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An Analysis Method of Symplectic Dual System for Decagonal Quasicrystal Plane Elasticity and Application

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Abstract: The symplectic solution system of decagonal quasicrystal elastic mechanics is considered. Hamiltonian dual equations together with the boundary conditions are investigated by utilizing the principle of minimum potential energy. Then the symplectic eigenvectors are given on the basis of the variable separation method. As application, analytical solution for decagonal quasicrystal cantilever beam with concentrated load is discussed. The analytical expressions of the stresses and displacements of the phonon field and phason field are obtained. The present method allows for the exploration of new analytic solutions of quasicrystal elasticity that are difficult to obtain by other analytic methods

Keywords: quasicrystal; Hamiltonian system; eigenvector; analytical solution

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1. Introduction

In 1984, Shechtman et al. initially discovered quasicrystals (QCs) [1]. QCs are non-periodic but ordered structural forms between crystals and glass. The elastic behavior of QCs varies from that of ordinary crystals [2]. Experimental and theoretical studies on mechanical and physical properties of QCs have been encouraged [3]. The dislocation of a straight and moving screw in one-dimensional hexagonal QCs was studied by Fan et al. [4]. The fracture mechanics problem of cubic QCs with a crack or an elliptical hole was solved by Gao et al. [5]. The Stroh formalism has been applied successfully to the study of two-dimensional deformation of quasicrystal materials [6,7]. One dimensional hexagonal QCs with planar cracks are mentioned in reference [8]. The Stroh-like formalism for the bending theory of decagonal quasicrystal plates was developed by Li et al. [9]. Fundamental quantities for the generalized elasticity and dislocation theory of QCs were provided by Lazar and Agiasofitou [10]. Ding et al. discussed two kinds of contact problems in three-dimensional icosahedral QCs by a complex variable function method [11]. Lazar and Agiasofitou derived material balance laws for quasicrystalline materials with dislocations [12]. The elastic field near the tip of an anticrack in a homogeneous decagonal quasicrystalline material was investigated by Wang et al. [13]. Guo et al. studied a mathematical model for nonlocal vibration and buckling of embedded two-dimensional decagonal quasicrystal layered nanoplates [14].

The Hamiltonian system exists extensively and is universally applicable. The symplectic approach is the variable separation method which is actually based on the Hamiltonian system [15]. This method does not assume the trial function in advance, but introduces the problem into the Hamiltonian system, and uses the variable separation method to solve the differential eigenvalue problem. Analytical solutions can then be obtained by the expansion of eigenfunctions. Many researchers use the symplectic method to study problems in mechanics and engineering science since it is helpful for finding analytical solutions of some basic elasticity problems. The symplectic approach was first used in computational solid mechanics by Feng [16]. Zhong introduced the symplectic approach in analytical solid mechanics [17]. In 2002, Zhong's group originated the symplectic elasticity approach and developed it to form a systematic methodology [18]. The symplectic approach was then used successfully to research elasticity [19,20], piezoelectricity [21,22], functionally graded effects [23] and differential equations [24–26], etc.

However, the symplectic approach for quasicrystal elasticity has not been developed in a systematic way due to the complexity of QCs' structure. In the present study, the basic equations of decagonal quasicrystals are first transferred to the Hamiltonian dual equations with the help of the variation principle. By introducing dual variables, the dual system is established directly, and a complete eigen-solution space is obtained. The solutions of the problem can be reduced to the zero-eigenvectors of the corresponding Hamiltonian operator matrix and all their Jordan form eigenvectors. Then the analytical solution for the problem is given by linear combination of these eigenvectors.

2. Theoretical Formulation

2.1. Basic Equations

Suppose that along the z direction, the decagonal quasicrystal is periodic, and quasi-periodic in the x-y plane. Based on the quasicrystal elasticity theory [2], the strain-displacement relations are

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), w_{ij} = \frac{\partial w_i}{\partial x_j}. \tag{1}$$

If ignoring the body forces, the equilibrium equations can be written as

$$\begin{gathered} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0, \frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0, \\ \frac{\partial H_{xx}}{\partial x} + \frac{\partial H_{xy}}{\partial y} = 0, \frac{\partial H_{yx}}{\partial x} + \frac{\partial H_{yy}}{\partial y} = 0. \end{gathered} \tag{2}$$

and the constitutive equations are as follows

$$\begin{aligned} \sigma_{xx} &= C_{11}\varepsilon_{xx} + C_{12}\varepsilon_{yy} + R(w_{xx} + w_{yy}), \\ \sigma_{yy} &= C_{12}\varepsilon_{xx} + C_{11}\varepsilon_{yy} - R(w_{xx} + w_{yy}), \\ \sigma_{xy} = \sigma_{yx} &= 2C_{66}\varepsilon_{xy} + R(w_{yx} - w_{xy}), \\ H_{xx} &= K_1 w_{xx} + K_2 w_{yy} + R(\varepsilon_{xx} - \varepsilon_{yy}), \\ H_{yy} &= K_1 w_{yy} + K_2 w_{xx} + R(\varepsilon_{xx} - \varepsilon_{yy}), \\ H_{xy} &= K_1 w_{xy} - K_2 w_{yx} - 2R\varepsilon_{xy}, \\ H_{yx} &= K_1 w_{yx} - K_2 w_{xy} + 2R\varepsilon_{xy}, \end{aligned} \tag{3}$$

in which $C_{66} = (C_{11} - C_{12})/2$. Here, $\sigma_{ij}(\sigma_{ij} = \sigma_{ji})$, u_i and $\varepsilon_{ij}(\varepsilon_{ij} = \varepsilon_{ji})$, respectively, represent the stress, displacement and strain of the phonon field. H_{ij}, w_i and w_{ij}, respectively, represent the stress, displacement and strain of the phason field. C_{ij} and K_i are the elastic constants of the phonon field and the phason field, respectively. R represents the coupling elastic constant of the phonon-phason field.

2.2. Variational Principle and Hamiltonian Dual Equation

Consider a decagonal quasicrystal rectangular domain with length l and width $2h$ in the x-y coordinates ($0 \leq x \leq l$ and $-h \leq y \leq h$). We use a dot to represent the differentiation with respect to x, i.e., $\dot{(.)} = \partial/\partial x$.

The strain energy density $U(u_x, u_y, w_x, w_y, \dot{u}_x, \dot{u}_y, \dot{w}_x, \dot{w}_y)$ can be expressed as

$$\begin{aligned}
&U(u_x, u_y, w_x, w_y, \dot{u}_x, \dot{u}_y, \dot{w}_x, \dot{w}_y) \\
&= \frac{1}{2}(\sigma_{xx}\varepsilon_{xx} + \sigma_{yy}\varepsilon_{yy} + 2\sigma_{xy}\varepsilon_{xy} + H_{xx}w_{xx} + H_{yy}w_{yy} + H_{xy}w_{xy} + H_{yx}w_{yx}) \\
&= \frac{1}{2}\left[C_{11}\left(\dot{u}_x^2 + \left(\frac{\partial u_y}{\partial y}\right)^2\right) + C_{66}\left(\frac{\partial u_x}{\partial y} + \dot{u}_y\right)^2 + K_1\left(\dot{w}_x^2 + \left(\frac{\partial w_x}{\partial y}\right)^2 + \dot{w}_y^2 + \left(\frac{\partial w_y}{\partial y}\right)^2\right)\right] \\
&+C_{12}\dot{u}_x\frac{\partial u_y}{\partial y} + R\left[\left(\dot{u}_x - \frac{\partial u_y}{\partial y}\right)\left(\dot{w}_x + \frac{\partial w_y}{\partial y}\right) + \left(\frac{\partial u_x}{\partial y} + \dot{u}_y\right)\left(\dot{w}_y - \frac{\partial w_x}{\partial y}\right)\right] \\
&+K_2\left(\dot{w}_x\frac{\partial w_y}{\partial y} - \frac{\partial w_x}{\partial y}\dot{w}_y\right).
\end{aligned} \tag{4}$$

The principle of minimum potential energy of the problem can be represented as

$$\delta\Pi = \delta\int_0^l \int_{-h}^h L(u_x, u_y, w_x, w_y, \dot{u}_x, \dot{u}_y, \dot{w}_x, \dot{w}_y)\mathrm{d}y\mathrm{d}x = 0 \tag{5}$$

where $L(u_x, u_y, w_x, w_y, \dot{u}_x, \dot{u}_y, \dot{w}_x, \dot{w}_y)$ is the Lagrange density function. If the body forces are neglected, we have

$$L(u_x, u_y, w_x, w_y, \dot{u}_x, \dot{u}_y, \dot{w}_x, \dot{w}_y) = U(u_x, u_y, w_x, w_y, \dot{u}_x, \dot{u}_y, \dot{w}_x, \dot{w}_y) \tag{6}$$

From the partial integration of Equation (5) in y direction, we can obtain

$$\left\{\begin{aligned}
\left[C_{12}\varepsilon_{xx} + C_{11}\varepsilon_{yy} - R(w_{xx} + w_{yy})\right]_{y=\pm h} &= \sigma_{yy}\big|_{y=\pm h}, \\
\left[2C_{66}\varepsilon_{xy} + R(w_{yx} - w_{xy})\right]_{y=\pm h} &= \sigma_{xy}\big|_{y=\pm h}, \\
\left[K_1 w_{yy} + K_2 w_{xx} + R(\varepsilon_{xx} - \varepsilon_{yy})\right]_{y=\pm h} &= H_{yy}\big|_{y=\pm h}, \\
\left[K_1 w_{xy} - K_2 w_{yx} - 2R\varepsilon_{xy}\right]_{y=\pm h} &= H_{xy}\big|_{y=\pm h}.
\end{aligned}\right. \tag{7}$$

Thus, the homogeneous boundary conditions are

$$\sigma_{yy} = 0,\ \sigma_{xy} = 0,\ H_{yy} = 0,\ H_{xy} = 0,\ \text{at}\ y = \pm h. \tag{8}$$

Let the displacement vector be

$$\mathbf{q} = (u_x, u_y, w_x, w_y)^{\mathrm{T}} \tag{9}$$

where the superscript T represents the transpose. Then the Lagrange density function can be written as $L(\mathbf{q}, \dot{\mathbf{q}})$, from which the dual variable $\mathbf{p}$ required by Hamiltonian form is derived as

$$\mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{q}}} = \begin{pmatrix}
C_{11}\dot{u}_x + C_{12}\frac{\partial u_y}{\partial y} + R(\dot{w}_x + \frac{\partial w_y}{\partial y}) \\
C_{66}(\frac{\partial u_x}{\partial y} + \dot{u}_y) + R(\dot{w}_y - \frac{\partial w_x}{\partial y}) \\
K_1\dot{w}_x + K_2\frac{\partial w_y}{\partial y} + R(\dot{u}_x - \frac{\partial u_y}{\partial y}) \\
K_1\dot{w}_y - K_2\frac{\partial w_x}{\partial y} + R(\frac{\partial u_x}{\partial y} + \dot{u}_y)
\end{pmatrix} = (\sigma_{xx}, \sigma_{yx}, H_{xx}, H_{yx})^{\mathrm{T}} \tag{10}$$

By Equations (2), (3) and (10), the Hamiltonian dual equation of the problem can be obtained as

$$\dot{v} = \mathbf{H}v \tag{11}$$

where $\mathbf{H} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & -\mathbf{A}^{\mathrm{T}} \end{bmatrix}$,$\mathbf{A} = \begin{bmatrix} 0 & a_1\frac{\partial}{\partial y} & 0 & a_2\frac{\partial}{\partial y} \\ -\frac{\partial}{\partial y} & 0 & a_3\frac{\partial}{\partial y} & 0 \\ 0 & a_4\frac{\partial}{\partial y} & 0 & a_5\frac{\partial}{\partial y} \\ 0 & 0 & a_6\frac{\partial}{\partial y} & 0 \end{bmatrix}$,$\mathbf{B} = \begin{bmatrix} b_1 & 0 & b_2 & 0 \\ 0 & b_3 & 0 & b_4 \\ b_2 & 0 & b_5 & 0 \\ 0 & b_4 & 0 & b_6 \end{bmatrix}$, $\mathbf{C} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & a_7\frac{\partial^2}{\partial y^2} & 0 & a_8\frac{\partial^2}{\partial y^2} \\ 0 & 0 & a_9\frac{\partial^2}{\partial y^2} & 0 \\ 0 & a_8\frac{\partial^2}{\partial y^2} & 0 & a_{10}\frac{\partial^2}{\partial y^2} \end{bmatrix}$, and $\boldsymbol{v} = (u_x, u_y, w_x, w_y, \sigma_{xx}, \sigma_{yx}, H_{xx}, H_{yx})^{\mathrm{T}}$. $\mathbf{A}^{\mathrm{T}}$ is the adjoint operator matrix of $\mathbf{A}$. $a_i (i = 1, 2, \cdots, 10)$ and $b_i (i = 1, 2, \cdots, 6)$ in the matrices are the elastic constants (see Appendix A). $\mathbf{H}$ satisfies $\mathbf{JHJ} = \mathbf{H}^{\mathrm{T}}$ is a Hamiltonian operator matrix [18], where $\mathbf{J} = \begin{bmatrix} \mathbf{0} & \mathbf{I}_4 \\ -\mathbf{I}_4 & \mathbf{0} \end{bmatrix}$ in which $\mathbf{I}_4$ is the 4×4 unit matrix. From Equations (3) and (11), Equation (8) can be written as

$$\begin{gathered} \sigma_{yy} = -a_7\frac{\partial u_y}{\partial y} - a_8\frac{\partial w_y}{\partial y} - a_1\sigma_{xx} - a_4 H_{xx} = 0\,, \\ \sigma_{xy} = 0\,, \\ H_{yy} = -a_8\frac{\partial u_y}{\partial y} - a_{10}\frac{\partial w_y}{\partial y} - a_2\sigma_{xx} - a_5 H_{xx} = 0\,, \\ H_{xy} = -a_9\frac{\partial w_x}{\partial y} - a_3\sigma_{yx} - a_6 H_{yx} = 0\,. \end{gathered} \tag{12}$$

at $y = \pm h$.

2.3. Symplectic Analysis and Eigenvectors

Let

$$\boldsymbol{v}(x, y) = X(x)\mathbf{Y}(y) \tag{13}$$

Substituting Equation (13) into Equation (11), we obtain

$$\dot{X}(x) = \mu X(x) \tag{14}$$

and the eigenvalue equation

$$\mathbf{H}\mathbf{Y}(y) = \mu\mathbf{Y}(y) \tag{15}$$

where μ and $\mathbf{Y}(y)$ are the eigenvalue and the corresponding eigenvector, respectively. Zero eigenvalue of the problem exists because the fact that boundaries at both sides ($y = \pm h$) are free. There are Jordan form eigen-solutions of different orders for the eigen-solutions of zero eigenvalue. The solution to the problem can be expressed by Jordan form eigen-solutions. Now the problem is to solve the zero-eigenvalue equation.

The eigenvalue equation is

$$\mathbf{H}\mathbf{Y}(y) = \mathbf{0} \tag{16}$$

when $\mu = 0$. Solving Equation (16) yields the eigen-solutions of zero eigenvalue

$$\mathbf{Y}_1^{(0)} = (1, 0, 0, 0, 0, 0, 0, 0)^{\mathrm{T}}, \tag{17}$$

$$\mathbf{Y}_2^{(0)} = (0, 1, 0, 0, 0, 0, 0, 0)^{\mathrm{T}}, \tag{18}$$

$$\mathbf{Y}_3^{(0)} = (0, 0, 1, 0, 0, 0, 0, 0)^{\mathrm{T}}, \tag{19}$$

$$\mathbf{Y}_4^{(0)} = (0, 0, 0, 1, 0, 0, 0, 0)^{\mathrm{T}}. \tag{20}$$

These eigenvectors are the solutions of the original Equation (11) with boundary conditions (8). Let

$$\boldsymbol{v}_1 = \mathbf{Y}_1^{(0)},\ \boldsymbol{v}_2 = \mathbf{Y}_2^{(0)},\ \boldsymbol{v}_3 = \mathbf{Y}_3^{(0)},\ \boldsymbol{v}_4 = \mathbf{Y}_4^{(0)}. \tag{21}$$

Next, solve the Jordan form eigen-solutions of zero eigenvalue.

2.3.1. The First-Order Jordan Form Eigen-Solutions

The governing equations for finding the first-order eigen-solutions of Jordan form are

$$\mathbf{H}\mathbf{Y}_i^{(1)} = \mathbf{Y}_i^{(0)}(i = 1, 2, 3, 4) \tag{22}$$

The solutions are

$$\mathbf{Y}_1^{(1)} = (0, a_1 y, 0, -a_4 y, -a_7, 0, a_8, 0)^{\mathrm{T}}, \tag{23}$$

$$\mathbf{Y}_2^{(1)} = (-y, 0, 0, 0, 0, 0, 0, 0)^{\mathrm{T}}, \tag{24}$$

$$\mathbf{Y}_3^{(1)} = (0, -a_2 y, 0, a_5 y, a_8, 0, -a_{10}, 0)^{\mathrm{T}}, \tag{25}$$

$$\mathbf{Y}_4^{(1)} = (-a_3 y, 0, a_6 y, 0, 0, 0, 0, -a_9)^{\mathrm{T}}. \tag{26}$$

These eigen-solutions are not directly the solutions of the original problem. The first-order Jordan form solutions of the original problem are

$$v_5 = \mathbf{Y}_1^{(1)} + x\mathbf{Y}_1^{(0)}, \tag{27}$$

$$v_6 = \mathbf{Y}_2^{(1)} + x\mathbf{Y}_2^{(0)}, \tag{28}$$

$$v_7 = \mathbf{Y}_3^{(1)} + x\mathbf{Y}_3^{(0)}, \tag{29}$$

$$v_8 = \mathbf{Y}_4^{(1)} + x\mathbf{Y}_4^{(0)}. \tag{30}$$

2.3.2. The SecondOrder Jordan Form Eigen-Solutions

Consider equations

$$\mathbf{H}\mathbf{Y}_i^{(2)} = \mathbf{Y}_i^{(1)}(i = 1, 2, 3, 4) \tag{31}$$

When $i = 2$, the solution of Equation (31) is

$$\mathbf{Y}_2^{(2)} = \left(0, -\frac{1}{2}a_1 y^2, 0, \frac{1}{2}a_4 y^2, a_7 y, 0, -a_8 y, 0\right)^{\mathrm{T}} \tag{32}$$

When $i = 1,3,4$, no solutions of Equation (31) exist due to the fact that solutions cannot satisfy the boundary conditions (8) at the same time. Hence, these chain of Jordan form eigen-solutions are terminated. The 2nd-order Jordan form solution of the original problem is

$$v_9 = \mathbf{Y}_2^{(2)} + x\mathbf{Y}_2^{(1)} + \frac{1}{2}x^2\mathbf{Y}_2^{(0)} \tag{33}$$

2.3.3. The Third-Order Jordan Form Eigen-Solutions

Solving equation

$$\mathbf{H}\mathbf{Y}_2^{(3)} = \mathbf{Y}_2^{(2)} \tag{34}$$

gives the eigen-solution

$$\mathbf{Y}_2^{(3)} = \left(a_{11}h^2 y + a_{12}y^3, 0, a_{13}h^2 y + \frac{1}{2}a_4 y^3, 0, 0, \frac{1}{2}a_7(h^2 - y^2), 0, a_{14}h^2 + a_{15}y^2\right)^{\mathrm{T}} \tag{35}$$

in which the constants $a_i(i = 11, 12, \cdots, 15)$ can be found in Appendix A. The 3rd-order Jordan form solution of the original problem can be composed in the same way

$$v_{10} = \mathbf{Y}_2^{(3)} + x\mathbf{Y}_2^{(2)} + \frac{1}{2}x^2\mathbf{Y}_2^{(1)} + \frac{1}{6}x^3\mathbf{Y}_2^{(0)} \tag{36}$$

It can be proven that there are no other high-order Jordan form solutions.

Up to here, the expressions of the general solution of Equation (11) can be written as

$$
\begin{aligned}
v &= \sum_{i=1}^{10} m_i v_i \\
&= \begin{pmatrix}
m_1 + m_5 x - (m_6 + a_3 m_8) y - m_9 x y + m_{10}\left(a_{11} h^2 y - \frac{1}{2} x^2 y + a_{12} y^3\right) \\
m_2 + m_6 x + (a_1 m_5 - a_2 m_7) y + m_9\left(\frac{1}{2} x^2 - \frac{a_1}{2} y^2\right) + m_{10}\left(\frac{1}{6} x^3 - \frac{a_1}{2} x y^2\right) \\
m_3 + m_7 x + a_6 m_8 y + m_{10}\left(a_{13} h^2 y + \frac{a_4}{2} y^3\right) \\
m_4 + m_8 x - (a_4 m_5 - a_5 m_7) y + \frac{a_4}{2} m_9 y^2 + \frac{a_4}{2} m_{10} x y^2 \\
-a_7 m_5 + a_8 m_7 + a_7 m_9 y + a_7 m_{10} x y \\
\frac{1}{2} a_7 m_{10}\left(h^2 - y^2\right) \\
a_8 m_5 - a_{10} m_7 - a_8 m_9 y - a_8 m_{10} x y \\
-a_9 m_8 + m_{10}\left(a_{14} h^2 + a_{15} y^2\right)
\end{pmatrix}
\end{aligned}
\tag{37}
$$

The constants $m_i (i = 1, 2, \cdots, 10)$ in Equation (37) can be determined according to the specific problem and boundary conditions. Then the stresses and displacements of the problem can be obtained.

3. Bending of Decagonal Quasicrystal Cantilever Beam with Concentrated Load

As an application of the symplectic approach for quasicrystal elasticity, an analytical solution for bending of a decagonal quasicrystal cantilever beam is discussed. Consider a decagonal quasicrystal cantilever beam with length l, width $2h$ and thickness b, which is under concentrated load P at the free end as pictured in Figure 1.

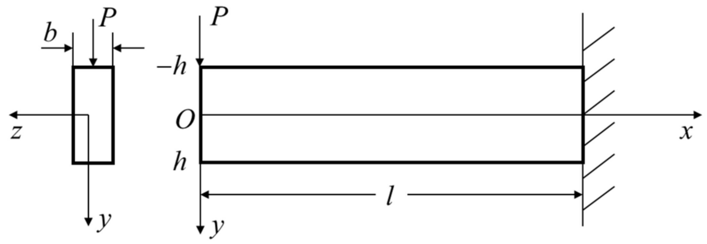

Figure 1. Decagonal quasicrystal cantilever beam.

The boundary conditions can be expressed as

$$
\begin{aligned}
&\sigma_{yy} = \sigma_{yx} = H_{yy} = H_{yx} = 0, \text{ for } y = \pm h, \\
&\sigma_{xx} = H_{xx} = 0, \; b\int_{-h}^{h} \sigma_{yx} \mathrm{d}y = -P, \text{ for } x = 0,
\end{aligned}
\tag{38}
$$

$$
u_x = u_y = w_x = w_y = 0, \; \frac{\partial u_y}{\partial x} = 0, \text{ for } y = 0 \text{ and } x = l.
$$

Substituting Equation (37) into the displacement and stress boundary conditions in Equation (38) respectively, we have

$$
\begin{aligned}
&m_2 = -\frac{l^3 P}{2 a_7 b h^3}, \; m_4 = -\frac{3 b_4 l P}{8 b h}, \; m_6 = \frac{3 l^2 P}{4 a_7 b h^3}, \\
&m_1 = m_3 = 0.
\end{aligned}
\tag{39}
$$

and

$$
\begin{aligned}
&m_8 = \frac{3 b_4 P}{8 b h}, \; m_{10} = -\frac{3P}{2 a_7 b h^3}, \\
&m_5 = m_7 = m_9 = 0.
\end{aligned}
\tag{40}
$$

Thus, the phonon and phason stresses are obtained as

$$\begin{aligned} &\sigma_{xx} = -\frac{3P}{2bh^3}xy\,,\ \sigma_{yy} = 0\,,\ \sigma_{xy} = \sigma_{yx} = \frac{3P}{4bh^3}(y^2 - h^2)\,, \\ &H_{xx} = \frac{3a_8P}{2a_7bh^3}xy\,,\ H_{yy} = 0\,, \\ &H_{xy} = \frac{3a_3P}{8bh^3}(y^2 - h^2)\,,\ H_{yx} = -\frac{3a_{15}P}{2a_7bh^3}(y^2 - h^2)\,. \end{aligned} \tag{41}$$

The displacements are obtained as

$$\begin{aligned} &u_x = \frac{3P}{4a_7bh^3}x^2y - \frac{3a_{12}P}{2a_7bh^3}y^3 - \frac{3P}{4a_7bh^3}l^2y - \frac{3b_3P}{4bh}y\,, \\ &u_y = \frac{3a_1P}{4a_7bh^3}xy^2 - \frac{P}{4a_7bh^3}x^3 + \frac{3P}{4a_7bh^3}l^2x - \frac{2P}{4a_7bh^3}l^3\,, \\ &w_x = -\frac{3b_4P}{8bh^3}y^3 + \frac{9b_4P}{8bh}y\,, \\ &w_y = -\frac{3b_4P}{8bh^3}xy^2 + \frac{3b_4P}{8bh}x - \frac{3b_4P}{8bh}l\,. \end{aligned} \tag{42}$$

Equation (41) shows that the expressions of the phonon stresses are exactly same as the stresses of the well-known classical elasticity theory [27].

The coupling constant R has great influence on the mechanical behaviors of QCs, and it has not been measured yet. Next, we consider the influence of the coupling elastic constant of the phonon-phason field on the displacements of phonon field and phason field. The phonon and phason elastic constants of the two-dimensional decagonal quasicrystals are $C_{11} = 234.33\text{GPa}$, $C_{12} = 57.41\text{GPa}$, $K_1 = 122\text{GPa}$ and $K_2 = 24\text{GPa}$ [28]. Take other parameters as $P = 200\text{KN}$, $l = 1\text{m}$, $h = 0.08\text{m}$ and $b = 0.1\text{m}$.

$\overline{u}_x = u_x/\left(10^{-3}\text{m}\right)$, $\overline{u}_y = u_y/\left(10^{-3}\text{m}\right)$, $\overline{w}_x = w_x/\left(10^{-4}\text{m}\right)$ and $\overline{w}_y = w_y/\left(10^{-4}\text{m}\right)$ are normalized displacements of phonon field and phason field, respectively. Figures 2 and 3 show that the displacements of phonon field and phason field both increase with the increase of the coupling constant R. The phonon field displacement is one order of magnitude larger than the phase field displacement.

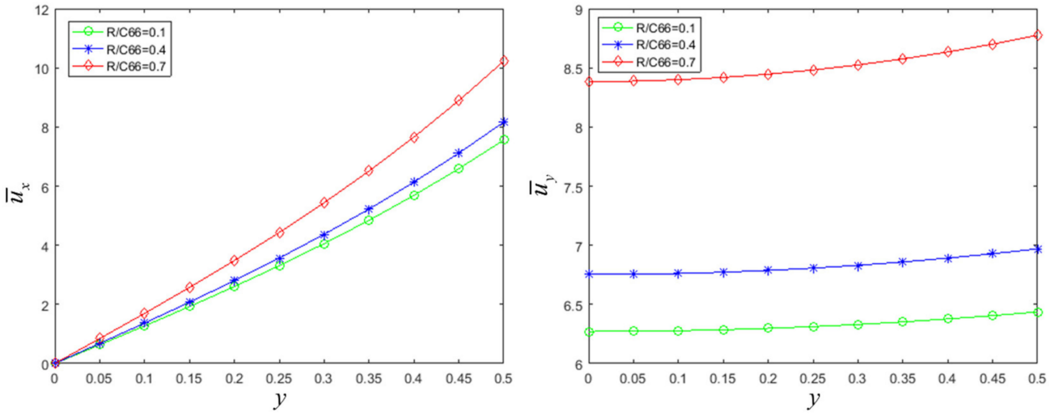

Figure 2. Normalized displacements of phonon field versus y at $x = 0.2$.

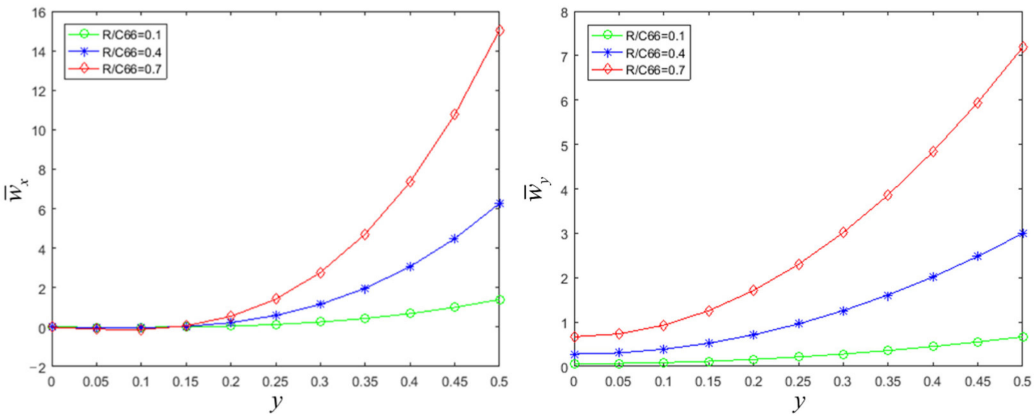

Figure 3. Normalized displacements of phason field versus y at $x = 0.2$.

4. Conclusions

The unified framework of the symplectic approach for quasicrystal elasticity problems is established. The problem is reduced to the zero eigenvalues with their Jordan forms, which are important in applications. Through working out an eigen-problem of the symplectic dual system, the solution of the Hamiltonian dual equation is obtained. The analytical solutions are obtained in a rigorous step-by-step manner, which is fundamentally different from the classical semi-inverse method with pre-determined trial functions. The stress and displacement can be calculated together. The symplectic approach is effective and provides a new channel for the research of quasicrystal elasticity theory.

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Conflicts of Interest: The authors declare no conflict of interest.

Appendix A

$$a_1 = -\frac{C_{12}K_1+R^2}{C_{11}K_1-R^2}, a_2 = \frac{R(K_2-K_1)}{C_{11}K_1-R^2}, a_3 = \frac{R(K_1-K_2)}{C_{66}K_1-R^2},$$

$$a_4 = \frac{R(C_{11}+C_{12})}{C_{11}K_1-R^2}, a_5 = -\frac{C_{11}K_2-R^2}{C_{11}K_1-R^2}, a_6 = \frac{C_{66}K_2-R^2}{C_{66}K_1-R^2},$$

$$a_7 = \frac{2a_4}{b_4}, a_8 = -\frac{a_3a_4}{b_4}, a_9 = \frac{a_3(1+a_6)}{b_4}, a_{10} = \frac{a_2(a_5-1)}{b_2},$$

$$a_{11} = -\frac{(C_{11}+C_{12})\left(2C_{66}K_1^2-(K_1+K_2)R^2\right)}{2(C_{11}K_1-R^2)(C_{66}K_1-R^2)}, a_{12} = \frac{2C_{11}K_1+C_{12}K_1-R^2}{6(C_{11}K_1-R^2)},$$

$$a_{13} = \frac{(C_{11}+C_{12})R(C_{66}(-3K_1+K_2)+2R^2)}{2(C_{11}K_1-R^2)(C_{66}K_1-R^2)},$$

$$a_{14} = -\frac{(C_{11}+C_{12})(K_1-K_2)R(2C_{66}K_1-C_{66}K_2-R^2)}{2(C_{11}K_1-R^2)(C_{66}K_1-R^2)},$$

$$a_{15} = \frac{3(C_{11}+C_{12})(K_1-K_2)R}{2(C_{11}K_1-R^2)},$$

$$b_1 = \frac{K_1}{C_{11}K_1-R^2}, b_2 = -\frac{R}{C_{11}K_1-R^2}, b_3 = \frac{K_1}{C_{66}K_1-R^2},$$

$$b_4 = -\frac{R}{C_{66}K_1-R^2}, b_5 = \frac{C_{11}}{C_{11}K_1-R^2}, b_6 = \frac{C_{66}}{C_{66}K_1-R^2}.$$

References

1. Shechtman, D.; Blech, I.; Gratias, D.; Cahn, J.W. Metallic phase with long-range orientational order and no translational symmetry. *Phys. Rev. Lett.* **1984**, *53*, 1951–1953. [CrossRef]
2. Hu, C.Z.; Wang, R.H.; Ding, D.H. Symmetry groups, physical property tensors, elasticity and dislocations in quasicrystals. *Rep. Prog. Phys.* **2000**, *63*, 1–39. [CrossRef]
3. Suck, J.B.; Schreiber, M.; Häussler, P. *Quasicrystals: An Introduction to Structure, Physical Properties and Applications*; Springer: Berlin/Heidelberg, Germany, 2002.
4. Li, X.F.; Fan, T.Y. Elastic analysis of a mode II crack in a decagonal quasi-crystal. *Chin. Phys. B* **2002**, *11*, 266–271.
5. Gao, Y.; Ricoeur, A.; Zhang, L.L. Plane problems of cubic quasicrystal media with an elliptic hole or a crack. *Phys. Lett. A* **2011**, *375*, 2775–2781. [CrossRef]
6. Radi, E.; Mariano, P.M. Stationary straight cracks in quasicrystals. *Int. J. Fract.* **2010**, *166*, 105–120. [CrossRef]
7. Yang, L.Z.; Gao, Y.; Pan, E.; Waksmanski, N. An exact solution for a multilayered two-dimensional decagonal quasicrystal plate. *Int. J. Solids Struct.* **2014**, *51*, 1737–1749. [CrossRef]
8. Li, X.Y. Elastic field in an infinite medium of one-dimensional hexagonal quasicrystal with a planar crack. *Int. J. Solids Struct.* **2014**, *51*, 1442–1455. [CrossRef]
9. Li, L.H.; Liu, T.G. Decagonal quasicrystal plate with elliptic holes subjected to out-of-plane bending moments. *Phys. Lett. A* **2014**, *378*, 839–844. [CrossRef]
10. Lazar, M.; Agiasofitou, E. Fundamentals generalized elasticity and dislocation theory of quasicrystals: Green tenser, dislocation key-formulas ang dislocation loops. *Philos. Mag.* **2014**, *94*, 4080–4101. [CrossRef]
11. Zhao, X.F.; Li, X.; Ding, S.H. Two kinds of contact problems in three-dimensional icosahedral quasicrystals. *Appl. Math. Mech.* **2015**, *36*, 1569–1580. [CrossRef]
12. Lazar, M.; Agiasofitou, E. Eshelbian mechanics of novel materials: Quasicrystals. *J. Micromech. Mol. Phys.* **2016**, *1*, 1640008. [CrossRef]
13. Wang, X.; Schiavone, P. Elastic field near the tip of an anticrack in a decagonal quasicrystalline material. *Appl. Math. Mech.* **2020**, *41*, 401–408. [CrossRef]
14. Sun, T.Y.; Guo, J.H.; Pan, E. Nonlocal vibration and buckling of two-dimensional layered quasicrystal nanoplates embedded in an elastic medium. *Appl. Math. Mech.* **2021**, *42*, 1077–1094. [CrossRef]
15. Zhong, W.X. *A New Systematic Methodology for Theory of Elasticity*; Dalian University of Technology Press: Dalian, China, 1995. (In Chinese)
16. Feng, K. On difference scheme and symplectic geometry. In *Proceedings of the Beijing Symposium Differential Geometry and Differential Equations*; Science Press: Beijing, China, 1985; pp. 42–58.
17. Zhong, W.X. Plane elasticity problem in strip domain and Hamiltonian system. *J. Dalian Univ. Technol.* **1991**, *31*, 373–384. (In Chinese)
18. Yao, W.A.; Zhong, W.X. *Symplectic Elasticity*; Higher Education Press: Beijing, China, 2002. (In Chinese)
19. Leung, A.Y.T.; Xu, X.S.; Zhou, Z.H.; Wu, Y.F. Analytic stress intensity factors for finite elastic disk using symplectic expansion. *Eng. Fract. Mech.* **2009**, *76*, 1866–1882. [CrossRef]
20. Liu, Y.M.; Li, R. Accurate bending analysis of rectangular plates with two adjacent edges free and the others clamped or simply supported based on new symplectic approach. *Appl. Math. Model.* **2010**, *34*, 856–865. [CrossRef]
21. Leung, A.Y.T.; Zheng, J.J.; Lim, C.W.; Zhang, X.C.; Xu, X.S.; Gu, Q. A new symplectic approach for piezoelectric cantilever composite plates. *Comput. Struct.* **2008**, *86*, 1865–1874. [CrossRef]
22. Xu, X.S.; Gu, Q.; Leung, A.Y.T.; Zheng, J.J. A symplectic eigensolution method in transversely isotropic piezoelectric cylindrical media. *J. Zhejiang Univ.-Sci. A* **2005**, *6*, 922–927.
23. Zhao, L.; Chen, W.Q. Symplectic analysis of plane problems of functionally graded piezoelectric materials. *Mech. Mater.* **2009**, *41*, 1330–1339. [CrossRef]
24. Jiménez, F.; Ober-Blöbaum, S. Fractional Damping through Restricted Calculus of Variations. *J. Nonlinear Sci.* **2021**, *31*, 46. [CrossRef]
25. Xu, C.H.; Leng, S.; Zhou, Z.H.; Xu, X.S.; Deng, Z.C. Accurate and straightforward symplectic approach for fracture analysis of fractional viscoelastic media. *Appl. Math. Mech.* **2022**, *43*, 403–416. [CrossRef]
26. Koyunbakan, H. The Transmutation Method and Schrödinger Equation with Perturbed Exactly Solvable Potential. *J. Comput. Acoust.* **2009**, *17*, 1–10. [CrossRef]
27. Timoshenko, S.P.; Goodier, J.N. *Theory of Elasticity*; Higher Education Press: Beijing, China, 2013. (In Chinese)
28. Fan, T.Y. Mathematical theory and methods of mechanics of quasicrystalline materials. *Engineering* **2013**, *5*, 407–448. [CrossRef]