atmosphere

# Supplementary Materials: A Detailed Limited-Area Atmospheric Energy Cycle for Climate and Weather Studies

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**Supplementary material 1: Effective thermodynamics equation**

When using the exact definition for $a_T$

$$\mathrm{a}_T(T) = C_p T_r\left[\left(\frac{T-T_r}{T_r}\right) - \ln\left(\frac{T}{T_r}\right)\right] \tag{S1.1}$$

the prognostic equation

$$\frac{d\mathrm{a}_T}{dt} - \frac{R\omega}{p}(T-T_r) - \left(1-\frac{T_r}{T}\right)Q = 0 \tag{S1.2}$$

is exact in the sense of resulting from the exact thermodynamic equation

$$C_p\frac{dT}{dt} - \frac{R}{p}\omega T - Q = 0 \tag{S1.3}$$

Let us define $\chi = \frac{T-T_r}{T_r} = \frac{T}{T_r} - 1$ which is <<1 for practical applications. Then

$$\begin{aligned}
\mathrm{a}_T(T) &= C_p T_r\left[\left(\frac{T-T_r}{T_r}\right) - \ln\left(\frac{T}{T_r}\right)\right] \\
&= C_p T_r\left[\chi - \ln(1+\chi)\right] \\
&= C_p T_r\left[\chi - \left(\chi - \frac{\chi^2}{2} + \frac{\chi^3}{3} - \ldots\right)\right] \\
&= C_p T_r\left[\frac{\chi^2}{2} - \frac{\chi^3}{3} + \ldots\right] \\
&= C_p T_r\left[\frac{\chi^2}{2}\left(1 - \frac{2\chi}{3} + \ldots\right)\right] \\
&\approx C_p T\frac{\chi^2}{2} \\
&\approx \frac{C_p T_r}{2}\left(\frac{T-T_r}{T_r}\right)^2
\end{aligned} \tag{S1.4}$$

which is the small-$\chi$ approximation used in defining the temperature component of available enthalpy

$$a_T\left(T\right) \approx \frac{C_p T_r}{2}\left(\frac{T-T_r}{T_r}\right)^2 \tag{S1.5}$$

Let us now return to the prognostic equation

$$\frac{da_T}{dt} - \frac{R\omega}{p}\left(T-T_r\right) - \left(1-\frac{T_r}{T}\right)Q = 0 \tag{S1.6}$$

and apply systematically the same small- $\chi$ approximation. Consider the diabatic contribution

$$\left(1-\frac{T_r}{T}\right) = \frac{T-T_r}{T_r}\frac{T_r}{T} = \frac{T-T_r}{T_r}\left(1+\chi\right)^{-1} \approx \frac{T-T_r}{T_r}\left(1-\chi\right) \approx \frac{T-T_r}{T_r} \tag{S1.7}$$

So in the small- $\chi$ approximation the corresponding approximate form of the prognostic equation is

$$\frac{da_T}{dt} - \frac{R\omega}{p}\left(T-T_r\right) - \left(\frac{T}{T_r}-1\right)Q \approx 0 \tag{S1.8}$$

Note that this equation is slightly different from eq. (S1.6).

Using this equation corresponds to using an approximate form of the thermodynamic equation, as we shall show here:

$$\begin{aligned} &\frac{da_T}{dt} - \frac{R\omega}{p}\left(T-T_r\right) - \left(\frac{T}{T_r}-1\right)Q \approx 0 \\ &\Rightarrow \frac{C_p}{T_r}\left(T-T_r\right)\left[\frac{dT}{dt} - \frac{\omega R T_r}{pC_p} - \frac{Q}{C_p} \approx 0\right] \\ &\Rightarrow \frac{dT}{dt} - \frac{\omega\alpha_r}{C_p} - \frac{Q}{C_p} \approx 0 \end{aligned} \tag{S1.9}$$

where

$$\alpha_r = \frac{RT_r}{p}$$

which gives the effective, approximate form of the thermodynamic equation implied by the small- $\chi$ approximation for $a_T$. We note that this approximate form of the thermodynamic equation is better than equation (65) in [1] [2] that had also a factor of order unity $\frac{T_r}{T}$ affecting the diabatic heating term.

**Supplementary material 2: The total kinetic energy equation**

The total kinetic energy equation is obtained as follows:

$$\begin{aligned}
\frac{\partial K}{\partial t} &= \frac{\partial\left(\frac{1}{2}V_h^{\;2}\right)}{\partial t} \\
&= \vec{V_h}\cdot\frac{\partial\vec{V_h}}{\partial t} \\
&= \vec{V_h}\cdot\left(-\left(\vec{V}\cdot\vec{\nabla}\right)\cdot\vec{V_h} \quad - f\vec{k}\times\vec{V_h} \quad -\vec{\nabla}\phi \quad + \quad \vec{F_h}\right) \\
&= -\left(\vec{V}\cdot\vec{\nabla}\right)K \quad - \quad \vec{V_h}\cdot\vec{\nabla_h}\phi \quad + \quad \vec{V_h}\cdot\vec{F_h}
\end{aligned} \tag{S2.1}$$

with

$$\begin{aligned}
\vec{V_h}\cdot\left(\vec{\nabla_h}\phi\right) &= \vec{\nabla_h}\cdot\left(\vec{V_h}\phi\right)-\phi\left(\vec{\nabla_h}\cdot\vec{V_h}\right) \\
&= \vec{\nabla_h}\cdot\left(\phi\vec{V_h}\right)+\phi\frac{\partial\omega}{\partial p} \\
&= \vec{\nabla_h}\cdot\left(\phi\vec{V_h}\right)+\frac{\partial\left(\phi\omega\right)}{\partial p}-\omega\frac{\partial\phi}{\partial p} \\
&= \vec{\nabla}\cdot\left(\phi\vec{V}\right)+\omega\alpha
\end{aligned}$$

**Supplementary material 3: Time-mean kinetic energy equation associated with time-mean wind**

An equation for $K_M$ is obtained by taking the dot product of the time-mean horizontal winds with the time-mean horizontal momentum equation as follows

$$\begin{aligned}\frac{\partial K_M}{\partial t} &= \left\langle\overrightarrow{V_h}\right\rangle\cdot\frac{\partial\left\langle\overrightarrow{V_h}\right\rangle}{\partial t} \\ &= -\left\langle\overrightarrow{V_h}\right\rangle\cdot\left\langle\left(\vec{V}\cdot\vec{\nabla}\right)\cdot\overrightarrow{V_h}\right\rangle - \left\langle\overrightarrow{V_h}\right\rangle\cdot\overrightarrow{\nabla\langle\phi\rangle} + \left\langle\overrightarrow{V_h}\right\rangle\cdot\left\langle\overrightarrow{F_h}\right\rangle\end{aligned} \tag{S3.1}$$

When applying the following identity from Reynolds averaging rules

$$\langle AB\rangle = \langle A\rangle\langle B\rangle + \langle A'B'\rangle \tag{S3.2}$$

we obtain

$$\begin{aligned}&\left\langle\overrightarrow{V_h}\right\rangle\cdot\left\langle\left(\vec{V}\cdot\vec{\nabla}\right)\cdot\overrightarrow{V_h}\right\rangle \\ &= \left\langle\overrightarrow{V_h}\right\rangle\cdot\left[\left(\left\langle\vec{V}\right\rangle\cdot\vec{\nabla}\right)\cdot\left\langle\overrightarrow{V_h}\right\rangle + \left\langle\left(\overrightarrow{V'}\cdot\vec{\nabla}\right)\cdot\overrightarrow{V_h'}\right\rangle\right] \\ &= \left\langle\overrightarrow{V_h}\right\rangle\cdot\left(\left\langle\vec{V}\right\rangle\cdot\vec{\nabla}\right)\cdot\left\langle\overrightarrow{V_h}\right\rangle + \left\langle\overrightarrow{V_h}\right\rangle\cdot\left\langle\left(\overrightarrow{V'}\cdot\vec{\nabla}\right)\cdot\overrightarrow{V_h'}\right\rangle \\ &= \left(\left\langle\vec{V}\right\rangle\cdot\vec{\nabla}\right)\cdot\left(\frac{1}{2}\left\langle\overrightarrow{V_h}\right\rangle^2\right) + \left\langle\left\langle\overrightarrow{V_h}\right\rangle\cdot\left(\overrightarrow{V'}\cdot\vec{\nabla}\right)\cdot\overrightarrow{V_h'}\right\rangle \\ &= \vec{\nabla}\cdot\left(\left\langle\vec{V}\right\rangle K_M\right) + \left\langle\left\langle\overrightarrow{V_h}\right\rangle\cdot\left(\overrightarrow{V'}\cdot\vec{\nabla}\right)\cdot\overrightarrow{V_h'}\right\rangle \\ &= \vec{\nabla}\cdot\left(\left\langle\vec{V}\right\rangle K_M\right) + \vec{\nabla}\cdot\left\langle\overrightarrow{V'}\left(\overrightarrow{V_h'}\cdot\left\langle\overrightarrow{V_h}\right\rangle\right)\right\rangle - \left\langle\overrightarrow{V_h'}\cdot\left(\overrightarrow{V'}\cdot\vec{\nabla}\right)\cdot\left\langle\overrightarrow{V_h}\right\rangle\right\rangle\end{aligned} \tag{S3.3}$$

so that

$$\begin{aligned}\frac{\partial K_M}{\partial t} = &-\left\{\vec{\nabla}\cdot\left(\left\langle\vec{V}\right\rangle K_M\right) + \vec{\nabla}\cdot\left\langle\overrightarrow{V'}\left(\overrightarrow{V_h'}\cdot\left\langle\overrightarrow{V_h}\right\rangle\right)\right\rangle - \left\langle\overrightarrow{V_h'}\cdot\left(\overrightarrow{V'}\cdot\vec{\nabla}\right)\cdot\left\langle\overrightarrow{V_h}\right\rangle\right\rangle\right\} \\ &-\left\langle\overrightarrow{V_h}\right\rangle\cdot\overrightarrow{\nabla_h}\langle\phi\rangle + \left\langle\overrightarrow{V_h}\right\rangle\cdot\left\langle\overrightarrow{F_h}\right\rangle\end{aligned} \tag{S3.4}$$

$A_M$ and $K_M$ energy reservoirs can be linked as follows:

$$\begin{aligned}\left\langle\overrightarrow{V_h}\right\rangle\cdot\left(\overrightarrow{\nabla_h}\langle\phi\rangle\right) &= \overrightarrow{\nabla_h}\cdot\left(\langle\phi\rangle\left\langle\overrightarrow{V_h}\right\rangle\right) - \langle\phi\rangle\left(\overrightarrow{\nabla_h}\cdot\left\langle\overrightarrow{V_h}\right\rangle\right) \\ &= \overrightarrow{\nabla_h}\cdot\left(\langle\phi\rangle\left\langle\overrightarrow{V_h}\right\rangle\right) - \langle\phi\rangle\left(-\frac{\partial\langle\omega\rangle}{\partial p}\right) \\ &= \overrightarrow{\nabla_h}\cdot\left(\langle\phi\rangle\left\langle\overrightarrow{V_h}\right\rangle\right) + \langle\phi\rangle\left(\frac{\partial\langle\omega\rangle}{\partial p}\right) \\ &= \overrightarrow{\nabla_h}\cdot\left(\langle\phi\rangle\left\langle\overrightarrow{V_h}\right\rangle\right) + \left\{\frac{\partial\left(\langle\phi\rangle\langle\omega\rangle\right)}{\partial p} - \langle\omega\rangle\frac{\partial\langle\phi\rangle}{\partial p}\right\} \\ &= \vec{\nabla}\cdot\left(\langle\phi\rangle\left\langle\vec{V}\right\rangle\right) + \langle\omega\rangle\langle\alpha\rangle\end{aligned} \tag{S3.5}$$

**Supplementary material 4: Time-mean kinetic energy associated with time-variability of the wind**

The easiest way to obtain the time-mean kinetic energy associated with time-variability of the wind is by subtracting the time-mean kinetic energy equation and the kinetic energy equation associated with time-mean wind as follows:

$$\begin{aligned}\frac{\partial K_E}{\partial t} &= \left\langle \overrightarrow{V_h}\cdot\frac{\partial \overrightarrow{V_h}}{\partial t}\right\rangle - \left\langle \overrightarrow{V_h}\right\rangle\cdot\frac{\partial\left\langle \overrightarrow{V_h}\right\rangle}{\partial t} \\ &= -\left\{\begin{matrix}\left\langle\left(\vec{V}\cdot\vec{\nabla}\right)K\right\rangle - \\ \left\langle\overrightarrow{V_h}\right\rangle\cdot\left\langle\left(\vec{V}\cdot\vec{\nabla}\right)\cdot\overrightarrow{V_h}\right\rangle\end{matrix}\right\} - \left\{\begin{matrix}\left\langle\overrightarrow{V_h}\cdot\overrightarrow{\nabla\phi}\right\rangle - \\ \left\langle\overrightarrow{V_h}\right\rangle\cdot\overrightarrow{\nabla\langle\phi\rangle}\end{matrix}\right\} + \left\{\begin{matrix}\left\langle\overrightarrow{V_h}\cdot\overrightarrow{F_h}\right\rangle - \\ \left\langle\overrightarrow{V_h}\right\rangle\cdot\left\langle\overrightarrow{F_h}\right\rangle\end{matrix}\right\}\end{aligned} \tag{S4.1}$$

Considering

$$\begin{aligned}&\left\langle\left(\vec{V}\cdot\vec{\nabla}\right)K\right\rangle \\ &= \vec{\nabla}\cdot\left\langle\vec{V}\;K\right\rangle \\ &= \vec{\nabla}\cdot\left(\left\langle\vec{V}\right\rangle\;\langle K\rangle\right) + \;\vec{\nabla}\cdot\left\langle\overrightarrow{V'}\;K_X\right\rangle \\ &= \vec{\nabla}\cdot\left\{\left\langle\vec{V}\right\rangle\;\left(\langle K_M\rangle + \langle K_E\rangle\right)\right\} \;+\vec{\nabla}\cdot\left\langle\overrightarrow{V'}\;\left(\frac{1}{2}V_h'^2 - \left\langle\frac{1}{2}V_h'^2\right\rangle + \left\langle\overrightarrow{V_h}\right\rangle\cdot\overrightarrow{V_h'}\right)\right\rangle \\ &= \vec{\nabla}\cdot\left(\left\langle\vec{V}\right\rangle\;K_M\right) + \vec{\nabla}\cdot\left(\left\langle\vec{V}\right\rangle\;K_E\right) + \vec{\nabla}\cdot\left\langle\overrightarrow{V'}\left(\frac{1}{2}V_h'^2\right)\right\rangle + \vec{\nabla}\cdot\left\langle\overrightarrow{V'}\left(\overrightarrow{V_h'}\cdot\left\langle\overrightarrow{V_h}\right\rangle\right)\right\rangle\end{aligned} \tag{S4.2}$$

and the identity obtained in eq. (S4.3), we obtain

$$\begin{aligned}&\left\langle\left(\vec{V}\cdot\vec{\nabla}\right)K\right\rangle - \left\langle\overrightarrow{V_h}\right\rangle\cdot\left\langle\left(\vec{V}\cdot\vec{\nabla}\right)\cdot\overrightarrow{V_h}\right\rangle \\ &= \left\{\begin{matrix}\left[\vec{\nabla}\cdot\left(\left\langle\vec{V}\right\rangle\;K_M\right) + \vec{\nabla}\cdot\left(\left\langle\vec{V}\right\rangle\;K_E\right) + \vec{\nabla}\cdot\left\langle\overrightarrow{V'}\left(\frac{1}{2}V_h'^2\right)\right\rangle + \vec{\nabla}\cdot\left\langle\overrightarrow{V'}\left(\overrightarrow{V_h'}\cdot\left\langle\overrightarrow{V_h}\right\rangle\right)\right\rangle\right] \\ -\left[\;\vec{\nabla}\cdot\left(\left\langle\vec{V}\right\rangle\;K_M\right) + \vec{\nabla}\cdot\left\langle\overrightarrow{V'}\left(\overrightarrow{V_h'}\cdot\left\langle\overrightarrow{V_h}\right\rangle\right)\right\rangle \;\; - \left\langle\overrightarrow{V_h'}\cdot\left(\overrightarrow{V'}\cdot\vec{\nabla}\right)\cdot\left\langle\overrightarrow{V_h}\right\rangle\right\rangle\right]\end{matrix}\right\} \\ &= \vec{\nabla}\cdot\left(\left\langle\vec{V}\right\rangle\;K_E\right) + \vec{\nabla}\cdot\left\langle\overrightarrow{V'}\left(\frac{1}{2}V_h'^2\right)\right\rangle - \left\langle\overrightarrow{V_h'}\cdot\left(\overrightarrow{V'}\cdot\vec{\nabla}\right)\cdot\left\langle\overrightarrow{V_h}\right\rangle\right\rangle\end{aligned} \tag{S4.3}$$

By doing the same for the frictional term

$$\left\langle\overrightarrow{V_h}\cdot\overrightarrow{F_h}\right\rangle - \left\langle\overrightarrow{V_h}\right\rangle\cdot\left\langle\overrightarrow{F_h}\right\rangle = \left\langle\overrightarrow{V_h'}\cdot\overrightarrow{F_h'}\right\rangle \tag{S4.4}$$

so that

$$\begin{aligned}\frac{\partial K_M}{\partial t} = &-\left\{\vec{\nabla}\cdot\left(\left\langle\vec{V}\right\rangle\;K_E\right) + \vec{\nabla}\cdot\left\langle\overrightarrow{V'}\left(\frac{1}{2}V_h'^2\right)\right\rangle - \left\langle\overrightarrow{V_h'}\cdot\left(\overrightarrow{V'}\cdot\vec{\nabla}\right)\cdot\left\langle\overrightarrow{V_h}\right\rangle\right\rangle\right\} \\ &- \left\langle\overrightarrow{V_h'}\cdot\overrightarrow{\nabla\phi'}\right\rangle + \left\langle\overrightarrow{V_h'}\cdot\overrightarrow{F_h'}\right\rangle\end{aligned} \tag{S4.5}$$

The link between $A_E$ and $K_E$ can be done by rewritten the term involving geopotential as follows:

$$
\begin{aligned}
&\left\langle \overrightarrow{V_h} \cdot \overrightarrow{\nabla \phi} \right\rangle - \left\langle \overrightarrow{V_h} \right\rangle \cdot \overrightarrow{\nabla \langle \phi \rangle} \\
&= \left\{ \vec{\nabla} \cdot \left\langle \phi \, \vec{V} \right\rangle + \langle \omega \alpha \rangle \right\} - \left\{ \vec{\nabla} \cdot \left( \langle \phi \rangle \left\langle \vec{V} \right\rangle \right) + \langle \omega \rangle \langle \alpha \rangle \right\} \\
&= \vec{\nabla} \cdot \left\{ \left\langle \phi \, \vec{V} \right\rangle - \left( \langle \phi \rangle \left\langle \vec{V} \right\rangle \right) \right\} + \left\{ \langle \omega \alpha \rangle - \langle \omega \rangle \langle \alpha \rangle \right\} \\
&= \vec{\nabla} \cdot \left\langle \phi' \, \vec{V}' \right\rangle + \langle \omega' \alpha' \rangle
\end{aligned}
\tag{S4.6}
$$

**Supplementary material 5: Time-mean available enthalpy equation associated with time-mean temperature**

An equation for $A_M$ is obtained by multiplying $\frac{C_p}{T_r}\left\langle T-T_r\right\rangle$ by the time-mean thermodynamics equation as follows:

$$\begin{aligned}\frac{\partial A_M}{\partial t} &= \frac{C_p}{T_r}\left\langle T-T_r\right\rangle\left\langle\frac{\partial T}{\partial t}\right\rangle \\ &= -\frac{C_p}{T_r}\left\langle T-T_r\right\rangle\left\langle\left(\vec{V}\cdot\vec{\nabla}\right)T\right\rangle+\left\langle\omega\right\rangle\frac{R}{p}\left\langle T-T_r\right\rangle+\left\langle\left(\frac{T}{T_r}-1\right)\right\rangle\left\langle Q\right\rangle\end{aligned} \tag{S5.1}$$

Once again, using Reynolds averaging rule eq. (S3.2), we obtain

$$\begin{aligned}&\frac{C_p}{T_r}\left\langle T-T_r\right\rangle\left\langle\left(\vec{V}\cdot\vec{\nabla}\right)\left(T-T_r\right)\right\rangle \\ &=\frac{C_p}{T_r}\left\langle T-T_r\right\rangle\left\{\left(\left\langle\vec{V}\right\rangle\cdot\vec{\nabla}\right)\left\langle T-T_r\right\rangle+\left\langle\left(\vec{V}'\cdot\vec{\nabla}\right)T'\right\rangle\right\} \\ &=\vec{\nabla}\cdot\left(\left\langle\vec{V}\right\rangle A_M\right)+\frac{C_p}{T_r}\left\langle\left\langle T-T_r\right\rangle\left(\vec{V}'\cdot\vec{\nabla}\right)T'\right\rangle \\ &=\vec{\nabla}\cdot\left(\left\langle\vec{V}\right\rangle A_M\right)+\vec{\nabla}\cdot\left\langle\vec{V}'\left(\frac{C_p}{T_r}T'\left\langle T-T_r\right\rangle\right)\right\rangle-\frac{C_p}{T_r}\left(\left\langle T'\vec{V}'\right\rangle\cdot\vec{\nabla}\right)\left\langle T-T_r\right\rangle \\ &=\vec{\nabla}\cdot\left(\left\langle\vec{V}\right\rangle A_M\right)+\vec{\nabla}\cdot\left\langle\vec{V}'\left(\frac{C_p}{T_r}T'\left\langle T-T_r\right\rangle\right)\right\rangle-\frac{C_p}{T_r}\left(\left\langle T'\vec{V}'\right\rangle\cdot\vec{\nabla}\right)\left\langle T\right\rangle\end{aligned} \tag{S5.2}$$

so that

$$\begin{aligned}\frac{\partial A_M}{\partial t} &= -\left\{\vec{\nabla}\cdot\left(\left\langle\vec{V}\right\rangle A_M\right)+\vec{\nabla}\cdot\left\langle\vec{V}'\left(\frac{C_p}{T_r}T'\left\langle T-T_r\right\rangle\right)\right\rangle-\frac{C_p}{T_r}\left(\left\langle T'\vec{V}'\right\rangle\cdot\vec{\nabla}\right)\left\langle T\right\rangle\right\} \\ &\quad+\left\langle\omega\right\rangle\left\langle\alpha\right\rangle-\left\langle\omega\right\rangle\alpha_r+\left\langle\left(\frac{T}{T_r}-1\right)\right\rangle\left\langle Q\right\rangle\end{aligned} \tag{S5.3}$$

**Supplementary material 6: Time-mean available enthalpy equation associated with time-variability of temperature**

An equation for $A_E$ is obtained by subtracting the time-mean enthalpy equation by the time-mean available enthalpy equation associated with time-mean temperature as follows

$$\frac{\partial A_E}{\partial t} = \frac{C_p}{T_r}\left\langle \left(T-T_r\right)\frac{\partial T}{\partial t}\right\rangle - \frac{C_p}{T_r}\left\langle T-T_r\right\rangle\left\langle\frac{\partial T}{\partial t}\right\rangle$$
$$= -\begin{bmatrix}\left\langle\left(\vec{V}\cdot\vec{\nabla}\right)A\right\rangle - \\ \frac{C_p}{T_r}\left\langle T-T_r\right\rangle\left\langle\left(\vec{V}\cdot\vec{\nabla}\right)T\right\rangle\end{bmatrix} + \left[\left\langle\omega\alpha\right\rangle - \left\langle\omega\right\rangle\left\langle\alpha\right\rangle\right] + \left[\left\langle\left(\frac{T}{T_r}-1\right)Q\right\rangle - \left\langle\left(\frac{T}{T_r}-1\right)\right\rangle\left\langle Q\right\rangle\right] \tag{S6.1}$$

Considering

$$\begin{aligned}&\left\langle\left(\vec{V}\cdot\vec{\nabla}\right)A\right\rangle \\ &= \vec{\nabla}\cdot\left\langle\vec{V}\ \mathrm{A}\right\rangle \\ &= \vec{\nabla}\cdot\left(\left\langle\vec{V}\right\rangle\ \left\langle A\right\rangle\right) + \vec{\nabla}\cdot\left\langle\vec{V}'\,A_X\right\rangle \\ &= \vec{\nabla}\cdot\left\{\left\langle\vec{V}\right\rangle\ \left(\left\langle A_M\right\rangle + \left\langle A_E\right\rangle\right)\right\} + \vec{\nabla}\cdot\left\langle\vec{V}'\left(\frac{C_p}{2T_r}T'^2 - \left\langle\frac{C_p}{2T_r}T'^2\right\rangle + \frac{C_p}{T_r}T'\left\langle T-T_r\right\rangle\right)\right\rangle \\ &= \vec{\nabla}\cdot\left(\left\langle\vec{V}\right\rangle\ A_M\right) + \vec{\nabla}\cdot\left(\left\langle\vec{V}\right\rangle\ A_E\right) + \vec{\nabla}\cdot\left\langle\vec{V}'\left(\frac{C_p}{2T_r}T'^2\right)\right\rangle + \vec{\nabla}\cdot\left\langle\vec{V}'\left(\frac{C_p}{T_r}T'\left\langle T-T_r\right\rangle\right)\right\rangle\end{aligned} \tag{S6.2}$$

and eq. (S6.2), we obtain

$$\begin{aligned}&\left\langle\left(\vec{V}\cdot\vec{\nabla}\right)A\right\rangle - \frac{C_p}{T_r}\left\langle T-T_r\right\rangle\left\langle\left(\vec{V}\cdot\vec{\nabla}\right)T\right\rangle \\ &= \left\{\begin{matrix}\left[\left[\vec{\nabla}\cdot\left(\left\langle\vec{V}\right\rangle\ A_M\right) + \vec{\nabla}\cdot\left(\left\langle\vec{V}\right\rangle\ A_E\right) + \vec{\nabla}\cdot\left\langle\vec{V}'\left(\frac{C_p}{2T_r}T'^2\right)\right\rangle + \vec{\nabla}\cdot\left\langle\vec{V}'\left(\frac{C_p}{T_r}T'\left\langle T-T_r\right\rangle\right)\right\rangle\right]\right] \\ -\left[\vec{\nabla}\cdot\left(\left\langle\vec{V}\right\rangle\ A_M\right) + \vec{\nabla}\cdot\left\langle\vec{V}'\left(\frac{C_p}{T_r}T'\left\langle T-T_r\right\rangle\right)\right\rangle - \frac{C_p}{T_r}\left(\left\langle T'\vec{V}'\right\rangle\cdot\vec{\nabla}\right)\left\langle T\right\rangle\right]\end{matrix}\right\} \\ &= \vec{\nabla}\cdot\left(\left\langle\vec{V}\right\rangle\ A_E\right) + \vec{\nabla}\cdot\left\langle\vec{V}'\left(\frac{C_p}{2T_r}T'^2\right)\right\rangle + \frac{C_p}{T_r}\left(\left\langle T'\vec{V}'\right\rangle\cdot\vec{\nabla}\right)\left\langle T\right\rangle\end{aligned} \tag{S6.3}$$

According to Reynolds decomposition rules

$$\left\langle\omega\alpha\right\rangle - \left\langle\omega\right\rangle\left\langle\alpha\right\rangle = \left\langle\omega'\alpha'\right\rangle \tag{S6.4}$$

$$\left\langle\left(\frac{T}{T_r}-1\right)Q\right\rangle - \left\langle\left(\frac{T}{T_r}-1\right)\right\rangle\left\langle Q\right\rangle = \frac{\left\langle T'Q'\right\rangle}{T_r} \tag{S6.5}$$

so that

$$\frac{\partial A_E}{\partial t} = -\left\{ \vec{\nabla} \cdot \left( \left\langle \vec{V} \right\rangle A_E \right) + \vec{\nabla} \cdot \left\langle \vec{V}' \left( \frac{C_p}{2T_r} T'^2 \right) \right\rangle + \frac{C_p}{T_r} \left( \left\langle T' \vec{V}' \right\rangle \cdot \vec{\nabla} \right) \left\langle T \right\rangle \right\} + \left\langle \omega' \alpha' \right\rangle + \frac{\left\langle T' Q' \right\rangle}{T_r} \quad \text{(S6.6)}$$

**Supplementary material 7: Instantaneous kinetic energy equation associated with variance of wind fluctuations**

An equation for $K_{X1}$ is obtained by taking the dot product of the time-variability horizontal wind $\overrightarrow{V_h}'$ with the time-variability momentum equation

$$\begin{aligned}\frac{\partial K_{X1}}{\partial t} &= \overrightarrow{V_h}'\cdot\frac{\partial \overrightarrow{V_h}'}{\partial t}-\left\langle \overrightarrow{V_h}'\cdot\frac{\partial \overrightarrow{V_h}'}{\partial t}\right\rangle \\ &= \overrightarrow{V_h}'\cdot\left(-\left\{\left(\vec{V}\cdot\vec{\nabla}\right)\cdot\vec{V_h}-\left\langle\left(\vec{V}\cdot\vec{\nabla}\right)\cdot\vec{V_h}\right\rangle\right\}-f\vec{k}\times\overrightarrow{V_h}'-\vec{\nabla}\phi'+\overrightarrow{F_h}'\right) \\ &\quad -\left\langle\overrightarrow{V_h}'\cdot\left(-\left\{\left(\vec{V}\cdot\vec{\nabla}\right)\cdot\vec{V_h}-\left\langle\left(\vec{V}\cdot\vec{\nabla}\right)\cdot\vec{V_h}\right\rangle\right\}-f\vec{k}\times\overrightarrow{V_h}'-\vec{\nabla}\phi'+\overrightarrow{F_h}'\right)\right\rangle\end{aligned} \tag{S7.1}$$

The first advection term may be rewritten using the following Reynolds identity

$$AB-\langle AB\rangle=\langle A\rangle B'+A'\langle B\rangle+A'B'-\langle A'B'\rangle \tag{S7.2}$$

$$\begin{aligned}&-\overrightarrow{V_h}'\cdot\left\{\left(\vec{V}\cdot\vec{\nabla}\right)\cdot\vec{V_h}-\left\langle\left(\vec{V}\cdot\vec{\nabla}\right)\cdot\vec{V_h}\right\rangle\right\} \\ &= -\overrightarrow{V_h}'\cdot\left\{\left(\overrightarrow{V'}\cdot\vec{\nabla}\right)\cdot\overrightarrow{\langle V_h\rangle}+\left(\left\langle\vec{V}\right\rangle\cdot\vec{\nabla}\right)\cdot\overrightarrow{V_h}'+\left(\overrightarrow{V'}\cdot\vec{\nabla}\right)\cdot\overrightarrow{V_h}'-\left\langle\left(\overrightarrow{V'}\cdot\vec{\nabla}\right)\cdot\overrightarrow{V_h}'\right\rangle\right\} \\ &= -\overrightarrow{V_h}'\cdot\left(\overrightarrow{V'}\cdot\vec{\nabla}\right)\cdot\left\langle\overrightarrow{V_h}\right\rangle-\vec{\nabla}\cdot\left\{\left\langle\vec{V}\right\rangle\left(\frac{1}{2}V_h'^2\right)\right\}-\vec{\nabla}\cdot\left\{\overrightarrow{V'}\left(\frac{1}{2}V_h'^2\right)\right\}+\overrightarrow{V_h}'\cdot\left\langle\left(\overrightarrow{V'}\cdot\vec{\nabla}\right)\cdot\overrightarrow{V_h}'\right\rangle \\ &= -\vec{\nabla}\cdot\left\{\vec{V}\left(\frac{1}{2}V_h'^2\right)\right\}-\overrightarrow{V_h}'\cdot\left(\overrightarrow{V'}\cdot\vec{\nabla}\right)\cdot\left\langle\overrightarrow{V_h}\right\rangle+\overrightarrow{V_h}'\cdot\left\langle\left(\overrightarrow{V'}\cdot\vec{\nabla}\right)\cdot\overrightarrow{V_h}'\right\rangle\end{aligned} \tag{S7.3}$$

Consequently, the second advection term is its time-average

$$\begin{aligned}&\left\langle\overrightarrow{V_h}'\cdot\left\{\left(\vec{V}\cdot\vec{\nabla}\right)\cdot\vec{V_h}-\left\langle\left(\vec{V}\cdot\vec{\nabla}\right)\cdot\vec{V_h}\right\rangle\right\}\right\rangle \\ &= \vec{\nabla}\cdot\left\{\left\langle\vec{V}\left(\frac{1}{2}V_h'^2\right)\right\rangle\right\}+\left\langle\overrightarrow{V_h}'\cdot\left(\overrightarrow{V'}\cdot\vec{\nabla}\right)\cdot\left\langle\overrightarrow{V_h}\right\rangle\right\rangle\end{aligned} \tag{S7.4}$$

Terms involving the Coriolis parameter are all equals to zero.
Finally,

$$\begin{aligned}\frac{\partial K_{X1}}{\partial t} &= -\vec{\nabla}\cdot\left\{\vec{V}\left(\frac{1}{2}V_h'^2\right)-\left\langle\vec{V}\left(\frac{1}{2}V_h'^2\right)\right\rangle\right\}-\left\{\overrightarrow{V_h}'\cdot\left(\overrightarrow{V'}\cdot\vec{\nabla}\right)\cdot\left\langle\overrightarrow{V_h}\right\rangle-\left\langle\overrightarrow{V_h}'\cdot\left(\overrightarrow{V'}\cdot\vec{\nabla}\right)\cdot\left\langle\overrightarrow{V_h}\right\rangle\right\rangle\right\} \\ &\quad +\overrightarrow{V_h}'\cdot\left\langle\left(\overrightarrow{V'}\cdot\vec{\nabla}\right)\cdot\overrightarrow{V_h}'\right\rangle-\left\{\overrightarrow{V_h}'\cdot\vec{\nabla}_h\phi'-\left\langle\overrightarrow{V_h}'\cdot\vec{\nabla}_h\phi'\right\rangle\right\}+\left\{\overrightarrow{V_h}'\cdot\overrightarrow{F_h}'-\left\langle\overrightarrow{V_h}'\cdot\overrightarrow{F_h}'\right\rangle\right\}\end{aligned} \tag{S7.5}$$

$A_{X1}$ and $K_{X1}$ energy reservoirs can be linked as follows:

$$
\begin{aligned}
&\overrightarrow{V_h}' \cdot \overrightarrow{\nabla_h}\phi' \\
&= \overrightarrow{V_h}' \cdot \overrightarrow{\nabla_h \phi'} + \omega' \frac{\partial \phi'}{\partial p} - \omega' \frac{\partial \phi'}{\partial p} \\
&= \overrightarrow{V}' \cdot \overrightarrow{\nabla \phi'} - \omega' \frac{\partial \phi'}{\partial p} \\
&= \left\{ \vec{\nabla} \cdot \left( \phi' \overrightarrow{V'} \right) - \phi' \left( \vec{\nabla} \cdot \overrightarrow{V'} \right) \right\} + \omega' \alpha' \\
&= \vec{\nabla} \cdot \left( \phi' \overrightarrow{V'} \right) + \omega' \alpha'
\end{aligned}
\tag{S7.6}
$$

**Supplementary material 8: Instantaneous kinetic energy equation associated with co-variance of wind fluctuations and wind time-average**

An equation for $K_{X2}$ is obtained by taking the dot product of time-variability horizontal wind $\overrightarrow{V_h}'$ with the time-mean momentum equation and the dot product of time-mean horizontal wind $\left\langle\overrightarrow{V_h}\right\rangle$ with the time-variability momentum equation as follows:

$$\begin{aligned}\frac{\partial K_{X2}}{\partial t} &= \overrightarrow{V_h}'\cdot\frac{\partial\left\langle\overrightarrow{V_h}\right\rangle}{\partial t}+\left\langle\overrightarrow{V_h}\right\rangle\cdot\frac{\partial\overrightarrow{V_h}'}{\partial t}\\ &= \overrightarrow{V_h}'\cdot\left[-\left\langle\left(\vec{V}\cdot\vec{\nabla}\right)\cdot\overrightarrow{V_h}\right\rangle - f\vec{k}\times\left\langle\overrightarrow{V_h}\right\rangle - \vec{\nabla}\left\langle\phi\right\rangle + \left\langle\overrightarrow{F_h}\right\rangle\right]\\ &+\left\langle\overrightarrow{V_h}\right\rangle\cdot\left[-\left\{\left(\vec{V}\cdot\vec{\nabla}\right)\cdot\overrightarrow{V_h}-\left\langle\left(\vec{V}\cdot\vec{\nabla}\right)\cdot\overrightarrow{V_h}\right\rangle\right\}-f\vec{k}\times\overrightarrow{V_h}' - \vec{\nabla}\phi' + \overrightarrow{F_h}'\right]\end{aligned} \tag{S8.1}$$

The total wind advection terms is:

$$\begin{aligned}&-\overrightarrow{V_h}'\cdot\left\langle\left(\vec{V}\cdot\vec{\nabla}\right)\cdot\overrightarrow{V_h}\right\rangle-\left[\left\langle\overrightarrow{V_h}\right\rangle\cdot\left(\vec{V}\cdot\vec{\nabla}\right)\cdot\overrightarrow{V_h}-\left\langle\overrightarrow{V_h}\right\rangle\cdot\left\langle\left(\vec{V}\cdot\vec{\nabla}\right)\cdot\overrightarrow{V_h}\right\rangle\right]\\ &= -\overrightarrow{V_h}'\cdot\left\langle\left(\vec{V}\cdot\vec{\nabla}\right)\cdot\overrightarrow{V_h}\right\rangle-\left[\left(\overrightarrow{V_h}-\overrightarrow{V_h}'\right)\cdot\left(\vec{V}\cdot\vec{\nabla}\right)\cdot\overrightarrow{V_h}-\left\langle\overrightarrow{V_h}\right\rangle\cdot\left\langle\left(\vec{V}\cdot\vec{\nabla}\right)\cdot\overrightarrow{V_h}\right\rangle\right]\\ &= \overrightarrow{V_h}'\cdot\left[\left(\vec{V}\cdot\vec{\nabla}\right)\cdot\overrightarrow{V_h}-\left\langle\left(\vec{V}\cdot\vec{\nabla}\right)\cdot\overrightarrow{V_h}\right\rangle\right]-\overrightarrow{V_h}\cdot\left(\vec{V}\cdot\vec{\nabla}\right)\cdot\overrightarrow{V_h}+\left\langle\overrightarrow{V_h}\right\rangle\cdot\left\langle\left(\vec{V}\cdot\vec{\nabla}\right)\cdot\overrightarrow{V_h}\right\rangle\end{aligned} \tag{S8.2}$$

The first term in bracket has already been decomposed in (S8.3), the second is the advection of the total kinetic energy by the three-dimensional wind vector and the last term is obtained by combining two Reynolds identities:

$$\begin{aligned}\langle A\rangle\langle BC\rangle &= \langle A(BC)\rangle-\langle A'(BC)'\rangle\\ &= \langle ABC\rangle-\left\langle A'\left(B'\langle C\rangle+\langle B\rangle C'+B'C'-\langle B'C'\rangle\right)\right\rangle\\ &= \langle ABC\rangle-\left\langle A'B'\langle C\rangle\right\rangle-\left\langle A'\langle B\rangle C'\right\rangle-\langle A'B'C'\rangle\end{aligned} \tag{S8.3}$$

we obtain

$$\overrightarrow{V_h}'\cdot\left[\left(\vec{V}\cdot\vec{\nabla}\right)\cdot\overrightarrow{V_h}-\left\langle\left(\vec{V}\cdot\vec{\nabla}\right)\cdot\overrightarrow{V_h}\right\rangle\right]-\overrightarrow{V_h}\cdot\left(\vec{V}\cdot\vec{\nabla}\right)\cdot\overrightarrow{V_h}+\left\langle\overrightarrow{V_h}\right\rangle\cdot\left\langle\left(\vec{V}\cdot\vec{\nabla}\right)\cdot\overrightarrow{V_h}\right\rangle$$
$$=\left\{\vec{\nabla}\cdot\left[\vec{V}\left(\frac{1}{2}V_h{'^2}\right)\right]+\overrightarrow{V_h}'\cdot\left(\vec{V}'\cdot\vec{\nabla}\right)\cdot\left\langle\overrightarrow{V_h}\right\rangle-\overrightarrow{V_h}'\cdot\left\langle\left(\vec{V}'\cdot\vec{\nabla}\right)\cdot\overrightarrow{V_h}'\right\rangle\right\}-\vec{\nabla}\cdot\left\{\vec{V}K\right\}$$
$$+\left\{\left\langle\overrightarrow{V_h}\cdot\left(\vec{V}\cdot\vec{\nabla}\right)\cdot\overrightarrow{V_h}\right\rangle-\left\langle\overrightarrow{V_h}'\cdot\left(\vec{V}'\cdot\vec{\nabla}\right)\cdot\left\langle\overrightarrow{V_h}\right\rangle\right\rangle-\left\langle\overrightarrow{V_h}'\cdot\left(\left\langle\vec{V}\right\rangle\cdot\vec{\nabla}\right)\cdot\overrightarrow{V_h}'\right\rangle-\left\langle\overrightarrow{V_h}'\cdot\left(\vec{V}'\cdot\vec{\nabla}\right)\cdot\overrightarrow{V_h}'\right\rangle\right\}$$
$$=\left\{\vec{\nabla}\cdot\left[\vec{V}\left(\frac{1}{2}V_h{'^2}\right)\right]+\overrightarrow{V_h}'\cdot\left(\vec{V}'\cdot\vec{\nabla}\right)\cdot\left\langle\overrightarrow{V_h}\right\rangle-\overrightarrow{V_h}'\cdot\left\langle\left(\vec{V}'\cdot\vec{\nabla}\right)\cdot\overrightarrow{V_h}'\right\rangle\right\}-\vec{\nabla}\cdot\left\{\vec{V}K\right\} \quad \text{(S8.4)}$$
$$+\left\{\vec{\nabla}\cdot\left\langle\vec{V}K\right\rangle-\left\langle\overrightarrow{V_h}'\cdot\left(\vec{V}'\cdot\vec{\nabla}\right)\cdot\left\langle\overrightarrow{V_h}\right\rangle\right\rangle-\vec{\nabla}\cdot\left\langle\vec{V}\left(\frac{1}{2}V_h{'^2}\right)\right\rangle\right\}$$
$$=-\vec{\nabla}\cdot\left\{\vec{V}\left(K-\frac{1}{2}V_h{'^2}\right)-\left\langle\vec{V}\left(K-\frac{1}{2}V_h{'^2}\right)\right\rangle\right\}$$
$$+\left\{\overrightarrow{V_h}'\cdot\left(\vec{V}'\cdot\vec{\nabla}\right)\cdot\left\langle\overrightarrow{V_h}\right\rangle-\left\langle\overrightarrow{V_h}'\cdot\left(\vec{V}'\cdot\vec{\nabla}\right)\cdot\left\langle\overrightarrow{V_h}\right\rangle\right\rangle\right\}-\overrightarrow{V_h}'\cdot\left\langle\left(\vec{V}'\cdot\vec{\nabla}\right)\cdot\overrightarrow{V_h}'\right\rangle$$

Terms involving geopotential may be rewritten as follows:

$$\frac{\partial K_{X2}}{\partial t}=-\vec{\nabla}\cdot\left\{\vec{V}\left(K-\frac{1}{2}V_h{'^2}\right)-\left\langle\vec{V}\left(K-\frac{1}{2}V_h{'^2}\right)\right\rangle\right\}+\left\{\overrightarrow{V_h}'\cdot\left(\vec{V}'\cdot\vec{\nabla}\right)\cdot\left\langle\overrightarrow{V_h}\right\rangle-\left\langle\overrightarrow{V_h}'\cdot\left(\vec{V}'\cdot\vec{\nabla}\right)\cdot\left\langle\overrightarrow{V_h}\right\rangle\right\rangle\right\}$$
$$-\overrightarrow{V_h}'\cdot\left\langle\left(\vec{V}'\cdot\vec{\nabla}\right)\cdot\overrightarrow{V_h}'\right\rangle-\vec{\nabla}\cdot\left\{\overrightarrow{V_h}'\cdot\nabla\left\langle\phi\right\rangle+\left\langle\overrightarrow{V_h}\right\rangle\cdot\overrightarrow{\nabla\phi'}\right\}+\left\{\overrightarrow{V_h}'\cdot\left\langle\overrightarrow{F_h}\right\rangle+\left\langle\overrightarrow{V_h}\right\rangle\cdot\overrightarrow{F_h}'\right\}$$

(S8.5)

$A_{X2}$ and $K_{X2}$ energy reservoirs can be linked as follows:

$$\overrightarrow{V_h}'\cdot\vec{\nabla}\left\langle\phi\right\rangle=\overrightarrow{\nabla}_h\cdot\left(\overrightarrow{V_h}'\left\langle\phi\right\rangle\right)-\left\langle\phi\right\rangle\left(\overrightarrow{\nabla}_h\cdot\overrightarrow{V_h}'\right)$$
$$=\overrightarrow{\nabla}_h\cdot\left(\overrightarrow{V_h}'\left\langle\phi\right\rangle\right)+\left\langle\phi\right\rangle\frac{\partial\omega'}{\partial p}$$
$$=\left\{\overrightarrow{\nabla}_h\cdot\left(\overrightarrow{V_h}'\left\langle\phi\right\rangle\right)+\frac{\partial\omega'\left\langle\phi\right\rangle}{\partial p}\right\}-\omega'\frac{\partial\left\langle\phi\right\rangle}{\partial p} \quad \text{(S8.6)}$$
$$=\vec{\nabla}\cdot\left(\vec{V}'\left\langle\phi\right\rangle\right)+\omega'\left\langle\alpha\right\rangle$$

Likewise

$$\left\langle\overrightarrow{V_h}\right\rangle\cdot\vec{\nabla}\phi'=\vec{\nabla}\cdot\left(\left\langle\vec{V}\right\rangle\phi'\right)+\left\langle\omega\right\rangle\alpha' \quad \text{(S8.7)}$$

**SUPPLEMENTARY MATERIAL 9: Instantaneous available enthalpy associated with variance of temperature fluctuations**

An equation for $A_{X1}$ is obtained by multiplying $\frac{C_p}{T_r}T'$ with the time-variability thermodynamics equation

$$\begin{aligned}\frac{\partial A_{X1}}{\partial t} &= \frac{C_p}{T_r}\left(T'\frac{\partial T'}{\partial t} - \left\langle T'\frac{\partial T'}{\partial t}\right\rangle\right)\\ &= \frac{C_p}{T_r}T'\left(-\left\{\left(\vec{V}\cdot\vec{\nabla}\right)T - \left\langle\left(\vec{V}\cdot\vec{\nabla}\right)T\right\rangle\right\} + \frac{\omega'\alpha_r}{C_p} + \frac{Q'}{C_p}\right)\\ &\quad - \frac{C_p}{T_r}\left\langle T'\left(-\left\{\left(\vec{V}\cdot\vec{\nabla}\right)T - \left\langle\left(\vec{V}\cdot\vec{\nabla}\right)T\right\rangle\right\} + \frac{\omega'\alpha_r}{C_p} + \frac{Q'}{C_p}\right)\right\rangle\end{aligned} \quad (S9.1)$$

The first advection term may be rewritten as follows:

$$\begin{aligned}&-\frac{C_p}{T_r}T'\left\{\left(\vec{V}\cdot\vec{\nabla}\right)T - \left\langle\left(\vec{V}\cdot\vec{\nabla}\right)T\right\rangle\right\}\\ &= -\frac{C_p}{T_r}T'\left\{\left(\left\langle\vec{V}\right\rangle\cdot\vec{\nabla}\right)T' + \left(\vec{V'}\cdot\vec{\nabla}\right)\langle T\rangle + \left(\vec{V'}\cdot\vec{\nabla}\right)T' - \left\langle\left(\vec{V'}\cdot\vec{\nabla}\right)T'\right\rangle\right\}\\ &= -\vec{\nabla}\cdot\left\{\left\langle\vec{V}\right\rangle\left(\frac{C_p}{2T_r}T'^2\right)\right\} - \frac{C_p}{T_r}T'\left(\vec{V'}\cdot\vec{\nabla}\right)\langle T\rangle - \vec{\nabla}\cdot\left\{\vec{V'}\left(\frac{C_p}{2T_r}T'^2\right)\right\} + \frac{C_p}{T_r}T'\left\langle\left(\vec{V'}\cdot\vec{\nabla}\right)T'\right\rangle\\ &= -\vec{\nabla}\cdot\left\{\vec{V}\left(\frac{C_p}{2T_r}T'^2\right)\right\} - \frac{C_p}{T_r}T'\left(\vec{V'}\cdot\vec{\nabla}\right)\langle T\rangle + \frac{C_p}{T_r}T'\left\langle\left(\vec{V'}\cdot\vec{\nabla}\right)T'\right\rangle\end{aligned} \quad (S9.2)$$

Consequently, the second advection term is its time-average

$$\begin{aligned}&\frac{C_p}{T_r}T'\left\{\left(\vec{V}\cdot\vec{\nabla}\right)T - \left\langle\left(\vec{V}\cdot\vec{\nabla}\right)T\right\rangle\right\}\\ &= \vec{\nabla}\cdot\left\{\left\langle\vec{V}\left(\frac{C_p}{2T_r}T'^2\right)\right\rangle\right\} + \frac{C_p}{T_r}\left\langle T'\left(\vec{V'}\cdot\vec{\nabla}\right)\langle T\rangle\right\rangle\end{aligned} \quad (S9.3)$$

so that

$$\begin{aligned}\frac{\partial A_{X1}}{\partial t} &= -\vec{\nabla}\cdot\left\{\vec{V}\left(\frac{C_p}{2T_r}T'^2\right) - \left\langle\vec{V}\left(\frac{C_p}{2T_r}T'^2\right)\right\rangle\right\} - \frac{C_p}{T_r}\left\{\left(T'\vec{V'}\cdot\vec{\nabla}\right)\langle T\rangle - \left\langle\left(T'\vec{V'}\cdot\vec{\nabla}\right)\langle T\rangle\right\rangle\right\}\\ &\quad + \frac{C_p}{T_r}T'\left\langle\left(\vec{V'}\cdot\vec{\nabla}\right)T'\right\rangle + \left\{\omega'\alpha' - \left\langle\omega'\alpha'\right\rangle\right\} + \frac{1}{T_r}\left\{T'Q' - \left\langle T'Q'\right\rangle\right\}\end{aligned} \quad (S9.4)$$

**SUPPLEMENTARY MATERIAL 10: Instantaneous available enthalpy associated with covariance of temperature fluctuations and wind time-average**

An equation for $A_{X2}$ is obtained by multiplying $\frac{C_p}{T_r}T'$ with the time-mean thermodynamics equation and $\frac{C_p}{T_r}\left\langle T-T_r\right\rangle$ with the time-variability thermodynamics equation as follows:

$$\begin{aligned}
\frac{\partial A_{X2}}{\partial t} &= \frac{C_p}{T_r}\left(T'\frac{\partial\left\langle T\right\rangle}{\partial t}+\left\langle T-T_r\right\rangle\frac{\partial T'}{\partial t}\right)\\
&= \frac{C_p}{T_r}T'\left(-\left\langle\left(\vec{V}\cdot\vec{\nabla}\right)\left(T-T_r\right)\right\rangle + \frac{\left\langle\omega\right\rangle\alpha_r}{C_p} + \frac{\left\langle Q\right\rangle}{C_p}\right)\\
&+\frac{C_p}{T_r}\left\langle T-T_r\right\rangle\left(-\left\{\left(\vec{V}\cdot\vec{\nabla}\right)\left(T-T_r\right)-\left\langle\left(\vec{V}\cdot\vec{\nabla}\right)\left(T-T_r\right)\right\rangle\right\}+\frac{\omega'\alpha_r}{C_p}+\frac{Q'}{C_p}\right)
\end{aligned} \tag{S10.1}$$

Considering

$$\begin{aligned}
&-T'\left\langle\left(\vec{V}\cdot\vec{\nabla}\right)\left(T-T_r\right)\right\rangle-\left[\left\langle T-T_r\right\rangle\left(\vec{V}\cdot\vec{\nabla}\right)\left(T-T_r\right)-\left\langle T-T_r\right\rangle\left\langle\left(\vec{V}\cdot\vec{\nabla}\right)\left(T-T_r\right)\right\rangle\right]\\
&=-T'\left\langle\left(\vec{V}\cdot\vec{\nabla}\right)T\right\rangle-\left[\left\langle T-T_r\right\rangle\left(\vec{V}\cdot\vec{\nabla}\right)T-\left\langle T-T_r\right\rangle\left\langle\left(\vec{V}\cdot\vec{\nabla}\right)T\right\rangle\right]\\
&=-T'\left\langle\left(\vec{V}\cdot\vec{\nabla}\right)T\right\rangle-\left[\left(T-T'-T_r\right)\left(\vec{V}\cdot\vec{\nabla}\right)T-\left\langle T-T_r\right\rangle\left\langle\left(\vec{V}\cdot\vec{\nabla}\right)T\right\rangle\right]\\
&=-T'\left\langle\left(\vec{V}\cdot\vec{\nabla}\right)T\right\rangle-\left[\left\{\left(T-T_r\right)-T'\right\}\left(\vec{V}\cdot\vec{\nabla}\right)T-\left\langle T-T_r\right\rangle\left\langle\left(\vec{V}\cdot\vec{\nabla}\right)T\right\rangle\right]\\
&=T'\left[\left(\vec{V}\cdot\vec{\nabla}\right)T-\left\langle\left(\vec{V}\cdot\vec{\nabla}\right)T\right\rangle\right]-\left(T-T_r\right)\left(\vec{V}\cdot\vec{\nabla}\right)T+\left\langle T-T_r\right\rangle\left\langle\left(\vec{V}\cdot\vec{\nabla}\right)T\right\rangle
\end{aligned} \tag{S10.2}$$

The first term into bracket has already been evaluated in eq. (S9.2), the second is proportional to the advection of the total available enthalpy by the three-dimensional wind vector and the last term is obtained using the combination of two Reynolds identity developed in eq. (S8.3) as follows:

$$\begin{aligned}
&\left\langle T-T_r\right\rangle\left\langle\left(\vec{V}\cdot\vec{\nabla}\right)T\right\rangle\\
&=\left\langle\left(T-T_r\right)\left(\vec{V}\cdot\vec{\nabla}\right)T\right\rangle+\left\langle T'\left(\vec{V}\cdot\vec{\nabla}\right)\left\langle T\right\rangle\right\rangle+\left\langle T'\left(\left\langle\vec{V}\right\rangle\cdot\vec{\nabla}\right)T'\right\rangle-\left\langle T'\left(\vec{V}'\cdot\vec{\nabla}\right)T'\right\rangle\\
&=\vec{\nabla}\cdot\left\langle\vec{V}\frac{\left(T-T_r\right)^2}{2}\right\rangle+\left\langle T'\left(\vec{V}\cdot\vec{\nabla}\right)\left\langle T\right\rangle\right\rangle+\vec{\nabla}\cdot\left\langle\left(\left\langle\vec{V}\right\rangle\frac{T'^2}{2}\right)\right\rangle-\vec{\nabla}\cdot\left\langle\left(\vec{V}'\frac{T'^2}{2}\right)\right\rangle
\end{aligned} \tag{S10.3}$$

Consequently, the total advection term becomes:

$$\frac{C_p}{T_r}T'\left[\left(\vec{V}\cdot\vec{\nabla}\right)T-\left\langle\left(\vec{V}\cdot\vec{\nabla}\right)T\right\rangle\right]-\frac{C_p}{T_r}\left(T-T_r\right)\left(\vec{V}\cdot\vec{\nabla}\right)T+\frac{C_p}{T_r}\left\langle T-T_r\right\rangle\left\langle\left(\vec{V}\cdot\vec{\nabla}\right)T\right\rangle$$

$$=\left\{\vec{\nabla}\cdot\left\{\vec{V}\left(\frac{C_p}{2T_r}T'^2\right)\right\}+\frac{C_p}{T_r}T'\left(\vec{V}'\cdot\vec{\nabla}\right)\left\langle T\right\rangle-\frac{C_p}{T_r}T'\left\langle\left(\vec{V}'\cdot\vec{\nabla}\right)T'\right\rangle\right\}-\vec{\nabla}\cdot\left\{\vec{V}A\right\}$$

$$+\left\{\vec{\nabla}\cdot\left\{\vec{V}A\right\}-\left\langle\frac{C_p}{T_r}T'\left(\vec{V}'\cdot\vec{\nabla}\right)\left\langle T\right\rangle\right\rangle-\left\langle\frac{C_p}{T_r}T'\left(\left\langle\vec{V}\right\rangle\cdot\vec{\nabla}\right)T'\right\rangle-\left\langle\frac{C_p}{T_r}T'\left(\vec{V}'\cdot\vec{\nabla}\right)T'\right\rangle\right\}$$

$$=\left\{\vec{\nabla}\cdot\left[\vec{V}\left(\frac{C_p}{2T_r}T'^2\right)\right]+\frac{C_p}{T_r}T'\left(\vec{V}'\cdot\vec{\nabla}\right)\left\langle T\right\rangle-\frac{C_p}{T_r}T'\left\langle\left(\vec{V}'\cdot\vec{\nabla}\right)T'\right\rangle\right\}-\vec{\nabla}\cdot\left\{\vec{V}A\right\}$$

$$+\left\{\vec{\nabla}\cdot\left\langle\vec{V}A\right\rangle-\left\langle\frac{C_p}{T_r}T'\left(\vec{V}'\cdot\vec{\nabla}\right)\left\langle T\right\rangle\right\rangle-\vec{\nabla}\cdot\left\langle\vec{V}\left(\frac{C_p}{2T_r}T'^2\right)\right\rangle\right\}$$

$$=-\vec{\nabla}\cdot\left\{\vec{V}\left(A-\frac{C_p}{2T_r}T'^2\right)-\left\langle\vec{V}\left(A-\frac{C_p}{2T_r}T'^2\right)\right\rangle\right\}+\frac{C_p}{T_r}\left\{\left(T'\vec{V}'\cdot\vec{\nabla}\right)\left\langle T\right\rangle-\left\langle\left(T'\vec{V}'\cdot\vec{\nabla}\right)\left\langle T\right\rangle\right\rangle\right\}$$

$$-\frac{C_p}{T_r}T'\left\langle\left(\vec{V}'\cdot\vec{\nabla}\right)T'\right\rangle \tag{S10.4}$$

Moreover

$$\frac{C_p}{T_r}T'\frac{\left\langle\omega\right\rangle\alpha_r}{C_p}=\left\langle\omega\right\rangle\alpha' \tag{S10.5}$$

$$\left\langle T-T_r\right\rangle\frac{\omega' R}{p}=\omega'\left(\left\langle\alpha\right\rangle-\alpha_r\right) \tag{S10.6}$$

so that

$$\frac{\partial A_{X2}}{\partial t}=-\vec{\nabla}\cdot\left\{\vec{V}\left(A-\frac{C_p}{2T_r}T'^2\right)-\left\langle\vec{V}\left(A-\frac{C_p}{2T_r}T'^2\right)\right\rangle\right\}+\frac{C_p}{T_r}\left\{\left(T'\vec{V}'\cdot\vec{\nabla}\right)\left\langle T\right\rangle-\left\langle\left(T'\vec{V}'\cdot\vec{\nabla}\right)\left\langle T\right\rangle\right\rangle\right\}$$

$$-\frac{C_p}{T_r}T'\left\langle\left(\vec{V}'\cdot\vec{\nabla}\right)T'\right\rangle+\left[\left(\left\langle\omega\right\rangle\alpha'+\omega'\left\langle\alpha\right\rangle\right)-\omega'\alpha_r\right]+\frac{1}{T_r}\left(T'\left\langle Q\right\rangle+\left\langle T-T_r\right\rangle Q'\right) \tag{S10.7}$$

**SUPPLEMENTARY MATERIAL 11: Total kinetic energy retrieval**

We ensure that the computation of time-mean and time-variability kinetic energy budget were correct by summing all their contributions to retrieve the total kinetic energy equation terms obtained in the "*Supplementary material*".

$$\frac{\partial K_M}{\partial t} = -\begin{bmatrix} -\left\langle \vec{V_h}' \cdot \left(\vec{V}' \cdot \vec{\nabla}\right) \cdot \left\langle \vec{V_h} \right\rangle \right\rangle \\ +\vec{\nabla} \cdot \left( \left\langle \vec{V} \right\rangle K_M \right) \\ +\vec{\nabla} \cdot \left\langle \vec{V}' \left( \vec{V_h}' \cdot \left\langle \vec{V_h} \right\rangle \right) \right\rangle \end{bmatrix} - \left[ \left\langle \vec{V_h} \right\rangle \cdot \nabla_h \left\langle \phi \right\rangle \right] + \left\langle \vec{V_h} \right\rangle \cdot \left\langle \vec{F_h} \right\rangle$$

$$\frac{\partial K_E}{\partial t} = -\begin{bmatrix} \left\langle \vec{V_h}' \cdot \left(\vec{V}' \cdot \vec{\nabla}\right) \cdot \left\langle \vec{V_h} \right\rangle \right\rangle \\ +\vec{\nabla} \cdot \left( \left\langle \vec{V} \right\rangle K_E \right) \\ +\vec{\nabla} \cdot \left\langle \vec{V}' \left( \dfrac{\vec{V_h}' \cdot \vec{V_h}'}{2} \right) \right\rangle \end{bmatrix} - \left[ \left\langle \vec{V_h}' \cdot \overline{\nabla_h \phi'} \right\rangle \right] + \left[ \left\langle \vec{V_h}' \cdot \vec{F_h}' \right\rangle \right]$$

$$\frac{\partial K_{X1}}{\partial t} = -\begin{bmatrix} \vec{\nabla} \cdot \left\{ \vec{V} \left( \frac{1}{2} V_h'^2 \right) - \left\langle \vec{V} \left( \frac{1}{2} V_h'^2 \right) \right\rangle \right\} + \\ \left\{ \vec{V_h}' \cdot \left(\vec{V}' \cdot \vec{\nabla}\right) \cdot \left\langle \vec{V_h} \right\rangle - \left\langle \vec{V_h}' \cdot \left(\vec{V}' \cdot \vec{\nabla}\right) \cdot \left\langle \vec{V_h} \right\rangle \right\rangle \right\} \\ -\vec{V_h}' \cdot \left\langle \left(\vec{V}' \cdot \vec{\nabla}\right) \cdot \vec{V_h}' \right\rangle \end{bmatrix} - \begin{bmatrix} \vec{V_h}' \cdot \overline{\nabla_h \phi'} - \\ \left\langle \vec{V_h}' \cdot \overline{\nabla_h \phi'} \right\rangle \end{bmatrix} + \left\{ \vec{V_h}' \cdot \vec{F_h}' - \left\langle \vec{V_h}' \cdot \vec{F_h}' \right\rangle \right\}$$

$$\frac{\partial K_{X2}}{\partial t} = -\begin{bmatrix} \vec{\nabla} \cdot \left\{ \vec{V} \left( K - \frac{1}{2} V_h'^2 \right) - \left\langle \vec{V} \left( K - \frac{1}{2} V_h'^2 \right) \right\rangle \right\} - \\ \left\{ \vec{V_h}' \cdot \left(\vec{V}' \cdot \vec{\nabla}\right) \cdot \left\langle \vec{V_h} \right\rangle - \left\langle \vec{V_h}' \cdot \left(\vec{V}' \cdot \vec{\nabla}\right) \cdot \left\langle \vec{V_h} \right\rangle \right\rangle \right\} \\ +\vec{V_h}' \cdot \left\langle \left(\vec{V}' \cdot \vec{\nabla}\right) \cdot \vec{V_h}' \right\rangle \end{bmatrix} - \begin{bmatrix} \vec{V_h}' \cdot \overline{\nabla_h \left\langle \phi \right\rangle} + \\ \left\langle \vec{V_h} \right\rangle \cdot \overline{\nabla_h \phi'} \end{bmatrix} + \left\{ \vec{V_h}' \cdot \left\langle \vec{F_h} \right\rangle + \left\langle \vec{V_h} \right\rangle \cdot \vec{F_H}' \right\}$$

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$$\frac{\partial K}{\partial t} = -\left(\vec{V} \cdot \vec{\nabla}\right) K - \vec{V_h} \cdot \overline{\nabla_h \phi} + \vec{V_h} \cdot \vec{F_h} \tag{S11.1}$$

In fact, the sum of conversion and boundary fluxes terms appearing in $K_M$, $K_E$, $K_{X1}$ and $K_{X2}$ tendency equations is:

$$
-\begin{bmatrix} -\left\langle \overrightarrow{V_h}'\cdot\left(\overrightarrow{V}'\cdot\vec{\nabla}\right)\cdot\left\langle\overrightarrow{V_h}\right\rangle\right\rangle \\ +\vec{\nabla}\cdot\left(\left\langle\vec{V}\right\rangle K_M\right) \\ +\vec{\nabla}\cdot\left\langle\overrightarrow{V}'\left(\overrightarrow{V_h}'.\left\langle\overrightarrow{V_h}\right\rangle\right)\right\rangle \end{bmatrix}
-\begin{bmatrix} \left\langle \overrightarrow{V_h}'\cdot\left(\overrightarrow{V}'\cdot\vec{\nabla}\right)\cdot\left\langle\overrightarrow{V_h}\right\rangle\right\rangle \\ +\vec{\nabla}\cdot\left(\left\langle\vec{V}\right\rangle K_E\right) \\ +\vec{\nabla}\cdot\left\langle\overrightarrow{V}'\left(\dfrac{\overrightarrow{V_h}'\cdot\overrightarrow{V_h}'}{2}\right)\right\rangle \end{bmatrix}
$$

$$
-\begin{bmatrix} \vec{\nabla}\cdot\left\{\vec{V}\left(\frac{1}{2}V_h'^2\right)-\left\langle\vec{V}\left(\frac{1}{2}V_h'^2\right)\right\rangle\right\}+ \\ \left\{\overrightarrow{V_h}'\cdot\left(\overrightarrow{V}'\cdot\vec{\nabla}\right)\cdot\left\langle\overrightarrow{V_h}\right\rangle-\left\langle\overrightarrow{V_h}'\cdot\left(\overrightarrow{V}'\cdot\vec{\nabla}\right)\cdot\left\langle\overrightarrow{V_h}\right\rangle\right\rangle\right\} \\ -\overrightarrow{V_h}'\cdot\left\langle\left(\overrightarrow{V}'\cdot\vec{\nabla}\right)\cdot\overrightarrow{V_h}'\right\rangle \end{bmatrix}
-\begin{bmatrix} \vec{\nabla}\cdot\left\{\vec{V}\left(K-\frac{1}{2}V_h'^2\right)-\left\langle\vec{V}\left(K-\frac{1}{2}V_h'^2\right)\right\rangle\right\}- \\ \left\{\overrightarrow{V_h}'\cdot\left(\overrightarrow{V}'\cdot\vec{\nabla}\right)\cdot\left\langle\overrightarrow{V_h}\right\rangle-\left\langle\overrightarrow{V_h}'\cdot\left(\overrightarrow{V}'\cdot\vec{\nabla}\right)\cdot\left\langle\overrightarrow{V_h}\right\rangle\right\rangle\right\} \\ +\overrightarrow{V_h}'\cdot\left\langle\left(\overrightarrow{V}'\cdot\vec{\nabla}\right)\cdot\overrightarrow{V_h}'\right\rangle \end{bmatrix}
$$

$$
\begin{aligned}
&= -\left\{\vec{\nabla}\cdot\left(\left\langle\vec{V}\right\rangle\left(K_M+K_E\right)\right)+\vec{\nabla}\cdot\left\langle\overrightarrow{V}'\left(\overrightarrow{V_h}'.\left\langle\overrightarrow{V_h}\right\rangle+\overrightarrow{V}'\left(\frac{\overrightarrow{V_h}'\cdot\overrightarrow{V_h}'}{2}\right)\right)\right\rangle\right\}-\vec{\nabla}\cdot\left\{\vec{V}\,K-\left\langle\vec{V}\,K\right\rangle\right\} \\
&= -\left\{\vec{\nabla}\cdot\left(\left\langle\vec{V}\right\rangle\left\langle K\right\rangle\right)+\vec{\nabla}\cdot\left\langle\overrightarrow{V}'\,K_X\right\rangle\right\}-\vec{\nabla}\cdot\left\{\vec{V}\,K-\left\langle\vec{V}\,K\right\rangle\right\} \\
&= -\vec{\nabla}\cdot\left\langle\vec{V}\,K\right\rangle-\vec{\nabla}\cdot\left\{\vec{V}\,K-\left\langle\vec{V}\,K\right\rangle\right\} \\
&= -\vec{\nabla}\cdot\left(\vec{V}\,K\right)
\end{aligned}
\tag{S11.2}
$$

The sum of contribution of geopotential heights appearing in $K_M$, $K_E$, $K_{X1}$ and $K_{X2}$ tendency equations is:

$$
\begin{aligned}
&-\left\langle\overrightarrow{V_h}\right\rangle\cdot\overline{\nabla_h\left\langle\phi\right\rangle}-\left\langle\overrightarrow{V_h}'\cdot\overline{\nabla_h\phi'}\right\rangle-\left(\overrightarrow{V_h}'\cdot\overline{\nabla_h\phi'}-\left\langle\overrightarrow{V_h}'\cdot\overline{\nabla_h\phi'}\right\rangle\right)-\left(\overrightarrow{V_h}'\cdot\overline{\nabla_h\left\langle\phi\right\rangle}+\left\langle\overrightarrow{V_h}\right\rangle\cdot\overline{\nabla_h\phi'}\right) \\
&= -\left\langle\overrightarrow{V_h}\right\rangle\cdot\overline{\nabla_h\left\langle\phi\right\rangle}-\overrightarrow{V_h}'\cdot\overline{\nabla_h\phi'}-\left(\overrightarrow{V_h}'\cdot\overline{\nabla_h\left\langle\phi\right\rangle}+\left\langle\overrightarrow{V_h}\right\rangle\cdot\overline{\nabla_h\phi'}\right) \\
&= -\left(\left\langle\overrightarrow{V_h}\right\rangle\cdot\overline{\nabla_h\left\langle\phi\right\rangle}+\left\langle\overrightarrow{V_h}\right\rangle\cdot\overline{\nabla_h\phi'}\right)-\left(\overrightarrow{V_h}'\cdot\overline{\nabla_h\phi'}+\overrightarrow{V_h}'\cdot\overline{\nabla_h\left\langle\phi\right\rangle}\right) \\
&= -\left\langle\overrightarrow{V_h}\right\rangle\cdot\overline{\nabla_h\phi}-\overrightarrow{V_h}'\cdot\overline{\nabla_h\phi} \\
&= -\overrightarrow{V_h}\cdot\overline{\nabla_h\phi}
\end{aligned}
\tag{S11.3}
$$

The sum of contribution to the dissipation of the kinetic energy appearing in $K_M$, $K_E$, $K_{X1}$ and $K_{X2}$ tendency equations is:

$$
\begin{aligned}
&\left\langle\overrightarrow{V_h}\right\rangle\cdot\left\langle\overrightarrow{F_h}\right\rangle+\left\langle\overrightarrow{V_h}'\cdot\overrightarrow{F_h}'\right\rangle+\left(\overrightarrow{V_h}'\cdot\overrightarrow{F_h}'-\left\langle\overrightarrow{V_h}'\cdot\overrightarrow{F_h}'\right\rangle\right)+\left(\overrightarrow{V_h}'\cdot\left\langle\overrightarrow{F_h}\right\rangle+\left\langle\overrightarrow{V_h}\right\rangle\cdot\overrightarrow{F_H}'\right) \\
&= \left\langle\overrightarrow{V_h}\right\rangle\cdot\left\langle\overrightarrow{F_h}\right\rangle+\overrightarrow{V_h}'\cdot\overrightarrow{F_h}'+\overrightarrow{V_h}'\cdot\left\langle\overrightarrow{F_h}\right\rangle+\left\langle\overrightarrow{V_h}\right\rangle\cdot\overrightarrow{F_h}' \\
&= \overrightarrow{V_h}\cdot\overrightarrow{F_h}
\end{aligned}
\tag{S11.4}
$$

Finally, we retrieve all terms present in "*Supplementary material*" (S1.1) and is a confirmation of the robustness of the kinetic energy equations developed in this paper.

**SUPPLEMENTARY MATERIAL 12: Total available potential energy retrieval**

We ensure that the computation of time-mean and time-variability available enthalpy energy budget were correct by summing all their contributions to retrieve the total kinetic energy equation terms obtained in (S3.8).

$$\frac{\partial A_M}{\partial t} = -\begin{bmatrix} -\frac{C_p}{T_r}\left(\left\langle T'\overrightarrow{V'}\right\rangle\cdot\vec{\nabla}\right)\left\langle T\right\rangle \\ +\vec{\nabla}\cdot\left(\left\langle\vec{V}\right\rangle A_M\right) \\ +\vec{\nabla}\cdot\left\langle\overrightarrow{V'}\left(\frac{C_p}{T_r}\left\langle T-T_r\right\rangle T'\right)\right\rangle \end{bmatrix} +\left(\left\langle\omega\right\rangle\left\langle\alpha\right\rangle-\left\langle\omega\right\rangle\alpha_r\right) + \left\langle\left(\frac{T}{T_r}-1\right)\right\rangle\left\langle Q\right\rangle$$

$$\frac{\partial A_E}{\partial t} = -\begin{bmatrix} \frac{C_p}{T_r}\left(\left\langle T'\ \overrightarrow{V'}\right\rangle\cdot\vec{\nabla}\right)\left\langle T\right\rangle \\ +\vec{\nabla}\cdot\left(\left\langle\vec{V}\right\rangle A_E\right) \\ +\vec{\nabla}\cdot\left\langle\overrightarrow{V'}\left(\frac{C_p}{2T_r}T'^2\right)\right\rangle \end{bmatrix} - \left[\left\langle\omega'\alpha'\right\rangle\right] + \left\langle\frac{T'Q'}{T_r}\right\rangle$$

$$\frac{\partial A_{X1}}{\partial t} = -\begin{bmatrix} \vec{\nabla}\cdot\left\{\vec{V}\left(\frac{C_p}{2T_r}T'^2\right)-\left\langle\vec{V}\left(\frac{C_p}{2T_r}T'^2\right)\right\rangle\right\}+ \\ \frac{C_p}{T_r}\left\{\left(T'\overrightarrow{V'}\cdot\vec{\nabla}\right)\left\langle T\right\rangle-\left\langle\left(T'\overrightarrow{V'}\cdot\vec{\nabla}\right)\left\langle T\right\rangle\right\rangle\right\} \\ -\frac{C_p}{T_r}T'\left\langle\left(\overrightarrow{V'}\cdot\vec{\nabla}\right)T'\right\rangle \end{bmatrix} -\left\{\omega'\alpha'-\left\langle\omega'\alpha'\right\rangle\right\} + \frac{1}{T_r}\left\{T'Q'-\left\langle T'Q'\right\rangle\right\}$$

$$\frac{\partial A_{X2}}{\partial t} = -\begin{bmatrix} \vec{\nabla}\cdot\left\{\vec{V}\left(A-\frac{C_p}{2T_r}T'^2\right)-\left\langle\vec{V}\left(A-\frac{C_p}{2T_r}T'^2\right)\right\rangle\right\}- \\ \frac{C_p}{T_r}\left\{\left(T'\overrightarrow{V'}\cdot\vec{\nabla}\right)\left\langle T\right\rangle-\left\langle\left(T'\overrightarrow{V'}\cdot\vec{\nabla}\right)\left\langle T\right\rangle\right\rangle\right\} \\ -\frac{C_p}{T_r}T'\left\langle\left(\overrightarrow{V'}\cdot\vec{\nabla}\right)T'\right\rangle \end{bmatrix} - \begin{bmatrix}\left(\left\langle\omega\right\rangle\alpha'+\omega'\left\langle\alpha\right\rangle\right) \\ -\omega'\alpha_r\end{bmatrix} + \frac{1}{T_r}\left(T'\left\langle Q\right\rangle+\left\langle T-T_r\right\rangle Q'\right)$$

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$$\frac{\partial A}{\partial t} = -\left(\vec{V}\cdot\vec{\nabla}\right)A - \omega\left(\alpha-\alpha_r\right) + \left(\frac{T}{T_r}-1\right)Q \qquad \text{(S12.1)}$$

The sum of conversion and boundary fluxes terms appearing in $A_M$, $A_E$, $A_{X1}$ and $A_{X2}$ tendency equations is:

$$
-\begin{bmatrix} -\frac{C_p}{T_r}\left(\left\langle T'\overrightarrow{V'}\right\rangle\cdot\vec{\nabla}\right)\langle T\rangle \\ +\vec{\nabla}\cdot\left(\left\langle\vec{V}\right\rangle A_M\right) \\ +\vec{\nabla}\cdot\left\langle\overrightarrow{V'}\left(\frac{C_p}{T_r}\langle T-T_r\rangle T'\right)\right\rangle \end{bmatrix} - \begin{bmatrix} \frac{C_p}{T_r}\left(\left\langle T'\ \overrightarrow{V'}\right\rangle\cdot\vec{\nabla}\right)\langle T\rangle \\ +\vec{\nabla}\cdot\left(\left\langle\vec{V}\right\rangle A_E\right) \\ +\vec{\nabla}\cdot\left\langle\overrightarrow{V'}\left(\frac{C_p}{2T_r}T'^2\right)\right\rangle \end{bmatrix}
$$

$$
-\begin{bmatrix} \vec{\nabla}\cdot\left\{\vec{V}\left(\frac{C_p}{2T_r}T'^2\right)-\left\langle\vec{V}\left(\frac{C_p}{2T_r}T'^2\right)\right\rangle\right\}+ \\ \frac{C_p}{T_r}\left\{\left(T'\overrightarrow{V'}\cdot\vec{\nabla}\right)\langle T\rangle-\left\langle\left(T'\overrightarrow{V'}\cdot\vec{\nabla}\right)\langle T\rangle\right\rangle\right\} \\ -\frac{C_p}{T_r}T'\left\langle\left(\overrightarrow{V'}\cdot\vec{\nabla}\right)T'\right\rangle \end{bmatrix} - \begin{bmatrix} \vec{\nabla}\cdot\left\{\vec{V}\left(A-\frac{C_p}{2T_r}T'^2\right)-\left\langle\vec{V}\left(A-\frac{C_p}{2T_r}T'^2\right)\right\rangle\right\}- \\ \frac{C_p}{T_r}\left\{\left(T'\overrightarrow{V'}\cdot\vec{\nabla}\right)\langle T\rangle-\left\langle\left(T'\overrightarrow{V'}\cdot\vec{\nabla}\right)\langle T\rangle\right\rangle\right\} \\ -\frac{C_p}{T_r}T'\left\langle\left(\overrightarrow{V'}\cdot\vec{\nabla}\right)T'\right\rangle \end{bmatrix}
$$

$$
\begin{aligned}
&=-\vec{\nabla}\cdot\left\{\left\langle\vec{V}\right\rangle\langle A\rangle-\left\langle\overrightarrow{V'}\ A_X\right\rangle\right\}-\vec{\nabla}\cdot\left\{\vec{V}\ A-\left\langle\vec{V}\ A\right\rangle\right\} \\
&=-\vec{\nabla}\cdot\left\langle\vec{V}\ A\right\rangle-\vec{\nabla}\cdot\left\{\vec{V}\ A-\left\langle\vec{V}\ A\right\rangle\right\} \\
&=-\vec{\nabla}\cdot\left(\vec{V}\ A\right)
\end{aligned}
$$

(S12.2)

The sum of terms contributing to the diabatic heating in $K_M$, $K_E$, $K_{X1}$ and $K_{X2}$ tendency equations is:

$$
\begin{aligned}
&\left\langle\left(\frac{T}{T_r}-1\right)\right\rangle\langle Q\rangle+\left\langle\frac{T'Q'}{T_r}\right\rangle+\frac{1}{T_r}\left\{T'Q'-\langle T'Q'\rangle\right\}+\frac{1}{T_r}\left(T'\langle Q\rangle+\langle T-T_r\rangle Q'\right) \\
&=\left\langle\left(\frac{T}{T_r}-1\right)\right\rangle\langle Q\rangle+\frac{T'Q'}{T_r}+\frac{1}{T_r}\left(T'\langle Q\rangle+\langle T-T_r\rangle Q'\right) \\
&=\frac{1}{T_r}\left\{\langle T\rangle\langle Q\rangle+T'Q'T'\langle Q\rangle+\langle T_r\rangle Q'\right\}-\left\{\langle Q\rangle+Q'\right\} \qquad \text{(S12.3)} \\
&=\frac{1}{T_r}TQ-Q \\
&=\left(\frac{T}{T_r}-1\right)Q
\end{aligned}
$$

We retrieve all terms present in (S1.8) for the available enthalpy; this confirms the robustness of the available enthalpy equations developed in this paper.

**References**

1. Marquet, P., *On the concept of exergy and available enthalpy: Application to atmospheric energetics.* Quarterly Journal of the Royal Meteorological Society, 1991. **117**(499): p. 449-475.
2. Nikiéma, O. and R. Laprise, *An approximate energy cycle for inter-member variability in ensemble simulations of a regional climate model.* Climate dynamics, 2013. **41**: p. 831-852.