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Review

Jackson R. Herring and the Statistical Closure Problem of Turbulence: A Review of Renormalized Perturbation Theories

David McComb

SUPA School of Physics and Astronomy, University of Edinburgh, Peter Guthrie Tait Road, Edinburgh EH9 3JZ, UK; wdm@ph.ed.ac.uk

Abstract: The pioneering applications of the methods of theoretical physics to the turbulence statistical closure problem are summarised. These are: the direct-interaction approximation (DIA) of Kraichnan, the self-consistent-field theory of Edwards, and the self-consistent-field theory of Herring. Particular attention is given to the latter, in terms of its elegance and its pedagogical value. We then concentrate on the assessment of these theories and take the historical route of Kraichnan's diagnosis of the failure of DIA, followed by Edwards's analysis of the failure of his self-consistent theory, when compared to the Kolmogorov spectrum. As all three theories are closely related, these analyses also shed light on Herring's theory. The second-generation theories that grew out of this assessment are then discussed. First, there were the Lagrangian theories, initially stemming from the work of Kraichnan and Herring, and later the purely Eulerian local energy-transfer (LET) theory. The latter is significant because its development exposes the underlying problems with the pioneering theories in terms of the basic physics of the inertial energy transfer. In particular, later work allows us to assign a unified explanation of the incompatibility of all three pioneering theories with the Kolmororov spectrum, in that they are all Markovian approximations (in wavenumber) to the non-Markovian phenomenon of fluid turbulence. In the interests of completeness, we briefly review the formalisms of Wyld and Martin, Siggia, and Rose. More recent developments are also discussed, in order to bring the subject up to the present day.

Keywords: isotropic turbulence; statistical closure approximations; renormalization

Citation: McComb, D. Jackson R. Herring and the Statistical Closure Problem of Turbulence: A Review of Renormalized Perturbation Theories. *Atmosphere* **2023**, *14*, 827. https://doi.org/10.3390/atmos14050827

Academic Editors: Boris Galperin, Annick Pouquet, Peter Sullivan and Paul D. Williams

Received: 15 March 2023
Accepted: 28 April 2023
Published: 4 May 2023

1. Introductory Remarks

In the 1960s, Jack Herring was one of the giants of turbulence theory. The other two were Bob Kraichnan and Sam Edwards. Their revolutionary contributions transformed the study of turbulence, taking it into the realm of theoretical physics. In chronological order, Kraichnan published his direct-interaction approximation (DIA) in 1959 [1] and followed this in 1964 with a numerical computation of the DIA equations for the two-point, two-time covariance and the renormalised response function, for freely decaying turbulence [2]. Edwards derived a form of the Liouville equation for the turbulence probability distribution functional (pdf) and obtained a self-consistent-field solution for the stationary, single-time convariance in 1964 [3] and followed this with some further remarks about his theory [4]. His approach was strongly influenced by a consideration of Brownian motion, but in 1965 Herring, using a much more abstract form of self-consistent-field theory, solved the Edwards–Liouville equation for the probability distribution for the stationary case [5], and extended this to the non-stationary case the following year [6].

When I began my postgraduate research, under the supervision of Sam Edwards in October 1966, the initial excitement about the new theories was tending to die down. Important points had been established about the relationships between the three approaches, but there were still some unanswered questions. It was clear that all three theories led to the same energy-balance equation, but there were differences between their results for the renormalized response function. Attempting to understand these differences was one of

the active topics of the time. However, the dominant problem was to understand why none of the three theories was compatible with the Kolmogorov–Obukhov'−5/3' wavenumber spectrum [7]. This aspect dominated the subject to such an extent that the 1964 computation of the DIA, which suggested that these theories were capable of making good qualitative and quantitative predictions about turbulence, tended to be overlooked. Indeed, my own first project was to rectify the Edwards theory by choosing the renormalized response such that the turbulent entropy was maximised. This did lead to a result which was compatible with the Kolmogorov–Obukhov spectrum, but the qualitative aspects were not satisfactory [8].

In the rest of this article, we shall review the subject in detail, discussing its failures and successes, and indicate how it has developed up until the present day. However, before finishing these preliminary remarks, I should point out two other contributions by Jack Herring. First, he introduced the use of negative damping to force the turbulence, rather than using stirring forces. Many years on, this is an important method of forcing numerical simulations of the Navier–Stokes equations. Secondly, he invented a simplified notation, combining wavenumber and tensor index labels, which has been used by others working in this field. For instance, Leslie made extensive use of it in his book [9], Qian used it in his work on closures [10] (see also the article by Shi [11]), and it is also used in Section 5.3 of the present work.

2. A Concise Overview of Statistical Closures

The statistical closure problem was first formulated for turbulent shear flows by Reynolds in the 1890s. He showed that the equation for the mean velocity $\bar{U}$ contains the unknown covariance of two fluctuating velocities $\langle uu \rangle$: the Reynolds stress. It was later formulated for isotropic turbulence by Taylor in the 1930s. In this case, the mean velocity is taken to be zero and the equation for the correlation of two velocities $\langle uu \rangle$ is found to contain the unknown correlation of three velocities $\langle uuu \rangle$. The equation for $\langle uuu \rangle$ contains the unknown fourth-order moment $\langle uuuu \rangle$, and so on. The problem is therefore seen as an open hierarchy of statistical equations (one-point for Reynolds, two-point for Taylor) which requires some means of closure. Theories used in engineering applications, such as the eddy-viscosity, mixing-length, and n-equation models, are all effectively statistical closure approximations for the single-point problem. In contrast, the Heisenberg eddy-viscosity and the quasi-normality theory were effectively closure approximations for the two-point problem. Further reading on this topic can be found in the following books [9,12–14].

The first *formal* treatment of the closure problem was the quasi-normality hypothesis (Proudman and Reid [15], Tatsumi [16]). The basic idea was to solve the next equation in the hierarchy for $\langle uuu \rangle$ by factorizing $\langle uuuu \rangle$ in terms of $\langle uu \rangle \times \langle uu \rangle$. This then gives a closed set of equations for the covariance. However, quasi-normality failed when computed numerically in the 1960s, as it predicted negative spectra: it was not *physically realizable* [17,18].

The attempt at quasi-normality illustrates an important aspect of all statistical theories of turbulence and indeed of all statistical field theories. In *all* theories, the averages are evaluated using the tractable properties of the Gaussian or normal distribution. In particular, even-order moments can be factorised into products of pair-correlations while odd-order moments cannot be evaluated that way, because they vanish identically by symmetry. In quasi-normality the fourth-order moment is evaluated in terms of pair correlations, whereas the triple moment is calculated from the resulting closed equation by inverting the linear operator of the Navier–Stokes equation. *Some such manoeuvre is needed in any theory if it is to be successful, with the degree of success depending on the choice of manoeuvre.*

2.1. The Pioneering Renormalized Perturbation Theories of Turbulence

The beginning of the modern age of turbulence theory was signalled by the publication in 1959 of Kraichnan's direct-interaction approximation (DIA) [1]. This was both a mean-field theory and a new kind of perturbation theory (it is an example of a renormalized perturbation theory, such as occurs in quantum field theory or many-body problems).

At the time, its most important feature was seen as being its physical realizability. It did not have the catastrophic behaviour of quasi-normality. Shortly afterwards, Wyld [19] showed that the DIA could be recovered from a conventional renormalized perturbation theory, thus establishing the validity of the mean-field assumption by Kraichnan. This work was couched in the language (and used the techniques) of quantum field theory, including the introduction of diagrams analogous to Feynmann diagrams. Although of considerable pedagogical interest, it contained some procedural errors. A discussion of this aspect may be found in Appendix A.

In 1964, Edwards published his self-consistent-field (SCF) theory [3], which he also described as a *random phase approximation*, and which was based on the probability distribution functional (pdf) of the fluctuating velocities. He began by introducing a model system with a Gaussian pdf and the same velocity covariance as the turbulence system. Then, he introduced a perturbation expansion in terms of the difference between the model system and the real system. The result was a set of equations for the single-time covariance and a renormalized response function. This result was later shown by Kraichnan to be cognate to the DIA theory and, like it, maintained the properties of the inertial term, such as conservation of energy [20].

Another attempt at a self-consistent-field theory was made by Herring [5] in 1965. This was a more abstract approach than that of Edwards, although it also started from the Edwards form of the Liouville equation. In strategy it lay somewhere between the Edwards SCF and the Kraichnan DIA, in that it formally renormalized a 'bare' operator. Herring's basic approach involved the introduction of single-mode projectors and single-mode distributions. Then, the perturbation expansion was in terms of the difference between the approximate single-mode forms and the exact coupled forms. There are similarities and differences between the two variants of SCF, and they both lead to the same form of covariance equation. However, the response function in Herring's theory was the same as the time-independent DIA form, and hence differed from that of Edwards. In the following year, Herring was able to extend his SCF theory to the two-time case [6], the result being a theory which was closely related to Kraichnan's DIA, rather than the Edwards theory.

2.2. The Later Renormalized Perturbation Theories of Turbulence

As early as 1961, Kraichnan realized that the experimental results for turbulence spectra (later published as [21]) supported the Obhukov–Kolmogorov prediction of $k^{-5/3}$, rather than his DIA prediction of $k^{-3/2}$. He began work on this and ultimately concluded that the DIA type of theory could not predict the time dependence of two-time correlations due to a violation of what he called *random Galilean invariance*. This led Kraichnan to formulate his theories in a mixed Eulerian–Lagrangian coordinate system [22]. This was not entirely satisfactory, and Herring and Kraichnan extended it to a theory based on the rate of strain, rather than on the velocity field [23]. This inspired a variety of Lagrangian renormalized perturbation theories, continuing up to the present time: [24–26]. We discuss these theories further in Section 6.1.

This was not seen as a problem for single-time correlations by Kraichnan, as such theories were invariant under the postulated random Galilean transformation. However, it should be recalled that single-time theories, such as those of Edwards [3] and Herring [5], are also incompatible with the Kolmogorov spectrum, as indeed also is the cognate single-time form of the DIA. The resolution of this problem was found in 1974, in the form of the local energy-transfer (LET) theory, and this is something that we discuss in greater detail in Section 6.2.

General background reading can be found in the books by Beran [12], Leslie [9], Monin and Yaglom [13], McComb [14], Sagaut and Cambon [27], McComb [28], Verma [29], and the review by McComb [30].

3. Basic Equations and Notation

In this review we follow a modern unified scheme of notation and employ the letter C for correlations and covariances, along with the letter R for response functions.

We consider the solenoidal Navier–Stokes equation (NSE) for the velocity field $u_\alpha(\mathbf{k},t)$ in wavenumber (k) space (see either [14] or [28]) as:

$$\left(\frac{\partial}{\partial t}+\nu k^2\right)u_\alpha(\mathbf{k},t)=M_{\alpha\beta\gamma}(\mathbf{k})\int d^3j u_\beta(\mathbf{j},t)\;ug(\mathbf{k}-\mathbf{j},t)+f_\alpha(\mathbf{k},t), \tag{1}$$

where ν is the fluid kinematic viscosity, $f_\alpha(\mathbf{k},t)$ is an arbitrarily chosen stirring force which we have to specify. Note that we follow Edwards [3] (and indeed Kolmogorov before him [31]) and use Greek letters to denote the usual Cartesian tensor indices, where these take the values 1, 2, or 3, as appropriate for a three-dimensional space. There should be no confusion with the conventional use of Greek indices in Minkowski four-space. The inertial transfer operator $M_{\alpha\beta\gamma}(\mathbf{k})$ is given by

$$M_{\alpha\beta\gamma}(\mathbf{k})=(2i)^{-1}\left[k_\beta P_{\alpha\gamma}(\mathbf{k})+k_\gamma P_{\alpha\beta}(\mathbf{k})\right], \tag{2}$$

where $i=\sqrt{-1}$, while the projector $P_{\alpha\beta}(\mathbf{k})$ is expressed in terms of the Kronecker delta as

$$P_{\alpha\beta}(\mathbf{k})=\delta_{\alpha\beta}-\frac{k_\alpha k_\beta}{k^2}. \tag{3}$$

Note that the use of the projector ensures that the velocity field remains solenoidal.

The covariance of the fluctuating velocity field may be introduced as

$$C_{\alpha\beta}(\mathbf{k},\mathbf{k}';t,t')=\langle u_\alpha(\mathbf{k},t)u_\beta(\mathbf{k}',t')\rangle, \tag{4}$$

and for isotropic, homogeneous turbulence we may write this as

$$\langle u_\alpha(\mathbf{k},t)u_\beta(-\mathbf{k},t')\rangle=P_{\alpha\beta}(\mathbf{k})\delta(\mathbf{k}+\mathbf{k}')C(k;t,t'), \tag{5}$$

where we anticipate the effects of homogeneity in writing the left-hand side. As is usual, the angle brackets $\langle\ldots\rangle$ denote the operation of taking an average. We shall discuss this further in the next sub-section, in connection with the introduction of the probability distribution functional.

If we consider the case $t=t'$, then we may introduce the spectral density function:

$$C(k,t)\equiv C(k;t,t), \tag{6}$$

and the energy spectrum:

$$E(k,t)=4\pi k^2 C(k,t). \tag{7}$$

We will also find it convenient to introduce the contracted notation:

$$C_k\equiv C(k), \tag{8}$$

for use with later time-independent theories.

4. The Statistical Formulation

4.1. The Stirring Forces

Edwards [3] introduced stirring forces in order to define the ensemble (and, when required, to study stationary turbulence). These are denoted by $f_\alpha(\mathbf{k},t)$ and can be added to the right-hand side of the equation of motion, as we have done here in Equation (1). These forces are chosen to be isotropic, homogeneous and (in order to maintain incompressibility) solenoidal. He considered random forces with a multivariate normal probability distribution, such that the associated functional integrals were analytically tractable.

He assumed that the autocorrelation of the forces was instantaneous in time and we represent this by choosing the time autocorrelation to be a delta function. A form of correlation which satisfies all these requirements is:

$$\left\langle f_\alpha(\mathbf{k},t) f_\beta(\mathbf{k}',t') \right\rangle = P_{\alpha\beta}(\mathbf{k}) F(k) \delta(\mathbf{k}+\mathbf{k}') \delta(t-t'). \tag{9}$$

Here, $F(k)$ is a spectral energy density which is related to the rate at which the force does work on the fluid. The rate at which the stirring forces do work on the fluid can be shown to take the form [3,32]:

$$\left\langle f_\alpha(\mathbf{k},t) u_\sigma(\mathbf{k}',t) \right\rangle = P_{\alpha\sigma}(\mathbf{k}) F(k) \delta(\mathbf{k}+\mathbf{k}'). \tag{10}$$

We may also introduce the energy-injection spectrum of the stirring forces $W(k)$ as:

$$W(k) = 4\pi k^2 F(k), \tag{11}$$

with associated rate of doing work ε_W, where:

$$\varepsilon_W = \int_0^\infty W(k) dk. \tag{12}$$

For stationary flows ε_W must be equal to the dissipation rate ε.

4.2. The Closure Problem

An equation for the covariance $C(k;t,t')$ may be obtained in the usual way. Multiply each term in Equation (1) by $u_\alpha(-\mathbf{k},t')$ and take the average, to obtain:

$$\left(\frac{\partial}{\partial t} + \nu k^2\right) C(k;t,t') = \tfrac{1}{2} M_{\alpha\beta\gamma}(\mathbf{k}) \int d^3 j \langle u_\beta(\mathbf{j},t) u_\gamma(\mathbf{k}-\mathbf{j},t) u_\alpha(-\mathbf{k},t') \rangle, \tag{13}$$

where we have used Equations (3) and (5), cancelled the factor $\delta(\mathbf{k}+\mathbf{k}')$ across, and invoked isotropy, along with the property $Tr\, P_{\alpha\beta}(\mathbf{k}) = 2$. We can also write this in the compact form:

$$\left(\frac{\partial}{\partial t} + \nu k^2\right) C(k;t,t') = P(k;t,t'), \tag{14}$$

where $P(k;t,t')$ is just the right-hand side of Equation (13). The problem of expressing this in terms of the covariance is the well-known statistical closure problem.

Further, using Equation (1), we can obtain an equation describing the energy balance. On the time diagonal, the covariance equation takes the form for isotropic turbulence:

$$\begin{aligned} \left(\frac{\partial}{\partial t} + 2\nu k^2\right) C(k,t) &= Re\left[M_{\alpha\beta\gamma}(\mathbf{k}) \int \mathrm{d}^3 j \langle u_\beta(\mathbf{j},t) u_\gamma(\mathbf{k}-\mathbf{j},t) u_\alpha(-\mathbf{k},t) \rangle \right] \\ &+ F(k), \end{aligned} \tag{15}$$

where $F(k)$ is the energy-injection spectral density, as defined in terms of the stirring forces by Equation (10). Again, we can write this in a more compact form as:

$$\left(\frac{\partial}{\partial t} + 2\nu k^2\right) C(k;t,t) = Q(k;t,t) + F(k), \tag{16}$$

where $Q(k;t,t)$ can be determined by comparison with the right-hand side of Equation (15).

The energy spectrum is related to the spectral density by Equation (7). Accordingly, if we multiply Equation (15) across by $4\pi k^2$, and rearrange terms, the governing equation of the energy spectrum takes the form:

$$\frac{\partial E(k,t)}{\partial t} = W(k) + T(k,t) - 2\nu k^2 E(k,t), \tag{17}$$

where the energy-transfer spectrum is $T(k,t) = 4\pi k^2 Q(k;t,t) \equiv 4\pi k^2 Q(k,t)$. This equation for the energy spectrum is well known, and is nowadays sometimes referred to as the *Lin equation* [27,28]. More detailed discussions of these equations can be found in Chapter 3 of the book [28].

5. The Pioneering Theories of Kraichnan, Edwards, and Herring

These theories are presented in chronological order of their first publication.

5.1. The Direct-Interaction Approximation of Kraichnan

The starting point for Kraichnan was the introduction of the infinitesimal response function $\widehat{R}_{\alpha\beta}(\mathbf{k},t)$, which arises when one assumes that a fluctuation in the stirring forces. thus:

$$f_\alpha(\mathbf{k},t) \to f_\alpha(\mathbf{k},t) + \delta f_\alpha(\mathbf{k},t), \tag{18}$$

leads to a corresponding fluctuation in the velocity field, thus:

$$u_\alpha(\mathbf{k},t) \to u_\alpha(\mathbf{k},t) + \delta u_\alpha(\mathbf{k},t). \tag{19}$$

The infinitesimal response function is then defined by the relationship:

$$\delta u_\alpha(\mathbf{k},t) = \int_{-\infty}^{t} \widehat{R}_{\alpha\beta}(\mathbf{k};t,t')\delta f_\beta(\mathbf{k},t')dt', \tag{20}$$

and its governing equation is obtained by functional differentiation of the Navier–Stokes equation.

Kraichnan obtained his theory by solving this governing equation simultaneously with the NSE, using perturbation theory. It was pointed out by Leslie (see page 56 of [9]) that the DIA was obtained by assuming that $\widehat{R}$ and u are uncorrelated, hence taking the step:

$$\langle \widehat{R}(t,t')u(t)u(t')\rangle = \langle \widehat{R}(t,t')\rangle\langle u(t)u(t')\rangle = R(t,t')\langle u(t)u(t')\rangle, \tag{21}$$

thus, introducing the ensemble-averaged response function $R(t,t') = \langle \widehat{R}(t,t')\rangle$.

This makes the DIA a form of mean-field theory, in that we replace the instantaneous response function by its average value. In fact, Kraichnan's result can be derived directly from conventional renormalized perturbation theory, as given by Wyld [19] (also see page 228 of [14]), thus validating the mean-field assumption. Applying these methods to the equations for the response function and the velocity covariance, leads to the DIA theory.

The governing equation for the response function is found to be,

$$\begin{aligned} &\left[\frac{\partial}{\partial t} + \nu k^2\right] R(k;t,t') \\ &+ \int d^3j L(\mathbf{k},\mathbf{j}) \int_{t'}^{t} dt'' R(k;t'',t')R(j;t,t'')C(|\mathbf{k}-\mathbf{j}|;t,t'') \\ &= \delta(t-t'). \end{aligned} \tag{22}$$

The second-order equation for the velocity covariance is:

$$\begin{aligned} &\left[\frac{\partial}{\partial t} + \nu k^2\right] C(k;t,t') \\ &= \int d^3j L(\mathbf{k},\mathbf{j}) \left[\int_0^{t'} ds R(k;t',s)C(j;t,s)C(|\mathbf{k}-\mathbf{j}|;t,s)\right. \\ &- \left.\int_0^{t} ds R(j;t,s)C(k;s,t')C(|\mathbf{k}-\mathbf{j}|;t,s)\right]; \end{aligned} \tag{23}$$

and on the time diagonal:

$$\left(\frac{\partial}{\partial t}+2\nu k^2\right)C(k,t)$$
$$=2\int d^3jL(\mathbf{k},\mathbf{j})\int_0^t dsR(k;t,s)R(j;t,s)R(|\mathbf{k}-\mathbf{j}|;t,s)\times$$
$$[C(j,s)C(|\mathbf{k}-\mathbf{j}|,s)-C(k,s)C(|\mathbf{k}-\mathbf{j}|,s)]. \quad (24)$$

The coefficient $L(\mathbf{k},\mathbf{j})$ is defined as:

$$L(\mathbf{k},\mathbf{j})=-2M_{\alpha\beta\gamma}(\mathbf{k})M_{\beta\alpha\delta}(\mathbf{j})P_{\gamma\delta}(\mathbf{k}-\mathbf{j}), \quad (25)$$

and can be evaluated as:

$$L(\mathbf{k},\mathbf{j})=-\frac{\left[\mu\left(k^2+j^2\right)-kj\left(1+2\mu^2\right)\right]\left(1-\mu^2\right)kj}{k^2+j^2-2kj\mu}, \quad (26)$$

where μ is the cosine of the angle between the vectors $\mathbf{k}$ and $\mathbf{j}$. It should be noted that these equations have been rendered in the formalism due to Edwards, rather than that of Kraichnan. For a discussion of this point, see the book [28].

These equations are an exact second-order truncation of the renormalized perturbation theory. Their derivation can rely on either the topology of Feynmann-type diagrams or reversion of power series. They satisfy all the required symmetries and in particular the closure conserves energy, displaying the correct antisymmetric behaviour of the transfer spectrum.

They can predict the free decay of isotropic turbulence, without invoking arbitrary constants, in quite good agreement with experiment and numerical simulation.

5.2. The Self-Consistent-Field (SCF) Theory of Edwards

Edwards used the prescription of the stirring forces to define a turbulence ensemble and made this the basis of a statistical mechanics approach. Taking these forces to have a multivariate normal distribution, and the properties shown in Equation (9), he obtained the mean distribution of the velocities, P, by averaging against the distribution of the forces and in this way obtained a generalization of the Liouville equation to the turbulent case, as:

$$\frac{\partial P}{\partial t}=-\sum_{\mathbf{k}}\frac{\partial}{\partial u_\alpha(k)}\Big\{-\nu k^2u_\alpha(k)+$$
$$+\sum_{\mathbf{j}}M_{\alpha\beta\gamma}(\mathbf{k})u_\beta(\mathbf{j})u_\gamma)(\mathbf{k}-\mathbf{j})-F(k)\frac{\partial}{\partial u_\alpha(-\mathbf{k})}\Big\}P, \quad (27)$$

where $F(k)$ is the force spectrum given by Equation (10). Although the non-linearity of the velocity field is incorporated, it is worth bearing in mind the fact that this equation is a linear equation for P and therefore its perturbation treatment is quite a different matter from that of the Navier–Stokes equation as discussed in the preceding section.

Edwards considered the stationary case, so we drop the time derivative in Equation (27) and, for later convenience, rearrange it as:

$$\sum_{\mathbf{k}}\frac{\partial}{\partial u_\alpha(k)}\left\{\nu k^2u_\alpha(k)+F(k)\frac{\partial}{\partial u_\alpha(-\mathbf{k})}\right\}P=$$
$$-\sum_{\mathbf{k}}\sum_{\mathbf{j}}M_{\alpha\beta\gamma}(\mathbf{k})u_\beta(\mathbf{j})u_\gamma)(\mathbf{k}-\mathbf{j})\frac{\partial P}{\partial u_\alpha(k)}. \quad (28)$$

In order to explain the perturbation approach, we introduce a more symbolic notation and write Equation (28) as:

$$LP-VP=0, \quad (29)$$

where the detailed forms of the operators L and V can be deduced from comparison with the above equation. In doing this, we are introducing the more symbolic notation of Herring [5], and this will also be useful when we discuss Herring's actual theory in the next section.

Edwards then sought a solution of Equation (29) in the form of an expansion in a bookkeeping parameter λ, which is put equal to one at the end of the calculation, and is used to order terms in an iterative procedure. For this purpose it is taken to be superficially of the same order as the non-linear terms in the NSE. Thus, expanding the probability distribution P as follows:

$$P = P_0 + \lambda P_1 + \lambda^2 P_2 + \ldots, \tag{30}$$

where the zero-order coefficient satisfies

$$L_0 P_0, \tag{31}$$

while the basis operator L_0 remains to be determined.

This is the key step in the Edwards theory and we will consider it presently. First, we introduce the general procedure, which begins by adding and subtracting $L_0 P$ from Equation (29) to obtain

$$L_0 P - VP - (L_0 - L)P = 0. \tag{32}$$

Further, we can associate the first order in λ with the term VP, as it contains the non-linearity of the NSE, second order in λ with $(L_0 - L)P$, as it represents the correction to the perturbation procedure. Hence, we can base the perturbation theory on Equation (32), rewritten as

$$L_0 P - \lambda VP - \lambda^2 (L_0 - L)P = 0, \tag{33}$$

where these assignments can later be shown to be self-consistent.

Now, we substitute the expansion Equation (30) into Equation (33) and equate coefficients of powers of λ. The result at zero order is just Equation (31), while at first order we have

$$L_0 P_1 = VP_0, \tag{34}$$

and at second order

$$L_0 P_2 = VP_1 + (L_0 - L)P_0; \tag{35}$$

and so on, to any order.

Logically, the next step is to decide on a form for the zero-order operator L_0, but in fact it is more intuitive to decide on the zero-order distribution first, and to choose this to be

$$P_0 = N \exp\left\{ -\sum_{\mathbf{k}} \frac{u_\alpha(\mathbf{k}) u_\alpha(-\mathbf{k})}{C(k)} \right\}, \tag{36}$$

where N is the normalization and $C(k)$ is the time-independent covariance of the velocity field. This form of P_0 is chosen to be Gaussian, because we need it to be mathematically tractable. It also means that the zero-order distribution is required to give the *exact* covariance. As formally the exact distribution leads to the exact covariance, we note that P_1 gives zero by symmetry and hence P_2 is required to give zero for self-consistency.

Edwards was guided by the statistical theory of Brownian motion. For instance, he saw the stirring forces as being analogous to the force due to random molecular impacts on the colloidal particles. Accordingly, he assumed that the force spectrum $F(k)$ and the modal decay rate νk^2 could be renormalized by writing:

$$d(k) = F(k) + s(k); \qquad \text{and} \qquad \omega(k) = \nu k^2 + r(k), \tag{37}$$

where the functions $s(k)$ and $r(k)$ remain to be determined; but must satisfy the condition:

$$2\omega(k) C(k) = d(k), \tag{38}$$

for the form Equation (36) for P_0 to hold. Correspondingly, the basis operator must take the Fokker–Planck form:

$$L_0 = \sum_{\mathbf{k}} \frac{\partial}{\partial u_\alpha(k)} \left\{ \omega(k) u_\alpha(k) + d(k) \frac{\partial}{\partial u_\alpha(-\mathbf{k})} \right\} P, \tag{39}$$

where the renormalized quantities $\omega(k)$ and $d(k)$ are the dynamical friction and generalized diffusivity, respectively.

In order to invert the basis operator, Edwards used an eigenfunction expansion, but we will not pursue that here. Details may be found in the original reference [3], and in the books [9,12,14]. The resulting equation for the stationary spectral density function $C(k)$ is

$$F(k) - 2\nu k^2 = -2 \int dj^3 L(\mathbf{k}, \mathbf{j}) \frac{C(\mathbf{k}-\mathbf{j})\{C(j) - C(k)\}}{\omega(k) + \omega(j) + \omega(\mathbf{k}-\mathbf{j})}. \tag{40}$$

The response equation is:

$$\omega(k) = \nu k^2 + \int dj^3 L(\mathbf{k}, \mathbf{j}) \frac{C(\mathbf{k}-\mathbf{j})}{\omega(k) + \omega(j) + \omega(\mathbf{k}-\mathbf{j})}, \tag{41}$$

and these two equations are analogous to Equations (24) and (23) of Kraichnan's DIA, in that order.

In order to complete this presentation of the Edwards theory, we specify the generalized diffusivity $s(k)$ from the constraint (38), thus:

$$s(k) = 2 \int dj^3 L(\mathbf{k}, \mathbf{j}) \frac{C(\mathbf{k}-\mathbf{j}) C(j)}{\omega(k) + \omega(j) + \omega(\mathbf{k}-\mathbf{j})}. \tag{42}$$

Equivalence of the Edwards Theory and Kraichnan's DIA

As we have noted, Edwards derived his closure approximation by inverting the Fokker–Planck form of L_0 in terms of an eigenfunction expansion, with the inverse modal lifetimes appearing as eigenvalues. However, instead of doing this, it is a simple matter to recover the Edwards SCF equation for $C(k)$ directly from the DIA Equation (24).

Our starting point is the recognition by Kraichnan in 1964 [20] that the DIA and Edwards covariance equations were equivalent if one assumed the same exponential time dependence for the covariance and the response function. That is,

$$C(k; t - t') = C(k) \exp[\omega(k)|t - t'|], \tag{43}$$

and

$$\begin{aligned} R(k; t - t') &= \exp[\omega(k)(t - t')], \qquad \text{for} \quad t' < t; \\ &= 0, \qquad \text{for} \quad t' > t, \end{aligned} \tag{44}$$

where $\omega(k)$ is the inverse lifetime for correlation and response of mode k. Its prescription is the second part of the Edwards closure approximation. Later, Kraichnan showed that the Edwards theory could be extended to statistically non-stationary states [33], while Herring and Kraichnan [34] demonstrated that the generalized Edwards covariance equation could be obtained from Equation (24), with $C(k,t) \equiv C(k;t,t)$, as:

$$\begin{aligned} \left(\frac{\partial}{\partial t} + 2\nu k^2 \right) C(k,t) = 2 \int d^3 j \, L(\mathbf{k}, \mathbf{j}) D(k, j, |\mathbf{k}-\mathbf{j}|; t) \times \\ \times [C(j,t) C(|\mathbf{k}-\mathbf{j}|, t) - C(k,t) C(|\mathbf{k}-\mathbf{j}|, t)], \end{aligned} \tag{45}$$

where they introduced a memory function D, given by

$$D(k, j, |\mathbf{k}-\mathbf{j}|; t) = \int_0^t ds\, R(k; t, s) R(j; t, s) R(|\mathbf{k}-\mathbf{j}|; t, s). \tag{46}$$

For the stationary case, this could be written [20]:

$$D(k, j, |\mathbf{k}-\mathbf{j}|) = \frac{1}{\omega(k) + \omega(j) + \omega(|\mathbf{k}-\mathbf{j}|)}. \tag{47}$$

However, when this procedure is applied to the DIA response Equation (22), the result is:

$$\omega(k) = \nu k^2 + \int dj^3 L(\mathbf{k}, \mathbf{j}) \frac{C(\mathbf{k}-\mathbf{j})}{\omega(j) + \omega(\mathbf{k}-\mathbf{j})}. \tag{48}$$

We see that the DIA form differs from the Edwards form, in that only two lifetimes appear in the denominator. That is, the labelling wavenumber is absent.

5.3. The Self-Consistent-Field (SCF) Theory of Herring

In 1965, Herring introduced a more abstract form of self-consistent-field theory. He took as his starting point the Liouville equation derived by Edwards. In order to discuss this self-consistent-field theory, we shall change our notation to that of Herring and rewrite Equation (29) as

$$L^{(0)} P - V P = 0, \tag{49}$$

where the change from L to $L^{(0)}$ is significant and reflects the fact that we shall be looking at a formal renormalization programme in which $L^{(0)}$ is the bare or viscous operator. In this sense, Herring's SCF will be seen to resemble DIA rather than the Edwards SCF, although there are in fact many procedural similarities between the theories of Herring and Edwards. Herring began by assuming that the basis operator for his expansion method was the sum of single-mode operators. The effect of mode coupling was then taken into account through an expansion in single-mode interactions. In this respect, it can be seen as less phenomenological than the Edwards form, in which the renormalized single-mode (i.e., quasi-particle) form is assumed from the outset.

We begin with some definitions relating to single-mode forms. First, we can generalize the concept of reduced distributions from particle statistical mechanics (e.g., see Section 4.1 of the book [14] or the relevant section of [35]) to single-mode distributions in the turbulent case. Let us write the the distribution of velocities for turbulence in $\mathbf{k}$ -space as

$$P[\mathbf{u}(\mathbf{k})] = \lim_{n\to\infty} P_n[\mathbf{u}(\mathbf{k}_1), \mathbf{u}(\mathbf{k}_2), \ldots, \mathbf{u}(\mathbf{k}_n)], \tag{50}$$

where the limit of infinite space volume (corresponding to $n \to \infty$) will not be taken until the end of the calculation. Here, the notation $\mathbf{k}_n$ is shorthand for the usual definition of a wavevector in a finite system. A particular value of n then corresponds to a particular set, e.g., n_1, n_2, n_3. In order to obtain a single-mode distribution, we integrate out all variables except (say) $\mathbf{u}(\mathbf{k}_n)$; thus

$$P_n[\mathbf{u}(\mathbf{k}_n)] = \int P[\mathbf{u}(\mathbf{k})] d\mathbf{u}(\mathbf{k}_1) d\mathbf{u}(\mathbf{k}_2) \ldots d\mathbf{u}(\mathbf{k}_{n-1}) d\mathbf{u}(\mathbf{k}_{n+1}) \ldots, \tag{51}$$

which defines the single-mode distribution $P_n[\mathbf{u}(\mathbf{k}_n)]$. Now, define the single-mode projection of an operator A by

$$\langle A \rangle = \sum_n \langle A \rangle_n, \tag{52}$$

where

$$\langle A \rangle_n = \int A \prod_{m\neq n} P[\mathbf{u}(\mathbf{k}_m)] d\mathbf{u}(\mathbf{k}_1) d\mathbf{u}(\mathbf{k}_2) \ldots d\mathbf{u}(\mathbf{k}_{n-1}) d\mathbf{u}(\mathbf{k}_{n+1}) \ldots. \tag{53}$$

Note that the notation $\langle \dots \rangle$ can be used here for this specialized purpose, as in this section there should be no confusion with its more usual meaning of ensemble average. Then, a single-mode operator can be defined as one which is invariant under the above averaging process, or one which satisfies the relation

$$\langle A \rangle = A. \tag{54}$$

Now, consider Equation (49). If $V = 0$, then it is clear that there will be no non-linear mixing and the solution P will be a product of single-mode distributions. However, if V is not zero, then non-linear mixing must result in some mode coupling and we should write the general solution as

$$P = \prod_n P_n + R, \tag{55}$$

where P_n is the exact single-mode distributions and R is a remainder term to take account of the mode couplings induced by V. It follows that R must satisfy the condition

$$\langle R \rangle_n = 0, \tag{56}$$

where the operation $\langle \dots \rangle_n$ is defined by Equation (53) and the general perturbation method then consists of expanding R about $V = 0$. In order to do this, it is convenient to introduce a new operator L, which should satisfy two requirements. First, it should be of single-mode form, or

$$\langle L \rangle = \sum_n L_n = L, \tag{57}$$

and second, in order to conserve probability, it should satisfy

$$\int L_n F[\mathbf{u}(\mathbf{k}_n)] d\mathbf{u}(\mathbf{k}_n) = 0 \tag{58}$$

for any well-behaved test function $F[\mathbf{u}(k_n)]$. We also introduce the associated function P', which is the solution of

$$LP' = 0, \tag{59}$$

such that

$$P' = \prod_n P'_n. \tag{60}$$

In other words, P' is the product of single-mode operators. Lastly, as V tends to zero, consistency requires that $L \to L_0$ and $P' \to P$. We now add the operator L to both sides of Equation (49), thereby leaving it unchanged, to obtain

$$LP = (L - L_0 + V)P. \tag{61}$$

The formal solution of this equation can be written as

$$P = P' + R', \tag{62}$$

and then we must solve Equation (61) perturbatively for R' in terms of L, L_0, V, and P_n. Our aim is now to find an operator L which (a) satisfies Equations (57)–(59) and (b) is such that $P'_n = P_n$: this is the self-consistency criterion. Our procedure is as follows. Noting Equation (59), we rewrite Equation (61) as

$$LP = LP' + (\Delta + V)P, \tag{63}$$

where

$$\Delta = L - L_0. \tag{64}$$

Then, inverting the operator on the left-hand side, we obtain

$$P = P' + L^{-1}(\Delta + V)P. \tag{65}$$

At this stage, there are two points worth mentioning. First, there is a technical point here which we are glossing over, in that we should invert the non-singular part of L: see Herring (1965) for a discussion. Second, comparison of Equations (65) and (62) indicates that

$$R' = L^{-1}(\Delta + V)P. \tag{66}$$

Our next step is to solve Equation (65) iteratively. As this is of pedagogic interest, we do this in detail in the next section. The result is:

$$P = P' + L^{-1}TP', \tag{67}$$

where T is given by

$$T = (L - L_0 + V)) + (L - L_0 + V)L^{-1}T. \tag{68}$$

Now, the probability distribution given by Equation (67) is in the following form: 'guessed probability + corrections'. For self-consistency we require that the effect of corrections will vanish, such that $P'_n = P_n$. This implies—along with Equation (55)—that we must have

$$\langle P \rangle_n = \langle \prod_n P_n \rangle + \langle R \rangle_n = \Pi_n P_n \tag{69}$$

where we have used Equations (54) and (56). In addition, from Equations (54) and (60), we note that

$$\langle P' \rangle_n = \prod_n P'_n. \tag{70}$$

Hence, if we operate on both sides of Equation 67 with the single-mode expectation value, we have

$$\prod_n P_n = \prod_n P'_n + \langle L^{-1}TP' \rangle_n \tag{71}$$

and, from the self-consistency condition,

$$\langle L^{-1}TP' \rangle_n = 0 \tag{72}$$

or

$$\langle L^{-1}T \rangle_n P' = 0 \tag{73}$$

as P' is the product of single-mode distributions and therefore is unaffected by the operation $\langle \dots \rangle_n$. This result can also be written as

$$[L_n - L_{0,n} + \langle VL^{-1}T \rangle]P' = 0, \tag{74}$$

where we have used Equation (68) and $L_{0,n}$ is defined by

$$L_0 = \sum_n L_{0,n}. \tag{75}$$

Herring has shown that the self-consistency condition can be satisfied, along with Equations (55) and (58), by choosing L_n such that

$$L_n = L_{0,n} - \langle VL^{-1}T \rangle_n, \tag{76}$$

and Equation (59) is retained as a separate constraint;

$$L_n P_n = 0 \tag{77}$$

where we have put $P'_n = P_n$ in Equation (59), in accordance with our self-consistency requirement. We can obtain the perturbation expansion for $\Delta = L - L_0$ by substituting for T from Equations (68) into (76) and then iterating about $\Delta = 0$. The result is

$$L = L_0 - \langle VL^{-1}V \rangle + O(V^4). \tag{78}$$

Similarly, we can obtain the perturbation expansion for P by substituting Equation (78) into (67) and expanding T:

$$P = P' + L^{-1}VP' + L_{-1}[VL^{-1}V - \langle VL_{-1}V \rangle]P' + \dots \tag{79}$$

It should be noted that truncation of the perturbation series at any order actually retains infinite powers of V. This is because of the non-linearity of Equation (63). It is at this point that we see the underlying resemblance to the renormalized perturbation theory of the DIA. On the other hand, Herring uses the same eigenfunction expansion as Edwards in order to invert the basis operator, and certainly Equation (65) is very similar (notational changes apart) to the Edwards form. The differences essentially lie in the requirement in Herring's SCF that a particular operator should vanish:

$$\left\{ L_n - L_n^{(0)} + \langle VL_{-1}T \rangle_n \right\} = 0, \tag{80}$$

where the Edwards version would require that the product of the same operator with the P_n should vanish. Furthermore, it involves a weaker assumption, that the basis operator should be the product of single-mode operators, without specifying the form these operators should take. In contrast, Edwards imposes the additional requirement that the basis operator should have the Fokker–Planck form. It should be noted that the basis operator in Herring's SCF is found not to be of this form, and leads to coupled equations for $C(k)$ and $\omega(k)$. The energy equation is identical to Equation (40), which is itself the same as the DIA equation when exponential time dependences are substituted. The equation for $\omega(k)$ is the same as the time-independent DIA response equation, as given by Equation (48).

5.3.1. Solution of Equation (65)

Equation (65) has to be solved by iteration. We write it in the form

$$P = P' + XP,$$

where

$$X = L^{-1}(\Delta + V).$$

Then, we proceed as follows:

- **zero-order approximation**

$$P_0 = P' + XP',$$

- **first-order approximation**

$$P_1 = P' + XP_0 = P' + X(P' + XP') = P' + XP' + X^2P'$$

- **second-order approximation**

$$P_2 = P' + XP' = P' + X(P' + XP' + X^2P') = P' + XP' + X^2P' + X^3P',$$

and so on.

The exact distribution P is given by

$$P = P' + X(1 + X + X^2 + \dots)P' = P' + X\frac{1}{1 - X}P'$$

provided that $1/(1-X)P'$ is non-singular. Then, substituting for X, we find

$$P = P' + L^{-1}TP',$$

which is just Equation (67), and the operator T is given by

$$T = \frac{(\Delta + V)}{1 - L^{-1}(\Delta + V)},$$

which can be shown to reduce to (68).

5.3.2. Time-Dependent SCF

The first step in extending SCF to non-stationary turbulence (Herring 1966) is to restore the time dependence to the Liouville equation. That is, we rewrite Equation (49) as

$$\frac{\partial P}{\partial t} + L^0 P = VP. \tag{81}$$

Then, we generalize Equation (68) to the form

$$\frac{\partial P}{\partial t} + \int_0^\infty L(t,t')P(t')dt' = \int_0^\infty \Omega(t,t')P(t')dt' \tag{82}$$

where

$$\Omega(t,t') = L(t,t') - \delta(t-t')(V - L_0) \tag{83}$$

such that

$$L(t,t') = 0 \quad \text{for} \quad t' > t. \tag{84}$$

It should be noted that this is all fully analogous to the procedure which led to Equation (61) for the stationary case and was based on the introduction of the operator L, as defined by Equation (57). Now, in order to obtain an iterative solution of Equation (82), Herring introduced a new operator, $U(k;t,t'$. The interested reader will find the details of these procedures in the original papers (Herring 1965, 1966); we shall only quote the defining relationship here for completeness, as we shall need it in order to write down the final governing equations. Accordingly, we have

$$\begin{aligned} U(k;t,t') &= 1 - \int_{t'}^t ds \int_0^\infty ds' L(s,s')U(k;s',t') \quad \text{for} \quad t > t' \\ &= 0 \quad \text{for} \quad t < t'. \end{aligned} \tag{85}$$

Then, the equation for the single-time correlation $C(k,t)$ is found to be

$$\begin{aligned} \left(\frac{d}{dt} + 2\nu k^2\right)C(k,t) &= 2\int d^3j L(\mathbf{k},\mathbf{j}) \int_0^t g(k;t,s)M(|\mathbf{k}-\mathbf{j}|;t,s)ds; \\ &- \int_0^t g(j;t,s)M(k;t,s)M(|\mathbf{k}-\mathbf{j}|;t,s)ds, \end{aligned} \tag{86}$$

where $g(k;t,t')$ can be interpreted as the response function for mode $\mathbf{k}$, and $M(k;t,t')$ plays the part of a two-time velocity covariance function. The two quantities arise, in fact, as non-trivial functional integrals, and can be written in a rather stylized notation in the form

$$g(k;t,t') = -\int \delta\mathbf{u}(\mathbf{k},t)\mathbf{u}(\mathbf{k},t)U(k;t,t')\frac{\partial P[\mathbf{u}(\mathbf{k},t')]}{\partial \mathbf{u}(\mathbf{k},t)} \tag{87}$$

and

$$M(k;t,t') = \int \delta\mathbf{u}(\mathbf{k},t)\mathbf{u}(\mathbf{k},t)U(k;t,t')\mathbf{u}(\mathbf{k},t)P[\mathbf{u}(\mathbf{k},t)], \tag{88}$$

and satisfy the conditions:

$$g(k;t,t) = 1; \tag{89}$$

$$M(k;t,t) = C(k,t). \tag{90}$$

Herring found governing equations for $g(k;t,t')$ and $M(k;t,t')$ which took the following forms:

$$\left(\frac{d}{dt} + \nu k^2\right) g(k;t,t') = -\int d^3j L(\mathbf{k},\mathbf{j}) \times \\ \times \int_{t'}^{t} ds\, g(j;t,s) M(|\ \mathbf{k}-\mathbf{j}\ |;t,s) g(k;s,t), \tag{91}$$

$$\left(\frac{d}{dt} + \nu k^2\right) M(k;t,t') = -\int d^3j L(\mathbf{k},\mathbf{j}) \times \\ \times \int_{t'}^{t} ds\, g(j;t,s) M(|\ \mathbf{k}-\mathbf{j}\ |;t,s) M(k;s,t'), \tag{92}$$

where, as in Equation (86), the coefficient $L(\mathbf{k},\mathbf{j})$ is as defined by Equation (25) and its analytical form is given by Equation (26). It can be deduced by inspection of Equations (91) and (92), that the following relationship holds

$$M(k;t,t') = g(k;t,t')C(k,t'). \tag{93}$$

This allows us to make contact with the DIA equations for $R(k;t,t')$ and $C(k;t,t')$, and $C(k;t,t) = C(k,t)$. It is readily verified that, if we replace the DIA Equation (23) for the two-time correlations $C(k;t,t')$ by Equation (93), then Equations (22) and (24) for $R(k;t,t')$ and $C(k,t)$ become identical with the SCF Equations, Equations (91) and (86) for $g(k;t,t')$ and $C(k,t)$.

5.4. Other Self-Consistent Methods

The Liouville equation was also been taken as a starting point by Balescu and Senatorski [36], who attempted to combine methods applied to two different fields of non-equilibrium statistical mechanics (i.e., statistical electrodynamics and Heisenberg spins on a lattice), arguing that in this way they could take account of the main features of the turbulence problem, i.e., an infinite number of degrees of freedom and strong coupling. The equations of motion are transformed by the introduction of action-angle variables and the basic statistical problem is reformulated in terms of a master equation. This is treated perturbatively and the summation of certain classes of diagrams leads to the recovery of Herring's SCF theory. Self-consistent methods have also been applied directly to the equation of motion (rather than the Liouville equation) by Phythian [37]. Strictly speaking, Phythian worked with the one-dimensional Burgers model equation, but his analysis also goes through for the Navier–Stokes equation. Although the methods used rather resemble those of Edwards and Herring, Phythian did formally introduce an infinitesimal response function which is identical in its basic definition with the corresponding DIA quantity as defined by Equation (20). It is perhaps not altogether surprising, therefore, that Phythian ended up with the DIA equations.

However, the importance of this work turned out to be more than just another way of deriving the DIA theory. Kraichnan found that Phythian's theory leads to a new model representation for the DIA equations and this in turn leads on to the development of 'almost Markovian' theories. It should be noted that the term 'Markovian' here refers to the time dependence, not to energy transfers in wavenumber.

6. Second-Generation Renormalized Perturbation Theories

6.1. The Lagrangian Theories

The Lagrangian theories arose out of Kraichnan's study of the failure of DIA to be compatible with the Kolmogorov $k^{-5/3}$ energy spectrum, which he ultimately interpreted as being due to the lack of invariance under random Galilean transformations. The resulting theories are very complicated and we shall only sketch out some general points here. The key element in Kraichnan's approach was the introduction of a generalized velocity field $\mathbf{u}(\mathbf{x}, t|s)$, with both Eulerian and Lagrangian characteristics. This is defined as the velocity measured at time s of a fluid particle which was at position $\mathbf{x}$ at time t. The two times are known as:

$$t = \text{labelling time}; \qquad \text{and} \qquad s = \text{measuring time}.$$

We may consider two limiting cases:

- Case 1: Set $s = t$ then $\mathbf{u}(\mathbf{x}, t|t) = \mathbf{u}(\mathbf{x}, t)$;
- Case 2: Fix $\mathbf{x} = \mathbf{x}_0$ and $t = t_0$ then $\mathbf{u}(\mathbf{x}_0, t_0|s) = V(\mathbf{x}_0, s)$,

where $V(\mathbf{x}_0, s)$ is the Lagrangian velocity field.

Kraichnan reworked the DIA in terms of this picture and called the result the Lagrangian-history direct-interaction approximation (or LHDI for short) [22]. This turned out to be very complicated and Kraichnan introduced a systematic way of reducing the number of time convolution integrals. This abridged set of equations was referred to, naturally enough, as the abridged LHDI, or ALHDI for short. In 1966, Kraichnan [38] reported the first theoretical value of the Kolmogrov spectral constant, which at $\alpha = 1.77$ lies within the range of experimental values, but nowadays would be regarded as rather large, as the consensus is that it should be about 1.62.

Kraichnan [39] later used the method of reversion (or inversion) of power series and rederived both LHDI and ALHDI. However, from a quantitative point of view, ALHDI did not prove altogether satisfactory, as it tended to overestimate the rate of inertial transfer of energy to higher wavenumbers. That is of interest, as the opposite tendency has been associated with the failure of DIA. This provided a motivation for Kraichnan and Herring [23] to use the same methods with the rate of strain tensor, rather than the velocity field. This led to strain-based ALHDI, or SBALHDI.

In 1981, Kaneda [24] introduced the Lagrangian position function and worked with measuring-time derivatives rather than labelling-time derivatives and derived the Lagrangian renormalized approximation (LRA), which predicted a spectral constant of $\alpha = 1.72$. Later, Kida and Goto [25] applied the sparse direct-interaction pertubation method, as originally used to derive Eulerian DIA, in the Lagrangian framework, to obtain the same result as Kaneda's LRA. More recently, Okumaru [26] has derived yet another Lagrangian theory, which he refers to as a 'Closure model', in a paper which gives a good overview of the theoretical work in this area.

We will return to Lagrangian theories in the section on computations, but their diversity does rather give added emphasis to the 1993 comment of Bazdenkov and Kukharin [40] to the effect that the results of Lagrangian theories depend on whether the velocity, the vorticity, or the rate of strain is taken as the basic field. See also the comments by Frederiksen and Davies [41].

Lastly, as we have diagnosed the problem with the historic Eulerian theories to be that they are Markovian with respect to energy transfer through wavenumber, the obvious question arises: what about the Lagrangian theories? Are they non-Markovian in this respect? In fact it is probably fairly obvious, merely be inspection, that they are in fact non-Markovian. However, perhaps the easiest way to answer this question is to consider the additional question: what steps would be needed to make them Markovian? In fact Kraichnan has answered this question for his Lagrangian-history direct-interaction theory. He has shown that LHDI can be reduced to an eddy-viscosity (i.e., Markovian) form if

the dummy wavenumber j is very much larger than the labelling wavenumber k [38]. For further discussion of this point, see Section 10.3.2 of the book [14].

6.2. The Local Energy-Transfer (LET) Theory in Two-Time Form

The LET theory was derived in 1974, in single-time form; and then heuristically in two-time form in 1978. More recently, the two-time LET theory was given a formal derivation in 2017. Here, we will make some brief remarks about this formal derivation [42].

We use the Edwards method, as described in Section 5.2, but we start from the Liouville equation with time-dependence included. The salient point is that we use the simplest possible basis operator that is compatible with a Gaussian distribution. There is still some element of arbitrariness involved in our choice of operator, but we do not follow Edwards and use the Fokker–Planck form as given by (39). It is this choice by Edwards which forces his theory to be Markovian with respect to energy transfers in wavenumber. Our approach is explained in Section 4.3 of [42], where we use Herring's reduced notation for the sake of simplicity. The subsequent perturbation theory is explained in Section 4.4 of [42], and the evaluation of the triple moment in Section 4.5 involves a functional differentiation that leads to the introduction of the response function.

LET Response Function

Kraichnan introduced his infinitesimal response tensor, which related an infinitesimal fluctuation in the velocity field to a corresponding fluctuation in the stirring forces (20). Instead of doing this, the LET introduces an infinitesimal velocity field propagator as determining the system response. In its most general form, this is defined by the relationship between two infinitesimal fluctuations of the velocity field, as:

$$\delta u_\alpha(\mathbf{k},t) = \widehat{R}_{\alpha\beta}(\mathbf{k},\mathbf{k}';t,t')\delta u_\beta(\mathbf{k}',t') \qquad \text{for} \qquad t' \le t, \tag{94}$$

where the hat symbol indicates that the propagator $\widehat{R}_{\alpha\beta}$ is a random variable. Further, we rewrite Equation (94) in terms of the functional derivative, in order to introduce the ensemble-averaged response function, thus:

$$R_{\alpha\beta}(\mathbf{k};t,t') = \left\langle \frac{\delta u_\alpha(\mathbf{k},t)}{\delta u_\beta(\mathbf{k},t')} \right\rangle = \left\langle \widehat{R}_{\alpha\beta}(\mathbf{k};t,t') \right\rangle = P_{\alpha\beta}(\mathbf{k})R(k;t,t')\,, \qquad t' \le t\,, \tag{95}$$

where we have in turn invoked homogeneity and then isotropy.

This analysis leads to a general fluctuation-response relation [42], relating the response function to the two-time and single-time covariances, thus:

$$C(k;t,t') = R(k;t,t')C(k;t',t'). \tag{96}$$

The quantity $R(k;t,t')$ is called the response function. We discuss some of the implications of this result in the next section.

6.3. Single-Time Models and Theories

Under this heading we begin with the eddy-damped quasi-normal Markovian (EDQNM) theory. This arose from Orszag's analysis of the failure of the original quasi-normality hypothesis and reflects a later and better understanding of turbulence phenomenology. The EDQNM theory is really a model (compare Kraichnan's similar test-field model [43]), in that it makes a particular assumption which requires the fitting of an adjustable constant. In stationary form it reduces to the Edwards SCF theory. Because it is a single-time theory, it is relatively easy to compute and has been widely applied to practical problems (e.g., [44–47]). For a recent discussion see the book by Sagaut and Cambon [27].

In the interests of completeness we should also mention that McComb and Kiyani [48] derived a Markovianised single-time closure from the LET theory, while Bos and Bertoglio [49]

derived a single-time closure from a Lagrangian perspective. Neither of these closures involve the fitting of adjustable constants, and so are classed as theories rather than models.

7. Application of Statistical Closure Theories

The statistical closure theories have been extensively computed for the case of freely decaying turbulence, starting with Kraichnan's pioneering work in 1964 [2]. A concise overview of this activity may be found in [42]. Here, we will just consider their application to passive scalar convection, along with some remarks on the possibility of more general applications.

7.1. Passive Scalar Convection

This is a very big subject and we will restrict our attention to the use of closures in the problem of passive scalar diffusion in isotropic turbulence. We note the pioneering work of Newman and Herring, who applied the test-field model (TFM) to this problem [50]; and later work of Herring et al. [51], which compared the TFM to EDQNM. This work, in particular, underlines the fact that this type of investigation is more about testing the performance of the closures than about practical calculations.

In 1992, as well as introducing the fluctuation–response relationship as the defining step of the LET theory, and providing a new simplified derivation (along with making some minor corrections to the equations), the work was extended to the case of passive scalar convection [52]. Results were obtained for free decay for $5 \leq R_\lambda \leq 1060$ and for Prandtl numbers of 0.1, 0.5, and 1.0. In both velocity and scalar spectra, Kolmogorov-type power laws were found at the higher Reynolds numbers.

For those wishing to pursue this topic further, the recent paper by Kitamura [53] tackles the more advanced problem of passive scalar diffusion in the presence of a mean scalar gradient, and also provides a comprehensive review of spectral methods applied to passive scalar convection.

7.2. Potential for Practical Applications

In view of the great importance of turbulent flow in engineering applications, it is surprising that the two-point statistical closures have been so little exploited for practical applications. To our knowledge, there have only been two exceptions to this. First, there is the activity on climate and atmospheric modelling by Frederiksen and co-workers (see [54] for a recent account), who have used DIA, Herring's SCF, and the LET theory over several decades for this purpose. This work also includes sub-grid modelling.

Secondly, there has been the widespread application of the EDQNM to many situations and over many decades. Its development by Orszag [55] took account of the greater understanding of turbulence phenomenology which had accrued during the 1960s and this is reflected in its success with practical applications. Because of its single-time nature it is easy to compute, and its ability to fit an adjustable constant leads to good quantitative results. As examples we may cite [44–47]; a more general account may be found in the book by Sagaut and Cambon [27].

In contrast, the LET theory, consisting as it does of the covariance equations, (23) for $C(k;t,t')$ and (24) for $C(k;t,t) \equiv C(k,t)$, which are identical to the DIA forms, plus the fluctuation–relaxation relation (96), is a fully two-time theory and has no adjustable parameters. Although rather more complicated than the EDQNM theory, and more demanding to compute, it nevertheless could be applied to non-isotropic and even inhomogeneous problems. Even if presenting too formidable a computational challenge, it could provide a basis for single-point modelling, which might offer a useful extension to existing approaches.

8. Conclusions

8.1. A Point-Wise Summary of the Main Conclusions

As most readers of this article are unlikely to be familiar with quantum field theory, I think that it will be helpful if I try to summarise its application to turbulence in as concise a

way as possible. Accordingly, I have set out a series of key points, which I have made as simple as I can without misrepresenting anything. In general my remarks are limited to isotropic turbulence (please note that homogeneity is a necessary condition for isotropy and does not therefore have to be specified). Furthermore, for the most part, I have restricted this summary to Eulerian theories. This is not to denigrate the Lagrangian approach, which is an entirely valid method of seeking a closure approximation, but to recognise that there is still the outstanding problem of converting results into an Eulerian form, which is of course the natural system for experimental work. One thinks of Taylor's theory of turbulent diffusion by continuous movements, which in the past has led to pragmatic attempts to convert it to Eulerian coordinates. In time, perhaps those who espouse the Lagrangian approach will address the problem of applying their results in an Eulerian coordinate system.

1. Every modern two-point closure approximation can be understood as a second-order truncation of a renormalized perturbation theory. The renormalization is achieved in terms of an approximate renormalized response function. All theories lead to the same equation for the energy spectrum, but the form of renormalized response function (and the method of obtaining it) varies from theory to theory.
2. The historic two-time closures (the DIA of Kraichnan and Herring's SCF) perform quite well, both quantitatively and qualitatively, on the test problem posed by freely decaying turbulence from an arbitrary initial state. However, they are not compatible with the Kolmogorov spectrum.
3. There has always been some uncertainty about this failure. Kraichnan argued that it arose because a two-time theory was unable to distinguish between sweeping effects and inertial transfer through wavenumber. In contrast, Edwards interpreted the failure of his single-time theory to be due to an infrared divergence at infinite Reynolds numbers. At the time, there seemed to be some degree of agreement between the two diagnoses, in that they both pointed to a problem at low wavenumbers, but this still left a feeling of uncertainty.
4. A corollary of Kraichnan's argument about the DIA is that no two-time Eulerian theory can be compatible with the Kolmogorov spectrum. However, the LET theory is both a two-time Eulerian theory *and* is compatible with the Kolmogorov spectrum.
5. A corollary of the Edwards diagnosis, in terms of an infrared divergence, is that there may be more to the failure of DIA than Kraichnan's diagnosis would suggest. This is because DIA reduces to a similar form to the Edwards SCF in time-independent form, and hence must also have an infrared divergence.
6. A corollary of the original derivation of the LET theory is that the relation of the DIA's response function to the energy balance is inconsistent with the experimental picture.
7. A corollary of the formal derivation of the LET theory in two-time form [42] is that the unifying explanation of why the historic closures fail to yield the Kolmogorov spectrum is that they are all Markovian with respect to energy transfer through wavenumber, when turbulence is non-Markovian.
8. A conjecture arising from this formal derivation [42] is that the underlying problem with the DIA may be its reliance on the random stirring forces to define the renormalized response function. While it is natural to study a dynamical system by applying a force, in the case of fluid motion it is the turbulent state which is the system. Hence, the applied force first has to create the dynamical system and then maintain it, while also determining its renormalized response. In contrast, the two-time LET determines its response in terms of a random, quasi-entropic force, which is related to the probability distribution of velocities.
9. Although the quantum-style formalisms set out to establish the general nature and existence of a statistical theory of turbulence, they both are dependent (in their different ways) on the use of the stirring forces to formalise their renormalized response. Further, they both reduce to the DIA when truncated at the lowest nontrivial order. Hence, to the extent that the DIA is incorrect, they also are incorrect, at least when applied to fluid turbulence.

10. The LET theory cannot claim to be the theory of turbulence, in that there are some question marks hanging over it. In particular, its prediction of the Kolmogorov spectral constant is almost certainly too large. However, a hybrid theory might be a possibility, with some vertex renormalization included by means of a restriction on wavenumber interaction ranges. In this way, it could, like EDQNM, benefit from the introduction of an adjustable constant.

8.2. Some Personal Reflections

I first met Jack Herring at the NASA-ICASE workshop on turbulence theory at Virginia Beach in October 1984. Taking part were physicists, mathematicians, and engineers, and I, as very much a new boy, was glad to be welcomed into the physicists' group with Jack, Bob Kraichnan, and various others. They were all, particularly Jack, very friendly and I immediately felt at home. In fact, his friendly open manner was the first impression one had of Jack, and I know that others felt the same about him. He was above all a good-humoured, relaxed man and, in view of his great abilities, remarkably unassuming. I shall return to that particular aspect later.

This was a long time ago, and I only have a few memories of the social interactions, but I do recall that there was a 'no-host cocktail hour' when the day's programme finished. I was quite amused by this phrase, which I had not met before, and which simply meant that the organisers were not about to pay for the drinks! I only remember one dinner, where we left the hotel and set off along the sea front to look for somewhere to eat. It turned out that a group of turbulence theorists could not ignore a seafood restaurant called 'Ocean Eddies'. I think we were the only customers and one of us did not like the loud music being played and either switched it off or asked someone to do so. After a few moments of pleasant silence, a small angry-looking man appeared (perhaps Eddy himself?), switched the music back on, glared at us, and then disappeared back behind the scenes. Despite the loud music, it was a pleasant evening and on our way back to the hotel Jack suddenly pounced and scooped something up from the road. It was a small crab that he had spotted scuttling along in front of us. After holding it in his cupped palms for us all to admire, he put it down again to resume its journey to the sea.

My first interaction with Jack was actually the previous year, when I submitted a paper reporting the first computations of my LET theory to the *Journal of Fluid Mechanics*. One morning I received a phone call from George Batchelor to say that Bob Kraichnan and Jack Herring were refereeing my paper and that they would like to discuss it with me. He added, 'You seem to have stirred them up a bit', and he sounded quite approving! In those days, 'discussion' was by means of air-mail letter, and was a relatively slow business. However, it was all very civil and constructive; I managed to satisfy both Jack and Bob, and the paper was duly accepted by JFM.

In 1990, my first book *The Physics of Fluid Turbulence* was published. It was reviewed by Jack for the *Journal of the Atmospheric Sciences*. He gave it a very favourable and quite lengthy review, but I was amused by one of his comments. He pointed out that I had gone into some detail in establishing the foundations of the many-body statistical approach, and it was 'very full, but not too tiresome'. Of course to someone like Jack, this would be very elementary material, but that would not necessarily be the case for most readers. I was glad that it had not been too tiresome!

I encountered Jack on a few other occasions over the years, and found that he was the sort of person with whom you felt that each time you met, you started again from where you last left off, as if it had been the previous day. Discussions were always amicable, even if we did not entirely agree.

While I have no further anecdotes to tell, I do have a couple of points that I would like to make. The first of these arises out of my earlier comment about Jack being unassuming. In his collaboration with Bob Kraichnan from the late 1960s onwards, I felt that Jack to some extent submerged his own interests in favour of Bob's interests. Of course, I have no sense of how the two men worked together on the derivation of the strain-based Lagrangian

theory. Nevertheless, the author ordering of the paper reporting the theory [23], and of the paper reporting its subsequent computation [56], rather suggests the minor role for Jack. However, others may disagree, and indeed be better informed than I am. So, I just offer this as an impression that I had at the time.

My second, and final, point is about the three pioneering theories and their authors. I was reflecting on these around the end of the previous millenium, and my conclusion was that these three remarkable theories would always be the foundation of any turbulence theory to come. Each was an outstanding intellectual achievement and they all, in effect, were in agreement. If they had led to a great wave of progress in turbulence, then they would have been in contention for the Nobel prize. However, unfortunately the cultural divide was too great. For instance, intelligent men like Phillip Saffman and John Lumley dismissed them as 'mathematically complicated', and I have seen the anonymous comment 'everyone knows that Kraichnan was wrong and this line of approach is finished.' Well, on the contrary, this line of approach is not finished and in my carefully considered view, Kraichnan, Edwards, and Herring, in their pioneering work on turbulence, were in the Nobel laureate class.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The author declares no conflict of interest.

Appendix A. Quantum-Style Formalisms

We will only touch briefly on this subject here, mainly for the sake of completeness. The formalisms of interest are the diagrammatic method due to Wyld [19], and the functional formalism of Martin et al. [57]. The latter is often referred to as MSR, after the names of its three authors. The aim of a formalism is to establish the existence and nature of a solution to the turbulence problem (this is in the sense of physics, rather than that of pure mathematics), rather than to produce a specific theory. In fact, both of these formalisms reduce to the direct-interaction approximation (DIA) when truncated at the lowest nontrivial order, and, by extension, also can be said to lead to the SCFs of Edwards and Herring.

The paper by Wyld in 1961 was the second publication (after Kraichnan's DIA) to apply the new renormalization methods of physics' turbulence problem, but, unlike the DIA, was presented very much using the terminology of quantum field theory. While this may be off-putting to those who are not themselves theoretical physicists (i.e., most fluid dynamicists), it is worth overcoming this feeling, because the diagrammatic expansion presents quite an intuitive picture of how renormalization can be expected to work. The basic idea is that, although the perturbation series is wildly divergent, some subclasses of terms in the expansion may be summed to all orders. In particular, summing one subclass (with its members determined on topological grounds) leads to an exact expression for the covariance of random stirring forces and the velocity field (also see Section 5.3.1 of the present work, which deals with an exact summation occurring in Herring's SCF). Essentially, the remaining terms lead to a renormalized perturbation series of unknown properties (at least, as regards convergence and possible truncation).

In 1965, Lee pointed out that there was a problem with the double counting of diagrams with Wyld's method and this could be cured by changing the left-hand emergent propagator from each diagram to the bare (i.e., viscous) form [58]. Later, McComb [59], in the course of deriving a two-time version of the LET theory, pointed out that the need for Lee's ad hoc procedure was due to Wyld carrying out more renormalizations than were actually required. A more detailed discussion of Wyld's theory may be found in Section 5.5 of the book [14].

Lastly, Martin et al. [57] argued that even with the improvements introduced by Lee, the Wyld theory was incorrect, due to the way in which it handled vertex corrections. As a

result, many people dismissed the Wyld–Lee theory out of hand. This is never a very wise thing to do. After all, the basic perturbation approach is the gold standard for this type of theory. More recently, Berera et al. [60] made a detailed comparison of the two formalisms and concluded that MSR had misinterpreted Wyld's way of handling the vertex corrections. They presented what they called the *Improved Wyld–Lee renormalized perturbation theory* in which there is no conflict with the MSR functional formalism. However, the reader should note the comment about both these formalisms which is made in the Conclusions of this article.

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