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Eshelby Tensors for Two-Dimensional Decagonal Piezoelectric Quasicrystal Composites

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Abstract: The Eshelby tensor for two-dimensional (2D) piezoelectric quasicrystal composites (QCs) is considered. The explicit expressions of Eshelby tensors for 2D piezoelectric QCs are given using the Green's function method and the interior polarization tensor method, respectively. On this basis, numerical examples of the Eshelby tensor for 2D piezoelectric QCs with ellipsoidal inclusions are discussed in detail.

Keywords: Eshelby tensor; Green's function; interior polarization tensor; 2D piezoelectric QCs

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1. Introduction

Quasicrystals are a new solid discovered by Shechtman et al. [1] in 1984 with unique physical and mechanical properties. As a new solid structure, quasicrystals have many ideal properties, such as low coefficient, low friction, low adhesion, low porosity and high wear resistance [2,3]. Hence, they have broad application prospects. Ding et al. [4] established the theory of quasicrystal linear elasticity. Fan [5] studied some problems of quasicrystal fracture mechanics. Li et al. [6] studied the elasticity and dislocations in quasicrystals with 18-fold symmetry. Wang et al. [7] investigated the elastic field near the tip of an anticrack in a homogeneous decagonal quasicrystal.

With the advent of quasicrystal composites materials (QCs), piezoelectric QCs are also favored by the majority of scholars. The piezoelectric effect is one of the important physical properties of quasicrystals. The structure of piezoelectric QCs is much more complex than that of QCs. Rao et al. [8] conducted theoretical research on the electro-elasticity of quasicrystals. Altay and Dökmeci [9] gave the basic equations for the elasticity problem, which laid a theoretical foundation for the study of piezoelectric QCs. Zhang et al. [10] gave the general solution of the plane elasticity of 1D QCs with the piezoelectric effect. Li et al. [11] expressed the 3D general solutions to 1D hexagonal piezoelectric quasicrystals. Fan et al. [12] investigated the three-dimensional cracks in one-dimensional hexagonal piezoelectric quasicrystals. Dang et al. [13] investigated the problem of anti-plane interface cracks in one-dimensional hexagonal quasicrystal coatings. Fu et al. [14] obtain the Green's functions of two-dimensional piezoelectric quasicrystal half-space and bimaterials.

In recent years, the inclusion problems of QCs have attracted widespread attention from many experts and scholars. Hence, many notable achievements have been made. Wang [15] obtained an analytic solution for Eshelby's problem of a two-dimensional inclusion of arbitrary shape in a decagonal quasicrystalline plane or half-plane. Gao et al. [16] studied the three-dimensional problem of a spheroidal quasicrystalline inclusion, which is embedded in an infinite matrix consisting of a two-dimensional quasicrystal subject to uniform loadings at infinity. Guo et al. [17] analysed an elliptical inclusion embedded in an infinite 1D hexagonal piezoelectric quasicrystal matrix. Guo and Pan [18] studied the three-phase cylinder model of 1D piezoelectric quasicrystal composites and predicted the effective moduli of the piezoelectric quasicrystalline composites. Wang and Guo [19] obtained the exact closed-form solution of phonon, phase and electric field stress in 1D piezoelectric quasicrystal composites with the confocal elliptic cylinder model. Zhai et al. [20] investigated the plane problem of two-dimensional decagonal quasicrystals with a rigid circular arc inclusion under infinite tension and concentrated force.

In this paper, the Eshelby tensors are considered in detail by the Green's function method and the polarization tensor method. The analytical expressions are given for the Eshelby tensors for elliptic cylinder and cylindrical inclusions embedded in a 2D decagonal piezoelectric quasicrystal matrix. Meanwhile, a numerical example of the Eshelby tensor for 2D piezoelectric QCs with ellipsoidal inclusion is also given. The effects of the inclusion aspect ratio and material constants on the Eshelby tensor are discussed in detail, which are critical to the research on the properties of the quasicrystal with inclusions. The results of the calculated Eshelby tensor are sufficient to demonstrate the effects of the two methods.

2. Mathematical Formulation

In a fixed rectangular coordinate system (x_1, x_2, x_3), the basic equations for the 2D decagonal piezoelectric QCs are as follows [9,21]. The constitutive equations without considering the body force are given by

$$
\begin{aligned}
&\sigma_{11} = C_{11}\varepsilon_{11} + C_{12}\varepsilon_{22} + C_{13}\varepsilon_{33} + R_1(w_{11} + w_{22}) + R_2(w_{12} - w_{21}) - e_{31}E_3, \\
&\sigma_{22} = C_{12}\varepsilon_{11} + C_{11}\varepsilon_{22} + C_{13}\varepsilon_{33} - R_1(w_{11} + w_{22}) - R_2(w_{12} - w_{21}) - e_{31}E_3, \\
&\sigma_{33} = C_{13}\varepsilon_{11} + C_{13}\varepsilon_{22} + C_{33}\varepsilon_{33} - e_{33}E_3, \\
&\sigma_{23} = \sigma_{32} = 2C_{44}\varepsilon_{23} - e_{15}E_2, \\
&\sigma_{13} = \sigma_{31} = 2C_{44}\varepsilon_{31} - e_{15}E_1, \\
&\sigma_{12} = \sigma_{21} = 2C_{66}\varepsilon_{12} + R_1(w_{21} - w_{12}) + R_2(w_{11} + w_{22}), \\
&H_{11} = R_1(\varepsilon_{11} - \varepsilon_{22}) + 2R_2\varepsilon_{12} + K_1 w_{11} + K_2 w_{22}, \\
&H_{22} = R_1(\varepsilon_{11} - \varepsilon_{22}) + 2R_2\varepsilon_{12} + K_2 w_{11} + K_1 w_{22}, \\
&H_{23} = K_4 w_{23}, \\
&H_{13} = K_4 w_{13}, \\
&H_{12} = -2R_1\varepsilon_{12} + R_2(\varepsilon_{11} - \varepsilon_{22}) + K_1 w_{12} - K_2 w_{21}, \\
&H_{21} = 2R_1\varepsilon_{12} - R_2(\varepsilon_{11} - \varepsilon_{22}) - K_2 w_{12} + K_1 w_{21}, \\
&D_1 = 2e_{15}\varepsilon_{31} + \xi_{11}E_1, \\
&D_2 = 2e_{15}\varepsilon_{23} + \xi_{22}E_2, \\
&D_3 = e_{31}\varepsilon_{11} + e_{31}\varepsilon_{22} + e_{33}\varepsilon_{33} + \xi_{33}E_3,
\end{aligned}
\tag{1}
$$

in which $C_{66} = (C_{11} - C_{12})/2$; ε_{kl} and $w_{\gamma l}$ respectively denote the phonon field and phason field strain; σ_{ij} and $H_{\alpha j}$ stand for the corresponding stress; D_i and E_k are the electric displacement and electric field, respectively; C_{ij}, K_α and R_i are the elastic constants of the phonon field, the phason field and the phonon-phason coupling field, respectively; e_{ij} and ξ_{ij} are the piezoelectric coefficient of phonon field and dielectric constant, respectively.

In addition, the geometric equations are

$$
\begin{aligned}
\varepsilon_{kl} &= \tfrac{1}{2}(u_{k,l} + u_{l,k}), \\
w_{\alpha l} &= w_{\alpha,l}, \\
E_i &= -\phi_{,i},
\end{aligned}
\tag{2}
$$

where u_k and w_α represent the displacement of the phonon field and the phason field, respectively; ϕ is the electric potential.

The equilibrium equations are

$$
\begin{aligned}
\sigma_{ij,j} &= 0, \\
H_{\alpha j,j} &= 0, \\
D_{i,i} &= 0,
\end{aligned}
\tag{3}
$$

where commas denote partial derivatives, and the summation convention applies to repeated subscripts.

Equations (1)–(3) can be compactly expressed with the notation of Lothe and Barnett as [22]

$$Z_{Kl} = \begin{cases} \varepsilon_{kl} = \frac{1}{2}(u_{k,l} + u_{l,k}), & K = 1,\ 2,\ 3, \\ w_{\gamma l} = w_{\gamma,l}, & K = 4,\ 5, \\ E_i = -\phi_{,i}, & K = 6, \end{cases} \quad U_K = \begin{cases} u_k, & K = 1,2,3, \\ w_\alpha, & K(= \alpha + 3) = 4,5, \\ \phi, & K = 6. \end{cases} \tag{4}$$

$$\Xi_{Ij} = \begin{cases} \sigma_{ij} & I = 1,\ 2,\ 3, \\ H_{\alpha j} & I = 4,\ 5, \\ D_i & I = 6. \end{cases} \quad L_{IjKl} = \begin{cases} C_{ijkl} & I,K = 1,\ 2,\ 3, \\ R_{ij\gamma l} & I = 1,\ 2,\ 3,\ K = 4,\ 5, \\ R_{kl\alpha j} & I = 4,\ 5,\ K = 1,\ 2,\ 3, \\ K_{\alpha j \gamma l} & I = 4,\ 5,\ K = 4,\ 5, \\ e_{lij} & I = 1,\ 2,\ 3,\ K = 6, \\ e_{jkl} & I = 4,5,\ K = 1,2,3, \\ \xi_{il} & I = 6, K = 6. \end{cases} \tag{5}$$

where Z_{Kl}, U_K, Ξ_{Ij} and L^0_{IjKl} are the matrices of the strain, the displacement, the stress, and the quasicrystal piezoelectric elastic modulus, respectively.

Then, Equation (1) can be rewritten as

$$\Xi_{Ij} = L_{IjKl} Z_{Kl}. \tag{6}$$

3. Problem Statement

In this section, an inclusion of elliptical shape Ω embedded in an infinite 2D decagonal piezoelectric quasicrystal matrix R^3 is considered (as pictured in Figure 1). The inclusion is defined by $(x_1/a_1)^2 + (x_2/a_2)^2 + (x_3/a_3)^2 = 1$, where $a_i (i = 1,2,3)$ are the lengths of semiaxes of the ellipsoid. The surface of the inclusion is denoted by $|\Omega|$.

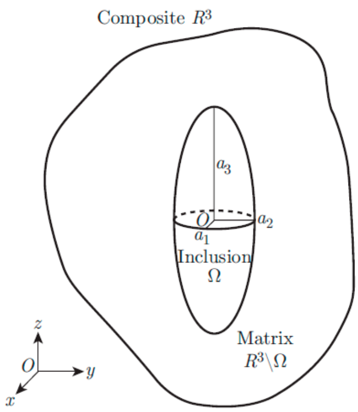

Figure 1. An ellipsoidal inclusion Ω in infinite 2D decagonal piezoelectric quasicrystal matrix R^3.

The inclusion is under a uniform stress-free strain and electric displacement-free electric field, represented by Z^*_{Kl}.

The phonon field displacement, phason field displacement and electric potential, U_K, due to the transformation of the inclusion can be expressed using the Green's function as [23,24].

$$U_K(\boldsymbol{x}) = \iint_{\partial\Omega} G_{KI}(\boldsymbol{x} - \boldsymbol{x}')\Xi^*_{Ij} n_j \mathrm{d}S(\boldsymbol{x}') - \iint_{\Omega}\int G_{KI}(\boldsymbol{x} - \boldsymbol{x}')\Xi^*_{Ij,j} \mathrm{d}V(\boldsymbol{x}'), \tag{7}$$

where n_j is the outward normal to $|\Omega|$ and Ξ_{Ij}^* is the stress and electric displacement, which is induced by the eigenstrain Z_{KI}^*, i.e., $\Xi_{Ij}^* = L_{IjKl} Z_{Kl}^*$. The Green's functions $G_{KI}(\boldsymbol{x}-\boldsymbol{x}')$ can be expressed as [25].

$$G_{KI}(\boldsymbol{x}-\boldsymbol{x}') = \frac{1}{8\pi^2|\boldsymbol{x}-\boldsymbol{x}'|}\int_{|z|=1} K_{KR}^{-1}\delta(\boldsymbol{z}\cdot\boldsymbol{t})\mathrm{d}S(\boldsymbol{z}), \tag{8}$$

where $\delta(\boldsymbol{x})$ is the Dirac delta function and $\boldsymbol{t}$ is the unit vector in the direction $\boldsymbol{x}-\boldsymbol{x}'$. $|\boldsymbol{z}| = 1$ is the surface of the unit sphere centred at $\boldsymbol{z} = 0$ and K_{KR}^{-1} is the inverse of

$$K_{IR} = z_j z_l L_{IjRl} \tag{9}$$

With the divergence theorem, we can obtain that

$$U_{K,l}(\boldsymbol{x}) = -L_{IjAb}Z_{Ab}^* \int\int_{\Omega}\int G_{KI,jl}(\boldsymbol{x}-\boldsymbol{x}')\mathrm{d}V(\boldsymbol{x}'). \tag{10}$$

Similar to the work of Ref. [25], $G_{KI,jl}(\boldsymbol{x}-\boldsymbol{x}')$ can be expressed as

$$G_{KI,jl}(\boldsymbol{x}-\boldsymbol{x}') = \frac{1}{8\pi^2}\frac{\partial^2}{\partial x_j \partial x_l}\int_{|z|=1} z_j z_l K_{KI}^{-1}(\boldsymbol{z})\delta[\boldsymbol{z}\cdot(\boldsymbol{x}-\boldsymbol{x}')]\mathrm{d}S(\boldsymbol{z}). \tag{11}$$

By using the properties of the Dirac delta function, we have

$$U_{K,l}(\boldsymbol{x}) = \frac{a_1a_2a_3}{4\pi}L_{IjAb}Z_{Ab}^*\int_{|z|=1} z_j z_l K_{KI}^{-1}(\boldsymbol{z})\zeta^{-3}\mathrm{d}S(\boldsymbol{z}). \tag{12}$$

The strain field Z_{Kl} and eigenstrain fields Z_{Ab}^* can be expressed as the following linear relationship

$$Z_{Kl} = S_{KlAb}Z_{Ab}^*, \tag{13}$$

where S_{KlAb} is the Eshelby tensor for the 2D decagonal piezoelectric quasicrystal.

In addition, by using the following variable transformations

$$\begin{aligned}
&a_1z_1 = \zeta_1,\, a_2z_2 = \zeta_2,\, a_3z_3 = \zeta_3,\, \tfrac{\zeta_1}{\zeta} = \overline{\zeta}_1,\, \tfrac{\zeta_2}{\zeta} = \overline{\zeta}_2,\, \tfrac{\zeta_3}{\zeta} = \overline{\zeta}_3,\\
&\zeta = (\zeta_1^2+\zeta_2^2+\zeta_3^2)^{\frac{1}{2}},\ \mathrm{dS}(\zeta) = a_1a_2a_3\zeta^{-3}\mathrm{dS}(z),\ \mathrm{dS}(\zeta) = \mathrm{d}\theta\mathrm{d}\overline{\zeta}_3,\\
&\overline{\zeta}_1 = (1-\overline{\zeta}_3^2)^{\frac{1}{2}}\cos\theta,\, \overline{\zeta}_2 = \left(1-\overline{\zeta}_3^2\right)^{\frac{1}{2}}\sin\theta,\, \overline{\zeta}_3 = \overline{\zeta}_3,
\end{aligned} \tag{14}$$

Equation (12) can be simplified as

$$U_{K,l}(\boldsymbol{x}) = \frac{1}{4\pi}L_{IjAb}Z_{Ab}^*\int_{-1}^{1}\int_0^{2\pi} z_j z_l K_{KI}^{-1}(\boldsymbol{z})\mathrm{d}\theta\mathrm{d}\overline{\zeta}_3. \tag{15}$$

Using Equations (2), (4), (13) and (15), S_{KlAb} can be expressed as

$$S_{KlAb} = \begin{cases} \frac{1}{8\pi}L_{IjAb}(\overline{G}_{kIjl}+\overline{G}_{lIjk}), & K = k = 1,2,3,\\ \frac{1}{4\pi}L_{IjAb}\overline{G}_{kIjl}, & K(=k+3) = 4,5,6, \end{cases} \tag{16}$$

where $\overline{G}_{kIjl}(\boldsymbol{z}) = \int_{-1}^{1}\int_0^{2\pi} z_j z_l K_{KI}^{-1}(\boldsymbol{z})\mathrm{d}\theta\mathrm{d}\overline{\zeta}_3$.

It is very helpful to express Equation (16) explicitly in terms of the matrix material constant.

$$
\begin{aligned}
S_{klab} &= \tfrac{1}{8\pi}(C_{ijab}(\overline{G}_{kijl}+\overline{G}_{lijk})+R_{abij}(\overline{G}_{kIjl}+\overline{G}_{lIjk})+e_{jab}(\overline{G}_{k6jl}+\overline{G}_{l6jk})),\\
S_{klAb} &= \tfrac{1}{8\pi}(R_{ijab}(\overline{G}_{kijl}+\overline{G}_{lijk})+K_{ijab}(\overline{G}_{kIjl}+\overline{G}_{lIjk})),\\
S_{klAb} &= \tfrac{1}{8\pi}(e_{bij}(\overline{G}_{kijl}+\overline{G}_{lijk})+\xi_{jb}(\overline{G}_{k6jl}+\overline{G}_{l6jk})),\\
S_{Klab} &= \tfrac{1}{4\pi}(C_{ijab}\overline{G}_{kijl}+R_{abij}\overline{G}_{kIjl}+e_{jab}\overline{G}_{k6jl}),\\
S_{KlAb} &= \tfrac{1}{4\pi}(C_{ijab}\overline{G}_{kijl}+K_{ijab}\overline{G}_{kIjl}),\\
S_{KlAb} &= \tfrac{1}{4\pi}(e_{bij}\overline{G}_{kijl}+\xi_{jb}\overline{G}_{k6jl}),\\
S_{Klab} &= \tfrac{1}{4\pi}(C_{ijab}\overline{G}_{6ijl}+R_{abij}\overline{G}_{6kIjl}+e_{jab}\overline{G}_{66jl}),\\
S_{KlAb} &= \tfrac{1}{4\pi}(R_{ijab}\overline{G}_{6ijl}+K_{ijab}\overline{G}_{6Ijl}),\\
S_{KlAb} &= \tfrac{1}{4\pi}(e_{bij}\overline{G}_{6ijl}+\xi_{jb}\overline{G}_{66jl}),
\end{aligned} \tag{17}
$$

in which $a, b, i, j, k, l = 1, 2, 3$ and $A, K, I = 4, 5, 6$.

According to Equation (17), the expression for the Eshelby tensor of a 2D decagonal piezoelectric QCs can be obtained. In order to improve computational efficiency, we can further obtain the Eshelby tensor by the polarization tensor method. The interior polarization tensors can be defined by [26].

$$
t_{iKjA} = -\int_{\Omega} G_{iK,jA}(\boldsymbol{x}-\boldsymbol{x}')dV(\boldsymbol{x}'). \tag{18}
$$

Similarly, using Equation (15), the above equation can be simplified as

$$
\begin{gathered}
t_{iKjA} = \tfrac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi} x_i x_j (L_{mKnA}x_m x_n)^{-1}\sin\theta d\phi d\theta,\\
x_1 = \tfrac{\sin\theta\cos\phi}{a_1},\ x_2 = \tfrac{\sin\theta\sin\phi}{a_2},\ x_3 = \tfrac{\cos\theta}{a_3}.
\end{gathered} \tag{19}
$$

The interior polarization tensor for 2D decagonal piezoelectric QCs can be written as

$$
\begin{aligned}
T_{iKjA} &= \tfrac{1}{4}(t_{iKjA}+t_{KijA}+t_{iKAj}+t_{KiAj}), && i, j, A, K = 1, 2, 3\\
T_{iKjA} &= \tfrac{1}{2}(t_{iKjA}+t_{iKAj}), && i, j, A = 1, 2, 3, K = 4, 5, 6\\
T_{iKjA} &= \tfrac{1}{2}(t_{iKjA}+t_{KijA}), && i, j, K = 1, 2, 3, A = 4, 5, 6\\
T_{iKjA} &= \tfrac{1}{2}(t_{iKjA}+t_{jKiA}), && i, j = 1, 2, 3, A, K = 4, 5, 6.
\end{aligned} \tag{20}
$$

Therefore, the Eshelby tensor for 2D decagonal piezoelectric QCs can be rewritten as

$$
S_{klab} = T_{klmn} : L_{mnab}, \tag{21}
$$

where the double dot product is used.

Based on Equations (17) and (21), the closed forms of the Eshelby tensor are given in Appendix A for cylindrical and elliptical cylindrical inclusions embedded in the 2D decagonal piezoelectric quasicrystal matrix. Comparing the expressions of the Eshelby tensor obtained by the two methods, the correctness of the results of this paper is verified.

When the contribution of the phason field is not taken into account, the results of the degradation are as follows:

$$\begin{aligned}
&S_{1111} = S_{2222} = \tfrac{5C_{11}+C_{12}}{8C_{11}},\ S_{1122} = S_{2211} = \tfrac{-C_{11}+3C_{12}}{8C_{11}},\\
&S_{1133} = S_{2233} = \tfrac{C_{13}}{2C_{11}},\ S_{1163} = S_{2263} = \tfrac{e_{13}}{2C_{11}},\\
&S_{1212} = S_{1221} = S_{2112} = S_{2121} = \tfrac{3C_{11}-C_{12}}{8C_{11}},\\
&S_{1313} = \tfrac{1}{4},\ S_{2323} = \tfrac{1}{4},\ S_{6161} = S_{6262} = \tfrac{1}{2}.
\end{aligned} \tag{22}$$

After comparing with the results in Ref. [27], it is found that the two results are completely consistent. The correctness of the method is further verified.

4. Numerical Examples and Discussion

In this section, numerical examples of the Eshelby tensor for 2D decagonal piezoelectric QCs with ellipsoidal inclusions are given. The material constants are shown in Table 1 [28,29].

Table 1. The material constants of 2D decagonal piezoelectric QCs.

Phonon (GPa)	C_{11} = 234.33, C_{12} = 57.41, C_{13} = 66.63, C_{33} = 232.22, C_{44} = 70.19,
Phason (GPa)	K_1 = 122, K_2 = 24, K_4 = 12,
Phonon-phason coupling (GPa)	R_1 = 8.846, R_2 = 8.846,
piezoelectric coefficient (C/m^2)	$e_{31} = -4.4$, e_{15} = 11.6, e_{33} = 18.6,
dielectric constant ($10^{-9}C^2/(Nm^2)$)	ξ_{11} = 11.2, ξ_{22} = 11.2, ξ_{33} = 12.6.

Furthermore, to avoid the pathology of the matrix caused by the difference of material parameter magnitude, the material constants can be treated as dimensionless quantities by the following formula

$$\widetilde{C}_{ijkl} = \frac{C_{ijkl}}{C_{11}},\ \widetilde{R}_{ijkl} = \frac{R_{ij\gamma l}}{R_1},\ \widetilde{e}_{ijk} = \frac{e_{ijk}}{e_{33}},\ \widetilde{K}_{\alpha j\gamma l} = \frac{K_{\alpha j\gamma l}C_{11}}{{R_1}^2},\ \widetilde{\xi}_{jk} = \frac{\xi_{jk}C_{11}}{{e_{33}}^2}. \tag{23}$$

where waves represent dimensionless quantities.

Observe from Figures 2–5 that there are 23 independent non-zero components of the Eshelby tensor in 2D decagonal piezoelectric QCs. The Eshelby tensors represented in the Equations (A1)–(A11) are depicted in Figures 2 and 3. It can be seen that the Eshelby tensors almost reach their asymptotic values at $a_3/a_1 = 10$. The curves in Figures 4 and 5 and S_{3333} in Figure 3 gradually vanish with the increasing aspect ratio of the inclusion. From Figures 2–5, we can see that the Eshelby tensors obtained by the Green's function method are in a good agreement with those obtained by the interior polarization method. Furthermore, our results are consistent with those of Ref. [27] when the contribution of the phason field is neglected.

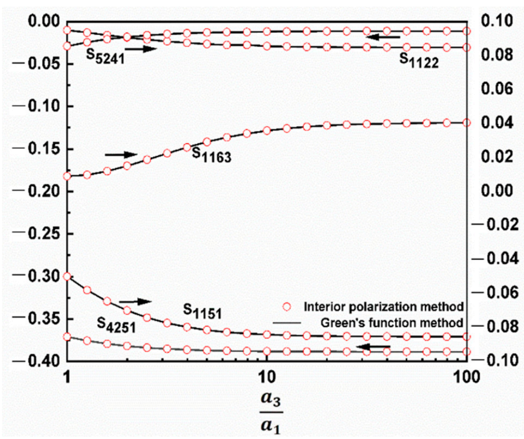

Figure 2. Eshelby tensors relating the piezoelectric response versus the aspect ratio.

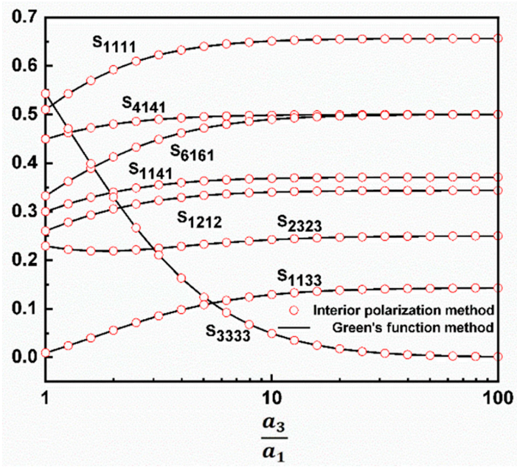

Figure 3. Eshelby tensors relating the phonon field eigenstrain response versus the aspect ratio.

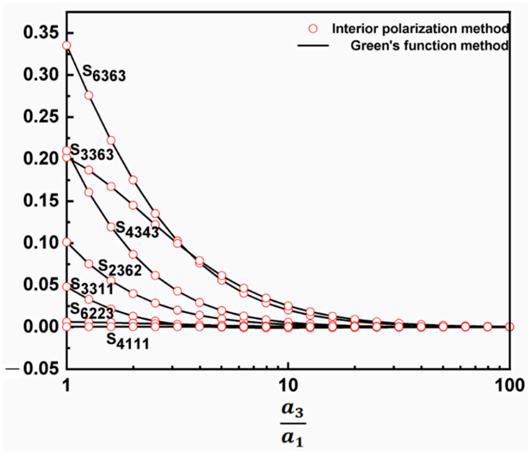

Figure 4. Eshelby tensors relating the phason field eigenstrain response versus the aspect ratio.

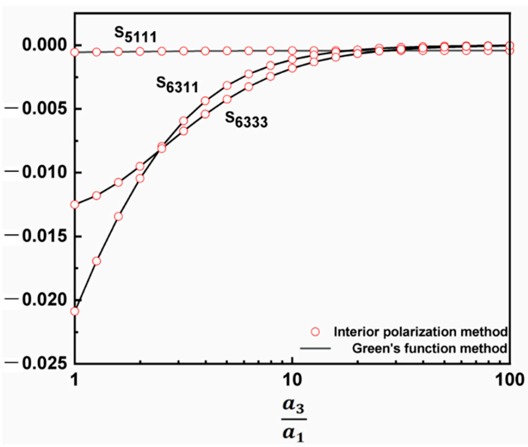

Figure 5. Eshelby tensors relating the interactive response between the phonon, phason and electric fields versus the aspect ratio.

5. Conclusions

In this paper, the Eshelby tensor for the ellipsoidal inclusion problem in the infinite 2D decagonal piezoelectric QCs matrix is investigated in detail. The explicit expressions of Eshelby tensors for 2D decagonal piezoelectric QCs are given with the help of the Green's function method and the polarization tensor method, respectively. On this basis, numerical examples of the Eshelby tensor are also presented. From the perspective of numerical results, the equivalence of the two methods is verified once again. It is also revealed that the inclusions aspect ratio and material constants have a significant effect on the Eshelby tensor.

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Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare that they have no conflict of interest to this work. We declare that we do not have any commercial or associative interest that represents a conflict of interest in connection with the work submitted.

Appendix A

Based on the Equations (17) and (21), the simplified and closed-form expressions of the Eshelby tensor are given by

(i) Cylindrical inclusions $(a_1 = a_2,\ a_3 \to \infty)$

$$S_{1111} = S_{2222} = \frac{4{R_1}^2 + 4{R_2}^2 - 5C_{11}K_1 - C_{12}K_1}{8({R_1}^2 + {R_2}^2 - C_{11}K_1)}, \tag{A1}$$

$$S_{1122} = S_{2211} = \frac{4{R_1}^2 + 4{R_2}^2 + 3C_{12}K_1 - C_{11}K_1}{8(C_{11}K_1 - {R_1}^2 + {R_2}^2)}, \tag{A2}$$

$$S_{1133} = S_{2233} = \frac{-C_{13}K_1}{2({R_1}^2 + {R_2}^2 - C_{11}K_1)}, \tag{A3}$$

$$S_{1212} = \frac{4{R_1}^2 + 4{R_2}^2 - 3C_{11}K_1 + C_{12}K_1}{8({R_1}^2 + {R_2}^2 - C_{11}K_1)}, \tag{A4}$$

$$S_{4111} = -S_{4122} = S_{5211} = -S_{5222} = -S_{4221} = S_{5121} = \frac{R_1(C_{11} + C_{12})}{8(C_{11}K_1 - {R_1}^2 + {R_2}^2)}, \tag{A5}$$

$$S_{4211} = -S_{4222} = -S_{5111} = S_{5122} = S_{4121} = S_{5221} = \frac{R_2(C_{11} + C_{12})}{8(C_{11}K_1 - {R_1}^2 + {R_2}^2)}, \tag{A6}$$

$$\begin{gathered} S_{1141} = -S_{1152} = -S_{2241} = S_{2252} = -S_{2142} = S_{2151} = \\ \frac{-R_1(K_1-K_2)(-4{R_1}^2-4{R_2}^2+3C_{11}K_1-C_{12}K_1)}{8({R_1}^2+{R_2}^2-C_{11}K_1)(-2{R_1}^2-2{R_2}^2+C_{11}K_1-C_{12}K_1)}, \end{gathered} \tag{A7}$$

$$\begin{gathered} S_{1142} = -S_{1151} = S_{2242} = -S_{2251} = S_{2141} = S_{2152} = \\ \frac{-R_2(K_1-K_2)(-4{R_1}^2-4{R_2}^2+3C_{11}K_1-C_{12}K_1)}{8({R_1}^2+{R_2}^2-C_{11}K_1)(2{R_1}^2+2{R_2}^2-C_{11}K_1+C_{12}K_1)}, \end{gathered} \tag{A8}$$

$$S_{1313} = S_{2323} = 2S_{4141} = 2S_{4242} = 2S_{5151} = 2S_{5252} = 2S_{6161} = 2S_{6262} = \frac{1}{4}, \tag{A9}$$

$$S_{1163} = S_{2263} = \frac{e_{31}K_1}{2C_{11}K_1 - 2({R_1}^2 + {R_2}^2)}, \tag{A10}$$

$$\begin{gathered} S_{4152} = S_{5241} = -S_{4251} = -S_{5142} = \\ \frac{(2{R_1}^2+2{R_2}^2)^2+2C_{11}K_1K_2(C_{11}-C_{12})}{4({R_1}^2+{R_2}^2-C_{11}K_1)(2{R_1}^2+2{R_2}^2-C_{11}K_1+C_{12}K_1)} + \\ \frac{(C_{12}-3C_{11})(K_1+K_2)({R_1}^2+{R_2}^2)}{4({R_1}^2+{R_2}^2-C_{11}K_1)(2{R_1}^2+2{R_2}^2-C_{11}K_1+C_{12}K_1)}. \end{gathered} \tag{A11}$$

(ii) Elliptic cylinder $(a_2/a_1 = a,\ a_3 \to \infty)$

$$S_{1111} = -\frac{a((3+2a)C_{11}K_1 + C_{12}K_1 - 2(1+a)(R_1{}^2 + R_2{}^2)}{2(1+a)^2(R_1{}^2 + R_2{}^2 - C_{11}K_1)},$$

$$S_{1122} = -\frac{a(-C_{11}K_1 + (1+2a)C_{12}K_1 + 2(1+a)(R_1{}^2 + R_2{}^2))}{2(1+a)^2(R_1{}^2 + R_2{}^2 - C_{11}K_1)},$$

$$S_{1133} = -\frac{aC_{13}K_1}{(1+a)(R_1{}^2 + R_2{}^2 - C_{11}K_1)},\ S_{2233} = -\frac{C_{13}K_1}{(1+a)(R_1{}^2 + R_2{}^2 - C_{11}K_1)},$$

$$S_{1141} = \frac{(K_2 + aK_2 - (1+a)K_1)R_1((2+a)C_{11}K_1 - aC_{12}K_1 - 2(1+a)(R_1{}^2 + R_2{}^2)}{(1+a)^3(R_1{}^2 + R_2{}^2 - C_{11}K_1)(C_{11}K_1 - C_{12}K_1 - 2(R_1{}^2 + R_2{}^2))},$$

$$S_{1142} = \frac{a^2(aK_1 + K_2 - (1+a)K_1)R_2((2+a)C_{11}K_1 - aC_{12}K_1 - 2(1+a)(R_1{}^2 + R_2{}^2)}{(1+a)^3(R_1{}^2 + R_2{}^2 - C_{11}K_1)(C_{11}K_1 - C_{12}K_1 - 2(R_1{}^2 + R_2{}^2))},$$

$$S_{1151} = -\frac{(K_1 + aK_2 - (1+a)K_1)R_2((2+a)C_{11}K_1 - aC_{12}K_1 - 2(1+a)(R_1{}^2 + R_2{}^2)}{(1+a)^3(R_1{}^2 + R_2{}^2 - C_{11}^0K_1)(C_{11}K_1 - C_{12}K_1 - 2(R_1{}^2 + R_2{}^2))},$$

$$S_{1152} = \frac{a^2(aK_1 + K_2 - (1+a)K_1)R_1((2+a)C_{11}K_1 - aC_{12}K_1 - 2(1+a)(R_1{}^2 + R_2{}^2)}{(1+a)^3(R_1{}^2 + R_2{}^2 - C_{11}K_1)(C_{11}K_1 - C_{12}K_1 - 2(R_1{}^2 + R_2{}^2))},$$

$$S_{1163} = -\frac{ae_{31}K_1}{(1+a)(-C_{11}K_1 + R_1^2 + R_2^2)},\ S_{2263} = -\frac{e_{31}K_1}{(1+a)(-C_{11}K_1 + R_1^2 + R_2^2)},$$

$$S_{2211} = -\frac{-aC_{11}K_1 + (2+a)C_{12}K_1 + 2(1+a)(R_1{}^2 + R_2{}^2)}{2(1+a)^2(R_1{}^2 + R_2{}^2 - C_{11}K_1)},\ S_{2323} = \frac{2}{4(1+a)},$$

$$S_{2222} = -\frac{a(2+3a)C_{11}K_1 + aC_{12}K_1 - 2(1+a)(R_1{}^2 + R_2{}^2)}{2(1+a)^2(R_1{}^2 + R_2{}^2 - C_{11}K_1)},\ S_{1313} = \frac{a}{4(1+a)},$$

$$S_{2241} = \frac{(K_1 - K_2)R_1((1+2a)C_{11}K_1 - C_{12}K_1 - 2(1+a)(R_1{}^2 + R_2{}^2)}{(1+a)^3(R_1{}^2 + R_2{}^2 - C_{11}K_1)(C_{11}K_1 - C_{12}K_1 - 2(R_1{}^2 + R_2{}^2))},$$

$$S_{2242} = -\frac{a(aK_1 + K_2 - (1+a)K_1)R_2((1+2a)C_{11}K_1 - C_{12}K_1 - 2(1+a)(R_1{}^2 + R_2{}^2)}{(1+a)^3(R_1{}^2 + R_2{}^2 - C_{11}K_1)(C_{11}K_1 - C_{12}K_1 - 2(R_1{}^2 + R_2{}^2))},$$

$$S_{2251} = \frac{(K_1 - K_2)R_2((1+2a)C_{11}K_1 - C_{12}K_1 - 2(1+a)(R_1{}^2 + R_2{}^2)}{(1+a)^3(R_1{}^2 + R_2{}^2 - C_{11}K_1)(C_{11}K_1 - C_{12}K_1 - 2(R_1{}^2 + R_2{}^2))},$$

$$S_{2252} = -\frac{a(aK_1 + K_2 - (1+a)K_1)R_1((1+2a)C_{11}K_1 - C_{12}K_1 - 2(1+a)(R_1{}^2 + R_2{}^2)}{(1+a)^3(R_1{}^2 + R_2{}^2 - C_{11}K_1)(C_{11}K_1 - C_{12}K_1 - 2(R_1{}^2 + R_2{}^2))},$$

$$S_{2121} = -\frac{(1+a+a^2)C_{11}K_1 - aC_{12}K_1 - (1+a)^2(R_1{}^2 + R_2{}^2)}{2(1+a)^2(R_1{}^2 + R_2{}^2 - C_{11}K_1)},$$

$$S_{2142} = -\frac{a(K_1 - K_2)R_1((1+a+a^2)C_{11}K_1 - aC_{12}K_1 - (1+a)^2(R_1{}^2 + R_2{}^2)}{(1+a)^3(R_1{}^2 + R_2{}^2 - C_{11}K_1)(C_{11}K_1 - C_{12}K_1 - 2(R_1{}^2 + R_2{}^2))},$$

$$S_{2141} = -\frac{(K_1 - K_2)R_2((1+a+a^2)C_{11}K_1 - aC_{12}K_1 - (1+a)^2(R_1{}^2 + R_2{}^2)}{(1+a)^3(R_1{}^2 + R_2{}^2 - C_{11}K_1)(C_{11}K_1 - C_{12}K_1 - 2(R_1{}^2 + R_2{}^2))},$$

$$S_{2151} = -\frac{(K_1 - K_2)R_1((1+a+a^2)C_{11}K_1 - aC_{12}K_1 - (1+a)^2(R_1{}^2 + R_2{}^2)}{(1+a)^3(R_1{}^2 + R_2{}^2 - C_{11}K_1)(C_{11}K_1 - C_{12}K_1 - 2(R_1{}^2 + R_2{}^2))},$$

$$S_{2152} = -\frac{a(K_1 - K_2)R_2((1+a+a^2)C_{11}K_1 - aC_{12}K_1 - (1+a)^2(R_1{}^2 + R_2{}^2)}{(1+a)^3(R_1{}^2 + R_2{}^2 - C_{11}K_1)(C_{11}K_1 - C_{12}K_1 - 2(R_1{}^2 + R_2{}^2))},$$

$$S_{4133} = -\frac{(-1+a)aC_{13}R_1}{(1+a)^2(R_1{}^2+R_2{}^2-C_{11}K_1)},\ S_{4121} = -\frac{a^2(C_{11}+C_{12})R_2}{(1+a)^3(R_1{}^2+R_2{}^2-C_{11}K_1)},$$

$$S_{4111} = -\frac{a(C_{11}+C_{12})R_1}{(1+a)^3(R_1{}^2+R_2{}^2-C_{11}K_1)},\ S_{4122} = \frac{a^3(C_{11}+C_{12})R_1}{(1+a)^3(R_1{}^2+R_2{}^2-C_{11}K_1)},$$

$$S_{4133} = -\frac{(-1+a)aC_{13}R_1}{(1+a)^2(R_1{}^2+R_2{}^2-C_{11}K_1)},\ S_{4121} = -\frac{a^2(C_{11}+C_{12})R_2}{(1+a)^3(R_1{}^2+R_2{}^2-C_{11}K_1)},$$

$$S_{4211} = -\frac{(C_{11}+C_{12})R_2}{(1+a)^3(R_1{}^2+R_2{}^2-C_{11}K_1)},\ S_{4221} = \frac{a(C_{11}+C_{12})R_1}{(1+a)^3(R_1{}^2+R_2{}^2-C_{11}K_1)},$$

$$S_{4151} = \frac{(-1+a)^2a(C_{11}+C_{12})(K_1-K_2)R_1R_2}{(1+a)^4(R_1{}^2+R_2{}^2-C_{11}K_1)(C_{11}K_1-C_{12}K_1-2(R_1{}^2+R_2{}^2))},$$

$$S_{4222} = \frac{a^2(C_{11}+C_{12})R_2}{(1+a)^3(R_1{}^2+R_2{}^2-C_{11}K_1)},\ S_{4233} = -\frac{(-1+a)C_{13}R_2}{2(1+a)^2(R_1{}^2+R_2{}^2-C_{11}K_1)},$$

$$S_{4241} = \frac{(-1+a)^2a(C_{11}+C_{12})(K_1+aK_2-(1+a)K_1)R_1R_2}{2(1+a)^4(R_1{}^2+R_2{}^2-C_{11}K_1)(C_{11}K_1-C_{12}K_1-2(R_1{}^2+R_2{}^2))},$$

$$S_{4252} = \frac{(-1+a)^2a(C_{11}+C_{12})(aK_1+K_2-(1+a)K_1)R_1R_2}{2(1+a)^4(R_1{}^2+R_2{}^2-C_{11}K_1)(C_{11}K_1-C_{12}K_1-2(R_1{}^2+R_2{}^2))},$$

$$S_{5111} = -\frac{a(C_{11}+C_{12})R_2}{(1+a)^3(R_1{}^2+R_2{}^2-C_{11}K_1)},\ S_{5122} = -\frac{a^3(C_{11}+C_{12})R_2}{(1+a)^3(R_1{}^2+R_2{}^2-C_{11}K_1)},$$

$$S_{5121} = -\frac{a^2(C_{11}+C_{12})R_1}{(1+a)^3(R_1{}^2+R_2{}^2-C_{11}K_1)},$$

$$S_{5141} = -\frac{(-1+a)^2(C_{11}+C_{12})(K_1+aK_2-(1+a)K_1)R_1R_2}{2(1+a)^4(R_1{}^2+R_2{}^2-C_{11}K_1)(C_{11}K_1-C_{12}K_1-2(R_1{}^2+R_2{}^2))},$$

$$S_{5211} = -\frac{a(C_{11}+C_{12})R_1}{(1+a)^3(R_1{}^2+R_2{}^2-C_{11}K_1)},\ S_{5233} = -\frac{(-1+a)C_{13}R_1}{(1+a)^2(-C_{11}K_1+R_1+R_2)},$$

$$S_{5221} = -\frac{a(C_{11}+C_{12})R_1}{(1+a)^3(R_1{}^2+R_2{}^2-C_{11}K_1)},\ S_{5222} = \frac{a^2(C_{11}+C_{12})R_1}{(1+a)^3(R_1{}^2+R_2{}^2-C_{11}K_1)},$$

$$S_{5152} = -\frac{(-1+a)^2(C_{11}+C_{12})(aK_1+K_2-(1+a)K_1)R_1R_2}{2(1+a)^4(R_1{}^2+R_2{}^2-C_{11}K_1)(C_{11}K_1-C_{12}K_1-2(R_1{}^2+R_2{}^2))},$$

$$S_{5242} = \frac{(-1+a)^2a(C_{11}+C_{12})(aK_1+K_2-(1+a)K_1)R_1R_2}{2(1+a)^4(R_1{}^2+R_2{}^2-C_{11}K_1)(C_{11}K_1-C_{12}K_1-2(R_1{}^2+R_2{}^2))},$$

$$S_{5251} = -\frac{(-1+a)^2a(C_{11}+C_{12})(K_1+aK_2-(1+a)K_1)R_1R_2}{2(1+a)^4(R_1{}^2+R_2{}^2-C_{11}K_1)(C_{11}K_1-C_{12}K_1-2(R_1{}^2+R_2{}^2))},$$

$$S_{5151} = S_{4141} = \frac{-(a(2(1+a)^3C_{11}{}^2K_1{}^2+f_0+C_{12}b_0-C_{11}c_0+K_1d_0}{2(1+a)^4(R_1{}^2+R_2{}^2-C_{11}K_1)(C_{11}K_1-C_{12}K_1-2(R_1{}^2+R_2{}^2)},$$

$$S_{4242} = S_{5252} = \frac{-((2(1+a)^3C_{11}{}^2K_1{}^2+f_0+C_{12}e_0-C_{11}f_0+K_1g_0}{2(1+a)^4(R_1{}^2+R_2{}^2-C_{11}K_1)(C_{11}K_1-C_{12}K_1-2(R_1{}^2+R_2{}^2))},$$

$$S_{5241} = -S_{4251} = \frac{-(2(1+a)^3C_{11}{}^2K_1{}^2K_2{}^2+g_1+C_{12}(b_1+c_1)+C_{11}(d_1+f_1)}{2(1+a)^4(R_1{}^2+R_2{}^2-C_{11}K_1)(C_{11}K_1-C_{12}K_1-2(R_1{}^2+R_2{}^2))},$$

$$S_{4152} = -S_{5142} = \frac{a(2(1+a)^3C_{11}{}^2K_1{}^2K_2{}^2+g_1+C_{12}(\mathrm{g}_1+h_1)+C_{11}(i_1+j_1)}{2(1+a)^4(R_1{}^2+R_2{}^2-C_{11}K_1)(C_{11}K_1-C_{12}K_1-2(R_1{}^2+R_2{}^2))},$$

$$S_{6161} = \frac{a}{1+a},\ S_{6262} = \frac{1}{1+a},$$

$$S_{1436} = \frac{(-1+a)ae_{31}R_1}{(1+a)^2(-C_{11}K_1+R_1{}^2+R_2{}^2)},\ S_{2536} = \frac{(-1+a)e_{31}R_1}{(1+a)^2(-C_{11}K_1+R_1{}^2+R_2{}^2)},$$

$$S_{2436} = \frac{(-1+a)e_{31}R_2}{(1+a)^2(-C_{11}K_1 + R_1{}^2 + R_2{}^2)},\ S_{1536} = -\frac{a(-1+a)e_{31}R_2}{(1+a)^2(-C_{11}K_1 + R_1{}^2 + R_2{}^2)},$$
$$b_0 = -(-1+a)^2K_2(R_1{}^2 - R_2{}^2) + K_1(3+4a+7a^2+2a^3)R_1{}^2 + (1+8a+5a^2+2a^3)R_2{}^2,$$
$$e_0 = (-1+a)^2aK_2(R_1{}^2 - R_2{}^2) + K_1(2+5a+8a^2+a^3)R_1{}^2 + (2+7a+4a^2+3a^3)R_2{}^2,$$
$$c_0 = 2(1+a)^3C_{12}K_1{}^2 + (-1+a)^2K_2(R_1{}^2 - R_2{}^2),$$
$$f_0 = 4(1+a)^3(R_1{}^2 + R_2{}^2)^2,$$
$$d_0 = (5+20a+17a^2+6a^3)R_1{}^2 + (7+16a+19a^2+6a^3)R_2{}^2,$$
$$f_0 = 2(1+a)^3C_{12}K_1{}^2 - (-1+a)^2aK_2(R_1{}^2 - R_2{}^2),$$
$$g_0 = K_2(a(5+2a+a^2)R_1{}^2 + (2+a+4a^2+5a^3)R_2{}^2,$$
$$g_1 = 4(1+a)^3(R_1{}^2 + R_2{}^2)^2,$$
$$b_1 = K_1((2+a+4a^2+a^3)R_1{}^2 + a(5+2a+a^2)R_2{}^2),$$
$$c_1 = K_2(a(5+2a+a^2)R_1{}^2 + (2+a+4a^2+5a^3)R_2{}^2),$$
$$d_1 = 2(1+a)^3C_{12}K_1K_2 + K_1((2+11a+8a^2+3a^3)R_1{}^2 + (4+7a+10a^2+3a^3)R_2{}^2,$$
$$f_1 = K_2((4+7a+10a^2+3a^3)R_1{}^2 + (2+11a+8a^2+3a^3)R_2{}^2),$$
$$g_1 = K_1((1+2a+5a^2)R_1{}^2 + (1+4a+a^2+2a^3)R_2{}^2,$$
$$h_1 = K_2((1+4a+a^2+2a^3)R_1{}^2 + (1+2a+5a^2)R_2{}^2),$$
$$i_1 = 2(1+a)^3C_{12}K_1K_2 + K_1((3+10a+7a^2+4a^3)R_1{}^2 + (3+8a+11a^2+2a^3)R_2{}^2,$$
$$j_1 = K_2((3+10a+7a^2+4a^3)R_1{}^2 + (3+8a+11a^2+2a^3)R_2{}^2).$$

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