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A New Exponential Distribution to Model Concrete Compressive Strength Data

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Abstract: Concrete mixtures can be developed to deliver a broad spectrum of mechanical and durability properties to satisfy the configuration conditions of construction. One technique for evaluating the compressive strength of concrete is to suppose that it pursues a probabilistic model from which it is reliability estimated. In this paper, a new technique to generate probability distributions is considered and a new three-parameter exponential distribution as a new member of the new family is presented in detail. The proposed distribution is able to model the compressive strength of high-performance concrete rather than some other competitive models. The new distribution delivers decreasing, increasing, upside-down bathtub and bathtub-shaped hazard rates. The maximum likelihood estimation approach is used to estimate model parameters as well as the reliability function. The approximate confidence intervals of these quantities are also obtained. To assess the performance of the point and interval estimations, a simulation study was conducted. We demonstrate the performance of the offered new distribution by investigating one high-performance concrete compressive strength dataset. The numerical outcomes showed that the maximum likelihood method provides consistent and asymptotically unbiased estimators. The estimates of the unknown parameters as well as the reliability function perform well as sample size increases in terms of minimum mean square error. The confidence interval of the reliability function has an appropriate length utilizing the delta method. Moreover, the real data analysis indicated that the new distribution is more suitable when compared to some well-known and some recently proposed distributions to evaluate the reliability of concrete mixtures.

Keywords: logarithmic transformed method; alpha power method; exponential distribution; maximum likelihood estimation; order statistics

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1. Introduction

In fact, concrete is a widely used construction material in the world. Concrete compressive strength is a criterion employed in specifying the portion of resistance a structural component can deliver to deformation. Compressive strength is a widely used standard to access the performance of a provided concrete mixture. This technique of assessing concrete is essential because it is the primary measure determining how sufficiently concrete can resist loads that impact its measure. It specifically informs us whether a distinct mixture is appropriate for encountering the conditions of a certain venture. Concrete can astoundingly stand up to compressive loading. This is a frequent reason for why it is useful for constructing arches, foundations, dams, columns and tunnel linings among other buildings.

Experimenters from various areas of science may endeavor to represent phenomena of interest, such as high-concrete concrete compressive strength, using probabilistic models. In recent years, many authors have exhibited significant appeal in representing new generalized families of probability distributions by adding one or more additional parameters to well-known distributions to yield new models with more incredible flexibility in modeling. The growth of proposing new statistical distributions is a major study area in the approach of distribution theory. Noteworthy methods include the following: the exponentiated method by Mudholkar and Srivastava [1], Marshall–Olkin method by Marshall and Olkin [2], beta-G family by Eugene et al.Olkin [3], Kumaraswamy-G family by Cordeiro and Castro [4], T-X family by Alzaatreh et al. [5], Weibull-G family by Bourguignon et al. [6], logarithmic transformed (LT) method by Pappas et al. [7], alpha power (AP) method by Mahdavi and Kundu [8], Marshall–Olkin AP method by Nassar et al. [9] and weighted AP transformed method by Alotaibi et al. [10]. For more details about other methods for generating distributions, one can refer to Lee et al. [11] and Jones [12].

Pappas et al. [7] proposed the LT approach, which uses a cumulative distribution function (CDF) and probability density function to induct a new parameter into well-known distributions (PDF), respectively.

$$F(x;\theta) = \begin{cases} 1 - \frac{\log[\theta-(\theta-1)G(x)]}{\log(\theta)} & \text{if } \theta > 0,\ \theta \neq 1 \\ G(x) & \text{if } \theta = 1 \end{cases} \tag{1}$$

$$f(x;\theta) = \begin{cases} \frac{(\theta-1)g(x)}{\log(\theta)[\theta-(\theta-1)G(x)]} & \text{if } \theta > 0,\ \theta \neq 1 \\ g(x) & \text{if } \theta = 1, \end{cases} \tag{2}$$

Pappas et al. [7] considered the modified Weibull extension distribution by Xie et al. [13] as a baseline distribution G in (1) and studied some characteristics of the new distribution. Nassar et al. [14] used the LT method to propose a new form for the Weibull distribution. Eltehiwy [15] introduced the LT inverse Lindley distribution. Alotaibi et al. [16] utilized the CDF in (1) to introduce a new generalization for the traditional Lomax distribution.

Lately, Mahdavi and Kundu [8] offered the AP method for yielding new probability distributions by introducing an extra parameter to bring about a more elastic family. The CDF of the AP method is provided as follows:

$$Q(x;\alpha) = \begin{cases} \frac{\alpha^{G(x)}-1}{\alpha-1} & \text{if } \alpha > 0,\ \alpha \neq 1 \\ G(x) & \text{if } \alpha = 1, \end{cases} \tag{3}$$

and the corresponding PDF is the following.

$$q(x;\alpha) = \begin{cases} \frac{\log(\alpha)g(x)\,\alpha^{G(x)}}{\alpha-1} & \text{if } \alpha > 0,\ \alpha \neq 1 \\ g(x) & \text{if } \alpha = 1. \end{cases} \tag{4}$$

Utilizing the AP method, a new AP Weibull (APW) distribution is proposed by Nassar et al. [17]. Dey et al. [18] introduced the AP inverse Lindley distribution. Ihtisham et al. [19] proposed the AP Pareto distribution. Eghwerido et al. [20,21] presented the AP Gompertz and AP Teissier distributions, respectively. Eghwerido et al. [22] proposed the AP-extended generalized exponential distribution.

The main goal of this article is to present a new method for generating probability distributions by using the AP CDF from (3) as the baseline CDF in (1). The logarithmic transformed alpha power (LTAP) family is the name given to the new family. The new LTAP family can be used to produce probability distributions with closed forms CDF and PDF. Firstly, we derive some structural properties of the LTAP family including the mixture representation for the PDF. Secondly, we assume the exponential as a baseline for the LTAP family and develop a new three-parameter LTAP-exponential (LTAPEx) distribution. The hazard rate functions (HRF) of the LTAPEx distribution accommodates monotonic, decreasing, increasing, upside-down bathtub and bathtub-shaped models. Therefore, the LTAPEx distribution can be used as a competitive model for many well-known distributions presented in the literature. Another motivation for the LTAPEx distribution is that it contains some sub-models such as exponential and alpha power exponential (APEx) distributions. Moreover, it can be considered as an appropriate model for modeling positively skewed data, which may not be suitably fitted by other standard distributions. The highly complex materials of high-performance concrete can render modeling its behavior an extremely difficult task. One of our practical objectives in this study is to evaluate the reliability of the high-performance concrete by assuming that the compressive strength of concrete follows the LTAPEx model. We then apply this model to a real high-performance concrete compressive strength dataset to check our findings. The outcomes of this analysis showed that the proposed model can be considered as an appropriate model when compared with some other competitive models to model high-performance concrete. The current study is an attempt to investigate and choose the most suitable model to assess the reliability of high-performance concrete, which we believe would be of outstanding appeal to reliability engineers.

The rest of the article is divided into the following sections: In Section 2, we present the LTAP family and provide a linear representation for the LTAP family PDF. In Section 3, we present the LTAPEx distribution. In Section 4, we study some properties of the LTAPEx distribution. The maximum likelihood method is considered to obtain the point and interval estimates for the model parameters as well as the corresponding reliability function (RF) in Section 5. A simulation study is conducted in Section 6. The investigation of one real dataset is provided in Section 7. Finally, Section 8 provides some conclusions.

2. The LTAP Family of Distributions

The CDF of the LTAP family is obtained by replacing $G(x)$ in Equation (1) by $Q(x)$ of the APT class given by (3). We have the following.

$$F_{LTAP}(x;\theta,\alpha) = \begin{cases} 1 - \frac{\log\left[\theta - \frac{\theta-1}{\alpha-1}\left(\alpha^{G(x)}-1\right)\right]}{\log(\theta)} & \text{if } \theta,\alpha > 0,\ \alpha,\theta \neq 1 \\ G(x) & \text{if } \alpha = \theta = 1. \end{cases} \tag{5}$$

Its PDF reduces to the following.

$$f_{LTAP}(x;\theta,\alpha) = \begin{cases} \frac{(\theta-1)\log(\alpha)}{\log(\theta)} \frac{g(x)\alpha^{G(x)}}{\left[(\theta\alpha-1)-(\theta-1)\alpha^{G(x)}\right]} & \text{if } \theta,\alpha > 0,\ \alpha,\theta \neq 1 \\ g(x) & \text{if } \alpha = \theta = 1. \end{cases} \tag{6}$$

Henceforth, we denotea random variable having PDF in (6) by X. In the next subsections, some general properties of the LTAP family are derived.

2.1. Mixture Representation of the LTAP Family

Following the same approach by Dey et al. [23], the PDF in (6) can be represented as follows.

$$f_{LTAP}(x;\theta,\alpha) = \frac{(\theta-1)\log(\alpha)}{\log(\theta)(\theta+1)} g(x)\alpha^{G(x)} \left[1 - \frac{1 + \frac{(\theta-1)}{(\alpha-1)}\left(\alpha^{G(x)}-1\right)}{\theta+1}\right]^{-1}. \tag{7}$$

Now, using the binomial expansion and the following two series:

$$(1-y)^{-1} = \sum_{k=0}^{\infty} y^k, |y| < 1, \quad \alpha^y = \sum_{m=0}^{\infty} \frac{(\log\alpha)^m y^m}{m!},$$

the PDF of the LTAP family in (6) can be written, after some simplifications, in the following form:

$$f_{LTAP}(x;\theta,\alpha) = \sum_{m=0}^{\infty} \phi_m w_{m+1}(x), \tag{8}$$

where $w_{c+1}(x) = (c+1)\, g(x)\, G^c(x)$ refers to the exponentiated-G (exp-G) PDF with shape parameter $c > 0$, and ϕ_m is given by the following.

$$\phi_m = \sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{i=0}^{j}\binom{k}{j}\binom{j}{i}\frac{(-1)^j(i+1)^m[\log(\alpha)]^{m+1}}{\log(\theta)(m+1)!}\left(\frac{1}{\theta+1}\right)^{k+1}\frac{(\theta-1)^{j+1}}{(\alpha-1)^j}. \tag{9}$$

The expansion in (8) provides the PDF of the LTAP family as a linear combination of the PDF of the exp-G family. Therefore, some structural properties of the new family can be obtained directly using this representation. In addition, the expansion of the CDF of the LTAP family can be derived by integrating (8), and it is stated as follows:

$$F_{LTAP}(x;\theta,\alpha) = \sum_{k=0}^{\infty}\phi_m W_{m+1}(x), \tag{10}$$

where $W_{c+1}(x)$ is the CDF of the exp-G family with shape parameter $c > 0$.

2.2. Quantile Function of the LTAP Family

For the LTAP family, the quantile function can be reached by inverting (5) as follows.

$$x_p = G^{-1}\left\{\frac{\log\left[1+\frac{\theta(\alpha-1)(1-\theta^{-p})}{\theta-1}\right]}{\log(\alpha)}\right\}, 0 < p < 1. \tag{11}$$

Many beneficial measures can be calculated from (11), including the first quartile, median and third quartile by placing p with $0.25, 0.5$ and 0.75, respectively. For example, the median of the LTAP family can be obtained as follows.

$$\texttt{Median} = G^{-1}\left\{\frac{\log\left[1+\frac{(\alpha-1)(\theta-\sqrt{\theta})}{\theta-1}\right]}{\log(\alpha)}\right\}. \tag{12}$$

Different significant applications relative to the quantile function in (11) are used to simulate random samples from the LTAP family. Let $U \sim Uniform(0,1)$, then one can generate a random sample consisting of n observations from the LTAP family as follows.

$$x_i \quad = \quad G^{-1}\left\{\frac{\log\left[1+\frac{\theta(\alpha-1)(1-\theta^{-u_i})}{\theta-1}\right]}{\log(\alpha)}\right\}, i = 1,\ldots,n. \tag{13}$$

3. The LTAPEx Distribution

In this section, we explain the LTAPEx distribution and some associated statistical features. By entering $G(x;\beta) = 1 - e^{-\beta x}$, of the exponential distribution in (5), one can reach the CDF of the LTAPEx distribution as follows.

$$F(x;\theta,\alpha,\beta) \quad = \quad 1 - \frac{\log\left[\theta - \frac{\theta-1}{\alpha-1}\left(\alpha^{1-e^{-\beta x}} - 1\right)\right]}{\log(\theta)}, \ x > 0, \ \theta,\alpha,\beta > 0. \tag{14}$$

By differentiating (14) with respect to x, we can obtain the PDF of the LTAPEx distribution as follows:

$$f(x;\theta,\alpha,\beta) = \frac{(\theta-1)\log(\alpha)}{\log(\theta)}\frac{\beta e^{-\beta x}}{1-\theta+(\theta\alpha-1)\alpha^{e^{-\beta x}-1}}, \tag{15}$$

where β is the scale parameter, and θ and α are the shape parameters. Moreover, the RF and the HRF of the LTAPEx distribution are furnished, respectively, by the following.

$$R(x;\theta,\alpha,\beta) \quad = \quad \frac{\log\left[\theta - \frac{\theta-1}{\alpha-1}\left(\alpha^{1-e^{-\beta x}} - 1\right)\right]}{\log(\theta)} \tag{16}$$

$$h(x;\theta,\alpha,\beta) = \frac{\log(\alpha)(\theta-1)\beta e^{-\beta x}}{\left[1-\theta+(\theta\alpha-1)\alpha^{e^{-\beta x}-1}\right]\log\left[\theta - \frac{\theta-1}{\alpha-1}\left(\alpha^{1-e^{-\beta x}} - 1\right)\right]}. \tag{17}$$

It can be observed here that when θ and α tend to one, the PDF in (15) reduces to the PDF of the exponential distribution. Moreover, when $\theta \to 1$, the LTAPEx distribution reduces to the alpha power exponential distribution proposed by Mahdavi and Kundu (2017). Another special case is when $\alpha \to 1$, the LTAPEx distribution reduces to the logarithmic transformed exponential distribution. Figure 1 layouts the various plots of the PDF of the LTAPEx distribution applying $\beta = 1$ in all cases and by considering several values for the shape parameters θ and α. In Figure 1, we can recognize that the new shape parameters θ and α afford more flexibility to the PDF of the LTAPEx distribution than the conventional exponential distribution. The LTAPEx distribution is a right-skewed distribution, and this characteristic encourages the use of this distribution to model right-skewed data rather than some other competitive distributions such as Weibull and gamma distributions. Figure 2 displays the various shapes of the HRF of the LTAPEx distribution. Figure 2 reveals that the HRF of the LTAPEx distribution has different shapes, including decreasing, increasing, upside-down and bathtub-shaped hazard rates.

Using the linear representation for the PDF in (8), one can write the PDF of the LTAPEx distribution given by (15) as follows:

$$f(x;\theta,\alpha,\beta) = \sum_{m=0}^{\infty}\sum_{a=0}^{m} \phi_{m,a}\, g(x;\beta(a+1)), \tag{18}$$

where $g(x;\beta(a+1))$ is the PDF of the exponential distribution with scale parameter $\lambda(a+1)$ and the following.

$$\phi_{m,a} = \phi_m \begin{pmatrix} m \\ a \end{pmatrix} \frac{(-1)^a(m+1)}{a+1}. \tag{19}$$

Various structural properties of the LTAPEx distribution can be acquired directly from (18) based on the well-known properties of exponential distribution. Integrating (18), one can write the expansion of the CDF of the LTAPEx distribution as follows:

$$F(x;\theta,\alpha,\beta) = \sum_{m=0}^{\infty}\sum_{a=0}^{m} \phi_{m,a}\, G(x;\beta(a+1)), \tag{20}$$

where $G(x;\beta(a+1))$ is the CDF of the exponential distribution.

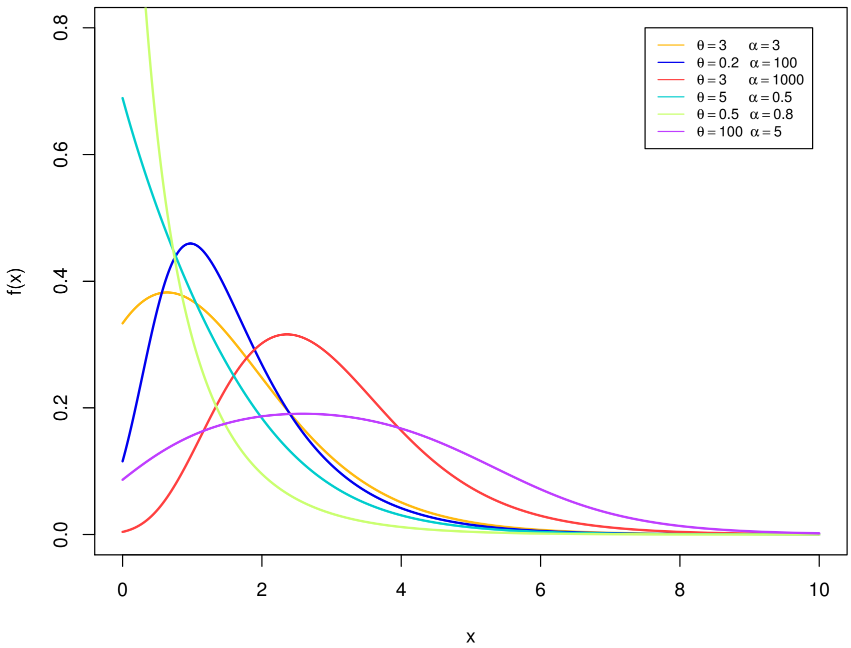

Figure 1. Different plots of the PDF of the LTAPEx distribution for various values of θ and α with $\beta = 1$.

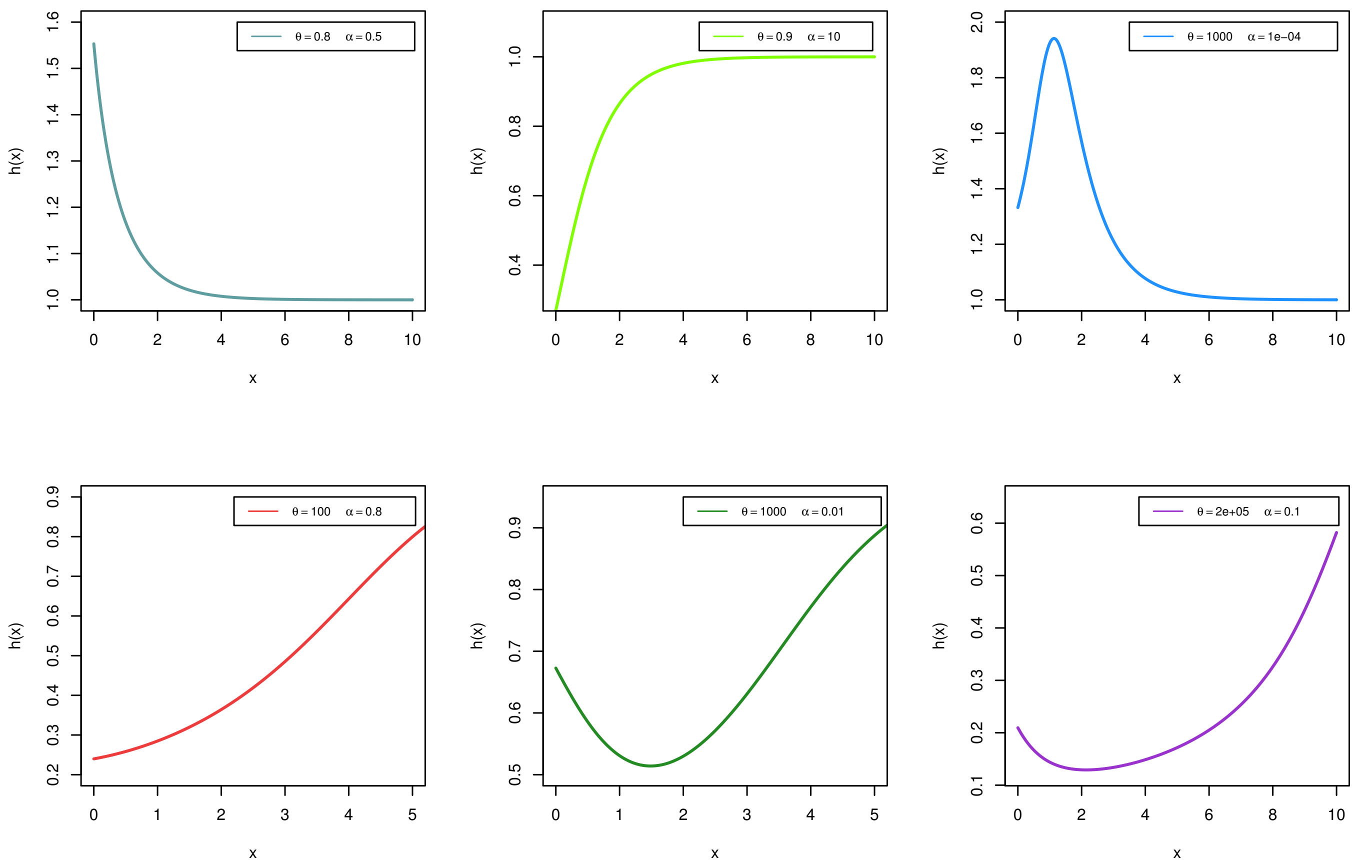

Figure 2. Different plots of the HRF of the LTAPEx distribution for various values of θ and α with $\beta = 1$.

4. Main Properties of the LTAPEx Distribution

In this section, we provide some necessary statistical mathematical properties for the LTAPEx distribution, such as the following: quantile, moments, moment generating function, quantile and order statistics.

4.1. Quantile and Random Number Generation

Quantiles are essential for estimation and simulation. For the LTAPEx distribution, the pth quantile x_p can be expresses as follows.

$$x_p = \frac{-1}{\beta} \log\left\{1 - \frac{\log\left[1 + \frac{\theta(\alpha-1)(1-\theta^{-p})}{\theta-1}\right]}{\log(\alpha)}\right\}, 0 < p < 1. \tag{21}$$

Let $U \sim Uniform(0,1)$, then (21) can be operated to generate a random sample containing n observations from the LTAPEx distribution as follows.

$$x_i = \frac{-1}{\beta} \log\left\{1 - \frac{\log\left[1 + \frac{\theta(\alpha-1)(1-\theta^{-u_i})}{\theta-1}\right]}{\log(\alpha)}\right\}, i = 1, \ldots, n. \tag{22}$$

4.2. Moments and Generating Function

Moments play an important role in statistics and its applications. Some significant properties of a probability distribution can be studied based on moments including, tendency, dispersion, skewness and kurtosis. For the LTAPEx distribution, the rth moment follows from (15) as follows:

$$\begin{aligned} \mu_r' &= \sum_{m=0}^{\infty} \sum_{a=0}^{m} \phi_{m,a} \int_0^{\infty} \beta(a+1) x^r e^{-\beta(a+1)x} \\ &= \sum_{m=0}^{\infty} \sum_{a=0}^{m} \phi_{m,a}\, E(Z_{a+1}^r) \end{aligned} \tag{23}$$

where Z follows the exponential distribution with scale parameter $\beta(a+1)$. It is known that for the exponential distribution with scale parameter β, the rth moment is $\Gamma(r+1)/\beta^r$, then it follows from (23) that

$$\mu_r' = \sum_{m=0}^{\infty} \sum_{a=0}^{m} \phi_{m,a} \frac{\Gamma(r+1)}{[\beta(a+1)]^r}. \tag{24}$$

Similarly, using the result that the moment generating function of the random variable Z is $M_Z(t) = (1 - t/\beta)^{-1}$, we can write the moment generating function of the LTAPEx distribution in the following expression.

$$M_X(t) = \sum_{m=0}^{\infty} \sum_{a=0}^{m} \phi_{m,a} \left(1 - \frac{t}{\beta(a+1)}\right)^{-1}. \tag{25}$$

Using (25) and for different values for θ and α with scale parameter $\beta = 1$, Figure 3 shows the plots for the mean, variance, skewness and kurtosis of the LTAPEx distribution. It is observed from Figure 3 that the mean and variance of the LTAPEx distribution increased as θ and α increased. On the other hand, as θ and α increased, the skewness and kurtosis decreased. It is also noted that the LTAPEx distribution is always a positively skewed distribution.

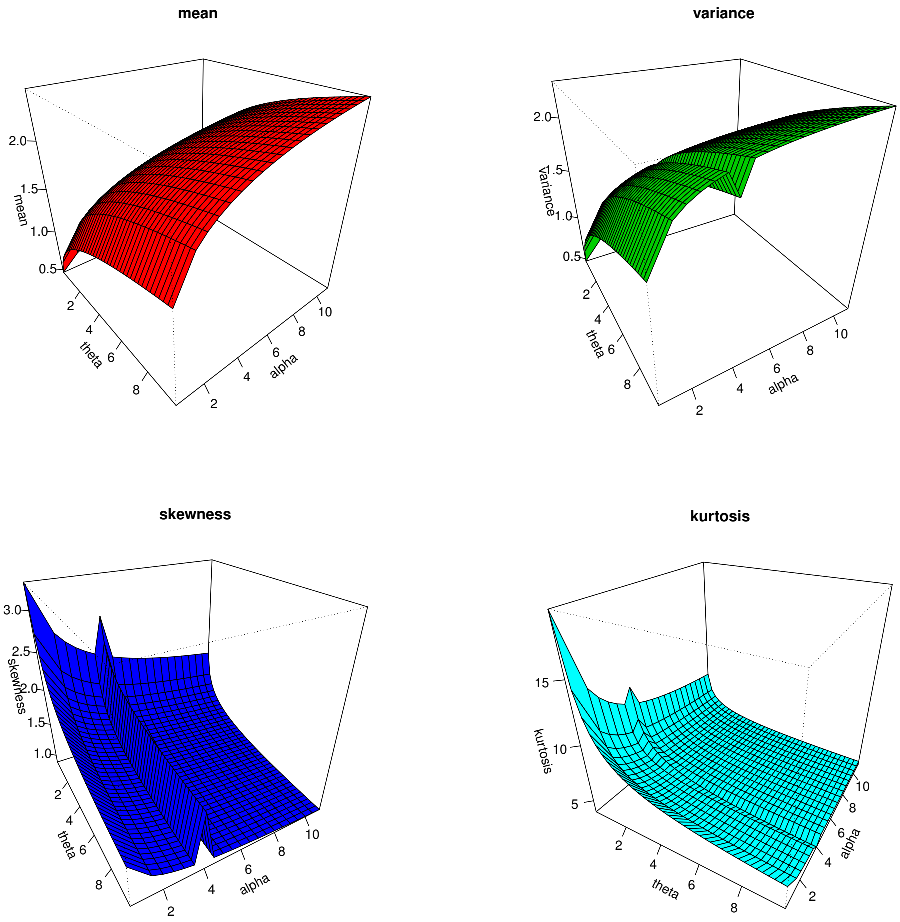

Figure 3. Mean, variance, skewness and kurtosis of LTAPEx distribution.

4.3. Entropies

Entropy has been employed in various cases and has numerous applications in different areas such as statistics and physics. For the random variable X, entropy measures uncertainty. The Rényi entropy (RE) is specified as follows.

$$E_\delta(x) = \frac{1}{1-\delta}\log\left\{\int_0^\infty [f(x)]^\delta dx\right\}, \delta >, \delta \neq 0. \tag{26}$$

From the LTAPEx distribution and using (7), we can write $[f(x)]^\delta$ as follows.

$$[f(x)]^\delta = \left[\frac{(\theta-1)\log(\alpha)\beta}{(\alpha-1)(\theta+1)\log(\theta)}\right]^\delta \left(e^{-\beta x}\,\alpha^{1-e^{-\beta x}}\right)^\delta \left[1-\frac{1+\frac{(\theta-1)}{(\alpha-1)}\left(\alpha^{1-e^{-\beta x}}-1\right)}{\theta+1}\right]^{-\delta}. \tag{27}$$

Using series $(1-z)^{-v} = \sum_{j=0}^\infty [\Gamma(v+j)/\Gamma(v)j!]z^j, v>0, |z|<1$, the power series and the binomial expansion, we can write (27) as follows:

$$[f(x)]^\delta = \left[\frac{(\theta-1)\log(\alpha)\beta}{(\alpha-1)(\theta+1)\log(\theta)}\right]^\delta \sum_{k=0}^\infty \sum_{t=0}^k \varphi_{k,t}\, e^{-\beta x(\delta+t)}, \tag{28}$$

where the following is the case.

$$\varphi_{k,t} = \sum_{j=0}^\infty \sum_{i=0}^j \sum_{m=0}^\infty \frac{(\log\alpha)^k(\delta+m)^k\Gamma(\delta+j)(\theta-1)^i(-1)^{i+t}}{(\theta+1)^j(\alpha-1)^i\Gamma(\delta)k!j!}\binom{j}{i}\binom{i}{m}\binom{k}{t}. \tag{29}$$

Then, from (26) and (28), we can write the following.

$$E_\delta(x) = \frac{1}{1-\delta}\log\left\{\left[\frac{(\theta-1)\log(\alpha)\beta}{(\alpha-1)(\theta+1)\log(\theta)}\right]^\delta \sum_{k=0}^{\infty}\sum_{t=0}^{k}\varphi_{k,t}\int_0^\infty e^{-\beta x(\delta+t)}dx\right\}. \quad (30)$$

After simplification, Equation (30) becomes the following.

$$E_\delta(x) = \frac{\delta}{1-\delta}\log\left[\frac{(\theta-1)\log(\alpha)\beta}{(\alpha-1)(\theta+1)\log(\theta)}\right] + \frac{1}{1-\delta}\log\left[\sum_{k=0}^{\infty}\sum_{t=0}^{k}\varphi_{k,t}\frac{1}{\beta(\delta+t)}\right]. \quad (31)$$

Another measure for entropy is the ξ-entropy, which is obtained as follows.

$$E_\xi(x) = \frac{1}{\xi-1}\log\left\{1-\int_0^\infty [f(x)]^\xi dx\right\}, \xi >, \xi \neq 0.$$

Then, it follows from (28) that the following is obtained.

$$E_\xi(x) = \frac{1}{\xi-1}\log\left\{1-\left[\frac{(\theta-1)\log(\alpha)\beta}{(\alpha-1)(\theta+1)\log(\theta)}\right]^\xi \sum_{k=0}^{\infty}\sum_{t=0}^{k}\varphi_{k,t}\frac{1}{\beta(\delta+t)}\right\}. \quad (32)$$

4.4. Order Statistics

Let $X_{(1)}, \ldots, X_{(n)}$ refer to the order statistics of a random sample of size n taken from a continuous probability distribution with PDF $f(x)$ and CDF $F(x)$, then the PDF of the sth order statistic, $X_{(s)}$, can be expressed as follows:

$$f_{X_{(s)}}(x) = f(x)\sum_{i=0}^{n-s} A_i F^{s+i-1}(x), \quad (33)$$

where the following is the case.

$$A_i = \frac{n!}{(s-1)!(n-s)!}(-1)^i \binom{n-s}{i}.$$

For the LTAPEx distribution with CDF and PDF given by (14) and (15), respectively, and based on (33), we can write the PDF of the sth order statistic as follows:

$$f_{X_{(s)}}(x) = \sum_{i=0}^{n-s}\sum_{k=0}^{s+i-1} A^*_{i,k}\frac{e^{-\beta x}}{1-\theta+(\theta\alpha-1)\alpha^{e^{-\beta x}-1}}\left\{\log\left[\theta-\frac{\theta-1}{\alpha-1}\left(\alpha^{1-e^{-\beta x}}-1\right)\right]\right\}^k, \quad (34)$$

where the following is the case.

$$A^*_{i,k} = A_i(-1)^k\frac{\beta(\theta-1)\log(\alpha)}{\log(\theta)[\log(\theta)]^k}\binom{s+i-1}{k}.$$

Moreover, the CDF of the sth order statistic can be expressed as follows.

$$F_{X_{(s)}}(x) = \sum_{i=0}^{n-s}\frac{A_i}{s+i}F^{s+i}(x). \quad (35)$$

Thus, from (14) and (36), the CDF of the sth-order statistic for the LTAPEx distribution is given by the following.

$$F_{X_{(s)}}(x) = \sum_{i=0}^{n-s}\sum_{k=0}^{s+i}\frac{A_i(-1)^k}{s+i}\binom{s+i}{k}\left\{\log\left[\theta-\frac{\theta-1}{\alpha-1}\left(\alpha^{1-e^{-\beta x}}-1\right)\right]\right\}^k. \quad (36)$$

5. Estimation of the Parameters and Reliability Function

In this section, we consider the maximum likelihood estimation method for estimating parameters $\hat{\theta}, \hat{\alpha}$ and $\hat{\beta}$ as well as the reliability function of the LTAPEx distribution. Let $x_1, \ldots, x_n$ be a random sample obtained from the LTAPEx distribution with PDF given by (15), then the likelihood function can be formulated as follows.

$$L(\theta,\alpha,\beta) = \left[\frac{\beta(\theta-1)\log(\alpha)}{\log(\theta)}\right]^n \prod_{i=1}^{n} \frac{e^{-\beta x_i}}{1-\theta+(\theta\alpha-1)\alpha^{e^{-\beta x_i}-1}}. \tag{37}$$

Taking the natural logarithm of (37), one can obtain the log-likelihood function, denoted by $\ell = \log L(\theta,\alpha,\beta)$, as follows.

$$\ell = n\log\left[\frac{\beta(\theta-1)\log(\alpha)}{\log(\theta)}\right] - \beta\sum_{i=1}^{n} x_i - \sum_{i=1}^{n}\log\left[1-\theta+(\theta\alpha-1)\alpha^{e^{-\beta x_i}-1}\right]. \tag{38}$$

Hence, the maximum likelihood estimates (MLEs) of θ, α and β, denoted by $\hat{\theta}, \hat{\alpha}$ and $\hat{\beta}$, can be computed by maximizing the objective log-likelihood function in (38) with respect to θ, α and β. Another useful approach to obtain these estimates is to solve the following three normal equations simultaneously.

$$\frac{\partial \ell}{\partial \theta} = \frac{n}{\theta-1} - \frac{n}{\theta\log(\theta)} - \sum_{i=1}^{n}\frac{\alpha^{e^{-\beta x_i}}-1}{1-\theta+(\theta\alpha-1)\alpha^{e^{-\beta x_i}-1}} = 0, \tag{39}$$

$$\frac{\partial \ell}{\partial \alpha} = \frac{n}{\alpha\log(\alpha)} - \sum_{i=1}^{n}\frac{\alpha^{e^{-\beta x_i}-2}\left[1+(\theta\alpha-1)e^{-\beta x_i}\right]}{1-\theta+(\theta\alpha-1)\alpha^{e^{-\beta x_i}-1}} = 0 \tag{40}$$

$$\frac{\partial \ell}{\partial \beta} = \frac{n}{\beta} - \sum_{i=1}^{n} x_i - (\theta\alpha-1)\log(\alpha)\sum_{i=1}^{n}\frac{x_i e^{-\beta x_i}\alpha^{e^{-\beta x_i}-1}}{1-\theta+(\theta\alpha-1)\alpha^{e^{-\beta x_i}-1}} = 0. \tag{41}$$

It is observed from (39)–(41) that there are no ended forms for the MLEs of θ, α and β. Therefore, to reach these estimates, we should adopt an iterative procedure for determining the numerical solution of (39)–(41). Now, based on the invariance property of MLEs, the MLEs of the RF at x_0 can be computed from (16) as follows.

$$\widehat{R}(x_0) \;=\; \frac{\log\left[\hat{\theta} - \frac{\hat{\theta}-1}{\hat{\alpha}-1}\left(\hat{\alpha}^{1-e^{-\hat{\beta}x_0}}-1\right)\right]}{\log(\hat{\theta})}. \tag{42}$$

The ACIs of θ, α and β are instantly obtained based on the asymptotic properties of the MLEs. It is known that $(\theta,\alpha,\beta) \sim N[(\hat{\theta},\hat{\alpha},\hat{\beta}), I_0^{-1}(\theta,\alpha,\beta)]$, where $I^{-1}(\theta,\alpha,\beta)$ is the asymptotic variance–covariance matrix of MLEs. Actually, it is not easy to reach $I^{-1}(\theta,\alpha,\beta)$; hence, the approximate asymptotic variance–covariance matrix of the MLEs expressed by $I^{-1}(\hat{\theta},\hat{\alpha},\hat{\beta})$ can be used alternatively as follows

$$I^{-1}(\hat{\alpha},\hat{\beta}) = \begin{bmatrix} -I_{\theta\theta} & -I_{\theta\alpha} & -I_{\theta\beta} \\ -I_{\alpha\theta} & -I_{\alpha\alpha} & -I_{\alpha\beta} \\ -I_{\beta\theta} & -I_{\beta\alpha} & -I_{\beta\beta} \end{bmatrix}^{-1}_{(\theta,\alpha,\lambda)=(\hat{\theta},\hat{\alpha},\hat{\lambda})} = \begin{bmatrix} \widehat{var}(\hat{\theta}) & \widehat{cov}(\hat{\theta},\hat{\alpha}) & \widehat{cov}(\hat{\theta},\hat{\beta}) \\ \widehat{cov}(\hat{\alpha},\hat{\theta}) & \widehat{var}(\hat{\alpha}) & \widehat{cov}(\hat{\alpha},\hat{\beta}) \\ \widehat{cov}(\hat{\beta},\hat{\theta}) & \widehat{cov}(\hat{\beta},\hat{\alpha}) & \widehat{cov}(\hat{\beta},\hat{\beta}) \end{bmatrix}. \tag{43}$$

The elements $I_{\theta\theta}, I_{\alpha\alpha}, I_{\lambda\beta}, I_{\theta\alpha} = I_{\alpha\theta}, I_{\theta\beta} = I_{\beta\theta}$ and $I_{\alpha\beta} = I_{\beta\alpha}$ are the second derivatives of the log-likelihood function in (38), and they can be expressed as follows:

$$I_{\theta\theta} = \frac{n[1+\log(\theta)]}{\theta^2\log^2(\theta)} - \frac{n}{(\theta-1)^2} + \sum_{i=1}^{n}\frac{\left(\alpha^{e^{-\beta x_i}}-1\right)^2}{\psi_i^2}, \tag{44}$$

$$I_{\alpha\alpha} = -\frac{n[1+\log(\alpha)]}{\alpha^2\log^2(\alpha)} + \sum_{i=1}^{n}\frac{\alpha^{2e^{-\beta x_i}-4}(1+\varphi_i)^2}{\psi_i^2} - \sum_{i=1}^{n}\frac{\alpha^{e^{-\beta x_i}-3}\left(\theta\alpha e^{-\beta x_i} + (1+\varphi_i)(e^{-\beta x_i}-2)\right)}{\psi_i}, \tag{45}$$

$$I_{\beta\beta} = -\frac{n}{\beta^2} + \sum_{i=1}^{n}\frac{x_i^{-2}\xi_i^2}{\psi_i^2} - \sum_{i=1}^{n}\frac{\xi_i}{\psi_i} - \log(\alpha)\sum_{i=1}^{n}\frac{\xi_i e^{-\beta x_i}}{\psi_i}, \tag{46}$$

$$I_{\theta\alpha} = I_{\alpha\theta} = \sum_{i=1}^{n}\frac{\left(\alpha^{e^{-\beta x_i}}-1\right)\alpha^{e^{-\beta x_i}-2}(1+\varphi_i)}{\psi_i^2} - \sum_{i=1}^{n}\frac{e^{-\beta x_i}\alpha^{e^{-\beta x_i}-1}}{\psi_i}, \tag{47}$$

$$I_{\theta\beta} = I_{\beta\theta} = \log(\alpha)\sum_{i=1}^{n}\frac{x_i e^{-\beta x_i}\alpha^{e^{-\beta x_i}}}{\psi_i} - \sum_{i=1}^{n}\frac{x_i^{-1}\xi_i\left(\alpha^{e^{-\beta x_i}}-1\right)}{\psi_i^2} \tag{48}$$

$$I_{\alpha\beta} = I_{\beta\alpha} = \sum_{i=1}^{n}\frac{x_i^{-1}\xi_i}{\alpha\log(\alpha)\psi_i} + \sum_{i=1}^{n}\frac{x_i^{-1}\xi_i(1+\varphi_i)}{\alpha(\theta\alpha-1)\psi_i} - \sum_{i=1}^{n}\frac{x_i^{-1}\xi_i(1+\varphi_i)\alpha^{e^{-\beta x_i}-2}}{\psi_i^2}, \tag{49}$$

where $\psi_i = 1-\theta+(\theta\alpha-1)\alpha^{e^{-\beta x_i}-1}$, $\varphi_i = (\theta\alpha-1)e^{-\beta x_i}$ and $\xi_i = (\theta\alpha-1)x_i^2\log(\alpha)e^{-\beta x_i}\alpha^{e^{-\beta x_i}-1}$. Now, the $(1-\upsilon)\%$ ACIs of parameters θ, α and β can be computed as follows:

$$\hat{\theta} \pm z_{\upsilon/2}\sqrt{\widehat{var}(\hat{\theta})}, \quad \hat{\alpha} \pm z_{\upsilon/2}\sqrt{\widehat{var}(\hat{\alpha})} \quad \text{and} \quad \hat{\beta} \pm z_{\upsilon/2}\sqrt{\widehat{var}(\hat{\beta})},$$

where $z_{\upsilon/2}$ is the upper $(\upsilon/2)$th percentile point of the standard normal distribution.

In order to obtain the ACIs of the reliability function, we are required to obtain its variance. One of the common significant adopted procedures for approximating the variance is the delta method. To practice this approach, suppose that $\Delta_R = (\partial R/\partial\theta, \partial R/\partial\alpha, \partial R/\partial\beta)|_{(\theta,\alpha,\beta)=(\hat{\theta},\hat{\alpha},\hat{\beta})}$, where the following is the case:

$$\frac{\partial R}{\partial\theta} = -\frac{\log(A)}{\theta\log^2(\theta)} - \frac{\alpha(\alpha^{-e^{-\beta x_0}}-1)}{(\alpha-1)\log(\theta)A}, \tag{50}$$

$$\frac{\partial R}{\partial\alpha} = \frac{(\theta-1)\left\{\alpha^{-e^{-\beta x_0}}\left[1+e^{-\beta x_0}(\alpha-1)\right]-1\right\}}{(\alpha-1)^2\log(\theta)A} \tag{51}$$

$$\frac{\partial R}{\partial\beta} = -\frac{x(\theta-1)\log(\alpha)e^{-\beta x_0}\alpha^{1-e^{-\beta x_0}}}{(\alpha-1)\log(\theta)A}, \tag{52}$$

where $A = \theta - \frac{\theta-1}{\alpha-1}\left(\alpha^{1-e^{-\beta x_0}}-1\right)$. Then, the approximate estimate for the variance of the reliability function is as follows, respectively:

$$\widehat{V}(\widehat{R}) \approx [\Delta_R I^{-1}(\hat{\alpha},\hat{\beta})\Delta_R^{\top}],$$

Therefore, the two-sided ACI for the reliability function is provided by the following.

$$\widehat{R}(x_0) \pm z_{\frac{\upsilon}{2}}\sqrt{\widehat{V}(\widehat{R})}.$$

6. Simulation Study

In this section, a simulation analysis is accomplished to assess the performance of the MLEs of the unknown parameters and the reliability function. The efficiency of the estimates is evaluated using their mean squared errors (MSEs) and the confidence interval lengths. We employ Equation (22) to yield samples from the LTAPEx distribution. The simulation experiment is replicated 1000 times, each for samples of 25, 50 and 100. These sample sizes are selected to reflect the impact of small, intermediate and large sample sizes on the estimates. Moreover, different values for the unknown parameters θ, α and β are considered. The selected values are $\theta = (0.5, 1.5)$, $\alpha = (0.5, 1.5, 2.5)$ and $\beta = (0.5, 1, 1.5, 2.5, 5)$. In a separate setting, we have acquired the MLE, MSE and confidence interval lengths (CILs). The simulation study is carried out based on following the steps:

1. Decide the values of n, θ, α, β and the distinct time x_0;
2. Generate n observations from the LTAPEx distribution using (22);
3. Use the generated sample to compute the MLEs of θ, α, β and $R(x_0)$;
4. Obtain the MSEs of θ, α, β and $R(x_0)$;
5. Obtain the confidence interval bounds of θ, α, β and $R(x_0)$;
6. Redo steps 2–5 M times;
7. Compute the the average values (AVs) of MLEs, MSEs, confidence interval bounds (CIBs) and CILs of the parameter $\lambda = [\theta, \alpha, \beta, R(x_0)]$ as follows:

$$\texttt{AV-MLE}(\lambda) = \frac{1}{M}\sum_{i=1}^{M}\hat{\lambda}_i, \qquad \texttt{AV-MSE}(\lambda) = \frac{1}{M}\sum_{i=1}^{M}(\hat{\lambda}_i - \lambda)^2$$

$$\texttt{AV-CIB}(\lambda) = \left[\frac{1}{M}\sum_{i=1}^{M}\lambda_i^L, \ \frac{1}{M}\sum_{i=1}^{M}\lambda_i^U\right], \qquad \texttt{AV-CIL}(\alpha) = \frac{1}{M}\sum_{i=1}^{M}(\lambda_i^U - \lambda_i^L),$$

where λ_i^L and λ_i^U are the lower and upper CIBs, respectively.

The simulation outcomes are presented in Tables 1–6. From these Tables, we can observe that as the sample size grows, the AV-MLEs of the different parameters and the reliability function are stable and relatively close to the actual parameter values. This implies that the MLEs act asymptotically unbiased estimators. Furthermore, the AV-MSEs reduce as the sample size increases in all issues, which indicate that the MLEs are consistent. Finally, one can observe that, as the sample size increases, CILs decrease in all the cases, as expected. This is because as the sample size raises, more additional information is collected.

Table 1. The AVs of estimates and the corresponding MSEs for $n = 25$.

Parameters			B				**MSE**			
$\boldsymbol{\theta}$	$\boldsymbol{\alpha}$	$\boldsymbol{\beta}$	$\boldsymbol{\theta}$	$\boldsymbol{\alpha}$	$\boldsymbol{\beta}$	$\boldsymbol{R(0.5)}$	$\boldsymbol{\theta}$	$\boldsymbol{\alpha}$	$\boldsymbol{\beta}$	$\boldsymbol{R(0.5)}$
0.5	0.5	0.5	0.694	0.670	0.648	0.592	0.347	0.343	0.149	0.033
		1.0	0.698	0.699	1.290	0.409	0.390	0.401	0.576	0.037
		1.5	0.684	0.716	1.938	0.302	0.335	0.432	1.319	0.033
		2.5	0.665	0.673	3.249	0.173	0.275	0.366	3.789	0.022
		5.0	0.687	0.672	6.455	0.064	0.314	0.329	14.69	0.008
	1.5	0.5	0.634	2.380	0.608	0.702	0.371	5.619	0.096	0.026
		1.0	0.590	1.945	1.221	0.511	0.290	2.865	0.385	0.037
		1.5	0.585	2.198	1.823	0.398	0.257	4.563	0.817	0.038
		2.5	0.617	2.133	3.054	0.251	0.298	4.129	2.490	0.031
		5.0	0.605	2.056	6.050	0.093	0.275	3.494	9.092	0.014
	2.5	0.5	0.574	3.970	0.600	0.736	0.261	16.93	0.076	0.023
		1.0	0.537	4.069	1.196	0.557	0.211	18.60	0.301	0.043
		1.5	0.599	3.960	1.793	0.441	0.347	17.21	0.696	0.042
		2.5	0.575	4.080	2.986	0.283	0.270	18.04	1.897	0.039
		5.0	0.559	3.822	6.005	0.107	0.247	15.24	8.021	0.020

Table 1. *Cont.*

Parameters			B				**MSE**			
θ	α	β	θ	α	β	$R(0.5)$	θ	α	β	$R(0.5)$
1.5	0.5	0.5	1.619	0.779	0.651	0.685	1.371	0.599	0.145	0.029
		1.0	1.597	0.833	1.299	0.502	1.408	0.752	0.562	0.044
		1.5	1.623	0.796	1.950	0.389	1.364	0.607	1.335	0.045
		2.5	1.593	0.779	3.275	0.233	1.292	0.628	3.834	0.033
		5.0	1.581	0.773	6.493	0.088	1.327	0.559	14.29	0.015
	1.5	0.5	1.688	2.136	0.629	0.768	2.385	4.508	0.101	0.021
		1.0	1.727	2.167	1.262	0.600	2.786	4.855	0.384	0.039
		1.5	1.609	1.964	1.899	0.460	2.328	3.382	0.911	0.049
		2.5	1.736	2.178	3.153	0.305	2.963	4.519	2.489	0.042
		5.0	1.673	2.038	6.354	0.109	2.421	3.808	10.53	0.017
	2.5	0.5	1.665	3.143	0.632	0.798	2.987	8.055	0.097	0.016
		1.0	1.522	3.336	1.253	0.650	1.480	9.600	0.379	0.031
		1.5	1.578	3.070	1.878	0.506	2.207	7.305	0.820	0.046
		2.5	1.493	3.280	3.148	0.332	1.324	8.840	2.293	0.041
		5.0	1.579	3.262	6.276	0.114	2.248	8.689	9.034	0.015

Table 2. The AVs of estimates and the corresponding MSEs for $n = 50$.

Parameters			**MLE**				**MSE**			
θ	α	β	θ	α	B	$R(0.5)$	θ	α	β	$R(0.5)$
0.5	0.5	0.5	0.636	0.631	0.556	0.657	0.119	0.115	0.059	0.015
		1.0	0.622	0.609	1.114	0.460	0.102	0.096	0.240	0.021
		1.5	0.596	0.622	1.680	0.336	0.069	0.096	0.525	0.018
		2.5	0.607	0.619	2.784	0.198	0.078	0.092	1.458	0.015
		5.0	0.608	0.618	5.584	0.068	0.077	0.094	6.716	0.007
	1.5	0.5	0.658	1.656	0.551	0.750	0.152	0.659	0.038	0.008
		1.0	0.621	1.728	1.102	0.573	0.110	0.928	0.160	0.017
		1.5	0.648	1.695	1.652	0.447	0.149	0.727	0.346	0.021
		2.5	0.662	1.642	2.744	0.279	0.163	0.647	0.926	0.021
		5.0	0.632	1.642	5.504	0.094	0.125	0.643	3.658	0.009
	2.5	0.5	0.570	2.997	0.552	0.790	0.077	3.055	0.032	0.006
		1.0	0.576	2.938	1.106	0.623	0.086	2.883	0.127	0.014
		1.5	0.584	2.849	1.649	0.492	0.094	2.685	0.284	0.019
		2.5	0.581	2.919	2.777	0.309	0.089	2.768	0.830	0.019
		5.0	0.611	2.803	5.525	0.107	0.133	1.878	3.259	0.010
1.5	0.5	0.5	1.816	0.584	0.571	0.744	0.818	0.089	0.053	0.010
		1.0	1.818	0.560	1.140	0.563	0.766	0.068	0.207	0.019
		1.5	1.821	0.582	1.713	0.437	0.746	0.097	0.471	0.023
		2.5	1.831	0.576	2.847	0.271	0.837	0.092	1.289	0.023
		5.0	1.824	0.567	5.694	0.095	0.768	0.078	5.130	0.011
	1.5	0.5	1.771	1.652	0.575	0.824	0.941	0.581	0.032	0.004
		1.0	1.874	1.752	1.147	0.677	1.378	1.037	0.126	0.011
		1.5	1.761	1.728	1.722	0.545	0.921	0.885	0.286	0.016
		2.5	1.722	1.635	2.869	0.348	0.799	0.536	0.780	0.018
		5.0	1.817	1.653	5.755	0.118	1.148	0.593	3.269	0.010
	2.5	0.5	1.807	2.932	0.579	0.859	1.249	2.651	0.027	0.003
		1.0	1.883	2.873	1.160	0.723	1.544	2.413	0.111	0.008
		1.5	1.776	2.837	1.744	0.596	1.130	2.013	0.252	0.013
		2.5	1.855	2.858	2.912	0.397	1.421	2.251	0.726	0.018
		5.0	1.739	2.884	5.817	0.129	0.930	2.396	2.957	0.007

Table 3. The AVs of estimates and the corresponding MSEs for $n = 25$.

Parameters			**MLE**				**MSE**			
θ	α	β	θ	α	β	$R(0.5)$	θ	α	β	$R(0.5)$
0.5	0.5	0.5	0.586	0.521	0.515	0.650	0.058	0.016	0.027	0.010
		1.0	0.573	0.523	1.026	0.458	0.048	0.016	0.109	0.015
		1.5	0.572	0.523	1.537	0.337	0.045	0.016	0.247	0.016
		2.5	0.576	0.521	2.589	0.193	0.048	0.016	0.677	0.013
		5.0	0.569	0.528	5.147	0.066	0.045	0.021	2.785	0.007
	1.5	0.5	0.568	1.581	0.521	0.757	0.033	0.174	0.022	0.004
		1.0	0.560	1.589	1.048	0.577	0.033	0.198	0.084	0.010
		1.5	0.557	1.565	1.573	0.445	0.028	0.119	0.189	0.012
		2.5	0.550	1.573	2.620	0.272	0.028	0.136	0.552	0.013
		5.0	0.553	1.609	5.223	0.092	0.033	0.196	2.086	0.009
	2.5	0.5	0.523	2.586	0.544	0.785	0.017	0.525	0.015	0.002
		1.0	0.526	2.545	1.085	0.613	0.019	0.428	0.062	0.006
		1.5	0.522	2.559	1.624	0.478	0.018	0.587	0.134	0.008
		2.5	0.536	2.592	2.729	0.291	0.021	0.567	0.382	0.008
		5.0	0.526	2.614	5.471	0.085	0.017	0.606	1.490	0.003
1.5	0.5	0.5	1.714	0.515	0.535	0.749	0.439	0.012	0.033	0.007
		1.0	1.741	0.520	1.064	0.576	0.452	0.012	0.133	0.015
		1.5	1.563	0.519	1.595	0.439	0.128	0.010	0.297	0.018
		2.5	1.560	0.512	2.669	0.267	0.128	0.012	0.813	0.019
		5.0	1.561	0.518	5.301	0.098	0.131	0.012	3.306	0.013
	1.5	0.5	1.555	1.483	0.563	0.822	0.109	0.034	0.020	0.002
		1.0	1.553	1.480	1.129	0.664	0.104	0.060	0.082	0.006
		1.5	1.548	1.488	1.690	0.534	0.086	0.067	0.175	0.009
		2.5	1.538	1.493	2.804	0.339	0.087	0.045	0.473	0.010
		5.0	1.539	1.473	5.615	0.102	0.086	0.042	1.848	0.004
	2.5	0.5	1.527	2.509	0.565	0.857	0.061	0.143	0.018	0.001
		1.0	1.528	2.524	1.122	0.722	0.043	0.205	0.064	0.004
		1.5	1.519	2.518	1.687	0.595	0.049	0.191	0.149	0.007
		2.5	1.522	2.499	2.807	0.390	0.047	0.154	0.399	0.009
		5.0	1.525	2.521	5.608	0.122	0.046	0.191	1.571	0.004

Table 4. The AVs of CIBs (in parentheses) and the corresponding CILs for $n = 25$.

Parameters			**ACIs**			
θ	α	β	θ	α	β	$R(0.5)$
0.5	0.5	0.5	[0,2.746]2.746	[0,2.916]2.916	[0.050,1.247]1.196	[0.422,0.761]0.338
		1.0	[0,2.510]2.510	[0,2.945]2.945	[0.263,2.316]2.053	[0.254,0.564]0.310
		1.5	[0,2.602]2.602	[0,3.159]3.159	[0.215,3.662]3.447	[0.163,0.441]0.278
		2.5	[0,2.502]2.502	[0,2.974]2.974	[0.531,5.967]5.436	[0.082,0.264]0.182
		5.0	[0,2.523]2.523	[0,2.908]2.908	[0.877,12.04]11.15	[0.013,0.115]0.102
	1.5	0.5	[0,3.160]3.160	[0,11.24]11.24	[0.271,0.945]0.674	[0.566,0.838]0.272
		1.0	[0,3.038]3.038	[0,10.07]10.07	[0.461,1.980]1.519	[0.343,0.679]0.336
		1.5	[0,2.814]2.814	[0,10.38]10.38	[0.742,2.905]2.163	[0.247,0.549]0.302
		2.5	[0,2.885]2.885	[0,9.831]9.831	[1.264,4.844]3.580	[0.144,0.357]0.213
		5.0	[0,2.833]2.833	[0,9.796]9.796	[2.287,9.812]7.525	[0.034,0.153]0.119
	2.5	0.5	[0,1.982]1.982	[0,16.45]16.45	[0.248,0.952]0.704	[0.588,0.884]0.296
		1.0	[0,1.739]1.739	[0,15.49]15.49	[0.536,1.855]1.319	[0.410,0.704]0.294
		1.5	[0,1.872]1.872	[0,15.85]15.85	[0.920,2.666]1.747	[0.297,0.586]0.288
		2.5	[0,1.823]1.823	[0,15.90]15.90	[1.467,4.505]3.039	[0.172,0.394]0.222
		5.0	[0,1.793]1.793	[0,15.01]15.01	[2.863,9.148]6.285	[0.047,0.167]0.120

Table 4. *Cont.*

Parameters			ACIs			
θ	α	β	θ	α	β	$R(0.5)$
1.5	0.5	0.5	[0,5.296]5.296	[0,2.951]2.951	[0.231,1.071]0.839	[0.554,0.817]0.263
		1.0	[0,5.235]5.235	[0,3.156]3.156	[0.479,2.119]1.640	[0.350,0.654]0.304
		1.5	[0,5.366]5.366	[0,3.082]3.082	[0.678,3.221]2.543	[0.247,0.530]0.283
		2.5	[0,5.269]5.269	[0,2.997]2.997	[1.150,5.399]4.249	[0.128,0.338]0.209
		5.0	[0,5.160]5.160	[0,2.970]2.970	[2.426,10.560]8.13	[0.035,0.141]0.106
	1.5	0.5	[0,5.380]5.380	[0,7.637]7.637	[0.283,0.976]0.692	[0.666,0.870]0.203
		1.0	[0,5.574]5.574	[0,7.905]7.905	[0.582,1.943]1.362	[0.463,0.737]0.275
		1.5	[0,5.668]5.668	[0,7.272]7.272	[0.838,2.959]2.121	[0.321,0.599]0.278
		2.5	[0,5.447]5.447	[0,7.710]7.710	[1.443,4.863]3.420	[0.182,0.429]0.246
		5.0	[0,6.181]6.181	[0,7.557]7.557	[2.531,10.17]7.645	[0.040,0.179]0.139
	2.5	0.5	[0,5.310]5.310	[0,12.37]12.37	[0.327,0.937]0.611	[0.701,0.895]0.194
		1.0	[0,4.956]4.956	[0,12.34]12.34	[0.656,1.849]1.193	[0.516,0.784]0.268
		1.5	[0,5.045]5.045	[0,11.89]11.89	[0.940,2.816]1.876	[0.361,0.650]0.289
		2.5	[0,4.763]4.763	[0,12.48]12.48	[1.660,4.636]2.975	[0.199,0.465]0.266
		5.0	[0,4.915]4.915	[0,13.08]13.08	[3.292,9.261]5.969	[0.038,0.191]0.153

Table 5. The AVs of CIBs (in parentheses) and the corresponding CILs for $n = 50$.

Parameters			ACIs			
θ	α	β	θ	α	β	$R(0.5)$
0.5	0.5	0.5	[0,1.920]1.920	[0,2.104]2.104	[0.234,0.878]0.644	[0.538,0.775]0.236
		1.0	[0,1.892]1.892	[0,2.103]2.103	[0.433,1.795]1.363	[0.341,0.579]0.238
		1.5	[0,1.749]1.749	[0,2.067]2.067	[0.680,2.680]2.000	[0.231,0.440]0.209
		2.5	[0,1.936]1.936	[0,2.201]2.201	[0.861,4.707]3.847	[0.117,0.278]0.161
		5.0	[0,1.785]1.785	[0,2.046]2.046	[2.275,8.892]6.617	[0.027,0.109]0.083
	1.5	0.5	[0,2.713]2.713	[0,6.503]6.503	[0.296,0.805]0.509	[0.648,0.851]0.203
		1.0	[0,2.781]2.781	[0,6.918]6.918	[0.599,1.606]1.007	[0.457,0.689]0.233
		1.5	[0,2.778]2.778	[0,6.921]6.921	[0.853,2.452]1.600	[0.329,0.566]0.237
		2.5	[0,2.967]2.967	[0,6.832]6.832	[1.440,4.048]2.608	[0.181,0.377]0.195
		5.0	[0,2.912]2.912	[0,6.876]6.876	[2.873,8.134]5.261	[0.037,0.151]0.114
	2.5	0.5	[0,1.854]1.854	[0,9.675]9.675	[0.315,0.789]0.474	[0.691,0.888]0.197
		1.0	[0,1.906]1.906	[0,9.730]9.730	[0.605,1.608]1.003	[0.485,0.761]0.276
		1.5	[0,1.846]1.846	[0,9.004]9.004	[0.928,2.371]1.444	[0.366,0.618]0.251
		2.5	[0,1.927]1.927	[0,9.722]9.722	[1.493,4.061]2.568	[0.196,0.422]0.226
		5.0	[0,1.882]1.882	[0,8.906]8.906	[3.170,7.880] 4.710	[0.051,0.162]0.111
1.5	0.5	0.5	[0,5.046]5.046	[0,1.655]1.655	[0.308,0.833]0.525	[0.661,0.828]0.166
		1.0	[0,5.189]5.189	[0,1.614]1.614	[0.593,1.688]1.095	[0.454,0.672]0.219
		1.5	[0,5.174]5.174	[0,1.664]1.664	[0.923,2.503]1.580	[0.332,0.541]0.209
		2.5	[0,5.034]5.034	[0,1.585]1.585	[1.549,4.144]2.595	[0.187,0.354]0.166
		5.0	[0,5.161]5.161	[0,1.608]1.608	[3.033,8.355]5.322	[0.046,0.144]0.098
	1.5	0.5	[0,4.447]4.447	[0,4.113]4.113	[0.352,0.798]0.447	[0.753,0.895]0.142
		1.0	[0,4.603]4.603	[0,4.239]4.239	[0.736,1.559]0.823	[0.582,0.772]0.190
		1.5	[0,4.344]4.344	[0,4.188]4.188	[1.098,2.347]1.249	[0.441,0.650]0.209
		2.5	[0,4.391]4.391	[0,4.093]4.093	[1.730,4.007]2.277	[0.246,0.450]0.203
		5.0	[0,4.619]4.619	[0,4.198]4.198	[3.590,7.920]4.330	[0.057,0.179]0.122
	2.5	0.5	[0,4.818]4.818	[0,8.072]8.072	[0.361,0.798]0.436	[0.790,0.927]0.137
		1.0	[0,5.083]5.083	[0,8.031]8.031	[0.718,1.601]0.883	[0.620,0.827]0.206
		1.5	[0,4.972]4.972	[0,8.218]8.218	[0.980,2.508]1.528	[0.463,0.729]0.267
		2.5	[0,4.857]4.857	[0,7.801]7.801	[1.811,4.013]2.202	[0.289,0.505]0.216
		5.0	[0,4.753]4.753	[0,7.941]7.941	[3.574,8.061]4.487	[0.063,0.195]0.132

Table 6. The AVs of CIBs (in parentheses) and the corresponding CILs for $n = 100$.

Parameters			ACIs			
θ	α	β	θ	α	β	$R(0.5)$
0.5	0.5	0.5	[0,1.426]1.426	[0,1.381]1.381	[0.272,0.758]0.486	[0.574,0.727]0.153
		1.0	[0,1.385]1.385	[0,1.356]1.356	[0.567,1.485]0.917	[0.382,0.533]0.151
		1.5	[0,1.361]1.361	[0,1.343]1.343	[0.853,2.221]1.368	[0.270,0.404]0.134
		2.5	[0,1.360]1.360	[0,1.315]1.315	[1.439,3.740]2.301	[0.139,0.246]0.107
		5.0	[0,1.479]1.479	[0,1.553]1.553	[2.616,7.677]5.061	[0.034,0.099]0.065
	1.5	0.5	[0,1.889]1.889	[0,5.153]5.153	[0.342,0.701]0.359	[0.682,0.831]0.148
		1.0	[0,1.909]1.909	[0,5.296]5.296	[0.678,1.419]0.741	[0.484,0.669]0.185
		1.5	[0,1.877]1.877	[0,5.115]5.115	[1.022,2.125]1.102	[0.359,0.530]0.171
		2.5	[0,1.849]1.849	[0,5.096]5.096	[1.706,3.535]1.829	[0.207,0.337]0.131
		5.0	[0,1.825]1.825	[0,5.130]5.130	[3.436,7.011]3.576	[0.055,0.129]0.074
	2.5	0.5	[0,1.577]1.577	[0,7.578]7.578	[0.381,0.706]0.326	[0.719,0.851]0.132
		1.0	[0,1.569]1.569	[0,7.172]7.172	[0.768,1.401]0.633	[0.529,0.698]0.169
		1.5	[0,1.560]1.560	[0,7.477]7.477	[1.149,2.098]0.948	[0.393,0.563]0.170
		2.5	[0,1.642]1.642	[0,7.908]7.908	[1.913,3.545]1.633	[0.221,0.361]0.140
		5.0	[0,1.620]1.620	[0,7.978]7.978	[3.820,7.121]3.301	[0.046,0.123]0.078
1.5	0.5	0.5	[0,5.042]5.042	[0,1.458]1.458	[0.339,0.731]0.393	[0.689,0.809]0.120
		1.0	[0,4.537]4.537	[0,1.460]1.460	[0.658,1.471]0.813	[0.491,0.647]0.156
		1.5	[0,4.454]4.454	[0,4.454]4.454	[0.964,2.226]1.262	[0.361,0.516]0.155
		2.5	[0,4.472]4.472	[0,1.438]1.438	[1.630,3.708]2.078	[0.206,0.329]0.123
		5.0	[0,4.625]4.625	[0,1.480]1.480	[3.289,7.314]4.025	[0.061,0.135]0.073
	1.5	0.5	[0,4.025]4.025	[0,3.764]3.764	[0.388,0.737]0.349	[0.768,0.876]0.108
		1.0	[0,3.976]3.976	[0,3.720]3.720	[0.790,1.467]0.677	[0.590,0.739]0.149
		1.5	[0,4.027]4.027	[0,3.771]3.771	[1.184,2.196]1.012	[0.453,0.614]0.161
		2.5	[0,3.960]3.960	[0,3.738]3.738	[1.935,3.674]1.739	[0.262,0.415]0.153
		5.0	[0,4.027]4.027	[0,3.755]3.755	[3.919,7.312]3.394	[0.056,0.148]0.092
	2.5	0.5	[0,3.816]3.816	[0,6.068]6.068	[0.400,0.729]0.329	[0.809,0.906]0.097
		1.0	[0,4.055]4.055	[0,6.515]6.515	[0.778,1.466]0.688	[0.646,0.797]0.151
		1.5	[0,3.902]3.902	[0,6.344]6.344	[1.185,2.188]1.004	[0.509,0.680]0.170
		2.5	[0,3.950]3.950	[0,6.332]6.332	[1.901,3.712]1.811	[0.303,0.477]0.174
		5.0	[0,4.044]4.044	[0,6.499]6.499	[3.807,7.408]3.601	[0.072,0.172]0.100

7. Real Data Analysis

In this section, one real dataset is considered to explain the flexibility of our offered LTAPEx distribution. We compare the results of the LTAPEx distribution with some competitive distributions, such as exponential (Ex), generalized exponential (GEx) by Gupta and Kundu [24], APEx by Mahdavi and Kundu [8], APW by Nassar et al. [17] and Marshall–Olkin alpha power exponential (MOAPEx) by Nassar et al. [9]. The PDFs of these distributions are shown in Table 7 (for $x > 0$). To compare the suitability of the different competitive models to fit the real datasets, we consider employing some different statistics including the following: The log-likelihood function is evaluated at the MLEs ($\hat{\ell}$), Anderson–Darling (A^*) and Cramér–Von Mises (W^*). Moreover, we use the Kolmogorov–Smirnov (KS) statistic in addition to the corresponding p-value.

Table 7. The PDFs of different competitive models.

Distribution	PDF
Ex	$f(x;\beta) = \beta e^{-\beta x}$
GEx	$f(x;\theta,\beta) = \theta\beta e^{-\beta x}(1-e^{-\beta x})^{\theta-1}$.
APEx	$f(x;\alpha,\beta) = \frac{\log(\alpha)}{\alpha-1}\beta e^{-\beta x}\alpha^{1-e^{-\beta x}}$.
MOAPEx	$f(x;\theta,\alpha,\beta) = \frac{\beta\theta\log(\alpha)}{\alpha-1}\frac{e^{\beta x}\alpha^{1-e^{-\beta x}}}{[\theta+(1-\theta)(\alpha-1)^{-1}(\alpha^{1-e^{-\beta x}}-1)]^2}$.
APW	$f(x;\theta,\alpha,\beta) = \frac{\log(\alpha)}{\alpha-1}\beta\theta x^{\theta-1}e^{-\beta x^{\theta}}\alpha^{1-e^{-\beta x^{\theta}}}$.

The dataset presents the high-performance concrete compressive strength, which is originally provided by Yeh [25] and analyzed recently by Alam and Nassar [26]. The dataset was established from 17 various sources to inspect the reliability of a proposed strength model. The data collected concrete comprising cement alongside fly ash, blast furnace slag and superplasticizer. The dataset consisted of a single dependent variable, namely the compressive strength of concrete (in MPa), and eight independent variables. The data contain 1030 instances. Our purpose here is to find a suitable model to fit the compressive strength of the concrete variable in order to evaluate reliability using some concrete compressive strength. Before analyzing this dataset, we first plot the corresponding histogram as well as the TTT plot. Figure 4 shows the corresponding histogram and the TTT plot. From this Figure, we can notice that the dataset is positively skewed. Furthermore, the TTT plot demonstrates that the empirical hazard rate function is an increasing function. Consequently, we can conclude that the LTAPEx distribution is appropriate to model this dataset.

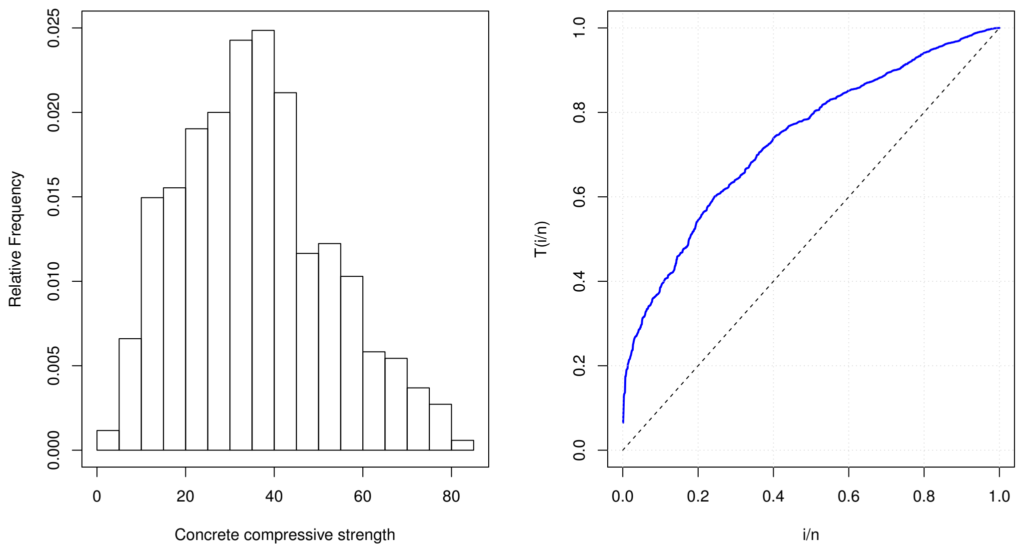

Figure 4. Histogram and TTT plots for the concrete compressive strength data.

The MLEs of the unknown parameters of the LTAPEx model and the other competitive models are obtained and displayed in Table 8. Moreover, the standard errors and the different goodness-of-fit statistics are computed and presented also in Table 8. From Table 8, one can observe that the LTAPEx model has the lowest values of A^*, W^* and KS distance with the highest p-value compared to all other competitive models. Consequently, we can deduce that the LTAPEx model is the most acceptable model to fit concrete compressive strength data. Figure 5 shows the fitted density and the estimated CDF, RF and probability–probability (PP) plots of the LTAPEx model for concrete compressive strength data. Figure 5 demonstrates that the LTAPEx model can deliver a tight fit to the dataset. Generally, we can infer that the LTAPEx model is appropriate for modeling concrete compressive strength data rather than some traditional and some recently proposed distributions.

Practically, the concrete compressive strength can vary from 17 MPa to 28 MPa for residential constructions, while it can be increased as 70 MPa in the case of commercial buildings. Accordingly, based on the results of the LTAPEx distribution in Table 8, reliability is estimated at 17 MPa, 28 MPa and 70 MPa. Table 9 shows the input and output values of the proposed model. The different estimates and the associated CIBs are displayed in Table 10. Figure 6 shows the ACIs of the reliability function at each point of the real dataset. Based on the reliability probabilities displayed in Table 10, one can infer that the tested instances were appropriate for commercial constructions.

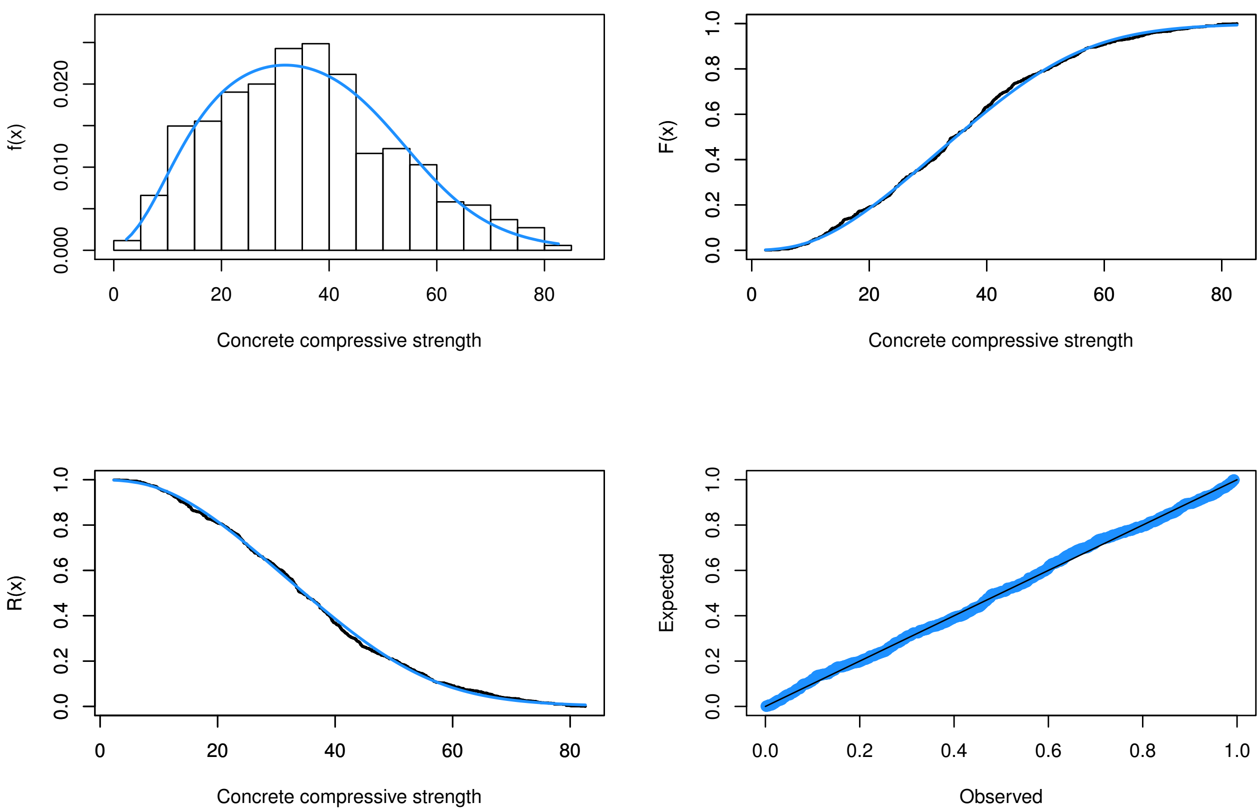

Figure 5. Plots of fitted functions and PP plot of the LTAPEx distribution for concrete compressive strength data.

Table 8. MLEs, standard errors (in parentheses) and goodness of fit statistics for real data.

Model	θ	α	β	$\hat{\ell}$	W	A	KS	p-Value
LTAPEx	111.953	376.044	0.1205	4327.724	0.1250	0.8956	0.0288	0.3589
	(88.023)	(186.23)	(0.0106)					
MOAPEx	4.7715	54.1317	0.0890	4337.01	0.1655	1.4423	0.0294	0.3355
	(1.1394)	(29.599)	(0.0034)					
APW	1.4428	30.956	0.0100	4335.53	0.1607	1.3842	0.0302	0.3030
	(0.0359)	(8.961)	(0.0016)					
APEx	-	474.873	0.0665	4350.909	0.5713	3.6236	0.0561	0.0030
		(116.849)	(0.0015)					
GEx	4.6603	0.0613	-	4361.66	1.1431	6.7256	0.0709	0.0001
	(0.2526)	(0.0017)						
Ex	-	-	0.02791	4715.80	0.6631	3.9067	0.2442	0.0000
			(0.0008)					

Table 9. Input and output values of the model for the concrete compressive strength data.

Input	**Output**
1. Concrete compressive strength data.	1. MLEs of θ, α and β.
2. Initial values of θ, α and β.	2. $\hat{R}(x_0)$.
3. $x_0 = 17, 28$ and 70.	3. CIBs of θ, α, β and $\hat{R}(x_0)$.

Table 10. MLEs and the CIBs (in parentheses) for the concrete compressive strength data.

Estimates	θ	α	β	$R(17)$	$R(28)$	$R(70)$
MLEs	111.953	376.044	0.1205	0.8694	0.6498	0.0284
CIBs	[0,284.47]	[11.03,741.06]	[0.099,0.141]	[0.854,0.885]	[0.629,0.669]	[0.020,0.036]

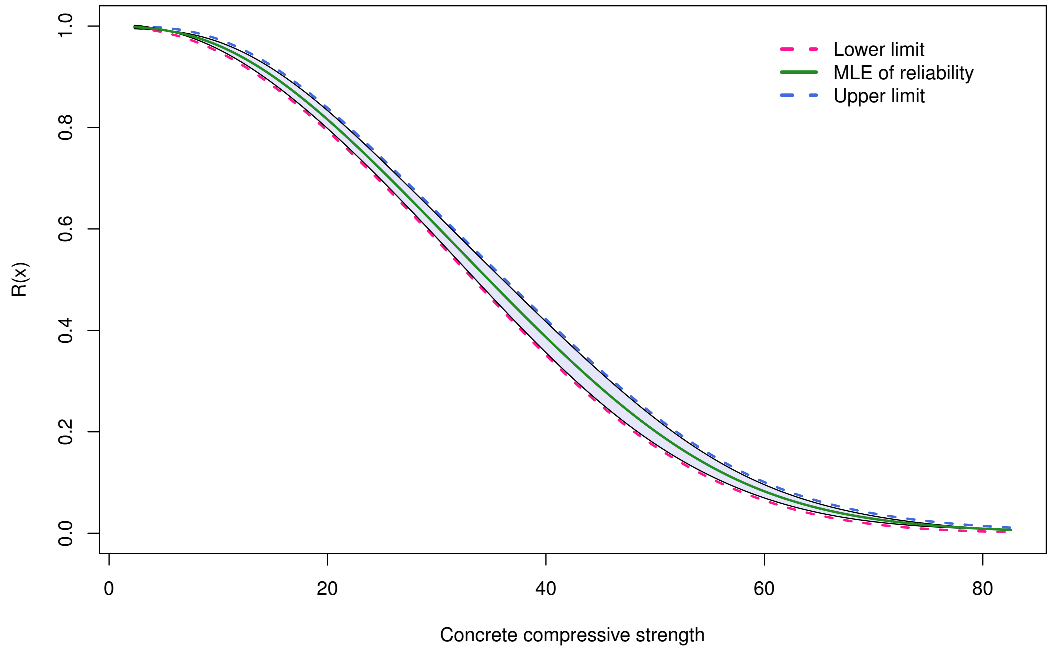

Figure 6. The ACIs of $R(x)$ for the concrete compressive strength data.

8. Conclusions

In this article, we have presented a new family of probability distributions called the logarithmic transformed alpha power family. The new family is obtained by taking the distribution function of the well-known alpha power method as the baseline distribution in the logarithmic transformed method. Some structural properties of the new family are derived. We have used the offered family to introduce a new three-parameter exponential distribution. We refer to the proposed distribution as the logarithmic transformed alpha power exponential distribution. The point and interval estimates of the unknown parameters and the reliability function are obtained via the maximum likelihood estimation method. By conducting a simulation study, the behavior of the point and interval estimates is studied based on mean square errors and confidence interval lengths, respectively. Moreover, one application relative to the high-performance concrete compressive strength dataset is considered. Based on theoretical and numerical results, we can conclude the following:

- The new distribution is able to provide a better fit for high-performance concrete compressive strength data rather than some other competitive distributions.
- The hazard rate function of the proposed model can bear various forms including decreasing, increasing, upside-down bathtub and bathtub-shaped rates.
- The proposed model can be regarded to be effective in modeling lifetime data.
- The simulation outcomes demonstrated that the estimates are asymptotically unbiased and consistent.
- The confidence interval of the reliability function performs well in terms of confidence length, which indicates that the delta method produces a small variation.
- Based on the empirical outcomes, we can infer that the logarithmic-transformed alpha power exponential distribution delivers a more satisfactory fit to the high-performance concrete compressive strength data than the traditional exponential and some other competitive models.

- As a future work, it is of interest to consider the same model described in this paper to assess the reliability of normal concrete.

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