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Three-Dimensional and Two-Dimensional Green Tensors of Piezoelectric Quasicrystals

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Abstract: In this work, within the framework of the linear piezoelectricity theory of quasicrystals, the three-dimensional and two-dimensional Green tensors for arbitrary piezoelectric quasicrystals are derived. In the piezoelectricity of quasicrystals, where phonon, phason and electric fields exist, we introduce the corresponding multifields by developing a hyperspace notation for piezoelectric quasicrystals. Using Fourier transform and the multifield formalism, the three-dimensional Green tensor for piezoelectric quasicrystals as well as its spatial gradient necessary for applications, are derived. The solutions for the “displacement”, “distortion” and “stress” multifields in the presence of a “force” multifield in a piezoelectric quasicrystal as well as the solution of the generalised Kelvin problem, are given. In addition, the two-dimensional Green tensors of piezoelectric quasicrystals as well as of quasicrystals, are determined.

Keywords: quasicrystals; piezoelectricity; Green tensor; fundamental solution; multifields; anisotropy



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1. Introduction

Nowadays, piezoelectric materials are an intriguing class of materials with broad use in technology, including energy conversion and sensing as well as in energy harvesting and transducer devices (see, e.g., [1]). For a piezoelectric material, the elastic and electric response is coupled and anisotropic (see, e.g., [2–4]). The Green functions or fundamental solutions of partial differential equations play an essential role in the solution of many problems in various scientific areas, like mathematics, physics, engineering science and material science. In general, the importance of Green functions is at least twofold. A Green function is the fundamental solution of a partial differential equation and can be used to clarify various approximative methods, such as finite element and boundary element methods for the study of defects like cracks, dislocations, disclinations, point defects and inclusions. The three-dimensional elastic Green tensor was derived by Lord Kelvin [5] for isotropic materials and by Lifshitz and Rosenzweig [6], Kröner [7] and Willis [8] for hexagonal materials. Lifshitz and Rosenzweig [6] and Synge [9] (see also [10,11]) derived the three-dimensional elastic Green tensor for arbitrary anisotropic materials. The two-dimensional Green tensor of anisotropic elasticity was derived by Wang and Achenbach [12], Wang [13] and Lazar [14] using Radon transform and Fourier transform, respectively. In analogy to the three-dimensional Green tensor of anisotropic elasticity given by Synge [9], the three-dimensional Green tensor for arbitrary piezoelectric materials was derived by Deeg [15], Chen [16], Dunn [17], Akamatsu and Tanuma [18], Pan and Tonon [19], Li and Wang [20] and Nowacki [21]. The three-dimensional Green tensor for hexagonal piezoelectric crystals with point group $6mm$ has been calculated by Michelitsch [22]. The two-dimensional Green tensor of piezoelectricity has been given by Wang [23].

To date, quasicrystals represent an interesting class of novel materials due to their particular properties, such as physical, structural, chemical, electronic and magnetic properties among others. Quasicrystals were discovered by Shechtman in 1982 and reported by

Shechtman et al. [24] in 1984. In 2011, Shechtman received the Nobel Prize in Chemistry for the discovery of quasicrystals. Quasicrystals belong to aperiodic crystals and possess long-range orientational order but no translational symmetry. The basis of the continuum theory of solid quasicrystals is set up by two elementary excitations: the phonons and the phasons [25,26]. Quasicrystals possess piezoelectric effects, attracting interest for their applications in technology, for example, in electronic systems as sensor and actuator devices by which an electric voltage can induce an elastic deformation and vice versa. There is considerable interest in the subject of piezoelectric quasicrystals, which is an ongoing important and active research field (see, e.g., [27–29]).

Concerning Green functions in quasicrystals, De and Pelcovits [30] found the two-dimensional Green functions for pentagonal quasicrystals and Ding et al. [31] found the explicit expressions of two-dimensional Green tensors for various forms of planar quasicrystals. Bachteler and Trebin [32] gave an approximative solution for the elastic Green tensor of three-dimensional icosahedral quasicrystals, assuming that the coupling between phonons and phasons is small. Lazar and Agiasofitou [33] derived the three-dimensional elastic Green tensor for arbitrary quasicrystals.

In the theory of piezoelectric quasicrystals, the phonon fields as well as the phason fields are coupled with the electric field (see, e.g., [34,35]). It is clear that, due to this fact, the piezoelectric behaviour of quasicrystals is much more complicated than that of conventional piezoelectric crystals. An improvement to the existing constitutive modelling of piezoelectric quasicrystals has been recently given by Agiasofitou and Lazar [36] with a focus on the tensor of phason piezoelectric moduli, which is proved to be fully asymmetric without any major or minor symmetry. Moreover, it has been shown that piezoelectric effects in quasicrystals can exist only due to phonons or only due to phasons or due to both phonon and phason fields, depending, in general, on the Laue class as well as on the point group (see [35,36]). Using group representation theory, Hu et al. [35] studied piezoelectric effects in all two-dimensional quasicrystals with crystallographic and non-crystallographic symmetries, and in three-dimensional icosahedral and cubic quasicrystals. Agiasofitou and Lazar [36] have investigated piezoelectric effects in all one-dimensional quasicrystals with crystallographic and non-crystallographic symmetries. The interested reader can find concrete results concerning which crystallographic point groups possess piezoelectric effects and which do not, and if these are induced only by phonons or only by phasons or due to both phonon and phason fields in [36] and the references therein.

Regarding Green functions in piezoelectric quasicrystals, Green functions have been given only for particular cases of one-dimensional piezoelectric quasicrystals in the literature (see, e.g., [37–39]). In this work, we derive exact analytical expressions for the three-dimensional and two-dimensional Green tensor functions for arbitrary piezoelectric quasicrystals, that means for one-dimensional, two-dimensional and three-dimensional piezoelectric quasicrystals.

The paper is organized as follows. In Section 2, the basic framework of the generalised linear piezoelectricity theory of quasicrystals is presented. In Section 3, the hyperspace notation of piezoelectric quasicrystals is introduced, being an essential tool for the derivation of the three-dimensional Green tensor for arbitrary piezoelectric quasicrystals, which is done in Section 4 using the Fourier transform and the multifield formalism. Additionally, Section 4 includes other fundamental properties, such as the solution of the generalised Kelvin problem for piezoelectric quasicrystals and the derivation of the two-dimensional Green tensors for arbitrary piezoelectric quasicrystals as well as for arbitrary quasicrystals. Conclusions are given in Section 5. In Appendix A, the major symmetry of the multifield tensor of the material moduli in the hyperspace notation is proved. In Appendix B, the multifield tensor of the material moduli in the hyperspace notation is explicitly given for one-dimensional, two-dimensional and three-dimensional piezoelectric quasicrystals.

2. Basic Framework of Piezoelectric Quasicrystals

In this section, the basic framework of the generalised linear piezoelectricity theory of quasicrystals [36], which is the generalisation of the linear (compatible) elasticity of quasicrystals [40] towards the piezoelectricity of quasicrystals, is given.

An $(n - 3)$ -dimensional quasicrystal can be generated by the projection of an n -dimensional periodic structure to the three-dimensional physical space ($n = 4, 5, 6$). The n -dimensional hyperspace E^n can be decomposed into the direct sum of two orthogonal subspaces,

$$E^n = E_{\parallel}^3 \oplus E_{\perp}^{(n-3)}, \quad (1)$$

where $E_{\parallel}^3$ is the three-dimensional physical or parallel space of the phonon fields and $E_{\perp}^{(n-3)}$ is the $(n - 3)$ -dimensional perpendicular space of the phason fields. Here, $\oplus$ denotes the direct sum. For $n = 4, 5, 6$, we speak of one-dimensional, two-dimensional and three-dimensional quasicrystals with the dimension of the corresponding hyperspace being 4D, 5D, and 6D, respectively. Here, indices in the parallel space are denoted by small letters i, j, k, l , with $i, j, k, l = 1, 2, 3$, and indices in the perpendicular space are denoted by Greek letters α, γ , with $\alpha, \gamma = 3$ for one-dimensional quasicrystals (with quasiperiodicity in the z -direction), $\alpha, \gamma = 1, 2$ for two-dimensional quasicrystals, and $\alpha, \gamma = 1, 2, 3$ for three-dimensional quasicrystals. Throughout the text, phonon fields will be denoted by $(\cdot)_{\parallel}$ and phason fields by $(\cdot)_{\perp}$. All field quantities (phonon and phason fields) depend on the so-called material space coordinates (or spatial coordinates) $\mathbf{x} \in \mathbb{R}^3$. Note that in the linear theory of quasicrystals, the material space coincides with the parallel space.

In the theory of the generalised compatible elasticity of quasicrystals, the (elastic) *phonon* and *phason* distortion tensors, $\beta_{kl}^{\parallel}$ and $\beta_{\gamma l}^{\perp}$, are defined as the gradient of the *phonon* displacement vector $u_k^{\parallel}$ and of the *phason* displacement vector $u_{\gamma}^{\perp}$, respectively,

$$\beta_{kl}^{\parallel} = u_{k,l}^{\parallel}, \quad \beta_{\gamma l}^{\perp} = u_{\gamma,l}^{\perp} \quad (2)$$

and they fulfill the *compatibility conditions*

$$\epsilon_{ijl} \beta_{kl,j}^{\parallel} = 0, \quad \epsilon_{ijl} \beta_{\gamma l,j}^{\perp} = 0, \quad (3)$$

where ϵ_{ijl} is the three-dimensional Levi–Civita tensor. The subscript comma denotes the partial differentiation $\partial_j = \partial/\partial x_j$ with respect to the spatial coordinates x_j . The *phonon strain tensor* is given by

$$e_{kl}^{\parallel} = \frac{1}{2} (u_{k,l}^{\parallel} + u_{l,k}^{\parallel}). \quad (4)$$

Using the two different coordinate systems and indices, it can be seen that a phason strain tensor cannot be defined (see the relative discussion in [36]). On the other hand, the *electric field strength vector* E_l is defined as the negative gradient of the *electrostatic potential* φ

$$E_l = -\varphi_{,l} \quad (5)$$

and it satisfies the *Bianchi identity*

$$\epsilon_{ijl} E_{l,j} = 0, \quad (6)$$

which is the compatibility condition for the electric field. Equation (6) is nothing but the *electrostatic version of the Faraday law* [41].

For piezoelectric quasicrystals, the (force) equilibrium conditions read [36,42]

$$\sigma_{ij,j}^{\parallel} + f_i^{\parallel} = 0, \quad (7)$$

$$\sigma_{\alpha j,j}^{\perp} + f_{\alpha}^{\perp} = 0, \quad (8)$$

and the Gauss law of electrostatics (see, e.g., [41]) reads

$$D_{j,j} = q. \quad (9)$$

Here, $\sigma_{ij}^{\parallel}$ and $\sigma_{\alpha j}^{\perp}$ are the phonon and phason stress tensors, respectively, D_j is the electric displacement vector or electric excitation, $f_i^{\parallel}$ is the conventional phonon body force density, $f_{\alpha}^{\perp}$ is the phason body force density (see, e.g., [33]) and q denotes the body charge density. Note that the phonon stress tensor is symmetric, $\sigma_{ij}^{\parallel} = \sigma_{ji}^{\parallel}$, whereas the phason stress tensor $\sigma_{\alpha j}^{\perp}$ is an asymmetric two-point tensor, $\sigma_{\alpha j}^{\perp} \neq \sigma_{j\alpha}^{\perp}$ (see also [33,42]).

Following the improved constitutive modelling for linear piezoelectric quasicrystals given recently by Agiasofitou and Lazar [36], the constitutive relations are

$$\sigma_{ij}^{\parallel} = C_{ijkl}e_{kl}^{\parallel} + D_{ij\gamma l}\beta_{\gamma l}^{\perp} - e_{lij}E_l, \quad (10)$$

$$\sigma_{\alpha j}^{\perp} = D_{kl\alpha j}e_{kl}^{\parallel} + E_{\alpha j\gamma l}\beta_{\gamma l}^{\perp} - f_{l\alpha j}E_l, \quad (11)$$

$$D_j = e_{jkl}e_{kl}^{\parallel} + f_{j\gamma l}\beta_{\gamma l}^{\perp} + \varepsilon_{jl}E_l. \quad (12)$$

In Equations (10)–(12), it can be seen that the electric field strength vector contributes to the phonon and phason stress tensors and that the elastic phonon strain tensor and phason distortion tensor contribute to the electric displacement vector for a linear piezoelectric quasicrystal. The constitutive tensors, which are the physical property tensors, possess the following symmetries [36]. The tensor of the elastic moduli of phonons C_{ijkl} possesses the major symmetry and the minor symmetries

$$C_{ijkl} = C_{klij}, \quad C_{ijkl} = C_{ijlk} = C_{jikl}. \quad (13)$$

The tensor of the elastic moduli of phasons $E_{\alpha j\gamma l}$ possesses only a major symmetry

$$E_{\alpha j\gamma l} = E_{\gamma l\alpha j}, \quad (14)$$

whereas the tensor of the elastic moduli of the phonon–phason coupling $D_{ij\gamma l}$ possesses only a minor symmetry with respect to the first two indices

$$D_{ij\gamma l} = D_{ji\gamma l}. \quad (15)$$

In addition, the tensor of the phonon piezoelectric moduli e_{kij} possesses a minor symmetry with respect to the last two indices

$$e_{kij} = e_{kji}, \quad (16)$$

the tensor of the phason piezoelectric moduli $f_{j\gamma l}$ is a fully asymmetric tensor possessing neither a major symmetry nor a minor symmetry with respect to the last two indices

$$f_{j\gamma l} \neq f_{jl\gamma} \quad (17)$$

and the tensor of the dielectric moduli ε_{ij} exhibits the major symmetry

$$\varepsilon_{ij} = \varepsilon_{ji}. \quad (18)$$

Substituting Equations (10)–(12) into Equations (7)–(9), and using Equations (2), (4) and (5), the coupled equations of equilibrium for the phonon and phason displacement fields and the electrostatic potential are given by

$$C_{ijkl}u_{k,lj}^{\parallel} + D_{ij\gamma l}u_{\gamma,lj}^{\perp} + e_{lij}\varphi_{,lj} = -f_i^{\parallel}, \quad (19)$$

$$D_{kl\alpha j}u_{k,lj}^{\parallel} + E_{\alpha j\gamma l}u_{\gamma,lj}^{\perp} + f_{l\alpha j}\varphi_{,lj} = -f_{\alpha}^{\perp}, \quad (20)$$

$$e_{jkl}u_{k,lj}^{\parallel} + f_{j\gamma l}u_{\gamma,lj}^{\perp} - \varepsilon_{jl}\varphi_{,lj} = q, \quad (21)$$

which are inhomogeneous partial differential equations with the body forces and body charges as source terms.

The electric enthalpy density Ψ for piezoelectric quasicrystals is given by [36]

$$\Psi(e_{ij}^{\parallel}, \beta_{\alpha j}^{\perp}, E_j) = \frac{1}{2} \sigma_{ij}^{\parallel} e_{ij}^{\parallel} + \frac{1}{2} \sigma_{\alpha j}^{\perp} \beta_{\alpha j}^{\perp} - \frac{1}{2} D_j E_j, \quad (22)$$

which, using Equations (10)–(12), takes the form

$$\begin{aligned} \Psi = & \frac{1}{2} C_{ijkl} e_{ij}^{\parallel} e_{kl}^{\parallel} + \frac{1}{2} D_{ij\gamma l} e_{ij}^{\parallel} \beta_{\gamma l}^{\perp} - \frac{1}{2} e_{lij} E_l e_{ij}^{\parallel} \\ & + \frac{1}{2} D_{kl\alpha j} e_{kl}^{\parallel} \beta_{\alpha j}^{\perp} + \frac{1}{2} E_{\alpha j\gamma l} \beta_{\alpha j}^{\perp} \beta_{\gamma l}^{\perp} - \frac{1}{2} f_{l\alpha j} E_l \beta_{\alpha j}^{\perp} \\ & - \frac{1}{2} e_{jkl} E_j e_{kl}^{\parallel} - \frac{1}{2} f_{j\gamma l} E_j \beta_{\gamma l}^{\perp} - \frac{1}{2} \varepsilon_{jl} E_j E_l. \end{aligned} \quad (23)$$

The form of the electric enthalpy density (23) is useful in order to write the constitutive tensors of the material moduli as a multifield tensor in the hyperspace notation, which follows in the next section.

3. Hyperspace Notation and Multifields for Piezoelectric Quasicrystals

Throughout this paper, we introduce the hyperspace notation appropriate for the description of piezoelectric quasicrystals. The *hyperspace notation for piezoelectric quasicrystals* is the unification of the notation introduced by Barnett and Lothe [43] for piezoelectric materials and of the hyperspace notation introduced by Lazar and Agiasofitou [33] for quasicrystals towards piezoelectric quasicrystals.

For a piezoelectric quasicrystal, the corresponding hyperspace is an $(n+1)$ -dimensional hyperspace $E^{(n+1)}$ which is decomposed into three orthogonal subspaces, $E_{\parallel}^3$, $E_{\perp}^{(n-3)}$, and E_{φ}^1 ,

$$E^{(n+1)} = E_{\parallel}^3 \oplus E_{\perp}^{(n-3)} \oplus E_{\varphi}^1, \quad (24)$$

where the additional space E_{φ}^1 is the one-dimensional space of the electrostatic potential. For $n = 4, 5, 6$, we speak of one-dimensional, two-dimensional and three-dimensional piezoelectric quasicrystals, and the dimension of the corresponding hyperspace being 5D, 6D, and 7D, respectively. The obtained fields which “live” in the hyperspace $E^{(n+1)}$ are called *multifields*. An index associated with a multifield is denoted by a capital letter, e.g., $I, K = 1, \dots, n+1$. In the hyperspace notation, we define the following multifields:

- The “displacement” multifield vector $U_K = (u_k^{\parallel}, u_{\gamma}^{\perp}, \varphi) \in E_{\parallel}^3 \oplus E_{\perp}^{(n-3)} \oplus E_{\varphi}^1$ or $U_K \in E^{(n+1)}$

$$U_K = \begin{cases} u_k^{\parallel}, & K = 1, 2, 3, \\ u_{\gamma}^{\perp}, & K = 4, \dots, n, \\ \varphi, & K = n+1. \end{cases} \quad (25)$$

- The “distortion” multifield tensor $B_{KI} \in (E_{\parallel}^3 \oplus E_{\perp}^{(n-3)} \oplus E_{\varphi}^1) \otimes E_{\parallel}^3$ or $B_{KI} \in E^{(n+1)} \otimes E_{\parallel}^3$

$$B_{KI} = \begin{cases} \beta_{kl}^{\parallel}, & K = 1, 2, 3, \\ \beta_{\gamma l}^{\perp}, & K = 4, \dots, n, \\ -E_l, & K = n + 1, \end{cases} \quad (26)$$

which has one multifield index and one index in the parallel space. Above, $\otimes$ denotes the tensor product. It is easy to see that

$$B_{KI} = U_{K,l}. \quad (27)$$

- The “stress” multifield tensor $\Sigma_{Ij} \in (E_{\parallel}^3 \oplus E_{\perp}^{(n-3)} \oplus E_{\varphi}^1) \otimes E_{\parallel}^3$ or $\Sigma_{Ij} \in E^{(n+1)} \otimes E_{\parallel}^3$

$$\Sigma_{Ij} = \begin{cases} \sigma_{ij}^{\parallel}, & I = 1, 2, 3, \\ \sigma_{\alpha j}^{\perp}, & I = 4, \dots, n, \\ D_j, & I = n + 1, \end{cases} \quad (28)$$

which has one multifield index and one index in the parallel space.

- The “body force density” multifield vector $F_I = (f_i^{\parallel}, f_{\alpha}^{\perp}, -q) \in E_{\parallel}^3 \oplus E_{\perp}^{(n-3)} \oplus E_{\varphi}^1$ or $F_I \in E^{(n+1)}$

$$F_I = \begin{cases} f_i^{\parallel}, & I = 1, 2, 3, \\ f_{\alpha}^{\perp}, & I = 4, \dots, n, \\ -q, & I = n + 1. \end{cases} \quad (29)$$

- The multifield tensor of the material moduli $C_{IjKI} \in (E_{\parallel}^3 \oplus E_{\perp}^{(n-3)} \oplus E_{\varphi}^1) \otimes E_{\parallel}^3 \otimes (E_{\parallel}^3 \oplus E_{\perp}^{(n-3)} \oplus E_{\varphi}^1) \otimes E_{\parallel}^3$ or $C_{IjKI} \in E^{(n+1)} \otimes E_{\parallel}^3 \otimes E^{(n+1)} \otimes E_{\parallel}^3$

$$C_{IjKI} = \begin{cases} C_{ijkl}, & I = 1, 2, 3; & K = 1, 2, 3, \\ D_{ij\gamma l}, & I = 1, 2, 3; & K = 4, \dots, n, \\ e_{lij}, & I = 1, 2, 3; & K = n + 1, \\ D_{kl\alpha j}, & I = 4, \dots, n; & K = 1, 2, 3, \\ E_{\alpha j\gamma l}, & I = 4, \dots, n; & K = 4, \dots, n, \\ f_{l\alpha j}, & I = 4, \dots, n; & K = n + 1, \\ e_{jkl}, & I = n + 1; & K = 1, 2, 3, \\ f_{j\gamma l}, & I = n + 1; & K = 4, \dots, n, \\ -\varepsilon_{jl}, & I = n + 1; & K = n + 1. \end{cases} \quad (30)$$

The tensor C_{IjKI} retains a major symmetry in the hyperspace notation (see Appendix A)

$$C_{IjKI} = C_{KlIj}. \quad (31)$$

In matrix form, Equation (30) reads

$$C_{IjKI} = \begin{pmatrix} C_{ijkl} & D_{ij\gamma l} & e_{lij} \\ D_{kl\alpha j} & E_{\alpha j\gamma l} & f_{l\alpha j} \\ e_{jkl} & f_{j\gamma l} & -\varepsilon_{jl} \end{pmatrix}. \quad (32)$$

The multifield tensor of the material moduli C_{IjKI} in the hyperspace notation is explicitly given for one-dimensional, two-dimensional and three-dimensional piezoelectric quasicrystals in the Appendix B.

Strictly speaking, the tensors B_{KI} , Σ_{Ij} and C_{IjKI} appearing in Equations (26), (28) and (30), respectively, are double tensor fields [44] or two-point tensors [45] since they have

indices in two spaces, namely the hyperspace and the parallel space. Specifically, B_{KL} and Σ_{Ij} are two-point tensors of rank two, and C_{IjKL} is a two-point tensor of rank four.

Using the hyperspace notation for piezoelectric quasicrystals, the basic equations can be now written in a simple “compact” form, which is easy to handle. The compatibility conditions (3) and (6) are simply written

$$\epsilon_{ijl} B_{KL,j} = 0. \quad (33)$$

The constitutive relations (10)–(12) can be written in the form

$$\Sigma_{Ij} = C_{IjKL} B_{KL} \quad (34)$$

and the equilibrium conditions (7)–(9) read

$$\Sigma_{Ij,j} + F_I = 0. \quad (35)$$

By substituting Equation (34) into Equation (35) and using Equation (27), the *coupled equations of equilibrium for the “displacement” multifield vector* U_K read as follows

$$C_{IjKL} U_{K,lj} + F_I = 0. \quad (36)$$

Equation (36) is the “compact” form of the coupled equations of the equilibrium for the phonon and phason displacement fields and the electrostatic potential, Equations (19)–(21), using the hyperspace notation, and it is a Navier-type partial differential equation for the “displacement” multifield vector U_K . Finally, the electric enthalpy density (23) in the hyperspace notation reads as

$$\Psi = \frac{1}{2} B_{Ij} C_{IjKL} B_{KL}, \quad (37)$$

where the major symmetry (31) can be easily seen.

4. Fundamental Properties of Piezoelectric Quasicrystals

In this section, we derive the three-dimensional Green tensor for homogeneous piezoelectric quasicrystals, which is used to calculate the “displacement”, “distortion” and “stress” multifields in the presence of a “force” multifield in a piezoelectric quasicrystal. By considering a Kelvin-type “force” multifield, the generalised Kelvin problem for piezoelectric quasicrystals is solved. The section ends with the derivation of the two-dimensional Green tensor for homogeneous piezoelectric quasicrystals. It should be mentioned that the Green functions and the corresponding Kelvin solutions serve, in general, as a pertinent basis of important numerical methods, such as the boundary element method [46].

4.1. Three-Dimensional Green Tensor of Piezoelectric Quasicrystals

The method of Green functions (see, e.g., [47]) is usually used to solve linear inhomogeneous partial differential equations like Equation (36). The Green tensor $G_{KM}(\mathbf{R})$, $K, M = 1, \dots, n+1$ of the three-dimensional Navier-type Equation (36), describing the electro-elastic fields in piezoelectric quasicrystals, is defined by

$$C_{IjKL} G_{KM,lj}(\mathbf{R}) + \delta_{IM} \delta(\mathbf{R}) = 0, \quad (38)$$

where $I, K, M = 1, \dots, n+1$, $j, l = 1, 2, 3$, δ_{IM} is the $(n+1) \times (n+1)$ unit matrix in the hyperspace, $\mathbf{R} = \mathbf{x} - \mathbf{x}'$, and $\delta(\mathbf{R})$ is the three-dimensional Dirac delta function. $G_{KM}(\mathbf{R})$ represents the “displacement” multifield in the hyperspace in the K direction at the point $\mathbf{x}$, arising from a unit point “force” multifield in the M direction applied at the point $\mathbf{x}'$.

Therefore, $G_{KM}(\mathbf{R})$ is also a *multifield Green tensor*. The Green tensor $G_{KM}(\mathbf{R})$ satisfies the symmetry relations

$$G_{KM}(\mathbf{R}) = G_{MK}(\mathbf{R}) = G_{KM}(-\mathbf{R}), \quad (39)$$

that is, $G_{KM}(\mathbf{R})$ is symmetric and is an even function of $\mathbf{R}$. The fact that the Green tensor $G_{KM}(\mathbf{R})$ is a symmetric tensor of rank two in the hyperspace is a direct consequence of the major symmetry (31).

Using the three-dimensional Fourier integral representation of the Green tensor

$$G_{KM}(\mathbf{R}) = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} G_{KM}(\mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{R}} d\mathbf{k} \quad (40)$$

and of the Dirac delta function

$$\delta(\mathbf{R}) = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} e^{i\mathbf{k} \cdot \mathbf{R}} d\mathbf{k}, \quad (41)$$

Equation (38) can be transformed to an algebraic equation in the Fourier space

$$C_{Ijkl} k_j k_l G_{KM} = \delta_{IM}. \quad (42)$$

If we introduce the unit vector in the three-dimensional Fourier space with $\mathbf{k} \in \mathbb{R}^3$

$$\boldsymbol{\kappa} = \mathbf{k} / |\mathbf{k}| \quad (43)$$

and the notation

$$(\kappa C \kappa)_{IK} = \kappa_j C_{Ijkl} \kappa_l = \begin{pmatrix} \kappa_j C_{ijkl} \kappa_l & \kappa_j D_{ij\gamma l} \kappa_l & \kappa_j e_{lij} \kappa_l \\ \kappa_j D_{kl\alpha j} \kappa_l & \kappa_j E_{\alpha j \gamma l} \kappa_l & \kappa_j f_{l\alpha j} \kappa_l \\ \kappa_j e_{jkl} \kappa_l & \kappa_j f_{j\gamma l} \kappa_l & -\kappa_j \varepsilon_{jl} \kappa_l \end{pmatrix}, \quad (44)$$

the multifield Green tensor can be written in the Fourier space as follows:

$$G_{KM}(\mathbf{k}) = \frac{1}{k^2} (\kappa C \kappa)_{KM}^{-1}, \quad (45)$$

which is a homogeneous function of $\mathbf{k}$ of degree -2 . The matrix $(\kappa C \kappa)_{KM}^{-1}$ is the inverse of $(\kappa C \kappa)_{KM}$ and is given by

$$(\kappa C \kappa)_{KM}^{-1} = \frac{A_{KM}(\boldsymbol{\kappa})}{D(\boldsymbol{\kappa})}, \quad (46)$$

where $D(\boldsymbol{\kappa})$ and $A_{KM}(\boldsymbol{\kappa})$ are the determinant and the adjoint of the matrix $(\kappa C \kappa)_{KM}$, respectively. Substituting Equation (45) into Equation (40), the three-dimensional Fourier integral can be reduced to a line integral along the unit circle in the plane orthogonal to $\mathbf{R}$ (see, e.g., [9,11,33])

$$\int_{-\infty}^{\infty} G_{KM}(\mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{R}} d\mathbf{k} = \frac{\pi}{R} \int_0^{2\pi} G_{KM}(\mathbf{n}) d\phi. \quad (47)$$

Here, $\mathbf{n}$ is a unit vector “scanning” the circle of integration and remaining orthogonal to $\mathbf{R}$ thus $\mathbf{n} \cdot \mathbf{R} = 0$ and $R = |\mathbf{R}|$. Note that $\mathbf{n}$ is a function of ϕ . Combining Equations (40), (45) and (47), the $(n+1) \times (n+1)$ Green tensor $G_{KM}(\mathbf{R})$ of piezoelectric quasicrystals can be written in the form

$$G_{KM}(\mathbf{R}) = \frac{1}{8\pi^2 R} \int_0^{2\pi} (n C n)_{KM}^{-1} d\phi, \quad (48)$$

which represents the *three-dimensional Green tensor of piezoelectric quasicrystals*. One can see that the Green tensor (48) is the generalisation of the three-dimensional Green tensors of the anisotropic elasticity (see, e.g., [9–11,48,49]), of piezoelectricity (see, e.g., [19–21]) and of the elasticity of quasicrystals [33] towards the piezoelectricity of quasicrystals. The integral in Equation (48) can be computed by standard numerical methods if C_{IjKm} is given.

The Green tensor $G_{KM}(\mathbf{R})$ of piezoelectric quasicrystals can be decomposed into its phonon, phason and electric parts, and their coupling

$$G_{KM}(\mathbf{R}) = \begin{pmatrix} G_{km}^{\parallel\parallel}(\mathbf{R}) & G_{k\mu}^{\parallel\perp}(\mathbf{R}) & G_{kn+1}^{\parallel\zeta}(\mathbf{R}) \\ G_{\gamma m}^{\perp\parallel}(\mathbf{R}) & G_{\gamma\mu}^{\perp\perp}(\mathbf{R}) & G_{kn+1}^{\perp\zeta}(\mathbf{R}) \\ G_{n+1m}^{\zeta\parallel}(\mathbf{R}) & G_{n+1\mu}^{\zeta\perp}(\mathbf{R}) & G_{n+1n+1}^{\zeta\zeta}(\mathbf{R}) \end{pmatrix} \quad (49)$$

with the symbol ζ denoting the electric contribution, and it has the following physical interpretation:

$G_{km}^{\parallel\parallel}(\mathbf{R})$ is the phonon displacement at x in the e_k caused by a unit phonon point force at x' in the e_m direction, $k, m = 1, 2, 3$;

$G_{k\mu}^{\parallel\perp}(\mathbf{R})$ is the phonon displacement at x in the e_k direction caused by a unit phason point force at x' in the e_μ direction, $k = 1, 2, 3$, $\mu = 4, \dots, n$;

$G_{kn+1}^{\parallel\zeta}(\mathbf{R})$ is the phonon displacement at x in the e_k direction caused by a unit electric point charge at x' , $k = 1, 2, 3$;

$G_{\gamma m}^{\perp\parallel}(\mathbf{R})$ is the phason displacement at x in the e_γ direction caused by a unit phonon point force at x' in the e_m direction, $m = 1, 2, 3$, $\gamma = 4, \dots, n$;

$G_{\gamma\mu}^{\perp\perp}(\mathbf{R})$ is the phason displacement at x in the e_γ direction caused by a unit phason point force at x' in the e_μ direction, $\gamma, \mu = 4, \dots, n$;

$G_{\gamma n+1}^{\perp\zeta}(\mathbf{R})$ is the phason displacement at x in the e_γ direction caused by a unit electric point charge at x' , $\gamma = 4, \dots, n$;

$G_{n+1m}^{\zeta\parallel}(\mathbf{R})$ is the electrostatic potential at x caused by a unit phonon point force at x' in the e_m direction, $m = 1, 2, 3$;

$G_{n+1\mu}^{\zeta\perp}(\mathbf{R})$ is the electrostatic potential at x caused by a unit phason point force at x' in the e_μ direction, $\mu = 4, \dots, n$;

$G_{n+1n+1}^{\zeta\zeta}(\mathbf{R})$ is the electrostatic potential at x caused by a unit electric point charge at x' .

4.2. “Displacement”, “Distortion” and “Stress” Multifields in the Presence of a “Force” Multifield

The three-dimensional Green tensor (48) of piezoelectric quasicrystals can be applied to calculate the phonon, phason and electric fields caused by phonon and phason body forces and a body charge in homogeneous piezoelectric quasicrystalline materials.

For an arbitrary “body force density” multifield vector F_M (see Equation (29)), the solution of the partial differential Equation (36) can be written as

$$U_K = G_{KM} * F_M, \quad (50)$$

where $*$ denotes the three-dimensional spatial convolution. Using the Green tensor (48), Equation (50) gives the “displacement” multifield vector caused by a “body force density” multifield in a piezoelectric quasicrystal

$$U_K(x) = \frac{1}{8\pi^2} \int_V \frac{F_M(x')}{R} \left(\int_0^{2\pi} (nCn)_{KM}^{-1} d\phi \right) dV'. \quad (51)$$

It is worth noticing that Equation (51) represents the generalisation towards the piezoelectricity of the quasicrystals in the following problems:

- The solution of the displacement for an arbitrary body force density known from anisotropic elasticity [9];

- The solution of the electrostatic potential for an arbitrary body charge density known from electrostatics [41];
- The solution of the “extended” displacement for an arbitrary “extended” force density known from piezoelectricity [21].

Combining Equations (27) and (50), the “distortion” multifield tensor reads in terms of the gradient of the Green tensor (48)

$$B_{KI} = G_{KM,I} * F_M, \quad (52)$$

where the spatial gradient of the Green tensor is given by (see also [33])

$$G_{KM,I}(\mathbf{R}) = -\frac{1}{8\pi^2 R^2} \int_0^{2\pi} \left(\tau_I (nCn)_{KM}^{-1} - n_I F_{KM} \right) d\phi \quad (53)$$

with

$$F_{KM} = (nCn)_{KN}^{-1} [(nC\tau)_{NP} + (\tau Cn)_{NP}] (nCn)_{PM}^{-1} \quad (54)$$

and $\tau = \mathbf{R}/R$ is the unit vector along $\mathbf{R}$. Substituting Equation (53) into Equation (52), we obtain the “distortion” multifield tensor caused by a “body force density” multifield in a piezoelectric quasicrystal

$$B_{KI}(\mathbf{x}) = -\frac{1}{8\pi^2} \int_V \frac{F_M(\mathbf{x}')}{R^2} \left(\int_0^{2\pi} \left(\tau_I (nCn)_{KM}^{-1} - n_I F_{KM} \right) d\phi \right) dV'. \quad (55)$$

Using the constitutive relation (34), the corresponding “stress” multifield tensor for piezoelectric quasicrystals reads

$$\Sigma_{Ij}(\mathbf{x}) = -\frac{1}{8\pi^2} \int_V \frac{F_M(\mathbf{x}')}{R^2} \left(\int_0^{2\pi} C_{IjKI} \left(\tau_I (nCn)_{KM}^{-1} - n_I F_{KM} \right) d\phi \right) dV'. \quad (56)$$

Generalised Kelvin Problem in Piezoelectric Quasicrystals

Here, we examine the generalised Kelvin problem towards piezoelectric quasicrystals. We specify a Kelvin-type “force” density multifield vector with strength $\hat{f}_M$

$$F_M(\mathbf{x}') = \hat{f}_M \delta(\mathbf{x}'), \quad (57)$$

which is a point “force” density multifield in three dimensions. Then, Equation (50) becomes

$$U_K(\mathbf{x}) = \hat{f}_M G_{KM} \quad (58)$$

and Equation (51) reduces to

$$U_K(\mathbf{x}) = \frac{\hat{f}_M}{8\pi^2 r} \int_0^{2\pi} (nCn)_{KM}^{-1} d\phi, \quad (59)$$

where $r = |\mathbf{x}| = \sqrt{x^2 + y^2 + z^2}$. Equation (59) is the Kelvin-type “force” multifield solution for the “displacement” multifield vector for piezoelectric quasicrystals. In this case, Equation (55) reduces to

$$B_{KI}(\mathbf{x}) = -\frac{\hat{f}_M}{8\pi^2 r^2} \int_0^{2\pi} \left(\tau_I (nCn)_{KM}^{-1} - n_I F_{KM} \right) d\phi, \quad (60)$$

which is the Kelvin-type “force” multifield solution for the “distortion” multifield tensor for piezoelectric quasicrystals. Analogously, Equation (56) yields to

$$\Sigma_{Ij}(\mathbf{x}) = -\frac{\hat{f}_M}{8\pi^2 r^2} \int_0^{2\pi} C_{Ijkl} \left(\tau_l (nCn)_{KM}^{-1} - n_l F_{KM} \right) d\phi, \quad (61)$$

which is the Kelvin-type “force” multifield solution for the “stress” multifield tensor for piezoelectric quasicrystals.

It is interesting to examine that for some particular cases, the solution (59) leads to well-known results from classical anisotropic elasticity, isotropic elasticity and electrostatics. Indeed:

- If the phason modes and the electric effects are absent, that is, considering only classical anisotropic elasticity, then Equation (59) reads

$$u_k(\mathbf{x}) = \frac{\hat{f}_m}{8\pi^2 r} \int_0^{2\pi} (nCn)_{km}^{-1} d\phi, \quad (62)$$

which is the displacement vector caused by a point force with strength $\hat{f}_m$ in anisotropic elasticity (see, e.g., [9]).

Next, if we consider additionally that the medium is isotropic with the tensor of elastic moduli to be given by

$$C_{ijkl} = \frac{2\mu\nu}{1-2\nu} \delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \quad (63)$$

where μ is the shear modulus and ν is the Poisson ratio, then Equation (62) reduces to

$$u_k(\mathbf{x}) = \frac{\hat{f}_m}{16\pi\mu(1-\nu)r} \left((3-4\nu)\delta_{km} + \frac{x_k x_m}{r^2} \right), \quad (64)$$

which is the well-known Kelvin solution of a point force with strength $\hat{f}_m$ in an infinite isotropic body (see, e.g., [5,50]).

- If only electric effects are considered, then Equation (59) gives

$$\varphi(\mathbf{x}) = \frac{\hat{q}}{8\pi^2 r} \int_0^{2\pi} (n\epsilon n)^{-1} d\phi, \quad (65)$$

which is an integral representation of the electrostatic potential φ caused by a point charge with strength $\hat{q}$ in anisotropic electrostatics.

Next, if we consider the tensor of the dielectric permittivity $\epsilon_{ij} = \delta_{ij}\epsilon_0$ with ϵ_0 the permittivity of the free space, then Equation (65) reduces to the well-known formula of the electrostatic potential of a point charge with strength $\hat{q}$ (see, e.g., [41])

$$\varphi(\mathbf{x}) = \frac{\hat{q}}{4\pi\epsilon_0} \frac{1}{r}. \quad (66)$$

4.3. Two-Dimensional Green Tensor of Piezoelectric Quasicrystals

In this section, the two-dimensional Green tensor for arbitrary piezoelectric quasicrystals is derived. Moreover, the two-dimensional Green tensor for arbitrary quasicrystals in the framework of elasticity is given here for the first time.

The Green tensor $G_{KM}(\mathbf{R})$ with $K, M = 1, \dots, n+1$ of the two-dimensional Navier-type equation is defined by

$$C_{Ijkl} G_{KM,lj}(\mathbf{R}) + \delta_{IM} \delta(\mathbf{R}) = 0, \quad (67)$$

where $I, K, M = 1, \dots, n+1, j, l = 1, 2, \mathbf{R} = \mathbf{x} - \mathbf{x}' \in \mathbb{R}^2$, and $\delta(\mathbf{R})$ is the two-dimensional Dirac delta function. The two-dimensional Fourier transform of the Green tensor in Equation (67) reads ($\mathbf{k} \in \mathbb{R}^2$)

$$G_{KM}(\mathbf{k}) = \frac{1}{k^2} (\kappa C \kappa)_{KM}^{-1}, \quad (68)$$

where κ denotes the unit vector in the $k_1 k_2$ -plane of the two-dimensional Fourier space (κ_j with $j = 1, 2$) defined by $\kappa = \mathbf{k}/k$ with $k = |\mathbf{k}|$. Here, the matrix $(\kappa C \kappa)_{KM}^{-1}$ is the inverse of $(\kappa C \kappa)_{KM}$ given in Equation (44) with $j, l = 1, 2$.

The two-dimensional Green tensor in the two-dimensional real space is obtained by the inverse Fourier transform of Equation (68) as follows:

$$\begin{aligned} G_{KM}(\mathbf{R}) &= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \frac{(\kappa C \kappa)_{KM}^{-1} \cos(\mathbf{k} \cdot \mathbf{R})}{k^2} d\mathbf{k} \\ &= \frac{1}{(2\pi)^2} \int_0^{2\pi} (\kappa C \kappa)_{KM}^{-1} \int_0^\infty \frac{\cos(k \kappa \cdot \mathbf{R})}{k} dk d\phi. \end{aligned} \quad (69)$$

In Equation (69), $d\mathbf{k} = k dk d\phi$ indicates the two-dimensional volume element in Fourier space in polar coordinates, and ϕ ($0 < \phi \leq 2\pi$) is an appropriate polar angle scanning a unit circle $\kappa^2 = 1$. The integration in k is performed using the relation

$$\int_0^\infty \frac{\cos(k \kappa \cdot \mathbf{R})}{k} dk = \frac{1}{2} \int_{-\infty}^\infty \frac{1}{k} e^{ik \kappa \cdot \mathbf{R}} dk \quad (70)$$

and the principal value integral [47]

$$\mathcal{P} \int_{-\infty}^\infty \frac{1}{k} e^{ikt} dk = -2\gamma_E - 2 \ln |t|, \quad (71)$$

where γ_E denotes the Euler constant and $\mathcal{P}$ means the principal value. The constant term can be neglected for the Green tensor (see, e.g., [47]). Using Equations (70) and (71), the two-dimensional Green tensor (69) can be expressed as integral over the unit circle in Fourier space

$$G_{KM}(\mathbf{R}) = -\frac{1}{(2\pi)^2} \int_0^{2\pi} (\kappa C \kappa)_{KM}^{-1} \ln |\kappa \cdot \mathbf{R}| d\phi. \quad (72)$$

Because $\kappa(\phi)$ varies with ϕ , the integrand $(\kappa C \kappa)_{KM}^{-1}$ is a function of the integration variable ϕ and $\mathbf{n} \equiv \kappa(\phi)$, and the $(n+1) \times (n+1)$ Green tensor $G_{KM}(\mathbf{R})$ becomes

$$G_{KM}(\mathbf{R}) = -\frac{1}{(2\pi)^2} \int_0^{2\pi} (n C n)_{KM}^{-1} \ln |\mathbf{n} \cdot \mathbf{R}| d\phi, \quad (73)$$

which represents the *two-dimensional Green tensor of piezoelectric quasicrystals*.

If the piezoelectric effects are not considered, that is, if only the elasticity of quasicrystals is considered, then the piezoelectric $(n+1) \times (n+1)$ Green tensor $G_{KM}(\mathbf{R})$, $K, M = 1, \dots, n+1$, Equation (73), reduces to the elastic $n \times n$ Green tensor of quasicrystals

$$G_{KM}(\mathbf{R}) = -\frac{1}{(2\pi)^2} \int_0^{2\pi} (n C n)_{KM}^{-1} \ln |\mathbf{n} \cdot \mathbf{R}| d\phi \quad (74)$$

with $K, M = 1, \dots, n$. Equation (74) represents the *two-dimensional Green tensor of quasicrystals*.

In the limit to piezoelectricity, that is, if the phason modes are absent, then the Green tensor (73) reduces to the two-dimensional Green tensor of piezoelectricity

$$G_{KM}(\mathbf{R}) = -\frac{1}{(2\pi)^2} \int_0^{2\pi} (nCn)_{\bar{K}\bar{M}}^{-1} \ln |\mathbf{n} \cdot \mathbf{R}| d\phi \quad (75)$$

with $K, M = 1, \dots, 4$, which is in agreement with the two-dimensional Green tensor of piezoelectricity given by Wang [23].

In the limit to anisotropic elasticity, that is, if the phason modes and electric fields are absent, then the Green tensor (73) reduces to the two-dimensional Green tensor of anisotropic elasticity

$$G_{km}(\mathbf{R}) = -\frac{1}{(2\pi)^2} \int_0^{2\pi} (nCn)_{\bar{k}\bar{m}}^{-1} \ln |\mathbf{n} \cdot \mathbf{R}| d\phi \quad (76)$$

with $k, m = 1, 2, 3$. Note that the Green tensor (76) is in agreement with the two-dimensional Green tensor of anisotropic elasticity given by Wang and Achenbach [12], Wang [13] and Lazar [14].

5. Conclusions

In this work, the three-dimensional Green tensor and its spatial gradient have been calculated for arbitrary piezoelectric quasicrystals. With this aim, the hyperspace notation for piezoelectric quasicrystals has been developed, and the corresponding multifields have been defined. Using the three-dimensional Green tensor, we have given a solution for the “displacement”, “distortion” and “stress” multifields in the presence of a “force” multifield in a piezoelectric quasicrystal as well as the solution of the generalised Kelvin problem. These solutions allow for the consideration of all elastic, dielectric and piezoelectric material properties in piezoelectric quasicrystals. Moreover, the two-dimensional Green tensors of piezoelectric quasicrystals and of quasicrystals have been calculated. The derived Green tensors may be used for the study of dislocation, fracture, interface and inclusion problems in piezoelectric quasicrystals.

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Appendix A. The Symmetry of the Multifield Tensor of the Material Moduli C_{IjKl}

In this Appendix, the major symmetry (see Equation (31)) of the multifield tensor of the material moduli C_{IjKl} in the hyperspace notation is proved. Indeed,

$$C_{KlIj} = \begin{cases} C_{klIj}, & K = 1, 2, 3; & I = 1, 2, 3, \\ D_{kl\alpha j}, & K = 1, 2, 3; & I = 4, \dots, n, \\ e_{jkl}, & K = 1, 2, 3; & I = n + 1, \\ D_{ij\gamma l}, & K = 4, \dots, n; & I = 1, 2, 3, \\ E_{\gamma l\alpha j}, & K = 4, \dots, n; & I = 4, \dots, n, \\ f_{j\gamma l}, & K = 4, \dots, n; & I = n + 1, \\ e_{lIj}, & K = n + 1; & I = 1, 2, 3, \\ f_{l\alpha j}, & K = n + 1; & I = 4, \dots, n, \\ -\varepsilon_{lj}, & K = n + 1; & I = n + 1, \end{cases} \quad (A1)$$

which corresponds to the electric enthalpy density

$$\begin{aligned}\Psi = & \frac{1}{2} C_{klij} e_{kl}^{\parallel} e_{ij}^{\parallel} + \frac{1}{2} D_{kl\alpha j} e_{kl}^{\parallel} \beta_{\alpha j}^{\perp} - \frac{1}{2} e_{jkl} E_j e_{kl}^{\parallel} \\ & + \frac{1}{2} D_{ij\gamma l} e_{ij}^{\parallel} \beta_{\gamma l}^{\perp} + \frac{1}{2} E_{\gamma l \alpha j} \beta_{\gamma l}^{\perp} \beta_{\alpha j}^{\perp} - \frac{1}{2} f_{j\gamma l} E_j \beta_{\gamma l}^{\perp} \\ & - \frac{1}{2} e_{lij} E_l e_{ij}^{\parallel} - \frac{1}{2} f_{l\alpha j} E_l \beta_{\alpha j}^{\perp} - \frac{1}{2} \varepsilon_{lj} E_l E_j.\end{aligned}\quad (A2)$$

Making use only of the major symmetries $C_{ijkl} = C_{klij}$, $E_{\gamma l \alpha j} = E_{\alpha j \gamma l}$ and $\varepsilon_{lj} = \varepsilon_{jl}$, it is easy to see that

$$C_{KLIj} = \begin{cases} C_{ijkl}, & I = 1, 2, 3; \quad K = 1, 2, 3, \\ D_{kl\alpha j}, & I = 4, \dots, n; \quad K = 1, 2, 3, \\ e_{jkl}, & I = n + 1; \quad K = 1, 2, 3, \\ D_{ij\gamma l}, & I = 1, 2, 3; \quad K = 4, \dots, n, \\ E_{\alpha j \gamma l}, & I = 4, \dots, n; \quad K = 4, \dots, n, \\ f_{j\gamma l}, & I = n + 1; \quad K = 4, \dots, n, \\ e_{lij}, & I = 1, 2, 3; \quad K = n + 1, \\ f_{l\alpha j}, & I = 4, \dots, n; \quad K = n + 1, \\ -\varepsilon_{jl}, & I = n + 1; \quad K = n + 1, \end{cases} = C_{IjKL}. \quad (A3)$$

Appendix B. The Multifield Tensor of the Material Moduli C_{IjKL}

In this Appendix, we give the multifield tensor of the material moduli C_{IjKL} in the hyperspace notation explicitly for one-dimensional, two-dimensional and three-dimensional piezoelectric quasicrystals.

(i) For one-dimensional piezoelectric quasicrystals

$$C_{IjKL} = \begin{cases} C_{ijkl}, & I = 1, 2, 3; \quad K = 1, 2, 3, \\ D_{ij\gamma l}, & I = 1, 2, 3; \quad K = 4, \\ e_{lij}, & I = 1, 2, 3; \quad K = 5, \\ D_{kl\alpha j}, & I = 4; \quad K = 1, 2, 3, \\ E_{\alpha j \gamma l}, & I = 4; \quad K = 4, \\ f_{l\alpha j}, & I = 4; \quad K = 5, \\ e_{jkl}, & I = 5; \quad K = 1, 2, 3, \\ f_{j\gamma l}, & I = 5; \quad K = 4, \\ -\varepsilon_{jl}, & I = 5; \quad K = 5, \end{cases} \quad (A4)$$

where $i, j, k, l = 1, 2, 3$ and $\alpha, \gamma = 3$.

(ii) For two-dimensional piezoelectric quasicrystals,

$$C_{IjKL} = \begin{cases} C_{ijkl}, & I = 1, 2, 3; \quad K = 1, 2, 3, \\ D_{ij\gamma l}, & I = 1, 2, 3; \quad K = 4, 5, \\ e_{lij}, & I = 1, 2, 3; \quad K = 6, \\ D_{kl\alpha j}, & I = 4, 5; \quad K = 1, 2, 3, \\ E_{\alpha j \gamma l}, & I = 4, 5; \quad K = 4, 5, \\ f_{l\alpha j}, & I = 4, 5; \quad K = 6, \\ e_{jkl}, & I = 6; \quad K = 1, 2, 3, \\ f_{j\gamma l}, & I = 6; \quad K = 4, 5, \\ -\varepsilon_{jl}, & I = 6; \quad K = 6, \end{cases} \quad (A5)$$

where $i, j, k, l = 1, 2, 3$ and $\alpha, \gamma = 1, 2$.

(iii) For three-dimensional piezoelectric quasicrystals,

$$C_{Ijkl} = \begin{cases} C_{ijkl}, & I = 1, 2, 3; \quad K = 1, 2, 3, \\ D_{ij\gamma l}, & I = 1, 2, 3; \quad K = 4, 5, 6, \\ e_{lij}, & I = 1, 2, 3; \quad K = 7, \\ D_{kl\alpha j}, & I = 4, 5, 6; \quad K = 1, 2, 3, \\ E_{\alpha j\gamma l}, & I = 4, 5, 6; \quad K = 4, 5, 6, \\ f_{l\alpha j}, & I = 4, 5, 6; \quad K = 7, \\ e_{jkl}, & I = 7; \quad K = 1, 2, 3, \\ f_{j\gamma l}, & I = 7; \quad K = 4, 5, 6, \\ -\varepsilon_{jl}, & I = 7; \quad K = 7, \end{cases} \quad (A6)$$

where $i, j, k, l = 1, 2, 3$ and $\alpha, \gamma = 1, 2, 3$.

References

1. Cook-Chennault, K.A.; Thambi, N.; Sastry, A.M. Powering MEMS portable devices—A review of non-regenerative and regenerative power supply systems with special emphasis on piezoelectric energy harvesting systems. *Smart Mater. Struct.* **2008**, *17*, 043001. [\[CrossRef\]](#)
2. Schouten, J.A. *Tensor Analysis for Physicists*; Oxford University Press: Oxford, UK, 1951.
3. Nye, J.F. *Physical Properties of Crystals*; Oxford University Press: Oxford, UK, 1957.
4. Landau, L.D.; Lifshitz, E.M. *Electrodynamics of Continuous Media*; Pergamon Press: Oxford, UK, 1960.
5. Kelvin, L. *Mathematical and Physical Papers*; Cambridge University Press: Cambridge, UK, 1882; Volume 1, p. 97.
6. Lifshitz, I.M.; Rosenzweig, L.N. On the construction of the Green's tensor for the basic equation of the theory of elasticity of an anisotropic medium. *Zh. Eksp. Teor. Fiz.* **1947**, *17*, 783–791.
7. Kröner, E. Das Fundamentalintegral der anisotropen elastischen Differentialgleichungen. *Z. Phys.* **1953**, *136*, 402–410. [\[CrossRef\]](#)
8. Willis, J.R. The elastic interaction energy of dislocation loops in anisotropic media. *Quart. J. Mech. Appl. Math.* **1965**, *18*, 419–433. [\[CrossRef\]](#)
9. Synge, J.L. *The Hypercircle in Mathematical Physics*; Cambridge University Press: Cambridge, UK, 1957.
10. Barnett, D.M. The precise evaluation of derivatives of the anisotropic elastic Green's functions. *Phys. Status Solidi B* **1972**, *49*, 741–748. [\[CrossRef\]](#)
11. Teodosiu, C. *Elastic Models of Crystal Defects*; Springer: Berlin/Heidelberg, Germany, 1982.
12. Wang, C.-Y.; Achenbach, J.D. Elastodynamic fundamental solutions for anisotropic solids. *Geophys. J. Int.* **1994**, *118*, 384–392. [\[CrossRef\]](#)
13. Wang, C.-Y. Two-dimensional elastostatic Green's functions for general anisotropic solids and generalization of Stroh's formalism. *Int. J. Solids Struct.* **1994**, *31*, 2591–2597. [\[CrossRef\]](#)
14. Lazar, M. Displacements and stress functions of straight dislocations and line forces in anisotropic elasticity: A new derivation and its relation to the integral formalism. *Symmetry* **2021**, *13*, 1721. [\[CrossRef\]](#)
15. Deeg, W.F.J. The analysis of dislocation, crack, and inclusion problems in piezoelectric solids. Ph.D. Thesis, Stanford University, Stanford, CA, USA, 1980.
16. Chen, T. Green's functions and the non-uniform transformation problem in a piezo electric medium. *Mech. Res. Commun.* **1993**, *20*, 271–278. [\[CrossRef\]](#)
17. Dunn, M.L. Electroelastic Green's functions for transversely isotropic piezoelectric media and their application to the solution of inclusion and inhomogeneity problems. *Int. J. Eng. Sci.* **1994**, *32*, 119–131. [\[CrossRef\]](#)
18. Akamatsu, M.; Tanuma, K. Green's function of anisotropic piezoelectricity. *Proc. R. Soc. Lond. A* **1997**, *453*, 473–487. [\[CrossRef\]](#)
19. Pan, E.; Tonon, F. Three-dimensional Green's functions in anisotropic piezoelectric solids. *Int. J. Solids Struct.* **2000**, *37*, 943–958. [\[CrossRef\]](#)
20. Li, X.; Wang, M. Three-dimensional Green's functions for infinite anisotropic piezoelectric media. *Int. J. Solids Struct.* **2007**, *44*, 1680–1684. [\[CrossRef\]](#)
21. Nowacki, J. *Static and Dynamic Coupled Fields in Bodies with Piezoeffects or Polarization Gradient*; Lecture Notes in Applied and Computational Mechanics; Springer: Berlin/Heidelberg, Germany, 2006.
22. Michelitsch, T. Calculation of the electroelastic Green's function of the hexagonal infinite medium. *Z. Phys. B* **1997**, *104*, 497–503. [\[CrossRef\]](#)
23. Wang, C.-Y. Green's functions and general formalism for 2D piezoelectricity. *Appl. Math. Lett.* **1996**, *9*, 1–7. [\[CrossRef\]](#)
24. Shechtman, D.; Blech, I.; Gratias, D.; Cahn, J.W. Metallic phase with long-range orientational order and no translational symmetry. *Phys. Rev. Lett.* **1984**, *53*, 1951–1953. [\[CrossRef\]](#)
25. Bak, P. Symmetry, stability, and elastic properties of icosahedral incommensurate crystals. *Phys. Rev. B* **1985**, *32*, 5764–5772. [\[CrossRef\]](#)

26. Levine, D.; Lubensky, T.C.; Ostlund, S.; Ramaswamy, S.; Steinhardt, P.J.; Toner, J. Elasticity and dislocations in pentagonal and icosahedral quasicrystals. *Phys. Rev. Lett.* **1985**, *54*, 1520–1523. [\[CrossRef\]](#)
27. Hu, K.; Meguid, S.A.; Wang, L.; Jin, H. Electro-elastic field of a piezoelectric quasicrystal medium containing two cylindrical inclusions. *Acta Mech.* **2021**, *232*, 2513–2533. [\[CrossRef\]](#)
28. Wang, Y.; Liu, C.; Zhang, L.; Pan, E.; Gao, Y. Mechanical analysis of functionally graded multilayered two-dimensional decagonal piezoelectric quasicrystal laminates with imperfect interfaces. *Crystals* **2024**, *14*, 170. [\[CrossRef\]](#)
29. Wang, Y.; Liu, C.; Zhu, Z.; Zhang, L.; Gao, Y. Analysis of multilayered two-dimensional decagonal piezoelectric quasicrystal beams with mixed boundary conditions. *Eur. J. Mech. A-Solids* **2024**, *106*, 105333. [\[CrossRef\]](#)
30. De, P.; Pelcovits, R.A. Linear elasticity theory of pentagonal quasicrystals. *Phys. Rev. B* **1987**, *35*, 8609–8620. [\[CrossRef\]](#) [\[PubMed\]](#)
31. Ding, D.-H.; Wang, R.; Yang, W.; Hu, C. General expressions for the elastic displacement fields induced by dislocations in quasicrystals. *J. Phys. Condens. Matter* **1995**, *7*, 5423–5436. [\[CrossRef\]](#)
32. Bachteler, J.; Trebin, H.-R. Elastic Green's function of icosahedral quasicrystals. *Eur. Phys. J. B* **1998**, *4*, 299–306. [\[CrossRef\]](#)
33. Lazar, M.; Agiasofitou, E. Fundamentals in generalized elasticity and dislocation theory of quasicrystals: Green tensor, dislocation key-formulas and dislocation loops. *Philos. Mag.* **2014**, *94*, 4080–4101. [\[CrossRef\]](#)
34. Yang, W.; Wang, R.; Ding, D.-H.; Hu, C. Elastic strains induced by electric fields in quasicrystals. *J. Phys. Condens. Matter* **1995**, *7*, L499–L502. [\[CrossRef\]](#)
35. Hu, C.; Wang, R.; Ding, D.-H.; Yang, W. Piezoelectric effects in quasicrystals. *Phys. Rev. B* **1997**, *56*, 2463–2468. [\[CrossRef\]](#)
36. Agiasofitou, E.; Lazar, M. On the constitutive modelling of piezoelectric quasicrystals. *Crystals* **2023**, *13*, 1652. [\[CrossRef\]](#)
37. Li, X.Y.; Li, P.D.; Wu, T.H.; Shi, M.X.; Zhu, Z.W. Three-dimensional fundamental solutions for one-dimensional hexagonal quasicrystal with piezoelectric effect. *Phys. Lett. A* **2014**, *378*, 826–834. [\[CrossRef\]](#)
38. Fan, C.Y.; Li, Y.; Xu, G.T.; Zhao, M.H. Fundamental solutions and analysis of three-dimensional cracks in one-dimensional hexagonal piezoelectric quasicrystals. *Mech. Res. Commun.* **2016**, *74*, 39–44. [\[CrossRef\]](#)
39. Zhang, L.; Wu, D.; Xu, W.; Yang, L.; Ricoeur, A.; Wanf, Z.; Gao, Y. Green's functions of one-dimensional quasicrystal bi-material with piezoelectric effect. *Phys. Lett. A* **2016**, *380*, 3222–3228. [\[CrossRef\]](#)
40. Ding, D.-H.; Wang, W.; Hu, C.; Yang, R. Generalized elasticity theory of quasicrystals. *Phys. Rev. B* **1993**, *48*, 7003–7010. [\[CrossRef\]](#) [\[PubMed\]](#)
41. Jackson, J.D. *Classical Electrodynamics*, 2nd ed.; Wiley: New York, NY, USA, 1975.
42. Lazar, M.; Agiasofitou, E. Eshelbian mechanics of novel materials: Quasicrystals. *J. Micromech. Mol. Phys.* **2016**, *1*, 1640008. [\[CrossRef\]](#)
43. Barnett, D.M.; Lothe, J. Dislocations and line charges in anisotropic piezoelectric insulators. *Phys. Status Solidi B* **1975**, *67*, 105–111. [\[CrossRef\]](#)
44. Ericksen, J.L. Appendix, Tensor Fields. In *Encyclopedia of Physics Vol. III/1*; Flügge, S., Ed.; Springer: Berlin/Heidelberg, Germany, 1960; pp. 794–858.
45. Marsden, J.E.; Hughes, T.J.S. *Mathematical Foundations of Elasticity*; Prentice Hall: Upper Saddle River, NJ, USA, 1982.
46. Becker, A.A. *The Boundary Element Method in Engineering: A Complete Course*; McGraw-Hill: New York, NY, USA, 1992.
47. Vladimirov, V.S. *Equations of Mathematical Physics*; Marcel Dekker, Inc.: New York, NY, USA, 1971.
48. Lazar, M.; Kirchner, H.O.K. Dislocation loops in anisotropic elasticity: Displacement field, stress function tensor and interaction energy. *Philos. Mag.* **2013**, *93*, 174–185. [\[CrossRef\]](#)
49. Li, S.; Wang, G. *Introduction to Micromechanics and Nanomechanics*; World Scientific: Singapore, 2008.
50. Gurtin, M.E. The linear theory of elasticity. In *Handbuch der Physik VIa/2*; Flügge, S., Ed.; Springer: Berlin/Heidelberg, Germany, 1972; pp. 1–345.

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