^{\alpha(2)}\Omega^{\alpha(2k-2)} + \tfrac{b_k}{k}\xi^\alpha{}_\beta f^{\alpha(2k-1)\beta} \\
\delta_\xi f^{\alpha(2k)} &= \xi^\alpha{}_\beta\Omega^{\alpha(2k-1)\beta} + a_k\xi_{\beta(2)}f^{\alpha(2k)\beta(2)} + \tfrac{(k+1)a_{k-1}}{k(k-1)(2k-1)}\xi^{\alpha(2)}f^{\alpha(2k-2)}
\end{aligned} \tag{75}$$

here a_k and b_k are defined by (6). For the massive spin-$(s \pm 1/2)$ field, the transformation laws look like:

$$\begin{aligned}
\delta_\eta\Phi^{\alpha(2k+1)} &= \eta^\alpha{}_\beta\Phi^{\alpha(2k)\beta} \\
\delta_\xi\Phi^{\alpha(2k+1)} &= \tfrac{d_k}{(2k+1)}\xi^\alpha{}_\beta\Phi^{\alpha(2k)\beta} + \tfrac{c_k}{k(2k+1)}\xi^{\alpha(2)}\Phi^{\alpha(2k-1)} \\
&\quad + c_{k+1}\xi_{\beta(2)}\Phi^{\alpha(2k+1)\beta(2)}
\end{aligned} \tag{76}$$

Here, c_k and d_k are defined by (18) for the spin-$(s+1/2)$ and (72) for the spin-$(s-1/2)$. To consider a structure of the AdS_3 algebra $Sp(2) \otimes Sp(2)$ in the left sector only, we introduce the new variables for the bosonic fields:

$$\hat{\Omega}^{\alpha(2k)} = \Omega^{\alpha(2k)} + \frac{sM}{2k(k+1)} f^{\alpha(2k)}, \qquad \hat{f}^{\alpha(2k)} = \Omega^{\alpha(2k)} - \frac{sM}{2k(k+1)} f^{\alpha(2k)} \tag{77}$$

so that the variables $\hat{\Omega}^{\alpha(2k)}$ correspond to the left sector. In terms of these variables, the transformations (74) and (75) have the form:

$$\begin{aligned}
\delta_\eta \hat{\Omega}^{\alpha(2k)} &= \eta^\alpha{}_\beta \hat{\Omega}^{\alpha(2k-1)\beta} \\
\delta_\xi \hat{\Omega}^{\alpha(2k)} &= \tfrac{(k+2)a_k}{k} \xi_{\beta(2)} \hat{\Omega}^{\alpha(2k)\beta(2)} + \tfrac{a_{k-1}}{k(2k-1)} \xi^{\alpha(2)} \hat{\Omega}^{\alpha(2k-2)} \\
&\quad + \tfrac{sM}{2k(k+1)} \xi^\alpha{}_\beta \hat{\Omega}^{\alpha(2k-1)\beta}
\end{aligned} \tag{78}$$

Now, let us consider the commutators of these transformations. The direct calculations lead to the following results:

$$\begin{aligned}
[\delta_{\eta_1}, \delta_{\eta_2}] \hat{\Omega}^{\alpha(2k)} &= (\eta_2{}^\alpha{}_\beta \eta_1{}^\beta{}_\gamma - \eta_1{}^\alpha{}_\beta \eta_2{}^\beta{}_\gamma) \hat{\Omega}^{\alpha(2k-1)\gamma}, \\
[\delta_{\xi_1}, \delta_{\xi_2}] \hat{\Omega}^{\alpha(2k)} &= \frac{\lambda^2}{4} (\xi_2{}^\alpha{}_\gamma \xi_1{}^\gamma{}_\beta - \xi_1{}^\alpha{}_\gamma \xi_2{}^\gamma{}_\beta) \hat{\Omega}^{\alpha(2k-1)\beta}, \\
[\delta_\eta, \delta_\xi] \hat{\Omega}^{\alpha(2k)} &= 2\frac{(k+2)a_k}{k} \xi_{\beta(2)} \eta^\beta{}_\gamma \hat{\Omega}^{\alpha(2k)\beta\gamma} + \frac{a_{k-1}}{k(2k-1)} \xi^\alpha{}_\gamma \eta^{\alpha\gamma} \hat{\Omega}^{\alpha(2k-2)} \\
&\quad + \frac{sM}{2k(k+1)} (\xi^\alpha{}_\beta \eta^\beta{}_\gamma - \eta^\alpha{}_\gamma \xi^\gamma{}_\beta) \hat{\Omega}^{\alpha(2k-1)}
\end{aligned}$$

Comparison with (78) shows that we do have the AdS-algebra:

$$[M_{\alpha(2)}, M_{\beta(2)}] \sim \varepsilon_{\alpha\beta} M_{\alpha\beta}, \qquad [P_{\alpha(2)}, P_{\beta(2)}] \sim \lambda^2 \varepsilon_{\alpha\beta} M_{\alpha\beta}$$

$$[M_{\alpha(2)}, P_{\beta(2)}] \sim \varepsilon_{\alpha\beta} P_{\alpha\beta}$$

The analogous results have a place for the commutators in the fermionic sector, as well.

5.4.2. *AdS* Supertransformations

Let us consider the supersymmetric theory. The supertransformations for the massive higher spin supermultiplets have the form of (61) and (66):

$$\begin{aligned}
\delta \Omega^{\alpha(2k)} &= \frac{isM}{2k(k+1)} \beta_k \Phi^{\alpha(2k-1)} \zeta^\alpha + \frac{isM}{2k(k+1)} \alpha_k \Phi^{\alpha(2k)\beta} \zeta_\beta \\
\delta f^{\alpha(2k)} &= i\beta_k \Phi^{\alpha(2k-1)} \zeta^\alpha + i\alpha_k \Phi^{\alpha(2k)\beta} \zeta_\beta \\
\delta \Phi^{\alpha(2k+1)} &= \frac{\alpha_k}{(2k+1)} \Omega^{\alpha(2k)} \zeta^\alpha + 2(k+1) \beta_{k+1} \Omega^{\alpha(2k+1)\beta} \zeta_\beta \\
&\quad + \frac{sM}{2k(k+1)(2k+1)} \alpha_k f^{\alpha(2k)} \zeta^\alpha + \frac{sM}{(k+2)} \beta_{k+1} f^{\alpha(2k+1)\beta} \zeta_\beta
\end{aligned}$$

where the parameters α_k and β_k are defined by (63) for the $(s, s+1/2)$ supermultiplets and (73) for the $(s, s-1/2)$ one. In terms of the new variables (77), the supertransformations look like:

$$
\begin{aligned}
\delta\hat{\Omega}^{\alpha(2k)} &= \frac{isM}{k(k+1)}\beta_k \Phi^{\alpha(2k-1)}\zeta^{\alpha} + \frac{isM}{k(k+1)}\alpha_k \Phi^{\alpha(2k)\beta}\zeta_{\beta} \\
\delta\hat{f}^{\alpha(2k)} &= 0 \\
\delta\Phi^{\alpha(2k+1)} &= \frac{\alpha_k}{(2k+1)}\hat{\Omega}^{\alpha(2k)}\zeta^{\alpha} + 2(k+1)\beta_{k+1}\hat{\Omega}^{\alpha(2k+1)\beta}\zeta_{\beta}
\end{aligned}
$$

One can see that the $\hat{f}^{\alpha(2k)}$ fields are inert under the supertransformations. This just means that we have (1,0) supersymmetry. Let us calculate the commutator of two supertransformations. We obtain:

$$
\begin{aligned}
[\delta_1,\delta_2]\hat{\Omega}^{\alpha(2k)} &= isM\hat{\alpha}^2[\frac{a_{k-1}}{k(2k-1)}\hat{\Omega}^{\alpha(2k-1)}{\zeta_1}^{\alpha}{\zeta_2}^{\alpha} + \frac{2(k+2)a_k}{k}\hat{\Omega}^{\alpha(2k)\gamma\beta}\zeta_{1\beta}\zeta_{2\gamma} \\
&\quad + \frac{sM}{k(k+1)}\hat{\Omega}^{\alpha(2k-1)\gamma}{\zeta_1}^{\alpha}\zeta_{2\gamma} + \lambda\hat{\Omega}^{\alpha(2k-1)\gamma}{\zeta_1}^{\alpha}\zeta_{2\gamma}] - (1\leftrightarrow 2)
\end{aligned}
$$

$$
\begin{aligned}
[\delta_1,\delta_2]\hat{\Phi}^{\alpha(2k+1)} &= isM\hat{\alpha}^2[\frac{c_k}{k(2k+1)}\Phi^{\alpha(2k-1)}{\zeta_1}^{\alpha}{\zeta_2}^{\alpha} + 2c_{k+1}\Phi^{\alpha(2k+1)\gamma\beta}\zeta_{1\beta}\zeta_{2\gamma} \\
&\quad + \frac{2d_k}{(2k+1)}\Phi^{\alpha(2k)\gamma}\zeta_1^{\alpha}\zeta_{2\gamma} + \lambda\Phi^{\alpha(2k)\gamma}\zeta_1^{\alpha}\zeta_{2\gamma}] - (1\leftrightarrow 2)
\end{aligned}
$$

Here, we use the explicit expressions for α_k and β_k and the conditions:

$$
\frac{2{\beta_k}^2}{(k+1)} + \frac{{\alpha_k}^2}{k(k+1)(2k+1)} = \hat{\alpha}^2[\frac{sM}{k(k+1)} + \lambda]
$$

$$
\frac{2{\beta_{k+1}}^2}{(k+2)} + \frac{{\alpha_k}^2}{k(k+1)(2k+1)} = \hat{\alpha}^2[\frac{2d_k}{(2k+1)} + \lambda]
$$

$$
\alpha_{k-1}\beta_k = (k+1)\hat{\alpha}^2 a_{k-1}, \quad \alpha_k\beta_{k+1} = (k+2)\hat{\alpha}^2 a_k, \quad \alpha_k\beta_k = (k+1)c_k\hat{\alpha}^2
$$

Comparing the commutators of the supertransformations with (76), we obtain the (1,0) AdS_3 superalgebra with the commutation relation:

$$
\{Q_\alpha, Q_\beta\} \sim P_{\alpha\beta} + \frac{\lambda}{2}M_{\alpha\beta} \tag{79}
$$

As we see, the algebra of the supertransformations is closed. It is worth emphasizing that we did not apply the equations of motion to obtain the relation (79) both in the bosonic and in the fermionic sectors. This situation is analogous to the one for the massless higher-spin fields in the three-dimensional frame-like formalism. Recall that in the massive supermultiplet cases, the invariance of the Lagrangians is achieved up to the terms proportional to the Spin-1 and Spin-0 auxiliary fields equations only. Note that in dimensions $d \geq 4$, one would have to use equations for the higher spin auxiliary fields, as well (though in odd dimensions, there exist examples of the theories where Lagrangians are invariant without any use of e.o.m [36]). The difference here comes from the well-known fact that no massless higher spin fields in three dimensions have any local degrees of freedom.

6. Summary

In this review, we have presented and discussed the component supersymmetric formulations of the higher spin fields in three dimensions. We applied these formulations for the Lagrangian construction of the on-shell $\mathcal{N} = 1$ massless and massive supermultiplets in 3D Minkowski and AdS spaces. Although the off-shell formulation of 3D massless higher spin supermultiplets has been known long enough, we discussed the on-shell massless formulation as well, since it is interesting by itself. Besides, such a formulation is a base for the deformation to the massive higher spin supermultiplets. To generate the massive terms in the Lagrangians, we have used the approach based on the gauge-invariant formulation of the massive higher spin fields where the massive fields are described as the system of the massless ones coupled to each other in a special way.

In 3D Minkowski space, we generalized the gauge-invariant formulation of massive higher spin fields to the case of the massive supermultiplets. In particular, we showed that the massive supermultiplets can be constructed from the extended massless one. This extended massless supermultiplet is defined, and its smooth mass deformation is explicitly constructed for the two cases of the massive supermultiplets $(s, s+1/2)$ and the $(s, s-1/2)$. Both of these massive supermultiplets have two bosonic and one fermionic (left) degrees of freedom, i.e., they possess (1,0) supersymmetry.

In 3D AdS space, the construction of the $\mathcal{N} = 1$ massive supermultiplets with (1,0) supersymmetry is realized by another more elegant way. We have used a technique of the gauge-invariant curvatures. Namely, we found their supersymmetric deformation by the background gravitino field. This means that the resulting theory describes the massive supermultiplets on the background of AdS supergravity. Constructed higher spin supermultiplets have a correct flat limit.

In this work, we restrict ourselves to the free models only. Let us stress however that such a gauge-invariant frame-like formalism is perfectly suited for the investigation of possible interactions. On the one hand, the presence of gauge invariance allows one to work with the so-called constructive approach requiring that the switching on an interaction must keep (though modified) all the initial gauge symmetries. On the other hand, many known interacting models have a much simpler form, namely in the frame-like formalism. As an instructive illustration, one can consider a paper [37] where the authors considered a simple model containing just massless Spin-2 and Spin-3 and tried to rewrite this in the metric-like formulation. Note that for the massless fields, switching on interaction leads to the Chern–Simons-like models, while for the massive fields, to the Fradkin–Vasiliev type [34]. At the same time, the construction of the interacting models containing both massless and massive fields is still an open question.

One more possible application of our results is related to the supersymmetric models in the context of the AdS/CFT correspondence. This class of models is investigated rather poorly, and one of the reasons is that at the bulk side, the spectrum usually contains massive fields and/or massive supermultiplets.

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