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Predicting Future Flood Risks in the Face of Climate Change: A Frequency Analysis Perspective

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Abstract: The frequency analysis of maximum flows represents a direct method to predict future flood risks in the face of climate change. Thus, the correct use of the tools (probability distributions and methods of estimating their parameters) necessary to carry out such analyzes is required to avoid possible negative consequences. This article presents four probability distributions from the generalized Beta families, using the L- and LH-moments method as parameter estimation. New elements are presented regarding the applicability of Dagum, Paralogistic, Inverse Paralogistic and the four-parameter Burr distributions in the flood frequency analysis. The article represents the continuation of the research carried out in the Faculty of Hydrotechnics, being part of larger and more complex research with the aim of developing a normative regarding flood frequency analysis using these methods. According to the results obtained, among the four analyzed distributions, the Burr distribution was found to be the best fit model because the theoretical values of the statistical indicators calibrated the corresponding values of the observed data. Considering the existence of more rigorous selection criteria, it is recommended to use these methods in the frequency analysis.

Keywords: flood; risks; frequency analysis; linear moments; statistical distributions

Citation: Anghel, C.G.; Ilinca, C. Predicting Future Flood Risks in the Face of Climate Change: A Frequency Analysis Perspective. *Water* **2023**, *15*, 3883. https://doi.org/10.3390/w15223883

Academic Editors: Qiting Zuo, Fuqiang Wang, Jiaqi Zhai, Xiuyu Zhang, Dunxian She, Lei Zou, Rong Gan and Zengliang Luo

Received: 23 September 2023
Revised: 31 October 2023
Accepted: 6 November 2023
Published: 7 November 2023

1. Introduction

The accurate determination of the maximum flows has the role of representing extremely important data in the design of hydrotechnical constructions as well as in the establishment of constructive and non-constructive measures to protect the areas subject to flooding, especially in the perspective of changes in the climatic conditions and the restoration of forested areas.

A direct method of determining these maximum flows with certain return periods is the frequency analysis [1–4], which can use the series of maximum annual flows or the partial series of these flows, their advantages and disadvantages being highlighted in previous materials [5].

Regardless of the analyzed series, the FFA is exclusively based on the use of statistical distributions and on various parameter estimation methods. Thus, the correct choice of probability distributions becomes particularly important.

The Gamma, the Generalized Pareto, and the GEV family distributions are some of the probability distributions that are most frequently employed in FFA or regional FFA [1,3–8]. But, in recent materials [9–13], other distributions and families of distributions have been introduced in FFA, such as distributions from the generalized beta family, generalized beta prime, and beta exponential [13], using the method of ordinary moments (MOM) and the L-moments method as parameter estimation.

Regarding the parameter estimation methods, the most used are the ordinary moments method (MOM), the L-moments method, the maximum likelihood method (MLE) and the least squares method (LSM). Among these, the L-moments method has received special attention, currently being one of the most popular methods, an aspect due to the advantage that this is a more robust method, which is less subject to bias and less affected by sampling variability [2,5,11,14–17].

Since 1997, Wang [18] postulated another parameter estimation method that has received a special attention, namely the higher order linear moments (LH-moments). This represents a generalization of the L-moments method, having the main advantage of assigning a lower importance to the small values of the maximum annual data series, knowing that these are not always floods. Thus, the "separation effect" stated by Matalas [1,14,19] is partially fulfilled. Since then, the LH-moments method was used for regional FFA [20–23], FFA [18,24–26], low-flow frequency analysis [27], and the annual maximum rainfall frequency analysis [28–30].

Important contributions, regarding the applicability of probability distributions using the LH-moments method, were made by Anghel and Ilinca, who presented all the elements necessary to apply a significant number of distributions from different families [31].

The relations and equations required to apply the L- and LH-moment approach to the Dagum (DG), Paralogistic (PR), Inverse Paralogistic (IPR), and four-parameter Burr (BR4) distributions are presented in this article. Table 1 summarizes all the novelty elements from this article.

Table 1. New elements of distributions.

New Elements	Distribution
Exact relationships for LH moments	DG, PR, IPR, BR4
Approximate relations for LH moments	PR, IPR
Approximate relations for L-moments	PR
LH moments diagrams and relationships	PR, IPR
Exact frequency factors	DG, PR, IPR, BR4
Approximate frequency factors	PR, IPR

The article's primary goal is to give researchers all the tools (approximate estimates, frequency factors, frequency factor approximations, etc.) they need to use these distributions in frequency analysis in hydrology. The presented analysis refers only to the pure Statistics component and not to the component of the analysis of the physical phenomena of the formation of maximum flows (physical systems, dynamics, etc.).

All these new elements are applied on four case studies, with the aim of verifying the relationships, determining the maximum flows for the usual annual exceedance probabilities.

2. Methods

The method for estimating the parameters of the analyzed distributions is the L- and the first level LH-moments methods.

2.1. Probability Distributions

In this section, only the inverse functions of the analyzed distributions are presented (see Table 2), the L- and LH-moments are based on the inverse function. Density functions and cumulative functions can be found in others materials [13,32,33].

Considering that these inverse functions can be expressed using frequency factors [11,13], the exact and approximate relationships of these frequency factors are presented in the Supplementary File.

Table 2. The quantile functions.

Probability Distribution	Quantile Functions $x(p)$
Dagum	$\beta \cdot \left((1-p)^{-\frac{1}{\gamma}} - 1\right)^{-\frac{1}{\alpha}}$
Burr	$\gamma + \lambda \cdot \left(\frac{1}{\left(\frac{1}{1-p}\right)^{\frac{1}{\alpha}} - 1}\right)^{\frac{1}{\beta}}$
Paralogistic	$\gamma + \beta \cdot \left(p^{-\frac{1}{\alpha}} - 1\right)^{\frac{1}{\alpha}}$
Inverse Paralogistic	$\gamma + \beta \cdot \left(\frac{(1-p)^{\frac{1}{\alpha}}}{1-(1-p)^{\frac{1}{\alpha}}}\right)^{\frac{1}{\alpha}}$

2.2. Determination of Distribution Parameters

This section presents the relationships for the L- and first level LH-moments. The sample L-moments and LH-moments are determined according to [1,6–9], and, respectively [18,24].

In general, the parameters are determined by solving systems of nonlinear equations. This is also the main disadvantage in using these distributions. This obstacle is overcome by presenting approximate relationships (for PR and IPR), characterized by very small errors.

Considering that in general, the L-skewness and the LH- skewness depend on a single parameter, the latter can be approximately determined using different functions (logarithmic, rational and exponential functions).

In the next section, the relations for the exact and approximate estimation of the parameters are presented.

2.2.1. Dagum Distribution (DG)

The equations for the L-moments are:

$$L_1 = \beta \cdot \frac{\Gamma\left(\gamma + \frac{1}{\alpha}\right) \cdot \Gamma\left(1 - \frac{1}{\alpha}\right)}{\Gamma(\gamma+1)} \tag{1}$$

$$L_2 = \gamma \cdot \beta \cdot \left(2 \cdot \frac{\Gamma\left(2\cdot\gamma + \frac{1}{\alpha}\right) \cdot \Gamma\left(1 - \frac{1}{\alpha}\right)}{\Gamma(2\cdot\gamma+1)} - \frac{\Gamma\left(\gamma + \frac{1}{\alpha}\right) \cdot \Gamma\left(1 - \frac{1}{\alpha}\right)}{\Gamma(\gamma+1)}\right) \tag{2}$$

$$L_3 = \gamma \cdot \beta \cdot \left(\begin{array}{l} 6 \cdot \frac{\Gamma\left(3\cdot\gamma+\frac{1}{\alpha}\right)\cdot\Gamma\left(1-\frac{1}{\alpha}\right)}{\Gamma(3\cdot\gamma+1)} - 6 \cdot \frac{\Gamma\left(2\cdot\gamma+\frac{1}{\alpha}\right)\cdot\Gamma\left(1-\frac{1}{\alpha}\right)}{\Gamma(2\cdot\gamma+1)} + \\ \frac{\Gamma\left(\gamma+\frac{1}{\alpha}\right)\cdot\Gamma\left(1-\frac{1}{\alpha}\right)}{\Gamma(\gamma+1)} \end{array}\right) \tag{3}$$

where, L_1, L_2 and L_3 represent the first three linear moments; α and γ are the shape parameters; β is the scale parameter.

The following equations apply to the first level LH-moments:

$$L_{H1} = \beta \cdot \frac{\Gamma\left(2\cdot\gamma + \frac{1}{\alpha}\right) \cdot \Gamma\left(1 - \frac{1}{\alpha}\right)}{\Gamma(2\cdot\gamma)} \tag{4}$$

$$L_{H2} = \frac{3}{2} \cdot \beta \cdot \Gamma\left(1 - \frac{1}{\alpha}\right) \cdot \left(\frac{\Gamma\left(3\cdot\gamma + \frac{1}{\alpha}\right)}{\Gamma(3\cdot\gamma)} - \frac{\Gamma\left(2\cdot\gamma + \frac{1}{\alpha}\right)}{\Gamma(2\cdot\gamma)}\right) \tag{5}$$

$$L_{H3} = 4 \cdot \beta \cdot \Gamma\left(1 - \frac{1}{\alpha}\right) \cdot \left(5 \cdot \frac{\Gamma\left(4 \cdot \gamma + \frac{1}{\alpha}\right)}{6 \cdot \Gamma(4 \cdot \gamma)} - \frac{4 \cdot \Gamma\left(3 \cdot \gamma + \frac{1}{\alpha}\right)}{3 \cdot \Gamma(3 \cdot \gamma)} + \frac{\Gamma\left(2 \cdot \gamma + \frac{1}{\alpha}\right)}{2 \cdot \Gamma(2 \cdot \gamma)}\right) \tag{6}$$

where, L_{H1}, L_{H2} and L_{H3} represent the first three high-order linear moments.

2.2.2. Paralogistic Distribution (PR)

The linear moments are:

$$L_1 = \gamma + \frac{\beta \cdot \Gamma\left(1 + \frac{1}{\alpha}\right) \cdot \Gamma\left(\alpha - \frac{1}{\alpha}\right)}{\Gamma(\alpha)} \tag{7}$$

$$L_2 = \beta \cdot \Gamma\left(1 + \frac{1}{\alpha}\right) \cdot \left(\frac{\Gamma\left(\alpha - \frac{1}{\alpha}\right)}{\Gamma(\alpha)} - \frac{\Gamma\left(2 \cdot \alpha - \frac{1}{\alpha}\right)}{\Gamma(2 \cdot \alpha)}\right) \tag{8}$$

$$L_3 = \beta \cdot \Gamma\left(1 + \frac{1}{\alpha}\right) \cdot \left(\frac{\Gamma\left(\alpha - \frac{1}{\alpha}\right)}{\Gamma(\alpha)} + \frac{2 \cdot \Gamma\left(3 \cdot \alpha - \frac{1}{\alpha}\right)}{\Gamma(3 \cdot \alpha)} - \frac{3 \cdot \Gamma\left(2 \cdot \alpha - \frac{1}{\alpha}\right)}{\Gamma(2 \cdot \alpha)}\right) \tag{9}$$

where, α, β and γ are the shape, the scale and the position parameters.

For the parameter α, the following approximation can be used:

$$\alpha = \exp\left(\begin{array}{l} 0.000006321 - 0.499775727 \cdot \ln(\tau_3) + 0.126690856 \cdot \ln(\tau_3)^2 + \\ 0.067638333 \cdot \ln(\tau_3)^3 + 0.002255123 \cdot \ln(\tau_3)^4 - 0.00279206 \cdot \ln(\tau_3)^5 - \\ 0.000477164 \cdot \ln(\tau_3)^6 - 0.000016444 \cdot \ln(\tau_3)^7 + 0.000001032 \cdot \ln(\tau_3)^8 \end{array}\right) \tag{10}$$

$$\beta = \frac{L_2 \cdot \alpha}{\Gamma\left(\frac{1}{\alpha}\right) \cdot \left(\frac{\Gamma\left(\alpha - \frac{1}{\alpha}\right)}{\Gamma(\alpha)} - \frac{\Gamma\left(2 \cdot \alpha - \frac{1}{\alpha}\right)}{\Gamma(2 \cdot \alpha)}\right)} \tag{11}$$

$$\gamma = L_1 - \frac{\beta \cdot \Gamma\left(1 + \frac{1}{\alpha}\right) \cdot \Gamma\left(\alpha - \frac{1}{\alpha}\right)}{\Gamma(\alpha)} \tag{12}$$

where, $\tau_3 = L_3 / L_2$ is the L-skewness.

The linear moments for the first level LH-moments are as follows:

$$L_{H1} = \gamma + \frac{\beta}{\alpha} \cdot \Gamma\left(\frac{1}{\alpha}\right) \cdot \left(\frac{2 \cdot \Gamma\left(\alpha - \frac{1}{\alpha}\right)}{\Gamma(\alpha)} - \frac{\Gamma\left(2 \cdot \alpha - \frac{1}{\alpha}\right)}{\Gamma(2 \cdot \alpha)}\right) \tag{13}$$

$$L_{H2} = \frac{3 \cdot \beta}{\alpha} \cdot \Gamma\left(\frac{1}{\alpha}\right) \cdot \left(\frac{\Gamma\left(3 \cdot \alpha - \frac{1}{\alpha}\right)}{2 \cdot \Gamma(3 \cdot \alpha)} - \frac{\Gamma\left(2 \cdot \alpha - \frac{1}{\alpha}\right)}{\Gamma(2 \cdot \alpha)} + \frac{\Gamma\left(\alpha - \frac{1}{\alpha}\right)}{2 \cdot \Gamma(\alpha)}\right) \tag{14}$$

$$L_{H3} = \frac{2 \cdot \beta}{\alpha} \cdot \Gamma\left(\frac{1}{\alpha}\right) \cdot \left(\frac{4 \cdot \Gamma\left(3 \cdot \alpha - \frac{1}{\alpha}\right)}{\Gamma(3 \cdot \alpha)} - \frac{5 \cdot \Gamma\left(4 \cdot \alpha - \frac{1}{\alpha}\right)}{3 \cdot \Gamma(4 \cdot \alpha)} - \frac{3 \cdot \Gamma\left(2 \cdot \alpha - \frac{1}{\alpha}\right)}{\Gamma(2 \cdot \alpha)} + \frac{2 \cdot \Gamma\left(\alpha - \frac{1}{\alpha}\right)}{3 \cdot \Gamma(\alpha)}\right) \tag{15}$$

The parameter α can be estimated, using a rational function ($0.1 < \tau_{H3} < 0.88$):

$$\alpha = \frac{30.67728772 + 43.85465205 \cdot \tau_{H3} + 17.39998218 \cdot {\tau_{H3}}^2 + 4.59707366 \cdot {\tau_{H3}}^3}{1 + 48.74542611 \cdot \tau_{H3} + 53.51830483 \cdot {\tau_{H3}}^2} \tag{16}$$

$$\beta = \frac{L_{H2}}{\frac{3}{\alpha} \cdot \Gamma\left(\frac{1}{\alpha}\right) \cdot \left(\frac{\Gamma\left(3\cdot\alpha - \frac{1}{\alpha}\right)}{2\cdot\Gamma(3\cdot\alpha)} - \frac{\Gamma\left(2\cdot\alpha - \frac{1}{\alpha}\right)}{\Gamma(2\cdot\alpha)} + \frac{\Gamma\left(\alpha - \frac{1}{\alpha}\right)}{2\cdot\Gamma(\alpha)}\right)} \tag{17}$$

$$\gamma = L_{H1} - \frac{\beta \cdot \Gamma\left(\frac{1}{\alpha}\right)}{\alpha} \cdot \left(\frac{2 \cdot \Gamma\left(\alpha - \frac{1}{\alpha}\right)}{\Gamma(\alpha)} - \frac{\Gamma\left(2 \cdot \alpha - \frac{1}{\alpha}\right)}{\Gamma(2 \cdot \alpha)}\right) \tag{18}$$

where, $\tau_{H3} = L_{H3}/L_{H2}$ is the LH-skewness.

2.2.3. Inverse Paralogistic (IPR)

The equations for the L-moments are:

$$L_1 = \gamma + \frac{\beta \cdot \Gamma\left(\frac{1}{\alpha} + \alpha\right) \cdot \Gamma\left(1 - \frac{1}{\alpha}\right)}{\Gamma(\alpha)} \tag{19}$$

$$L_2 = \beta \cdot \Gamma\left(1 - \frac{1}{\alpha}\right) \cdot \left(\frac{\Gamma\left(\frac{1}{\alpha} + 2 \cdot \alpha\right)}{\Gamma(2 \cdot \alpha)} - \frac{\Gamma\left(\frac{1}{\alpha} + \alpha\right)}{\Gamma(\alpha)}\right) \tag{20}$$

$$L_3 = \beta \cdot \Gamma\left(1 - \frac{1}{\alpha}\right) \cdot \left(\frac{\Gamma\left(\alpha + \frac{1}{\alpha}\right)}{\Gamma(\alpha)} + \frac{2 \cdot \Gamma\left(3 \cdot \alpha + \frac{1}{\alpha}\right)}{\Gamma(3 \cdot \alpha)} - \frac{3 \cdot \Gamma\left(2 \cdot \alpha + \frac{1}{\alpha}\right)}{\Gamma(2 \cdot \alpha)}\right) \tag{21}$$

where, α, β and γ are the shape, the scale and the position parameters.

An approximate form for parameter α can be adopted:

If $0.17 \le \tau_3 \le \frac{1}{3}$:

$$\alpha = \frac{5.07675 \cdot 10^3 - 7.55684 \cdot 10^4 \cdot \tau_3 + 5.36797 \cdot 10^5 \cdot \tau_3^2 - 1.35970 \cdot 10^6 \cdot \tau_3^3 + 1.30233 \cdot 10^6 \cdot \tau_3^4}{1 - 4.59437 \cdot 10^3 \cdot \tau_3 + 2.70167 \cdot 10^4 \cdot \tau_3^2} \tag{22}$$

If $\frac{1}{3} < \tau_3 < 1$:

$$\begin{aligned} \alpha = 5.37952 \cdot 10 - 4.84016 \cdot 10^2 \cdot \tau_3 + 2.02215 \cdot 10^3 \cdot \tau_3^2 - 4.79644 \cdot 10^3 \cdot \tau_3^3 + \\ 6.86370 \cdot 10^3 \cdot \tau_3^4 - 5.88301 \cdot 10^3 \cdot \tau_3^5 + 2.78592 \cdot 10^3 \cdot \tau_3^6 - 5.61108 \cdot 10^2 \cdot \tau_3^7 \end{aligned} \tag{23}$$

$$\beta = \frac{L_2}{\Gamma\left(1 - \frac{1}{\alpha}\right) \cdot \left(\frac{\Gamma\left(\frac{1}{\alpha} + 2\cdot\alpha\right)}{\Gamma(2\cdot\alpha)} - \frac{\Gamma\left(\frac{1}{\alpha} + \alpha\right)}{\Gamma(\alpha)}\right)} \tag{24}$$

$$\gamma = L_1 - \frac{\beta \cdot \Gamma\left(\frac{1}{\alpha} + \alpha\right) \cdot \Gamma\left(1 - \frac{1}{\alpha}\right)}{\Gamma(\alpha)} \tag{25}$$

For the first level LH-moments, the equations are:

$$L_{H1} = \gamma + \beta \cdot \frac{\Gamma\left(2 \cdot \alpha + \frac{1}{\alpha}\right)\left(1 - \frac{1}{\alpha}\right)}{\Gamma(2 \cdot \alpha)} \tag{26}$$

$$L_{H2} = \frac{3 \cdot \beta}{2} \cdot \Gamma\left(1 - \frac{1}{\alpha}\right) \cdot \left(\frac{\Gamma\left(3 \cdot \alpha + \frac{1}{\alpha}\right)}{\Gamma(3 \cdot \alpha)} - \frac{\Gamma\left(2 \cdot \alpha + \frac{1}{\alpha}\right)}{\Gamma(2 \cdot \alpha)}\right) \tag{27}$$

$$L_{H3} = 2 \cdot \beta \cdot \Gamma\left(1 - \frac{1}{\alpha}\right) \cdot \left(\frac{5 \cdot \Gamma\left(4 \cdot \alpha + \frac{1}{\alpha}\right)}{3 \cdot \Gamma(4 \cdot \alpha)} - \frac{8 \cdot \Gamma\left(3 \cdot \alpha + \frac{1}{\alpha}\right)}{3 \cdot \Gamma(3 \cdot \alpha)} + \frac{\Gamma\left(2 \cdot \alpha + \frac{1}{\alpha}\right)}{\Gamma(2 \cdot \alpha)}\right) \tag{28}$$

An approximate form for parameter α can be adopted:
If $0.25 \leq \tau_{H3} \leq 1/3$:

$$\alpha = \frac{16517.60198 - 82738.36913 \cdot \tau_{H3} + 644148.00047 \cdot {\tau_{H3}}^2 - 645765.89383 \cdot {\tau_{H3}}^3}{1 - 54573.65663 \cdot \tau_{H3} + 224419.26179 \cdot {\tau_{H3}}^2} \quad (29)$$

If $1/3 < \tau_{H3} < 89$:

$$\alpha = \frac{5.9986867 - 60.4311085 \cdot \tau_{H3} - 29.7201729 \cdot {\tau_{H3}}^2}{1 + 18.5442652 \cdot \tau_{H3} - 81.3230745 \cdot {\tau_{H3}}^2 - 35.3549804 \cdot {\tau_{H3}}^3} \quad (30)$$

$$\beta = \frac{L_{H2}}{\frac{3}{2} \cdot \Gamma\left(1 - \frac{1}{\alpha}\right) \cdot \left(\frac{\Gamma\left(\frac{1}{\alpha} + 3 \cdot \alpha\right)}{\Gamma(3 \cdot \alpha)} - \frac{\Gamma\left(\frac{1}{\alpha} + 2 \cdot \alpha\right)}{\Gamma(2 \cdot \alpha)}\right)} \quad (31)$$

$$\gamma = L_{H1} - \frac{\beta \cdot \Gamma\left(\frac{1}{\alpha} + 2 \cdot \alpha\right) \cdot \Gamma\left(1 - \frac{1}{\alpha}\right)}{\Gamma(2 \cdot \alpha)} \quad (32)$$

2.2.4. The Four Parameters Burr Distribution (BR4)

For the L-moments, the equations are:

$$L_1 = \gamma + \frac{\lambda \cdot \Gamma\left(1 - \frac{1}{\beta}\right) \cdot \Gamma\left(\alpha + \frac{1}{\beta}\right)}{\Gamma(\alpha)} \quad (33)$$

$$L_2 = \lambda \cdot \Gamma\left(1 - \frac{1}{\beta}\right) \cdot \left(\frac{\Gamma\left(2 \cdot \alpha + \frac{1}{\beta}\right)}{\Gamma(2 \cdot \alpha)} - \frac{\Gamma\left(\alpha + \frac{1}{\beta}\right)}{\Gamma(\alpha)}\right) \quad (34)$$

$$L_3 = \lambda \cdot \Gamma\left(1 - \frac{1}{\beta}\right) \cdot \left(\frac{2 \cdot \Gamma\left(3 \cdot \alpha + \frac{1}{\beta}\right)}{\Gamma(3 \cdot \alpha)} - \frac{3 \cdot \Gamma\left(2 \cdot \alpha + \frac{1}{\beta}\right)}{\Gamma(2 \cdot \alpha)} + \frac{\Gamma\left(\alpha + \frac{1}{\beta}\right)}{\Gamma(\alpha)}\right) \quad (35)$$

$$L_3 = \lambda \cdot \Gamma\left(1 - \frac{1}{\beta}\right) \cdot \left(\frac{5 \cdot \Gamma\left(4 \cdot \alpha + \frac{1}{\beta}\right)}{\Gamma(4 \cdot \alpha)} - \frac{10 \cdot \Gamma\left(3 \cdot \alpha + \frac{1}{\beta}\right)}{\Gamma(3 \cdot \alpha)} + \frac{6 \cdot \Gamma\left(2 \cdot \alpha + \frac{1}{\beta}\right)}{\Gamma(2 \cdot \alpha)} - \frac{\Gamma\left(\alpha + \frac{1}{\beta}\right)}{\Gamma(\alpha)}\right) \quad (36)$$

where, α and β are the shape parameters; λ is the scale parameter; γ is the position parameter.
The equations for the first level LH-moments are:

$$L_{H1} = \gamma + \frac{\lambda \cdot \Gamma\left(1 - \frac{1}{\beta}\right) \cdot \Gamma\left(2 \cdot \alpha + \frac{1}{\beta}\right)}{\Gamma(2 \cdot \alpha)} \quad (37)$$

$$L_{H2} = \frac{3}{2} \cdot \lambda \cdot \Gamma\left(1 - \frac{1}{\beta}\right) \cdot \left(\frac{\Gamma\left(3 \cdot \alpha + \frac{1}{\beta}\right)}{\Gamma(3 \cdot \alpha)} - \frac{\Gamma\left(2 \cdot \alpha + \frac{1}{\beta}\right)}{\Gamma(2 \cdot \alpha)}\right) \quad (38)$$

$$L_{H3} = 2 \cdot \lambda \cdot \Gamma\left(1 - \frac{1}{\beta}\right) \cdot \left(\frac{5 \cdot \Gamma\left(4 \cdot \alpha + \frac{1}{\beta}\right)}{3 \cdot \Gamma(4 \cdot \alpha)} - \frac{8 \cdot \Gamma\left(3 \cdot \alpha + \frac{1}{\beta}\right)}{3 \cdot \Gamma(3 \cdot \alpha)} + \frac{\Gamma\left(2 \cdot \alpha + \frac{1}{\beta}\right)}{\Gamma(2 \cdot \alpha)}\right) \quad (39)$$

$$L_{H4} = \frac{5}{2} \cdot \lambda \cdot \Gamma\left(1 - \frac{1}{\beta}\right) \cdot \left(\frac{21 \cdot \Gamma\left(5 \cdot \alpha + \frac{1}{\beta}\right)}{6 \cdot \Gamma(5 \cdot \alpha)} - \frac{15 \cdot \Gamma\left(4 \cdot \alpha + \frac{1}{\beta}\right)}{2 \cdot \Gamma(4 \cdot \alpha)} + \frac{5 \cdot \Gamma\left(3 \cdot \alpha + \frac{1}{\beta}\right)}{\Gamma(3 \cdot \alpha)} - \frac{\Gamma\left(2 \cdot \alpha + \frac{1}{\beta}\right)}{\Gamma(2 \cdot \alpha)}\right) \quad (40)$$

3. Case Studies

The case studies consist in determining the maximum flows with the annual exceedance probabilities of 0.01%, 0.1%, 0.5%, 1%, 40% and 80%, on the Jijia, Buhai, Miletin and Sitna Rivers from the Prut River basin, Romania.

The Jijia river is a tributary of the Prut River, originating from the Hiliseu-Horia locality, Botosani. The rivers Buhai, Miletin and Sitna are part of the hydrographic basin of the River Jijia, being its right tributaries [34].

The analyzed rivers are located in the eastern part of Romania with a general northeast orientation, as shown in Figure 1 (47°55′59.6″ N 26°24′17.1″ E).

Figure 1. The positioning of the rivers: Jijia, Buhai, Miletin and Sitna; and the positioning of the hydrometric stations: Dorohoi, Padureni, Sipote and Todireni.

The Prut River has its sources in the Wooded Carpathians (Ukraine); its length on the territory of Romania is 742 km; and it has a hydrographic basin of 10,990 km^2, representing about 4.6% of Romania's surface [34].

A temperate continental climate characterizes the Prut-Barlad hydrographic space. In terms of thermal regime and precipitation, a multiannual average temperature of 9.0 °C and multiannual average precipitation quantities ranging from 400 mm to 600 mm per year are reported. From a geological perspective, siliceous features dominate the landscape of the analyzed river's watershed.

The morphometric elements of the four analyzed rivers are presented centrally in Table 3 [34].

Table 3. The morphometric elements for the analyzed rivers.

River	Length [km]	Average Stream Slope [‰]	Sinuosity Coefficient [-]	Average Altitude, [m]	Catchments Area, [km^2]
Jijia	275	1.0	1.45	152	5757
Buhai	18	10	1.17	279	134
Miletin	90	3.0	1.24	166	675
Sitna	78	2.0	1.4	166	943

The four monitoring stations are positioned so that the regime of maximum recorded flows is a natural one. Appendix A presents a tabular and graphical series of the maximum yearly flows recorded at each of the four stations over time. The analysis period varies between 37 years (the Buhai and Miletin Rivers) and 57 years (the Jijia and Sitna Rivers).

The statistical indicators specific to these recorded data are presented in Table 4.

Table 4. The statistical indicators for the analyzed rivers.

River	MOM		L-Moments Method							LH-Moments Method						
	μ	C_v	L_1	L_2	L_3	L_4	τ_2	τ_3	τ_4	L_{H1}	L_{H2}	L_{H3}	L_{H4}	τ_{H2}	τ_{H3}	τ_{H4}
	[m^3/s]	[-]	[m^3/s]	[m^3/s]	[m^3/s]	[m^3/s]	[-]	[-]	[-]	[m^3/s]	[m^3/s]	[m^3/s]	[m^3/s]	[-]	[-]	[-]
Jijia	32.1	1.22	32.1	18.3	8.22	4.47	0.5703	0.4483	0.2436	50.5	19.9	8.46	4.59	0.3945	0.4247	0.2307
Buhai	14.4	1.452	14.4	8.78	4.93	3.28	0.6102	0.5607	0.3731	23.2	10.3	5.47	3.40	0.4436	0.5319	0.3303
Miletin	42.5	0.883	42.5	18.1	5.52	4.89	0.264	0.3041	0.2698	60.7	17.8	6.94	5.63	0.2924	0.3912	0.3175
Sitna	53.9	0.931	53.9	24.3	8.53	5.96	0.4508	0.3513	0.2451	78.2	24.6	9.66	6.15	0.3149	0.3923	0.2496

Notes: μ and C_v represents the arithmetic mean and the coefficient of variation; L_1, L_2, L_3 and L_4 represents the first four L-moments; τ_2, τ_3 and τ_4 represents the coefficient of L-variation, L-skewness and L-kurtosis. Analyzing the values of the coefficient of variation, we can observe an average torrentiality, with the exception of the Buhai river with a more pronounced torrentiality.

Considering that the first stage consists of checking the homogeneity of the data as well as identifying the possible outliers, the testing of these two conditions was carried out with the help of the von Neumann and Grubb-Beck tests with a confidence level of 10%. No extreme values were identified, and the analyzed data are homogeneous (see Table 5).

Table 5. Results of statistical tests for homogeneity and outliers.

River	Homogeneity	Outliers	Q_{max} for the Observed Data
	von Newman	Grubb-Beck	
	[-]	[m^3/s]	[m^3/s]
Jijia	2.0712	548	190
Buhai	2.3503	159	96
Miletin	2.1681	353	204
Sitna	2.4471	507	290

4. Results and Discussions

In general, FFA involves the determination of maximum flows, regardless of the length of the analyzed data series, for the annual exceedance probabilities corresponding to rare and very rare events (the flow with a return period of 10,000 years, the value used in the design of hydrotechnical dam-type retention constructions—First class of importance, Category A). It is important that the values are characterized by as small as possible errors and uncertainties, depending (from a statistical point of view) on the statistical distributions and the method of estimating the parameters used.

The four analyzed probability distributions were applied to determine the maximum flows on the Jijia, Buhai, Miletin and Sitna Rivers, using the L- and first order LH-moments methods, as parameter estimation methods.

In the analysis, the annual maximum flow series (AMS) was used, its main advantage being the ease of data selection, the data which represents the maximum flows characteristic of each year of analysis (block maxima). The major disadvantage is the fact that the lower maximum values do not, in many cases, also represent floods. There are values higher than this in the chronological series of maximum flows, which naturally should be taken into consideration, but whose selection requires additional and often difficult operations. This principle is the basis of the analysis with partial series, both Peak Over Threshold (POT) [35–38] and Annual Exceedance Series (AES) [39,40]. A similar principle is also the basis of the LH-moments method, by reducing the importance of the lower maximum flows.

4.1. Estimated Parameters and Quantiles

The resulting values of the parameters are summarized in Table 6. Their presentation is necessary so that the results can be reproduced, thus ensuring the objectivity of the analysis.

Table 6. Estimated parameters using L-moments and LH-moments.

Parameter	Distribution							
	DG	PR	IPR	BR4	DG	PR	IPR	BR4
	L-Moments				LH-Moments (First Level)			
	Jijia River							
α	2.3353	1.5672	2.3426	0.1799	2.5609	1.6837	2.9441	0.1437
β	48.1	34.5	22.7	2.6613	59.0	43.8	35.0	2.8001
γ	0.3093	−4.27	−16.1	2.32	0.2334	−9.86	−35.1	3.95
λ	-	-	-	66.6	-	-	-	74.6
	Bahna River							
α	1.798	1.3757	1.8029	0.2621	1.906	1.4172	1.9788	0.1528
β	12.1	10.3	6.57	1.9499	16.2	11.9	8.2	2.1129
γ	0.5647	−0.298	−2.56	1.34	0.3994	−1.345	−5.34	2.67
λ	-	-	-	20.7	-	-	-	30.1
	Miletin River							
α	3.1175	1.9546	4.2066	1.0611	2.8317	1.7967	3.5184	0.761
β	57.5	58.9	56.7	3.2033	46.8	47.3	42.4	2.6568
γ	0.3811	−4.56	−51.8	−16.34	0.5509	2.12	−31.99	0.25
λ	-	-	-	48.7	-	-	-	38.8
	Sitna River							
α	2.7591	1.8033	3.297	22.676	2.7679	1.7925	3.4943	5.3107
β	66.5	66.3	52.7	3.7366	67.0	65.2	58.2	3.5955
γ	0.4376	−3.47	−41.6	−63.8	0.4323	−2.83	−49.5	−55.3
λ	-	-	-	41.0	-	-	-	55.2

The values of the derived quantiles are displayed in Table 7 using the two parameter estimation methods. The quantile values are presented only for the annual exceedance probabilities interested in flood frequency analysis, namely for rare events (left-hand, upper part of the graph) where, in most cases, there are no recorded data.

In general, all these annual exceedance probabilities are used for the design of important hydrotechnical constructions, especially dams for water storage and for bankfull discharge. For high annual exceedance probabilities (>80%), the values are not of interest in the analysis of maximum flows.

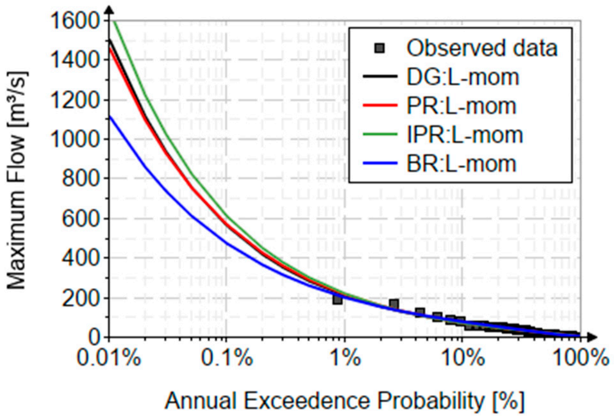
Figures 2–5 show the fitting models of the four analyzed rivers. The Hazen empirical probability was used [1,3,4,41]. In general, for the L-moments, the most suitable empirical probabilities are the Hazen empirical probability ($P = (i - 0.5)/n$) and the IEC 56 empirical probability ($P = (i - 0.5)/(n + 25)$) [42].

Table 7. Estimated flood discharge (m^3/s), for 0.01%, 0.1%, 0.5%, 1%, 40% and 80%.

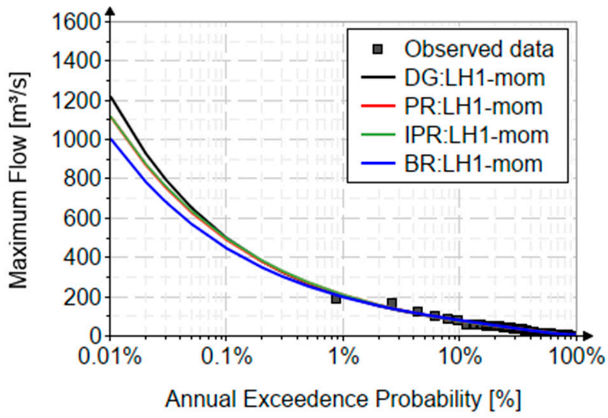
Distribution	Annual Exceedance Probabilities [%]											
	L-Moments Method						LH-Moments (First Level)					
	0.01	0.1	0.5	1	40	80	0.01	0.1	0.5	1	40	80
	Jijia River											
DG	1503	560	280	207	26.0	5.20	1219	496	263	199	26.3	4.0
PR	1460	566	287	213	25.5	6.14	1117	487	267	204	26.3	3.89
IPR	1652	608	297	217	25.5	6.76	1118	492	270	206	26.4	3.86
BR4	1116	471	257	197	25.8	4.63	1005	443	250	195	25.1	5.32

Table 7. *Cont.*

Distribution	Annual Exceedance Probabilities [%]											
	L-Moments Method						LH-Moments (First Level)					
	0.01	0.1	0.5	1	40	80	0.01	0.1	0.5	1	40	80
	Buhai River											
DG	1471	409	166	113	9.72	2.55	1257	375	161	111	9.84	1.98
PR	1337	394	166	114	9.60	2.62	1160	366	162	113	9.74	2.06
IPR	1505	418	169	114	9.65	2.81	1211	374	163	113	9.86	1.96
BR4	1176	362	158	111	9.60	2.23	968	327	153	110	8.95	2.87
	Miletin River											
DG	810	387	230	184	41.2	14.9	981	435	246	192	40.3	17.0
PR	649	349	223	182	40.1	15.4	820	399	239	191	39.8	17.5
IPR	660	360	229	186	40.4	16.1	799	400	241	192	39.9	17.5
BR4	863	412	242	192	40.2	16.4	1122	471	257	198	39.7	18.6
	Sitna River											
DG	1388	602	335	260	49.9	17.7	1379	600	335	260	49.9	17.6
PR	1119	545	325	258	49.3	18.1	1140	550	327	259	49.2	18.3
IPR	1195	574	336	264	49.0	19.1	1112	551	329	261	49.3	18.4
BR4	1049	537	326	260	49.1	18.7	1083	545	328	261	49.2	18.4

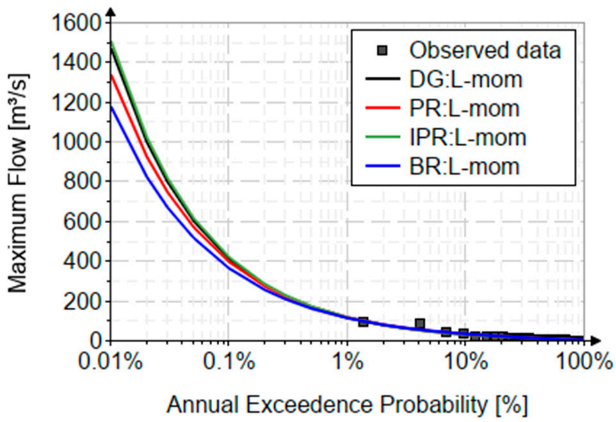


(**a**) L-moments

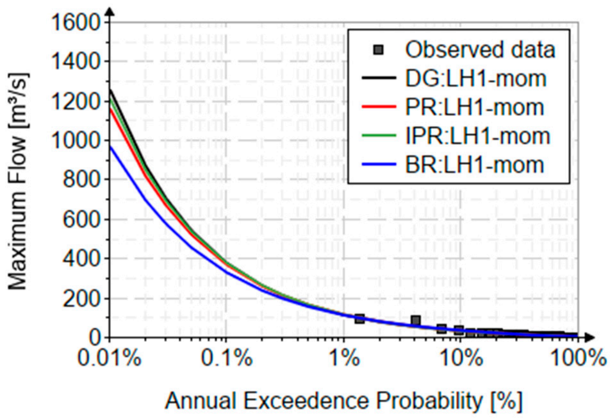


(**b**) LH-moments (first level)

Figure 2. The graphical results for the Jijia River—Dorohoi Station.

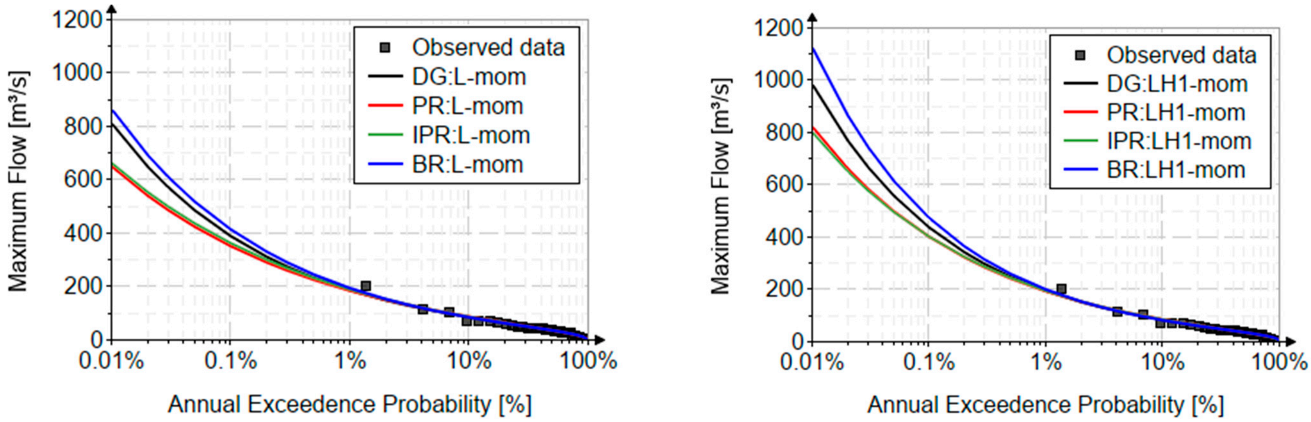


(**a**) L-moments



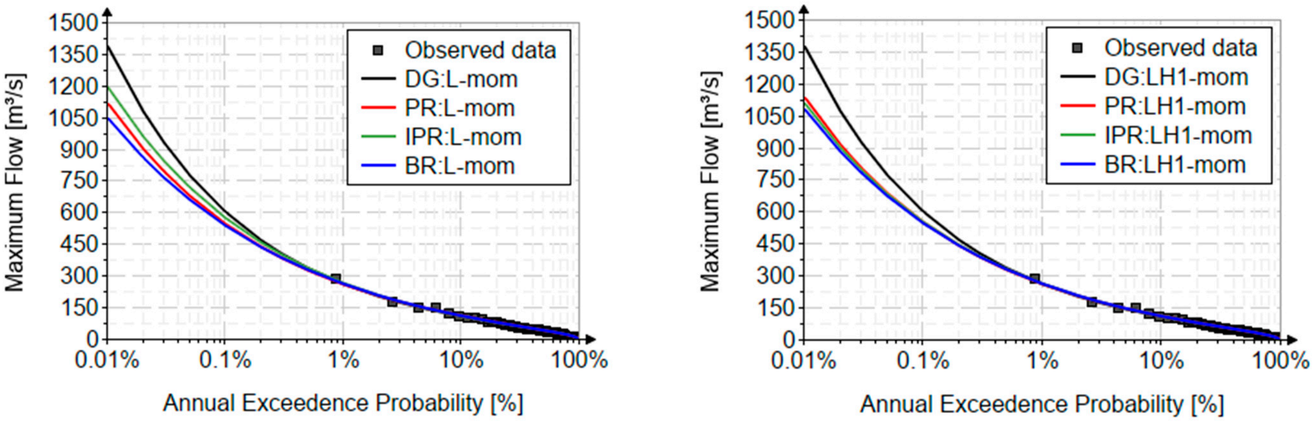
(**b**) LH-moments (first level)

Figure 3. The graphical results for the Buhai River—Padureni Station.



(**a**) L-moments (**b**) LH-moments (first level)

Figure 4. The graphical results for the Miletin River—Sipote Station.



(**a**) L-moments (**b**) LH-moments (first level)

Figure 5. The graphical results for the Sitna River—Todireni Station.

In order to emphasize the heavy tail (the domain of rare events), the decimal logarithmic scale was employed on the horizontal axis.

As can be seen from the results, the values generated in the field of annual excess probabilities less than 1%, have a variability depending on the particularities of each analyzed distribution, the influence being given by the number of parameters that characterize each distribution. The DG, PR and IPR distributions have three parameters, properly calibrating the L-skewness, but generating different values of the L-kurtosis, an extremely important aspect in choosing the best distribution using these estimation methods. The four-parameter Burr distribution properly calibrates all four linear moments specific to the two analyzed methods.

Examining the results for the probability of exceeding 0.01%, it can be observed that, for Jijia River, they vary between 1652 m^3/s (IPR distribution) and 1116 m^3/s (BR4 distribution) using L-moments, between 1219 m^3/s (DG distribution) and 1005 m^3/s (BR4 distribution) using LH-moments. In the case of the Buhai river, the maximum flows vary between 1505 m^3/s (IPR distribution) and 1176 m^3/s (BR4 distribution) using L-moments, between 1257 m^3/s (DG distribution) and 968 m^3/s (BR4 distribution) using LH-moments. For the Miletin river, these maximum values vary between 863 m^3/s (BR4 distribution) and 649 m^3/s (PR distribution) using L-moments, between 1122 m^3/s (BR4 distribution) and 799 m^3/s (IPR distribution) using LH-moments. The results in the case of the Sitna river vary between 1388 m^3/s (DG distribution) and 1049 m^3/s (BR4 distribution) using L-moments, between 1379 m^3/s (DG distribution) and 1083 m^3/s (BR4 distribution) using LH-moments.

The differences in the distribution curves and the resulting quantile values, for the two estimation methods, are mainly due to the variation of the parameter that characterizes L-skewness, which imposes a different behavior, especially in the area of rare and very rare events, with a more or less pronounced heavy tail. As could be observed in the case of other distributions from other families [5,11] using the L-moments method, this stability between methods of some distributions, is due to the reduced variability of the distribution parameter that characterizes L-skewness, LH-skewness, around the value of the sample L-skewness.

4.2. Best-Fit Distribution Selection

The main advantage of the analyzed methods, compared to other estimation methods (the method of ordinary moments, the method of maximum likelihood, the method of least squares, the principle of maximum entropy, etc.), is that there is a more rigorous selection criterion.

In general, due to the small lengths of the data series, statistical tests and performance metrics are only valid in the field of empirical probabilities (recorded data). Outside of this field, indicators and statistical tests lose their relevance (especially in the case of small and medium data series), because it is desired to determine the maximum values for small annual probabilities, where generally there is no recorded data.

Table 8 shows the performance quotes used in the case of the four case studies, with the mention that the RAE (the relative absolute error) and RME (the relative mean error) indicators [43–45] are relevant only in the conditions described previously:

$$\mathrm{RME} = \frac{1}{n} \cdot \sqrt{\sum_{i=1}^{n} \left(\frac{x_i - x(p)}{x_i} \right)^2} \tag{41}$$

$$\mathrm{RAE} = \frac{1}{n} \cdot \sum_{i=1}^{n} \left| \frac{x_i - x(p)}{x_i} \right| \tag{42}$$

where n, x_i and $x(p)$ represent the length of the recorded series, the observed value, and the estimated value for a given probability.

According to the results of the statistical indicators of the probability distributions, for the two analyzed methods (L- and LH-moments), the BR4 distribution has the best results, since it is a four-parameter distribution which calibrates accordingly all linear moments. The theoretical values of the statistical indicators of the distribution best approximate those of the data set.

Although the DG, PR and IPR distributions have a lower number of parameters than the BR4 distribution, it can be seen that these distributions have applicability in FFA, as long as the use of the distributions respects as much as possible the selection criteria imposed by the analyzed methods.

Thus, it can be observed that in certain situations the three-parameter distributions can represent alternatives to the four-parameter distributions whose applicability requires more laborious calculations. This aspect is all the more important since for many distributions there are approximate relationships for parameter estimation, characterized by very small errors, which greatly simplifies the calculation. The same can be observed in the case of the Siret and Buhai Rivers, where the PR distribution can be used as an alternative to the BR4 distribution, similarly with the DG distribution in the case of the Miletin River, the values generated for $Q_{0.01\%}$ being characterized by a bias of less than 20%, a more than acceptable error regarding the rarity of this event.

Considering that this represents the main preselection and selection criterion of the best fit distribution, the approximate relationships of the L-kurtosis–L-skewness variation are necessary. This information is presented in detail in Supplementary File.

Table 8. Performance measurement for the Jijia, Buhai, Miletin and Sitna Rivers.

Distribution	L-Moments				Observed Data		LH-Moments				Observed Data	
	RME	RAE	τ_3	τ_4	τ_3	τ_4	RME	RAE	τ_{H3}	τ_{H4}	τ_{H3}	τ_{H4}
Jijia River												
DG	0.0319	0.1724	0.4483	0.2881	0.4483	0.2436	0.0415	0.2125	0.4247	0.2586	0.4247	0.2307
PR	0.0773	0.2668		0.3106			0.1867	0.5742		0.2717		
IPR	0.119	0.3504		0.3327			0.2714	0.7564		0.2777		
BR4	0.0287	0.1476		0.2436			0.0651	0.2426		0.2307		
Buhai River												
DG	0.0327	0.1252	0.5607	0.4157	0.5607	0.3731	0.0478	0.1986	0.5319	0.3689	0.5319	0.3303
PR	0.0341	0.1186		0.4115			0.0799	0.2493		0.3684		
IPR	0.061	0.166		0.4309			0.14	0.3625		0.375		
BR4	0.0305	0.1312		0.3731			0.0899	0.2695		0.3303		
Miletin River												
DG	0.0292	0.1275	0.3041	0.2143	0.3041	0.2698	0.0399	0.1424	0.3912	0.2596	0.3912	0.3175
PR	0.0402	0.1544		0.2125			0.0552	0.1992		0.2457		
IPR	0.0619	0.1858		0.2300			0.0473	0.1706		0.2507		
BR4	0.0635	0.1852		0.2698			0.0603	0.2095		0.3175		
Sitna River												
DG	0.0174	0.0781	0.3513	0.2470	0.3513	0.2451	0.0175	0.0781	0.3923	0.257	0.3923	0.2496
PR	0.0221	0.089		0.2406			0.0205	0.0872		0.2465		
IPR	0.0399	0.1172		0.2606			0.0482	0.1276		0.2516		
BR4	0.0341	0.1084		0.2451			0.0435	0.1202		0.2496		

4.3. Confidence Intervals

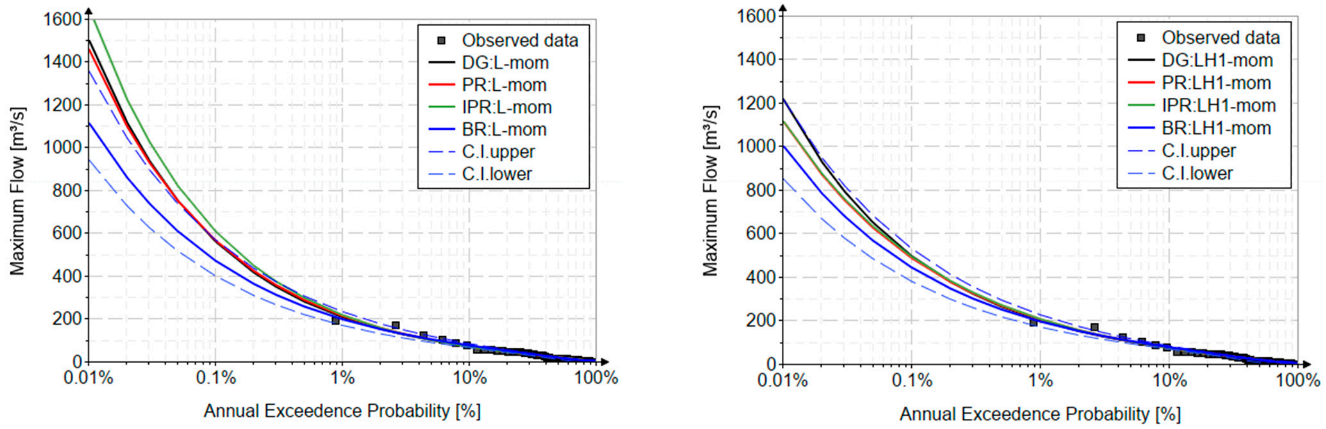
Taking into account the relatively short length of the analyzed series, the RME and RAE performance indicators are presented as indicative, making a performance classification valid only for the probability area of the recorded data, observation also valid for statistical tests such as Kolmogorov–Smirnov [46], Anderson-Darling, Akaike Information Criteria and Bayesian Information Criteria.

It is noteworthy that the quantile results exhibit some degree of uncertainty, mostly due to the short data length and the inability of three-parameter distributions to accurately calibrate the fourth-order linear moment.

The statistical uncertainties resulting from the variability of the observed data length, as demonstrated by other materials [47], need to be emphasized on three levels that are particular to the parameter estimation method: the estimation of statistical indicators, the estimation of parameters, and—most importantly—the estimation of quantiles.

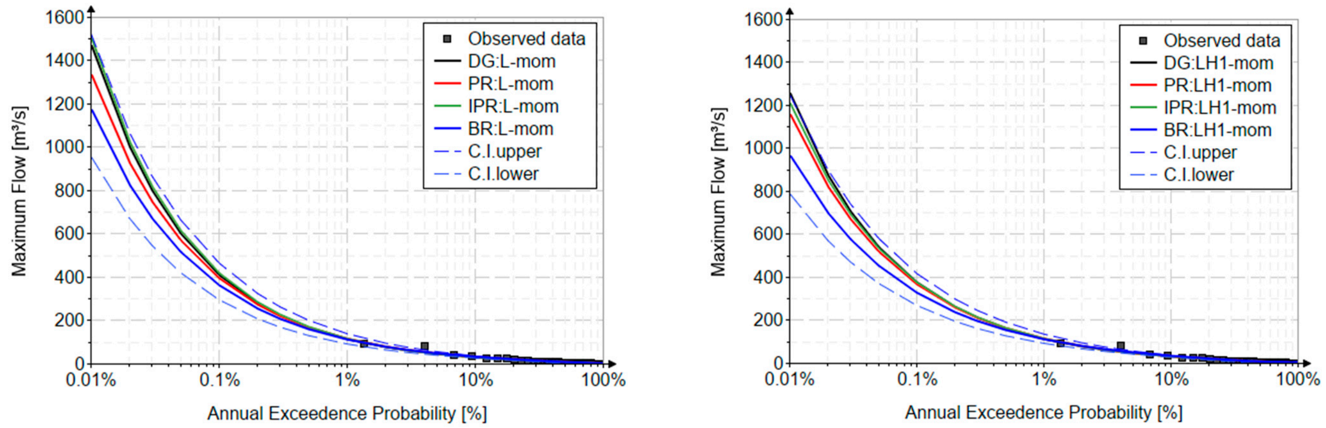
The confidence interval (C.I) for these distributions must be shown, in light of all these statistical uncertainties. In this article, the interval is built based on Chow's relation [48], presented and promoted by another research such as those in Bulletin 17B, Bulletin 17 C, Rao et al. [1,3,4,48], being a simplified approach, using the quantile and the frequency factor specific to the LH-moments, information also presented in previous materials [4,11–13]. Of course, there are other ways to calculate the C.I as well, including the conventional Bootstrap procedure [49–51], but these still have certain drawbacks, need a more involved analysis, and are not available to everyone.

Figures 6–9 highlights the quantiles of the distributions with the analyzed estimation methods, as well as the confidence interval for the L- and LH-moments.



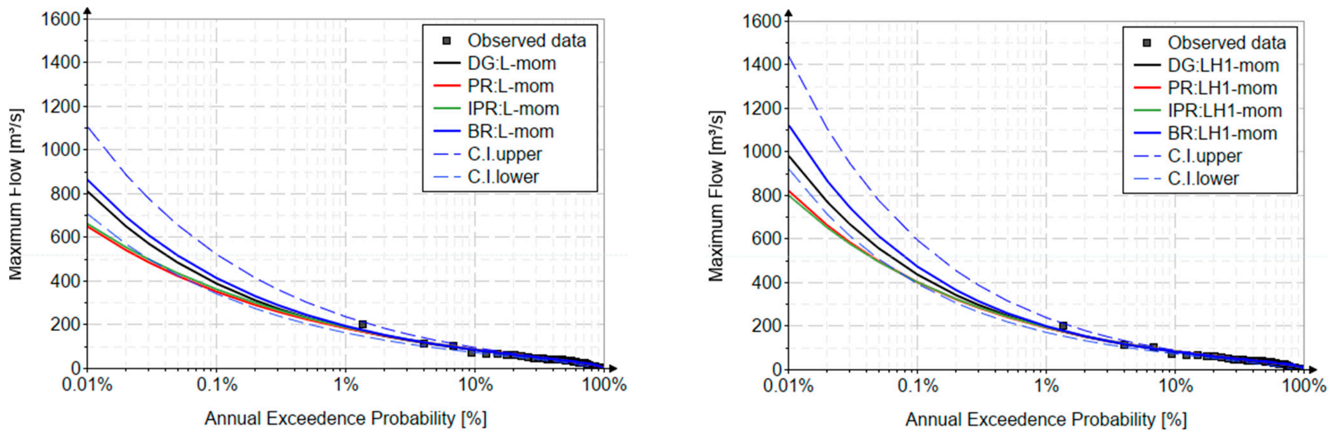
(**a**) L-moments (**b**) LH-moments (first level)

Figure 6. The confidence interval for the best fit model: Jijia River—Dorohoi Station.



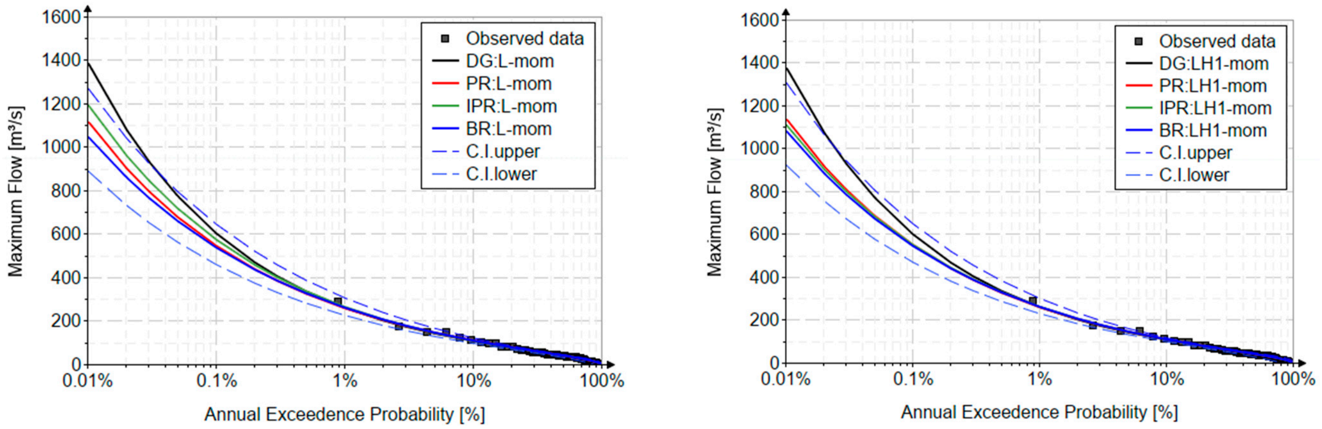
(**a**) L-moments (**b**) LH-moments (first level)

Figure 7. The confidence interval for the best fit model: Buhai River—Padureni Station.



(**a**) L-moments (**b**) LH-moments (first level)

Figure 8. The confidence interval for the best fit model: Miletin River—Sipote Station.



(**a**) L-moments (**b**) LH-moments (first level)

Figure 9. The confidence interval for the best fit model: Sitna River—Todireni Station.

It is recommended to include an easy method for the confidence interval, even based on a Gaussian assumption, if we consider that the existing Romanian legislation [52] for determining the maximum flows suffers from great deficiencies, both in terms of the use of distributions and parameter estimation methods, but especially of the recommendations regarding the determination of the confidence interval (the respective normative contains non-technical elements, such as the uncertainty interval [52,53]).

5. Conclusions

In the recent period, the use of these distributions from the generalized beta families has been greatly simplified with the presentation of important elements to apply in FFA, using MOM, and similarly in L-moments.

Considering the main advantage of the LH-moments, namely the fulfillment of the so-called "separation effect" of the maximum flow rates, this article presents all the exact and approximate relationships necessary for their use using this parameter estimation method.

The approximate relations for the frequency factors are a significant benefit in determining the maximum values of the flows for the required probability.

Another element of particular importance is the relationships and variation diagrams of the higher-order statistical indicators; they represent the main selection criterion of the best-fit distribution in the case of these parameter estimation methods. It also constitutes the basis for the preselection of certain distributions in FFA, especially regarding three-parameter distributions, using the two methods.

The performances of the analyzed distributions are checked on the four case studies, i.e., the Jijia, Buhai, Miletin and Sitna Rivers, from Romania, to find the maximum flows corresponding to the interested probabilities.

Following the results obtained, the BR4 distribution gives the best results for all four case studies, with the values of their theoretical statistical indicators correspondingly calibrating the similar values of the analyzed series. Among the distributions with three parameters, the best results were obtained by the distribution DG (Jijia and Sitna), followed by the distributions PR (Buhai) and IPR (Miletin). However, it should be highlighted that the results do not differ much from the BR4 distribution, the values being in general (for the maximum flow with the annual probability of exceeding 0.01%) below 20%, which represent a more than acceptable error considering this very rare event. The advantage of using PR and IPR distributions is that the parameters are considerably easier to estimate due to the availability of approximate relations for their estimation, thus avoiding solving systems of nonlinear equations, which are frequently an impediment.

The novelty elements presented regarding these methods will help researchers in frequency analysis, thus offering, in addition to classical approaches, other means of analyzing extreme events in hydrology.

The article does not rule out the use of different parameter estimation methods and probability distributions, especially as their relevant information has already been covered in prior papers, both in terms of the presentation of inverse functions, exact and approximate estimation relations of the parameters for different methods, diagrams and relationships of variation of high-order indicators characteristic of the methods, graphs of variation of shape parameters for different methods (and their comparative presentation),

The research presented in this article supplements the broader research begun within the Faculty of Hydrotechnics on the proposal to develop some norms regarding frequency analysis in hydrology for maximum flows, average flows, low flows, and hydrological drought, the results of which have been presented in previous materials [5,9–13,31].

All of these new aspects will be provided in separate open-source applications, making it easier to apply these probability distributions and parameter estimate methods in extreme event frequency investigations.

Supplementary Materials: The following supporting information can be downloaded at: https://www.mdpi.com/article/10.3390/w15223883/s1. Figure S1: The variation diagram for L-skewness and L-kurtosis; Figure S2: The variation diagram for LH-skewness and LH-kurtosis. Table S1: Frequency factors; Table S2: The frequency factor of the PR distribution, for the L-moments method; Table S3: The frequency factor of the PR distribution, for the first level LH-moments; Table S4: The frequency factor of the IPR distribution, for the L-moments method; Table S5: The frequency factor of the IPR distribution, for the first level LH-moments.

Author Contributions: Conceptualization, C.G.A. and C.I.; Methodology, C.G.A. and C.I.; Software, C.G.A. and C.I.; Validation, C.G.A. and C.I.; Formal analysis, C.G.A. and C.I.; Investigation, C.G.A. and C.I.; Resources, C.G.A. and C.I.; Data curation, C.G.A. and C.I.; Writing—original draft, C.G.A. and C.I.; Writing—review & editing, C.G.A. and C.I.; Visualization, C.G.A. and C.I.; Supervision, C.G.A. and C.I.; Project administration, C.G.A. and C.I.; Funding acquisition, C.G.A. and C.I. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Data are contained within the article and Supplementary Materials.

Conflicts of Interest: The authors declare no conflict of interest.

Appendix A. The Observed Data for Jijia, Buhai, Miletin and Sitna Rivers

Table A1 shows the data observed for the four analyzed rivers.

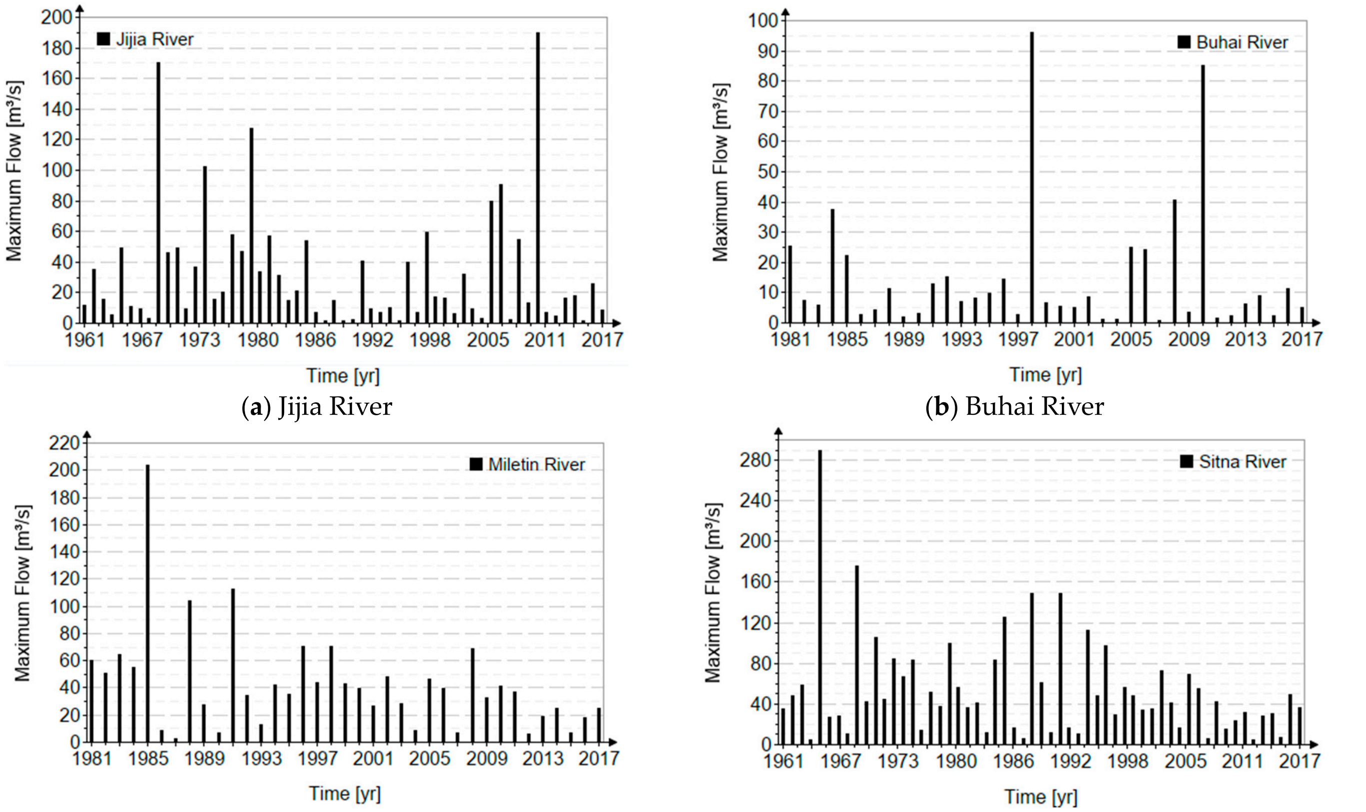
Table A1. The annual maximum series for the four analyzed rivers.

Jijia River				Buhai River				Miletin River				Sitna River			
Date	Flow	Date	Flow	Date	Flow	Date	Flow	Date	Flow	Date	Flow	Date	Flow	Date	Flow
[yr]	[m^3/s]	[yr]	[m^3/s]	[yr]	[m^3/s]	[yr]	[m^3/s]	[yr]	[m^3/s]	[yr]	[m^3/s]	[yr]	[m^3/s]	[yr]	[m^3/s]
1961	12.1	1989	1.44	1981	25.4	2010	85	1981	60.4	2010	41.6	1961	35.3	1989	61.2
1962	35.4	1990	2.29	1982	7.31	2011	1.58	1982	50.4	2011	36.9	1962	47.6	1990	11.6
1963	15.8	1991	40.5	1983	5.68	2012	2.34	1983	64.5	2012	6.21	1963	58.7	1991	149
1964	5.75	1992	9.5	1984	37.6	2013	6.14	1984	55.4	2013	18.5	1964	5.27	1992	16.7
1965	49.1	1993	7.28	1985	22.4	2014	9.09	1985	204	2014	25	1965	290	1993	10.5
1966	10.8	1994	9.83	1986	2.75	2015	2.15	1986	9.02	2015	6.58	1966	26.9	1994	113
1967	9.6	1995	1.51	1987	4.4	2016	11.2	1987	2.68	2016	17.7	1967	28.2	1995	48
1968	3.27	1996	39.9	1988	11.2	2017	5.05	1988	104	2017	25.3	1968	11	1996	97
1969	170	1997	7.3	1989	1.8			1989	27.7			1969	176	1997	28.9
1970	45.9	1998	59.2	1990	3.2			1990	6.81			1970	42.5	1998	56.8
1971	49.1	1999	17.2	1991	12.9			1991	113			1971	105	1999	48.1
1972	9.2	2000	16.4	1992	15.2			1992	34.4			1972	44.9	2000	34.4
1973	36.6	2001	6.43	1993	6.86			1993	12.8			1973	84.5	2001	35.4
1974	102	2002	32.2	1994	8.14			1994	42.1			1974	66.4	2002	72.5
1975	16	2003	9.06	1995	9.6			1995	35.5			1975	82.7	2003	41.4

Table A1. *Cont.*

Jijia River				Buhai River				Miletin River				Sitna River			
Date	**Flow**	**Date**	**Flow**	**Date**	**Flow**	**Date**	**Flow**	**Date**	**Flow**	**Date**	**Flow**	**Date**	**Flow**	**Date**	**Flow**
[yr]	**[m^3/s]**	**[yr]**	**[m^3/s]**	**[yr]**	**[m^3/s]**	**[yr]**	**[m^3/s]**	**[yr]**	**[m^3/s]**	**[yr]**	**[m^3/s]**	**[yr]**	**[m^3/s]**	**[yr]**	**[m^3/s]**
1976	20.4	2004	3.02	1996	14.5			1996	70.8			1976	14.2	2004	16.2
1977	57.5	2005	79.5	1997	2.87			1997	44.2			1977	51.2	2005	69.5
1978	47	2006	90.6	1998	96			1998	70.1			1978	37.2	2006	55.2
1979	127	2007	2.47	1999	6.68			1999	42.7			1979	100	2007	6.2
1980	33.5	2008	54.38	2000	5.53			2000	39.8			1980	56.3	2008	41.8
1981	56.7	2009	13.32	2001	4.96			2001	26.6			1981	36.5	2009	15.6
1982	31.4	2010	190	2002	8.55			2002	47.9			1982	41	2010	23
1983	14.8	2011	7.304	2003	1.02			2003	28.6			1983	12.2	2011	31.6
1984	20.9	2012	4.5	2004	1.34			2004	8.73			1984	82.8	2012	4.65
1985	54.2	2013	16.4	2005	25			2005	46.5			1985	125	2013	28.5
1986	7.21	2014	17.82	2006	24.2			2006	39.56			1986	15.9	2014	30.4
1987	1.34	2015	1.636	2007	0.77			2007	6.81			1987	5.74	2015	7.4
1988	14.9	2016	25.5	2008	40.6			2008	68.6			1988	149	2016	48.74
		2017	8.306	2009	3.644			2009	32.8					2017	36.4

Figure A1 shows the graphic representation of the four series of annual maximum flows.



(**a**) Jijia River (**b**) Buhai River

(**c**) Miletin River (**d**) Sitna River

Figure A1. The graph of data recorded for the analyzed rivers.

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