water

Appendix A: Derivation of Equations for Up-flow SPB

To simplify Equations (19-20) in the manuscript into a one-dimensional form similar to the equations given in Frishfelds et al. [26] and for further deduction of equations for the motion of the liquid fronts in the inner and the outer annular media, a trial function, which fulfils the boundary conditions, is introduced for the radial pressure variation. After an averaging procedure over the cross-section, equations similar to those in Frishfelds et al. [26] are obtained with the difference that in the present model the finite widths of the channels are taken into account. The trial functions considered are chosen from truncated solutions using a separation of variables method. Hence, the trial functions are chosen as

$$\begin{aligned}\varphi_i(x,r,t) &= A_0(x,t) + A_1(x,t)r^2 + A_2(x,t)\ln r \\ \varphi_o(x,r,t) &= B_0(x,t) + B_1(x,t)r^2 + B_2(x,t)\ln r\end{aligned} \tag{1}$$

To find the functions $A_0(x,t), A_1(x,t), A_2(x,t), B_0(x,t), B_1(x,t)$ and $B_2(x,t)$ the following boundary conditions are adopted as being valid in the region where the liquids in the two channels are in contact

$$\begin{aligned}&\left.\frac{\partial \varphi_i}{\partial r}\right|_{r=\delta} = 0 \quad \left.\frac{\partial \varphi_o}{\partial r}\right|_{r=b} = 0 \\ &\varphi_i(x,a,t) = \varphi_o(x,a,t) \\ &\phi_i K_{i,\perp} \left.\frac{\partial \varphi_i}{\partial r}\right|_{r=a} = \phi_o K_{o,\perp} \left.\frac{\partial \varphi_o}{\partial r}\right|_{r=a}\end{aligned} \tag{2}$$

From the boundary conditions the functions $A_1(x,t), A_2(x,t), B_1(x,t), B_2(x,t)$ can be eliminated in terms of $A_0(x,t), B_0(x,t)$ as

$$\begin{aligned}A_1 &= \frac{(A_0 - B_0)(a^2 - b^2)}{N} \\ B_1 &= \frac{(A_0 - B_0)(a^2 - \delta^2)X}{N} \\ X &= \frac{\phi_i K_{i,\perp}}{\phi_o K_{o,\perp}} \\ N &= (a^2 - 2b^2 \ln a)X(a^2 - \delta^2) + (b^2 - a^2)(a^2 - 2\delta^2 \ln a) \\ A_2 &= -2A_1\delta^2 \\ B_2 &= -2B_1 b^2\end{aligned} \tag{3}$$

Equations (1) are averaged over respective channels introducing the definition of average modified pressures according to

$$\overline{\varphi}_i = \frac{1}{\pi(a^2-\delta^2)}\int_{\delta}^{a}\varphi_i\, 2\pi r\, dr$$
$$\overline{\varphi}_o = \frac{1}{\pi(b^2-a^2)}\int_{a}^{b}\varphi_o\, 2\pi r\, dr \tag{4}$$

Averaging of Equations (1) yields the following equations

$$K_{i,\|}(a^2-\delta^2)\frac{d^2\overline{\varphi}_i}{dx^2} + 4K_{i,\perp}\frac{(a^2-\delta^2)(a^2-b^2)(A_0-B_0)}{N} = 0$$
$$K_{o,\|}(b^2-a^2)\frac{d^2\overline{\varphi}_i}{dx^2} - 4K_{o,\perp}X\frac{(a^2-\delta^2)(a^2-b^2)(A_0-B_0)}{N} = 0 \tag{5}$$

Evaluating $\overline{\varphi}_i - \overline{\varphi}_o$ in terms of $A_0 - B_0$ the system of Equations (5) can be written as

$$K_{i,\|}(a^2-\delta^2)\frac{d^2\overline{\varphi}_i}{dx^2} - K_{i,\perp}M\,(\overline{\varphi}_i-\overline{\varphi}_o) = 0$$
$$K_{o,\|}(b^2-a^2)\frac{d^2\overline{\varphi}_o}{dx^2} + K_{o,\perp}\frac{\phi_i}{\phi_o}M\,(\overline{\varphi}_i-\overline{\varphi}_o) = 0 \tag{6}$$

where

$$M = \frac{4(a^2-\delta^2)^2(a^2-b^2)^2}{(a^2-\delta^2)(a^2-b^2)N + 2(a^2-\delta^2)^2\alpha_2 X + 2(a^2-b^2)^2\alpha_1}$$
$$\alpha_1 = \frac{a^4}{4} - 3\frac{\delta^4}{4} - \delta^2 a^2 \ln a + \frac{a^2\delta^2}{2} + \delta^4 \ln\delta \tag{7}$$
$$\alpha_2 = \frac{3b^4}{4} - \frac{a^4}{4} - \frac{a^2b^2}{2} - b^4 \ln b + b^2a^2 \ln a$$

From Equations (6) one finds that the combination $\overline{\varphi}' = \alpha\,\overline{\varphi}_o + \beta\,\overline{\varphi}_i$ satisfies $d^2\overline{\varphi}'/dx^2 = 0$, if

$$\alpha = \frac{K_{o,\|}(b^2-a^2)}{K_{o,\|}(b^2-a^2) + \frac{\phi_i}{\phi_o}K_{i,\|}(a^2-\delta^2)}$$
$$\beta = \frac{K_{i,\|}(a^2-\delta^2)\frac{\phi_i}{\phi_o}}{K_{o,\|}(b^2-a^2) + \frac{\phi_i}{\phi_o}K_{i,\|}(a^2-\delta^2)} \tag{8}$$

Another equation can be derived for $\Delta\overline{\varphi} = \overline{\varphi}_i - \overline{\varphi}_o$ so that

$$\frac{d^2\Delta\overline{\varphi}}{dx^2} - \frac{\Delta\overline{\varphi}}{\lambda^2} = 0 \tag{9}$$

in which λ is a characteristic length of the pressure variation defined from

$$\frac{1}{\lambda^2} = \frac{K_{i,\perp} M}{K_{i,||}(a^2 - \delta^2)} + \frac{K_{o,\perp} M\, \phi_i / \phi_o}{K_{o,||}(b^2 - a^2)} \tag{10}$$

Combining Equations (8) and (9), the solution for the modified pressures in the inner and outer porous channels results in

$$\begin{aligned} \overline{\varphi}_i &= -\Gamma(t) x + \alpha \Lambda(t) \sinh(x/\lambda) \\ \overline{\varphi}_o &= -\Gamma(t) x - \beta \Lambda(t) \sinh(x/\lambda) \end{aligned} \tag{11}$$

Here $\Gamma(t)$ and $\Lambda(t)$ depend on the position of the liquid fronts in the two channels.

In the part, where there is no contact between the fluids, the pressure gradient is uniform and the pressure variation linear. The time dependent functions $\Gamma(t)$ and $\Lambda(t)$ are then derived from the condition of continuity of the volume flux at the position where the linear pressure distribution and the distribution given by Equation (11) meet each other. They depend on whether the front in the outer channel is in the lead, or the front in the inner channel is in the lead. If it is assumed that initially there is a higher permeability in the outer channel then the liquid front in this channel is in the lead. Due to the exchange, fluid from the outer channel is gradually transported into the inner channel. The front in the inner channel then finally catches the front in the outer channel. After the merging point, where the position of each front is equal, the front in the inner channel will be leading. Hence, for an initial period of time the liquid front of the outer channel $X_o(t)$ is leading so that $X_o(t) > X_i(t)$. Applying continuity condition to the volume flux in the outer channel at the uppermost position, where the two fluids are in contact, i.e. at $x = X_i(t)$, the point where the pressure distribution given by Equation (11) changes into the linear pressure distribution, and the functions *Γ(t)* and $\Lambda(t)$ can be obtained as shown below. The modified pressure at the free surface in the inner channel is given by

$$\overline{\varphi}_i(X_i(t)) = p_{in} - \Gamma^{(1)}(t) X_i(t) + \alpha\, \Lambda^{(1)}(t) \sinh(X_i(t)/\lambda) = p_1^o + \rho g X_i(t) \tag{12}$$

where p_{in} is the pressure at the inlet and $p_1^0 = -\gamma/R_i$ is the capillary pressure at the free surface of the inner channel and γ is the surface tension. Solving for $\Gamma(t)$ in Equation (12) gives

$$\Gamma^{(1)}(t) = \frac{p_{in} + \alpha\, \Lambda^{(1)}(t) \sinh(X_i(t)/\lambda) - p_1^o - \rho g X_i(t)}{X_i(t)} \tag{13}$$

From the condition of continuity of the volume flux at the point of contact between the two fluids at $x = X_i(t)$ it follows, after some calculations, that

$$\begin{aligned} \Lambda^{(1)}(t) &= \frac{(p_1^0 - p_2^0) X_i(t) + (p_1^0 - p_{in})(X_o(t) - X_i(t))}{X_o(t)\sinh(X_i(t)/\lambda) - (\beta \sinh(X_i(t)/\lambda) - \beta/\lambda X_i(t) - B_2^{(1)} X_i(t))(X_o(t) - X_i(t))} \\ B_2^{(1)}(t) &= \frac{\phi_i K_{\perp,i} \lambda M}{\phi_o K_{||,o}(b^2 - a^2)} (\cosh(X_i(t)/\lambda) - 1) \end{aligned} \tag{14}$$

where $p_2^0 = -\gamma/R_o$ is the capillary pressure at the free surface $X_o(t)$ of the outer channel. The equation of motion for the outer channel surface can then be stated as

$$\begin{aligned} \phi_o(b^2 - a^2)\frac{dX_o}{dt} &= \frac{K_{||,o}}{\mu}(b^2 - a^2)\frac{\varphi_o(X_o(t)) - \varphi_o(X_i(t))}{X_o(t) - X_i(t)} = \\ &= (b^2 - a^2)\frac{K_{||,o}}{\mu}\frac{p_2^0 + \rho g X_o(t) - (p_{in} - \Gamma^{(1)}(t) X_o(t) - \beta \Lambda^{(1)}(t)\sinh(X_o(t)/\lambda) - \rho g X_i(t)}{X_o(t) - X_i(t)} \end{aligned} \tag{15}$$

The singularity on the right-hand side at $X_i(t) = X_o(t)$ can be eliminated by rewriting the expression as

$$\phi_o \frac{dX_o}{dt} = -\frac{K_{\|,o}}{\mu}\rho g - \frac{K_{\|,o}(p_2^0 - p_{in})\sinh(X_i(t)/\lambda) + (p_2^0 - p_1^0)(\beta g(X_i(t)) + B_2^{(1)}X_i(t))}{\mu(X_o(t)\sinh(X_i(t)/\lambda) + (\beta g(X_i(t)) + B_2^{(1)}X_i(t))(X_o(t) - X_i(t))} \tag{16}$$

where $g(X_i(t)) = X_i(t)/\lambda - \sinh(X_i(t)/\lambda)$.

In turn, the equation of motion for the inner free surface can be written as

$$\phi_i(a^2 - \delta^2)\frac{dX_i}{dt} = -\frac{K_{\|,i}}{\mu}(a^2 - \delta^2)\left(\left.\frac{d\overline{\varphi}_i}{dx}\right|_{x=X_i(t)} - \left.\frac{d\overline{\varphi}_i}{dx}\right|_{x=0}\right) \tag{17}$$

Using the expression for the modified pressure distribution in the inner channel Equation(13) then finally yields

$$\begin{aligned} \phi_i \frac{dX_i}{dt} &= \frac{K_{\|,i}}{\mu}\left(\frac{p_{in} - p_1^0}{X_i(t)} - \rho g\right) + \frac{K_{\|,i}}{\mu}\Lambda\left(-\frac{\alpha}{\lambda} + \frac{\alpha\sinh(X_i(t)/\lambda)}{X_i(t)} + B_1(t)\right) \\ B_1(t) &= \frac{MK_{\perp,i}\lambda}{(a^2 - \delta^2)K_{\|,i}}(\cosh(X_i(t)/\lambda) - 1) \end{aligned} \tag{18}$$

The set of Equations (16-18) is valid as long as $X_o(t) > X_i(t)$. After some time t_1, the growth of the inner channel catches up the front in the outer channel so that $X_o(t_1) = X_i(t_1)$. After the merging of the fronts the front of the inner channel takes the lead and therefore another set of equations governing the motion of the fronts has to be derived. The modified pressure of the outer free surface can then be written as

$$\overline{\varphi}_o(X_o(t)) = p_{in} - \Gamma^{(2)}(t)X_o(t) - \beta\Lambda^{(2)}(t)\sinh(X_o(t)/\lambda) = p_2^0 + \rho g X_o(t) \tag{19}$$

From which $\Gamma^{(2)}(t)$ is found as

$$\Gamma^{(2)}(t) = \frac{p_{in} - \beta\Lambda^{(2)}(t)\sinh(X_o(t)/\lambda) - p_2^0 - \rho g X_o(t)}{X_o(t)} \tag{20}$$

and the corresponding value of $\Lambda^{(2)}(t)$ becomes

$$\begin{aligned} \Lambda^{(2)}(t) &= \frac{(p_{in} - p_2^0)(X_i(t) - X_o(t) + (p_1^0 - p_2^0)X_o(t)}{X_i(t)\sinh(X_o(t)/\lambda) - (\alpha\sinh(X_o(t)/\lambda) - \alpha/\lambda X_o(t) - B_1X_o(t))(X_o(t) - X_i(t))} \\ B_1^{(2)}(t) &= \frac{K_{\perp,i}\lambda M}{K_{\|,o}(a^2 - \delta^2)}(\cosh(X_o(t)/\lambda) - 1) \end{aligned} \tag{21}$$

The equation governing the front of the inner channel then becomes

$$\begin{aligned} \phi_i \frac{dX_i}{dt} &= -\frac{K_{\|,i}}{\mu}\rho g - \frac{K_{\|,o}(p_1^0 - p_{in})\sinh(X_o(t)/\lambda) + (p_1^0 - p_2^0)(\alpha g(X_o(t)) + B_1X_o(t))}{\mu(X_i(t)\sinh(X_o(t)/\lambda) + (\alpha g(X_o(t)) + B_1^{(2)}X_o(t))(X_i(t) - X_o(t))} \\ B_1^{(2)}(t) &= \frac{K_{\perp,i}\lambda M}{K_{\|,i}(a^2 - \delta^2)}(\cosh(X_o(t)/\lambda) - 1) \\ g(X_o(t)) &= X_o(t)/\lambda - \sinh(X_o(t)/\lambda) \end{aligned} \tag{22}$$

For the front in the outer channel the result is

$$\begin{aligned} &\phi_o \frac{dX_o}{dt} = \frac{K_{\|,o}}{\mu}\left(\frac{p_{in} - p_2^0}{X_o(t)} - \rho g\right) + \frac{K_{\|,o}}{\mu}\Lambda^{(2)}\left(\frac{\beta}{\lambda} - \frac{\beta \sinh(X_o(t)/\lambda)}{X_o(t)} + B_2^{(2)}(t)\right) \\ &B_2^{(2)}(t) = \frac{MK_{\perp,o}\lambda}{(b^2 - a^2)K_{\|,o}}(\cosh(X_o(t)/\lambda) - 1) \end{aligned} \tag{23}$$

Initially, if $X_o(t) > X_i(t)$ and $X_o, X_i \to 0$, Equations (16) and (18) reduce to

$$\begin{aligned} &\phi_o \frac{dX_o}{dt} = \frac{K_{\|,o}}{\mu}\frac{p_{in} - p_2^0}{X_o(t)} \\ &\phi_i \frac{dX_i}{dt} = \frac{K_{\|,i}}{\mu}\frac{p_{in} - p_1^0}{X_i(t)} \end{aligned} \tag{24}$$

which then shows no interaction between the two channels. The gravity term can also be neglected initially. The system of equations is integrated numerically using MATLAB. If the permeability is, for instance, chosen to be higher in the outer channel the numerical procedure starts by integrating Equations (16) and (18) numerically, with the initial conditions given by Equation(24). The integration is continued until the inner channel front merges with the outer channel front. From this position, Equations (22) and (23) are used to integrate the development of the front, in which now the inner channel front is in the lead. The equations are integrated until fronts reach their maximum amplitude, which turns out to be given by

$$\begin{aligned} &\lim_{t\to\infty} X_i(t) = \frac{(p_{in} - p_1^0)}{\rho g} = \frac{(p_{in} + \gamma/R_i)}{\rho g} \\ &\lim_{t\to\infty} X_o(t) = \frac{(p_{in} - p_2^0)}{\rho g} = \frac{(p_{in} + \gamma/R_o)}{\rho g} \end{aligned} \tag{25}$$