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Frankl–Wilson Dividend Games: Compact Support and Tractable Shapley, Core, and Least-Core Computation

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Abstract

Cooperative games with transferable utility are a standard framework for modeling collaboration, fair division, cost sharing, and coalition formation. Computing central solution concepts such as the Shapley value, the core, the least core, and the nucleolus is often computationally difficult because a general game contains exponentially many coalitions. Recent work on extremal dividend structures, such as sunflower-dividend games, shows that restricting the Harsanyi dividend support to a structured set family can restore substantial algorithmic tractability. This paper introduces Frankl–Wilson dividend games, also called L-intersecting dividend games, a class of cooperative games in which the nonzero non-singleton Harsanyi dividends are supported on a structured intersection family. We distinguish two precise cases. In the uniform ordinary case, the dividend support is an r-uniform L-intersecting family: every two distinct dividend coalitions have an intersection size belonging to a fixed set L. In the modular Frankl–Wilson case, the dividend support satisfies prescribed intersection congruences modulo a prime p, together with the admissibility condition that coalition sizes avoid the allowed intersection residues modulo p. The algorithmic tools used in this paper are classical: the Shapley value is computed from Harsanyi dividends, and coalition-excess separation for non-negative dividends is performed via the standard maximum-weight closure construction. The paper does not propose a new Shapley algorithm, a new maximum-flow algorithm, or a new linear-programming method. Its main contribution is structural: it identifies Frankl–Wilson admissible dividend supports as a cooperative-game condition under which these classical procedures receive a polynomially compact input and therefore become polynomial-time algorithms in the number of players when the intersection complexity is fixed. We prove several results. First, for a modular Frankl–Wilson dividend game with residue set A and $s = |A|$, the non-singleton dividend support has size at most $\sum_{i=0}^{s} \binom{n}{i}$, which is polynomial in n for fixed s. In the uniform ordinary case with $|L| = s$, the support size is at most $\binom{n}{s}$. Second, when the dividend support is explicitly given, the Shapley value is computable in time polynomial in n, the support size, and the input bit length. Third, for non-negative dividends, core and least-core separation can be performed through a maximum-weight closure construction whose graph size is polynomial in the dividend support. Hence, for fixed intersection complexity s, the least core is polynomial-time computable under explicit support. Fourth, we state a conditional nucleolus result. If the fixed constraints in the lexicographic scheme have a polynomial-size explicit representation and a polynomial-time constrained K-best closure oracle is available, then the nucleolus is polynomial-time computable. Without these additional assumptions, nucleolus computation for general Frankl–Wilson dividend games

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remains open. Fifth, if a game's Harsanyi dividends can be approximated in dividend mass by an L-intersecting or Frankl–Wilson-admissible subfamily, then its Shapley value can be approximated with additive error bounded by the omitted dividend mass. We also show that total dividend approximation error gives explicit approximate least-core and approximate-core stability bounds. Frankl–Wilson dividend games are closely related to several extremal dividend structures. Sunflower active coalitions with common kernel size κ form a $\{\kappa\}$-intersecting family; when the active coalitions are uniform, they fit the Type I case, and under additional modular admissibility conditions they may also fit Type II. Pairwise disjoint dividend coalitions correspond to $L = \{0\}$. Finite-geometric designs, block designs, and modular intersection families provide natural examples. The paper therefore extends the research program of using extremal set theory to create cooperative-game classes with exact or certified approximate tractability.

Keywords: cooperative games; transferable utility; Harsanyi dividends; Shapley value; core; least core; nucleolus; Frankl–Wilson theorem; Ray-Chaudhuri–Wilson theorem; L-intersecting families; modular intersection families; extremal set theory; algorithmic game theory

1. Introduction

1.1. Cooperative Game Theory Introduction

A transferable-utility cooperative game consists of a finite player set N and a characteristic function

$$v{:}2^N \to \mathbb{R},\ v(\emptyset) = 0,$$

where $v(S)$ is the value generated by coalition $S \subseteq N$. Central solution concepts include the Shapley value, the core, the least core, and the nucleolus. These concepts are fundamental in economics, operations research, political science, and multi-agent systems (Aumann & Hart, 1992; Peleg & Sudhölter, 2007). The Shapley value was introduced by Shapley (1953), while the nucleolus originates with Schmeidler (1969) and was further developed by Maschler et al. (1979).

The main computational difficulty is that a general cooperative game has exponentially many coalitions. Exact Shapley computation is computationally intractable in many succinct and oracle-based representations, and naive computation over all coalitions is exponential in the number of players. Core and least-core computation require handling exponentially many coalition constraints. Nucleolus computation requires solving a sequence of lexicographic linear programs over coalition excesses (Maschler et al., 1979; Peleg & Sudhölter, 2007).

A common response is to restrict the structure of feasible coalitions. Graph-restricted games (Myerson, 1977), permission structures, convex geometries, antimatroids, and sparse marginal-contribution networks have all been studied. A different approach, developed in the sunflower-dividend games framework (Vu et al., 2026), is to restrict not the feasible coalitions but the Harsanyi dividend support (Harsanyi, 1959). Every cooperative game can be written uniquely as

$$v(S) = \sum_{T \subseteq S} \lambda_v(T),$$

where $\lambda_v(T)$ is the Harsanyi dividend of coalition T (Harsanyi, 1959). If the nonzero dividends are supported on a structured set family, many solution concepts become computationally tractable.

This paper studies two precise extremal structures for dividend support: uniform ordinary L-intersecting families and proper modular Frankl–Wilson families. Let $\mathcal{F}$ be a family of coalitions. We say that $\mathcal{F}$ is ordinary L-intersecting, for a set L of non-negative integers, if

$$|F \cap F'| \in L$$

for every pair of distinct $F, F' \in \mathcal{F}$. Thus, every pair of distinct dividend coalitions intersects in one of a small number of allowed sizes. If all sets in $\mathcal{F}$ have the same size r, then $\mathcal{F}$ is an r-uniform ordinary L-intersecting family.

In the modular version, let p be a prime and let $A \subseteq \mathbb{Z}_p$. We say that $\mathcal{F}$ is modular A-intersecting if

$$|F \cap F'| \quad p \in A$$

for every pair of distinct $F, F' \in \mathcal{F}$. The Frankl–Wilson theorem applies under the additional properness or admissibility condition

$$|F| \quad p \notin A$$

for every $F \in \mathcal{F}$ (Frankl & Wilson, 1981). That is, the allowed intersection residues and the forbidden coalition-size residues must be separated modulo p. Under these admissibility conditions, the Frankl–Wilson theorem provides the bound:

$$|\mathcal{F}| \leq \sum_{i=0}^{|A|} \binom{n}{i}.$$

For fixed $|A|$, this is polynomial in n. In the uniform ordinary case, the Ray-Chaudhuri–Wilson theorem provides a related polynomial bound when $|L|$ is fixed (Ray-Chaudhuri & Wilson, 1971). We do not claim that arbitrary non-uniform ordinary L-intersecting families are polynomially bounded without either uniformity or modular admissibility.

1.1.1. Cooperative Game Theory and Classical Solution Concepts

Transferable-utility cooperative game theory provides a formal framework for analyzing collective value creation, stability, and fair allocation. Standard references include the Handbook of Game Theory (Aumann & Hart, 1992) and the monograph by Peleg and Sudhölter (2007). The Shapley value (Shapley, 1953), the core, the least core, and the nucleolus (Schmeidler, 1969; Maschler et al., 1979) are among the central solution concepts.

The Shapley value assigns to each player the expected marginal contribution under a uniformly random ordering of the players. It is characterized by efficiency, symmetry, the dummy-player property, and additivity (Shapley, 1953). The core is the classical stability concept: no coalition should be able to obtain more by deviating (Peleg & Sudhölter, 2007). The least core minimizes the maximum coalition excess when the core is empty. The nucleolus refines the least-core idea by lexicographically minimizing the sorted excess vector (Schmeidler, 1969; Maschler et al., 1979).

1.1.2. Harsanyi Dividends and Sparse Cooperative Representations

The Harsanyi dividend representation is especially useful for structured games (Harsanyi, 1959). The dividend $\lambda_v(S)$ measures the irreducible surplus generated by coalition S after subtracting the contributions of its subcoalitions. The Shapley value linearizes in this representation:

$$\varphi_i(v) = \sum_{S \ni i} \frac{\lambda_v(S)}{|S|}.$$

This linearity is central to the present paper. It allows exact Shapley computation when the dividend support is compact, and it allows approximation guarantees to be expressed directly in terms of omitted dividend mass (Harsanyi, 1959; Shapley, 1953).

1.1.3. Sunflower-Dividend Games and Extremal Set Theory

The sunflower-dividend games framework uses the Erdős–Rado sunflower structure (Erdős & Rado, 1960) to obtain fixed-parameter tractability for Shapley, core, least-core, and nucleolus computation (Vu et al., 2026). In a sunflower-dividend game, the non-kernel positive dividends are supported on coalitions $K \cup P_j$, where K is a common kernel and the petals P_j are pairwise disjoint.

A sunflower with kernel size κ is a $\{\kappa\}$-intersecting family at the level of active coalitions: any two active coalitions intersect exactly in the kernel. Thus, sunflower-active coalitions form a special case of ordinary L-intersecting families with $L = \{\kappa\}$. However, sunflower-dividend games are not merely a special case of Frankl–Wilson dividend games. They also allow an arbitrary internal kernel game and, crucially, exploit the pairwise disjointness of petals to obtain specialized aggregation and separation procedures. Frankl–Wilson dividend games generalize the intersection pattern, but they do not inherit the petal-disjointness structure. Consequently, the source of tractability is different. Sunflower-dividend games rely on kernel-petal decomposition, whereas Frankl–Wilson dividend games rely on polynomial support-size bounds from extremal set theory and on maximum-weight closure separation.

1.1.4. Frankl–Wilson and Ray-Chaudhuri–Wilson Theorems

The Frankl–Wilson theorem (Frankl & Wilson, 1981) is a fundamental result in extremal set theory. It provides polynomial bounds on the size of set families satisfying modular intersection constraints. The Ray-Chaudhuri–Wilson theorem (Ray-Chaudhuri & Wilson, 1971) yields related bounds for uniform families with restricted intersection sizes. These theorems have been widely used in combinatorics, coding theory, complexity theory, and geometry, but their systematic use in cooperative game theory is new.

1.1.5. Algorithmic Game Theory and Compact Representations

Algorithmic game theory studies the computational complexity of game-theoretic solution concepts and the design of compact representations for large games (Nisan et al., 2007). Multi-agent systems similarly require efficient methods for coalition formation, payoff division, and mechanism design (Shoham & Leyton-Brown, 2008). The Frankl–Wilson dividend framework contributes a compact representation based on modular intersection constraints.

Recent years have seen substantial progress in identifying tractable subclasses of cooperative games, often driven by structural restrictions that break the exponential wall of general coalitional representations. Two broad lines of work are directly relevant here.

Recent advances in Shapley-value computation. Beyond the classical bounded-treewidth results for graph-restricted games, several recent works have identified new polynomial-time computable classes. Besner (2022) provides exact polynomial-time algorithms for the Shapley level value and related hierarchical values, using relevant coalition functions whose irrelevant coalitions have zero Harsanyi dividends. Martínez Mori (2024) provides a polynomial-time algorithm for the Shapley value of parking games, a natural cost-sharing model. Tan et al. (2024) studied the Shapley value for maximum independent set games on perfect graphs, characterizing #P-hardness and identifying tractable special cases via parameterized and polynomial-time algorithms. Molinero et al. (2022) introduced linearly representable games, including weighted voting, airport, and bankruptcy games, and showed that the Shapley value is pseudo-polynomial for this class, becoming strictly polynomial when the representation parameters are polynomial in the number of players. Bienvenu et al. (2025) further showed that a broad family of weighted-sum-of-minimal-supports responsibility measures coincide with Shapley values of suitably defined cooperative games and enjoy tractable data complexity for large query classes.

Recent advances in core, least-core, and nucleolus computation. Structural tractability has also advanced for stability concepts. Xiao and Fang (2023) studied arboricity games and developed an efficient nucleolus algorithm for when the core is nonempty, using a prime partition built on the densest-subgraph lattice. Benedek et al. (2023) surveyed the complexity of matching games, including b-matching and partitioned matching games, for the core, nucleolus, and Shapley value, consolidating a decade of hardness and tractability results. Subsequent work (Benedek et al., 2023, 2025) extended these results to large kidney-exchange and partitioned matching settings. These results reinforced a central lesson: structural restrictions on the coalitional value function, whether hierarchical, graph-based, or dividend-sparse, can convert exponentially large optimization problems into polynomial-time procedures. The present paper contributes to this program by identifying Frankl–Wilson admissibility as a new structural condition.

1.2. Research Gaps

The literature reveals several research gaps.

Gap 1: Limited use of intersection constraints on dividend supports. Sunflower-dividend games impose a very specific intersection pattern: all active coalitions share the same kernel. More general L-intersecting dividend supports have not been systematically studied.

Gap 2: Lack of compactness results from Frankl–Wilson theory. The Frankl–Wilson theorem implies polynomial bounds on modular intersection families, but this has not been used to obtain compact cooperative-game representations.

Gap 3: Core separation for non-sunflower intersection structures. The sunflower separation oracle exploits petal disjointness and kernel aggregation. For general L-intersecting supports, a different separation technique is required.

Gap 4: Approximation guarantees for L-intersecting substructures. If a game can be approximated by an L-intersecting dividend subgame, one would like a certificate bounding the Shapley error. The linearity of the Shapley value suggests such a guarantee, but it needs to be formalized.

Gap 5: Relationship between Frankl–Wilson structure and stability concepts. The effect of modular intersection constraints on core, least core, and nucleolus computation has not been studied.

The present paper should be read primarily as a structural contribution rather than an algorithmic one. We do not introduce a new formula for the Shapley value, a new maximum-weight-closure algorithm, or a new linear-programming solver. Instead, we identify a new extremal condition on the Harsanyi dividend support, Frankl–Wilson admissibility, under which the dividend representation becomes polynomially compact for fixed intersection complexity. Once the support is explicit and compact, known dividend-based algorithms for the Shapley value, core separation, and least-core separation become polynomial-time procedures. The main novelty is therefore the bridge between extremal intersection theorems and the compact representability of cooperative games.

1.3. Contributions

This paper makes the following contributions.

Contribution 1 (Frankl–Wilson dividend games)**.** *We define Frankl–Wilson dividend games, also called L-intersecting dividend games, as games whose nonzero non-singleton dividends are supported on a Frankl–Wilson admissible L-intersecting family.*

Contribution 2 (Compactness of dividend support)**.** *Using the Frankl–Wilson and Ray-Chaudhuri–Wilson theorems, we show that Frankl–Wilson-admissible dividend supports are polynomially bounded when the intersection complexity s is fixed. In the modular case,*

$$|\mathcal{D}| \leq \sum_{i=0}^{s} \binom{n}{i}.$$

In the uniform ordinary case,

$$|\mathcal{D}| \leq \binom{n}{s}.$$

These bounds require the precise admissibility conditions stated in Section 3. If s is not fixed, the bounds may be exponential in n.

Contribution 3 (Polynomial-time Shapley computation under explicit support)**.** *Given an explicit dividend support D_v, the Shapley value can be computed in time polynomial in n, $|D_v|$, and the input bit length using the dividend formula. If the support is Frankl–Wilson admissible with fixed intersection complexity s, then the Frankl–Wilson or Ray-Chaudhuri–Wilson bound implies $|D_v| = \mathrm{poly}(n)$; hence, the Shapley value is computable in polynomial time in n.*

Contribution 4 (Core and least-core separation for non-negative dividends)**.** *For non-negative dividends, coalition-excess maximization reduces to maximum-weight closure. Since the closure graph has size polynomial in the explicit dividend support, the least core can be computed in time polynomial in n, $|D_v|$, and the input bit length. For fixed intersection complexity s, the Frankl–Wilson or Ray-Chaudhuri–Wilson bound implies $|D_v| = \mathrm{poly}(n)$, yielding polynomial-time computability in n.*

Contribution 5 (Nucleolus computation under additional oracle assumptions)**.** *Using the standard lexicographic linear-programming scheme, the nucleolus is polynomial-time computable provided that (i) the set of coalition constraints fixed at previous lexicographic stages has polynomial size, and (ii) the maximum-weight-closure separation oracle can be extended to a polynomial-time constrained K-best closure oracle for polynomial K. Without these additional assumptions, nucleolus computation for general Frankl–Wilson dividend games remains open.*

Contribution 6 (Approximation and stability certificates)**.** *If a game is approximated by an L-intersecting dividend subgame with omitted dividend mass R, then the Shapley error is at most R. More generally, total dividend error Echanges the preimputation least-core excess by at most 2E, and a core allocation of the approximating game yields a 2E-approximate core allocation of the original game in the sense of Definition 2 (efficiency exact, coalitional excess bounded by 2E, individual rationality up to 2E).*

Contribution 7 (Cryptographic protocol instantiation)**.** *We demonstrate that Frankl–Wilson dividend games arise naturally in block-design compartmented secret-sharing protocols. In particular, the Fano-plane multi-secret sharing protocol induces a game whose Harsanyi dividends are supported exactly on the seven lines of the Fano plane, a 3-uniform $\{1\}$-intersecting family that is both Type I and Type II Frankl–Wilson admissible. This generalizes to all finite projective planes and symmetric BIBDs.*

What Is New and What Is Transposed?

The following ingredients are known:

1. **The Frankl–Wilson theorem** and the **Ray-Chaudhuri–Wilson theorem** are classical extremal set-theoretic bounds.
2. **The Harsanyi dividend representation** and the dividend formula for the Shapley value are classical.
3. **The maximum-weight closure construction** for coalition-excess maximization with non-negative dividends is standard.

4. **The ellipsoid method** and separation-oracle-based linear programming are standard tools for polynomial-time optimization.
5. **The Maschler-style lexicographic scheme** for nucleolus computation is classical.

The new contributions of the paper are as follows:

1. **Definition of Frankl–Wilson dividend games.** We define two precise classes of dividend games: uniform ordinary L-intersecting dividend games and modular Frankl–Wilson dividend games.
2. **Compactness consequence for cooperative games.** We show that, under Type I or Type II admissibility with fixed intersection complexity s the non-singleton dividend support is polynomially bounded.
3. **Tractability consequence under explicit support.** We explain how this compactness turns classical explicit-support dividend algorithms into polynomial-time algorithms in the number of players.
4. **Closure-based separation for non-sunflower dividend supports.** Unlike sunflower-dividend games, Frankl–Wilson dividend games do not admit petal aggregation. We show that maximum-weight closure still provides a valid separation oracle when the dividend support is explicit and non-negative.
5. **Approximation certificates.** We provide additive error bounds showing that omitted dividend mass controls the Shapley error, and that total dividend error controls approximate least-core and approximate-core stability.
6. **Design-theoretic and cryptographic instantiations.** We show that finite projective planes, symmetric BIBDs, and compartmented secret-sharing protocols naturally induce Frankl–Wilson dividend games.

Thus, the paper's contribution is not the creation of new algorithmic primitives, but the identification of a new extremal support structure under which known algorithmic primitives become polynomial-time tractable. A summary is provided in Table 1.

Table 1. Summary of gaps and contributions.

Research Gap	Contribution	Where Addressed
Gap 1: Limited use of intersection constraints on dividend supports.	C1: Definition of Frankl–Wilson dividend games: Type I uniform ordinary L-intersecting and Type II modular Frankl–Wilson-admissible dividend supports.	Section 3.2; Assumptions 3–5 in Section 3.3
Gap 2: Lack of compactness results from Frankl–Wilson and Ray–Chaudhuri–Wilson theory.	C2: Polynomial support-size bounds for admissible dividend supports with fixed intersection complexity s.	Theorems 3 and 4; Corollary 1
Gap 2: Algorithmic consequence: compactness must translate into explicit-support tractability.	C3: Polynomial-time Shapley computation under explicit dividend support; symmetric dividend-design formula.	Theorem 5; Corollary 2; Algorithm 1 and its complexity paragraph in Section 5
Gap 3: Core/least-core separation for non-sunflower intersection structures.	C4: Maximum-weight closure separation oracle for non-negative dividends; polynomial-time least core under explicit support.	Sections 6.2–6.4; Algorithm 2; Theorems 6 and 7; Corollary 3; Figure 1

Table 1. *Cont.*

Research Gap	Contribution	Where Addressed
Gap 5: Relationship between Frankl–Wilson structure and stability concepts, especially the nucleolus.	C5: Conditional nucleolus tractability under Assumptions 7 and 8; concrete constrained K-best closure oracle for the Fano-plane instance.	Section 7; Theorem 8; Section 7.5; Assumptions 7 and 8
Gap 4: Approximation guarantees for L-intersecting substructures.	C6: Omitted-dividend-mass Shapley bound; total-dividend-error approximate least-core and approximate-core stability bounds.	Section 8; Proposition 2; Definitions 2 and 3; Theorems 9 and 10
Application/instantiation gap: concrete domains producing admissible dividend supports.	C7: Design-theoretic and cryptographic instantiations: Fano plane, projective planes, symmetric BIBDs, block-design secret sharing, and coding/quantum connections; exact symbolic verification.	Section 10; Theorems 11–15; Section 10.3.9; Table 2

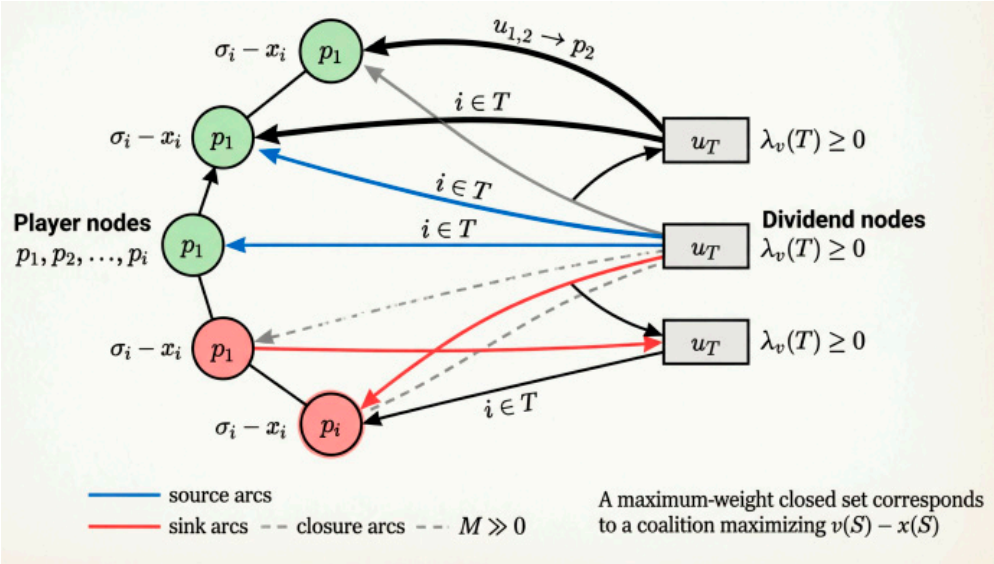


Figure 1. Closure graph for coalition-excess maximization. Player nodes p_i have weights $\sigma_i - x_i$. Dividend nodes u_T have weights $\lambda_v(T) \geq 0$. Each dividend node u_T has directed arcs to the player nodes p_i for $i \in T$. Selecting u_T forces selection of all players in T. In the minimum-cut implementation, source arcs enter positive-weight nodes, negative-weight nodes connect to the sink, and closure arcs are assigned sufficiently large capacity. A maximum-weight closed set corresponds to a coalition maximizing $v(S) - x(S)$.

2. Preliminaries

Notation. Throughout the paper, N denotes a finite player set and $n = |N|$. The Harsanyi dividend of a coalition $T \subseteq N$ is denoted by $\lambda_v(T)$. The non-singleton dividend support is

$$D_v := \{T, \subseteq N : |T| \geq 2\, \lambda_v(T) \neq 0\}.$$

The singleton dividend of player i is denoted by

$$\sigma_i := \lambda_v(\{i\}).$$

For ordinary intersection constraints, we use L to denote a finite set of allowed intersection sizes. For modular Frankl–Wilson constraints, we use p for a prime and $A \subseteq \mathbb{Z}_p$ for the set of allowed intersection residues modulo p. The intersection complexity is

$$s := \begin{cases} |\, L \,|, & \text{in the ordinary uniform Type I case,} \\ |\, A \,|, & \text{in the modular Frankl–Wilson Type II case.} \end{cases}$$

If the family is uniform, we use r for the common coalition size. The symbol k is not used for intersection complexity.

2.1. Transferable-Utility Games

Let N be a finite set of players with $|N| = n$. A transferable-utility cooperative game is a pair (N, v), where

$$v{:}2^N \to \mathbb{R},\ v(\emptyset) = 0.$$

For an allocation $x \in \mathbb{R}^N$, write

$$x(S) = \sum_{i \in S} x_i.$$

The excess of coalition S at allocation x is

$$e_v(S, x) = v(S) - x(S).$$

An allocation is efficient if

$$x(N) = v(N).$$

The imputation set is

$$I(v) = \{x, \in \mathbb{R}^N{:}x(N) = v(N)\ x_i \geq v(\{i\})\ \forall i \in N\}.$$

The core is

$$\mathrm{Core}(v) = \{x \in I(v){:}e_v(S, x) \leq 0\ \forall \emptyset \neq S \subsetneq N\}.$$

The least-core excess over imputations is

$$\varepsilon^*(v) = \min_{x \in I(v) \emptyset \neq S \subsetneq N} \max e_v(S, x).$$

The core is nonempty if and only if

$$\varepsilon^*(v) \leq 0.$$

The nucleolus is the unique imputation that lexicographically minimizes the nonincreasingly sorted vector of excesses.

The Shapley value is

$$\varphi_i(v) = \sum_{S \subseteq N{:}\, i \in S} \frac{(|\, S \,| - 1)!(n - |\, S \,|)!}{n!} (v(S) - v(S \setminus \{i\})).$$

2.2. Harsanyi Dividends

The Harsanyi dividend of a coalition S is defined recursively by

$$v(S) = \sum_{T \subseteq S} \lambda_v(T),$$

or explicitly by Möbius inversion:

$$\lambda_v(S) = \sum_{T \subseteq S} (-1)^{|S| - |T|} v(T).$$

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Let u_S be the unanimity game on S:

$$u_S(T) = \begin{cases} 1, & S \subseteq T, \\ 0, & \text{otherwise.} \end{cases}$$

Then,

$$v = \sum_{S \subseteq N} \lambda_v(S) u_S.$$

The Shapley value is linear and satisfies

$$\varphi_i(u_S) = \begin{cases} 1/\mid S \mid, & i \in S, \\ 0, & i \notin S. \end{cases}$$

Therefore,

$$\varphi_i(v) = \sum_{S \subseteq N : i \in S} \frac{\lambda_v(S)}{\mid S \mid}.$$

This dividend representation is central to the paper.

2.3. Ordinary and Modular Intersecting Families

Let $\mathcal{F} \subseteq 2^N$. Let L be a finite set of non-negative integers. The family $\mathcal{F}$ is ordinary L-intersecting if

$$|F \cap F'| \in L$$

for every pair of distinct sets $F, F' \in \mathcal{F}$.

If all sets in $\mathcal{F}$ have the same size r, then $\mathcal{F}$ is called an r-uniform ordinary L-intersecting family.

For modular intersection constraints, let p be a prime and let

$$A \subseteq \{0, 1, \dots, p-1\}.$$

The family $\mathcal{F}$ is modular A-intersecting modulo p if

$$|F \cap F'| \quad p \in A$$

for every pair of distinct sets $F, F' \in \mathcal{F}$.

It is proper modulo p, or Frankl–Wilson admissible modulo p, if, in addition,

$$|F| \quad p \notin A$$

for every $F \in \mathcal{F}$.

The properness condition is essential for the Frankl–Wilson bound. Intersection constraints alone are not sufficient.

2.4. Frankl–Wilson and Ray-Chaudhuri–Wilson Bounds

We use two standard extremal set-theoretic bounds.

Theorem 1 (Frankl–Wilson modular bound)**.** *Let p be prime and let $A \subseteq \{0, 1, \dots, p-1\}$ with $|A| = s$. Let $\mathcal{F} \subseteq 2^N$ be a family of subsets of N. Suppose the following two hypotheses hold.*

(FW1) Properness/admissibility. *For every $F \in \mathcal{F}$,*

$$|F| \quad p \notin A.$$

(FW2) Modular intersection constraint. *For every pair of distinct $F, F' \in \mathcal{F}$,*

$$|F \cap F'| \quad p \in A.$$

Then,

$$|\mathcal{F}| \leq \sum_{i=0}^{s} \binom{n}{i}.$$

In particular, for fixed s,

$$|\mathcal{F}| = O(n^s).$$

Theorem 2 (Ray-Chaudhuri–Wilson uniform bound)**.** *Let $\mathcal{F}$ be an r-uniform family of subsets of N. Suppose there is a set L of sintegers such that*

$$|F \cap F'| \in L$$

for every pair of distinct $F, F' \in \mathcal{F}$. Then,

$$|\mathcal{F}| \leq \binom{n}{s}.$$

Thus, for fixed s, the family size is $O(n^s)$.

These bounds are polynomial in n only when the intersection complexity s is fixed. If s is part of the input, the bounds may be exponential.

3. Frankl–Wilson Dividend Games

We now define the central object of the paper.

3.1. Dividend Support Decomposition

Let v be a cooperative game with Harsanyi dividends λ_v. Define the singleton dividend of player i by

$$\sigma_i := \lambda_v(\{i\}).$$

Let

$$D_v := \{T, \subseteq N : |T| \geq 2\, \lambda_v(T) \neq 0\}$$

be the non-singleton dividend support. The game can be written as

$$v(S) = \sum_{i \in S} \sigma_i + \sum_{T \in D_v : T \subseteq S} \lambda_v(T).$$

We treat singleton dividends separately because they represent standalone values and do not naturally participate in pairwise intersection constraints.

3.2. Definition

Let L be a finite set of allowed ordinary intersection sizes or let $A \subseteq \mathbb{Z}_p$ be a set of allowed modular intersection residues. A cooperative game v is a Frankl–Wilson dividend game if its non-singleton dividend support

$$D_v := \{T, \subseteq N : |T| \geq 2\, \lambda_v(T) \neq 0\}$$

satisfies one of the following Frankl–Wilson admissibility conditions.

Type I: Uniform ordinary L-intersection. There exists an integer r and a set L with $|L| = s$ such that every coalition in D_v has size r, and

$$|T \cap T'| \in L$$

for every pair of distinct $T, T' \in D_v$.

Type II: Modular Frankl–Wilson intersection. There exists a prime p and a residue set $A \subseteq \mathbb{Z}_p$ with $|A| = s$ such that the following are true:

1. for every $T \in D_v$,
$$|T| \quad p \notin A;$$
2. for every pair of distinct $T, T' \in D_v$,
$$|T \cap T'| \quad p \in A.$$

In either case, the parameter s is the intersection complexity of the dividend support.

Singleton dividends $\sigma_i = \lambda_v(\{i\})$ are treated separately and are not required to satisfy the intersection constraints.

3.3. Assumptions

We use the following assumptions.

Assumption 1 (Rational data)**.** *All game values, dividends, and payoff parameters are rational numbers.*

Assumption 2 (Reduced player set)**.** *Null players have been removed. Every remaining player either has a nonzero singleton dividend or belongs to at least one coalition in* $\mathcal{D}_v$*.*

Assumption 3 (Frankl–Wilson admissible dividend support)**.** *The non-singleton dividend support* $\mathcal{D}_v$ *satisfies Type I or Type II admissibility with intersection complexity* s*.*

Assumption 4 (Explicit support for exact algorithms)**.** *For exact Shapley, core, and least-core algorithms, the support* $\mathcal{D}_v$ *and the dividends* $\lambda_v(T)$ *are explicitly given or can be enumerated.*

Assumption 5 (Non-negative dividends for stability results)**.** *For core and least-core tractability, assume*
$$\sigma_i \geq 0 \ \forall i \in N,$$
and
$$\lambda_v(T) \geq 0 \ \forall T \in \mathcal{D}_v.$$

Assumption 6 (Nonempty imputation set)**.** *For results over imputations, assume* $I(v) \neq \emptyset$*.*

3.4. Value Representation

For every coalition $S \subseteq N$,
$$v(S) = \sum_{i \in S} \sigma_i + \sum_{T \in \mathcal{D}_v : T \subseteq S} \lambda_v(T).$$

In particular,
$$v(N) = \sum_{i \in N} \sigma_i + \sum_{T \in \mathcal{D}_v} \lambda_v(T).$$

3.5. Relationship to Sunflower-Dividend Games

In a sunflower-dividend game (Vu et al., 2026) with kernel K of size k and petals $P_1, \ldots, P_m$, the active coalitions are
$$K \cup P_1, \ldots, K \cup P_m.$$

For any two active coalitions,
$$(K \cup P_i) \cap (K \cup P_j) = K,$$
so the active coalitions form a $\{k\}$-intersecting family. Thus, at the level of active non-kernel dividend coalitions, sunflower structure is a special case of ordinary L-intersection with
$$L = \{k\}.$$

However, the relationship to the Type I and Type II classes defined in this paper requires additional qualifications.

First, Type I requires the dividend support to be uniform. Therefore, sunflower active coalitions fit Type I only when all active coalitions $K \cup P_j$ have the same cardinality, equivalently when all petals P_j have the same size. If the petal sizes differ, the active coalitions are still $\{k\}$-intersecting, but they are not Type I admissible unless they also satisfy a modular Type II condition.

Second, sunflower-dividend games allow an arbitrary internal game on the kernel K. The Frankl–Wilson dividend framework treats singleton dividends separately and imposes no structure on arbitrary higher-order kernel dividends unless those dividends are either absent or themselves supported on an admissible family.

Third, sunflower games exploit pairwise disjointness of the petals. This disjoint-petal structure enables specialized aggregation and separation procedures that are not available for general L-intersecting families. Frankl–Wilson dividend games instead obtain tractability from support compactness and maximum-weight closure separation.

Thus, Frankl–Wilson dividend games generalize the intersection pattern of sunflower active coalitions in many cases, especially when the active coalitions are uniform or satisfy modular admissibility. They do not, however, automatically capture all structural and algorithmic properties of sunflower-dividend games.

In summary, sunflower-dividend games and Frankl–Wilson dividend games belong to the same research program of extremal dividend structures, but they use different structural mechanisms. Sunflower games impose a common kernel and disjoint petals; Frankl–Wilson games impose ordinary or modular pairwise intersection constraints. The former enables specialized petal-based aggregation, while the latter enables polynomial support-size bounds through the Frankl–Wilson and Ray-Chaudhuri–Wilson theorems. Therefore, Frankl–Wilson dividend games should be viewed as a broader intersection-theoretic extension of extremal dividend games, not as a direct algorithmic strengthening of the sunflower framework.

Remark 1. *What is due to Frankl–Wilson compactness?*

It is useful to distinguish between two types of results in this paper.

The algorithms for the Shapley value, core separation, and least-core computation are primarily algorithms for explicit dividend representations. If the non-singleton dividend support D_v and the dividends $\lambda_v(T)$ are explicitly given, then the Shapley value can be computed by summing dividends over the support, and for non-negative dividends the coalition-excess separation problem can be solved by a maximum-weight closure construction. These algorithmic statements hold for any explicitly given dividend support, whether or not that support satisfies Frankl–Wilson or Ray-Chaudhuri–Wilson intersection constraints.

The specific Frankl–Wilson contribution is the compactness guarantee. When D_v satisfies Type I or Type II admissibility and the intersection complexity s is fixed, the Ray-Chaudhuri–Wilson theorem or the Frankl–Wilson theorem implies that

$$|D_v| = \mathrm{poly}(n).$$

Thus, for fixed s, the explicit-support algorithms become polynomial-time algorithms in the number of players n.

The extremal support-size bound by itself does not enumerate the support. If the dividend support is given only implicitly, or if the nonzero dividends must be recovered by Möbius inversion from a value oracle, additional assumptions are required. Exact tractability therefore relies on Assumption 4: explicit support or efficient enumerability.

4. Compactness of the Dividend Support

The first structural result is that Frankl–Wilson admissibility yields a polynomial bound on the number of non-singleton dividend coalitions.

Theorem 3 (Modular Frankl–Wilson compactness)**.** *Let v be a Frankl–Wilson dividend game of Type II with intersection complexity $s = |A|$. Then,*

$$| D_v | \leq \sum_{i=0}^{s} \binom{n}{i}.$$

For fixed s,

$$|D_v| = O(n^s).$$

Proof. We verify that D_v satisfies exactly the hypotheses of the Frankl–Wilson theorem (Theorem 1). By the Type II definition, there exist a prime p and a residue set $A \subseteq \mathbb{Z}_p$ with $|A| = s$ such that

1. for every $T \in D_v$,
$$|T| \quad p \notin A;$$
2. for every pair of distinct $T, T' \in D_v$,
$$|T \cap T'| \quad p \in A.$$

These are precisely the two hypotheses of Theorem 1. The first is the properness/admissibility condition; the second is the modular intersection condition. Therefore, applying Theorem 1 to the family D_v yields

$$| D_v | \leq \sum_{i=0}^{s} \binom{n}{i}.$$

If $D_v = \emptyset$, the bound is trivial. Otherwise, the bound follows directly. □

Theorem 4 (Uniform Ray-Chaudhuri–Wilson compactness)**.** *Let v be a Frankl–Wilson dividend game of Type I with uniform coalition size r and intersection complexity $s = |L|$. Then,*

$$| \mathcal{D}_v | \leq \binom{n}{s}.$$

For fixed s,

$$|\mathcal{D}_v| = O(n^s).$$

Proof. Using Type I admissibility, the dividend support D_v is an r-uniform family. Moreover, there exists a set L with $|L| = s$ such that

$$|\mathcal{D}_v| = O(n^s)$$

for every pair of distinct $T, T' \in D_v$. These are exactly the hypotheses of the Ray-Chaudhuri–Wilson theorem (Theorem 2). Applying that theorem to $\mathcal{F} = D_v$ yeilds

$$| D_v | \leq \binom{n}{s}.$$

For fixed s,

$$\binom{n}{s} = O(n^s).$$

□

Corollary 1 (Polynomial dividend representation for fixed s)**.** *If the intersection complexity s is fixed, where $s = |L|$ in the Type I case and $s = |A|$ in the Type II case, then the non-singleton dividend support has polynomial size. Including singleton dividends, the total number of nonzero dividend coalitions is at most*

$$n + \sum_{i=0}^{s} \binom{n}{i}$$

in the modular case, or

$$n + \binom{n}{s}$$

in the uniform case.

Thus, for fixed intersection complexity, the dividend representation is polynomially compact.

5. Shapley Value Tractability

The Shapley value is especially simple in the Harsanyi-dividend representation.

Theorem 5 (Shapley formula for Frankl–Wilson dividend games)**.** *Let v be a Frankl–Wilson dividend game with singleton dividends σ_i and non-singleton dividend support $\mathcal{D}_v$. Then, for every player $i \in N$,*

$$\varphi_i(v) = \sigma_i + \sum_{T \in \mathcal{D}_v : i \in T} \frac{\lambda_v(T)}{|T|}.$$

Proof. Using the dividend formula,

$$\varphi_i(v) = \sum_{S \ni i} \frac{\lambda_v(S)}{|S|}.$$

Separate the singleton dividend $\lambda_v(\{i\}) = \sigma_i$ from the non-singleton dividends. The singleton term contributes σ_i. Every non-singleton dividend coalition $T \in \mathcal{D}_v$ containing i contributes $\lambda_v(T)/|T|$. □

The Algorithm 1 which illustrate the Shapley value for explicit Frankl–Wilson dividend games is provided below.

Algorithm 1: Shapley value for explicit Frankl–Wilson dividend games

Input:

- player set N;
- singleton dividends σ_i;
- non-singleton dividend support $\mathcal{D}_v$;
- dividends $\lambda_v(T)$ for $T \in \mathcal{D}_v$.

Output: Shapley value $\varphi(v)$.

Procedure:

1. Initialize $\varphi_i \leftarrow \sigma_i$ for every $i \in N$.
2. For each $T \in \mathcal{D}_v$:
   - compute $a_T = \lambda_v(T)/|T|$;
   - for every $i \in T$, set $\varphi_i \leftarrow \varphi_i + a_T$.
3. Return φ.

Complexity of Algorithm 1. *The running time is*

$$O(n + I(\mathcal{D}_v)),$$

where

$$I(\mathcal{D}_v) = \sum_{D \in \mathcal{D}_v} |D|,$$

assuming unit-cost arithmetic on rational dividends. If all coalitions in $\mathcal{D}_v$ have size at most r, this becomes $O(n + r|\mathcal{D}_v|)$.

This complexity statement holds for any explicitly given dividend support. The Frankl–Wilson or Ray-Chaudhuri–Wilson compactness bounds imply that $|\mathcal{D}_v| = poly(n)$ only for Type I or Type II admissible supports with fixed intersection complexity s; hence, only in that case does Algorithm 1 become polynomial in n. If the game is provided only by a value oracle or by an implicit rule, the dividends are not automatically available: recovering them may require Möbius inversion over exponentially many coalitions. Thus, Assumption 4—explicit support or efficient enumerability—is essential. See also Remark 1 and Section 12.7.

Corollary 2 (Symmetric dividend designs)**.** *Suppose all non-singleton dividend coalitions have the same dividend d, the same size r, and each player belongs to exactly ρ dividend coalitions. Then,*

$$\varphi_i(v) = \sigma_i + \rho\frac{d}{r}.$$

If all singleton dividends are also equal, the Shapley value is symmetric.
This applies, for example, to block designs and finite-geometric dividend structures.

6. Core and Least-Core Tractability for Non-Negative Dividends

We now study stability concepts. The cleanest results hold under Assumption 5: all dividends are non-negative.

6.1. Convexity

Proposition 1 (Non-negative dividends imply convexity)**.** *If*

$$\sigma_i \geq 0 \ \forall i \in N$$

and

$$\lambda_v(T) \geq 0 \ \forall T \in \mathcal{D}_v,$$

then v is a convex game.

Proof. Every unanimity game is convex (Shapley, 1971). Since

$$v = \sum_{i \in N} \sigma_i u_{\{i\}} + \sum_{T \in \mathcal{D}_v} \lambda_v(T) u_T,$$

and all coefficients are non-negative, v is a non-negative linear combination of convex games, and hence convex. □

Therefore, under non-negative dividends, the core is nonempty, and the Shapley value belongs to the core.

Nevertheless, least-core computation still requires a separation oracle.

Remark 2. *The Frankl–Wilson condition constrains the dividend support, not a family of feasible coalitions; the support need not be closed under intersection. When all dividends are non-negative, v is a non-negative combination of unanimity games, hence convex and totally balanced, and the core is nonempty (Proposition 1). With signed dividends, convexity and balancedness are not guaranteed (Section 12.3).*

6.2. Excess Maximization as Maximum-Weight Closure

Given an allocation x, the separation problem is

$$\max_{\emptyset \neq S \subsetneq N} e_v(S, x).$$

For non-negative dividends, this can be solved by maximum-weight closure (Picard, 1976). Construct a directed graph G_x as follows.

1. For each player $i \in N$, create a player node p_i with weight
$$w(p_i) = \sigma_i - x_i.$$
2. For each non-singleton dividend coalition $T \in \mathcal{D}_v$, create a dividend node u_T with weight
$$w(u_T) = \lambda_v(T) \geq 0.$$
3. For each dividend node u_T and each player $i \in T$, add a directed arc
$$u_T \to p_i.$$

A subset of nodes C is closed if whenever it contains a node, it also contains all successors of that node. In this graph, selecting a dividend node u_T forces selection of all players in T.

Let S be the set of selected player nodes. If S is selected, then every dividend node u_T with $T \subseteq S$ can also be selected without violating closure. Since all such dividend nodes have non-negative weight, an optimal closure will select all of them.

Thus, the maximum closure weight is
$$\max_{S \subseteq N} \left[\sum_{i \in S} (\sigma_i - x_i) + \sum_{T \in \mathcal{D}_v : T \subseteq S} \lambda_v(T) \right].$$

But, this is exactly
$$\max_{S \subseteq N} [v(S) - x(S)] = \max_{S \subseteq N} e_v(S, x).$$

Therefore, maximum-weight closure solves the unconstrained excess maximization problem.

6.3. Proper Nonempty Coalitions

The core and least core exclude the empty coalition and the grand coalition. The maximum-closure problem may return $S = \emptyset$ or $S = N$. To obtain a separation oracle for proper nonempty coalitions, we can solve a polynomial number of forced closure problems.

For each ordered pair (i, j) with $i \neq j$, solve a maximum-weight closure problem with player i forced into the closure and player j forced out. Every proper nonempty coalition contains at least one player and omits at least one player, so this enumeration covers all proper nonempty coalitions.

There are $O(n^2)$ such pairs. Since maximum-weight closure is polynomial, the proper nonempty separation oracle is polynomial.

The Algorithm 2 which illustrate the least-core separation oracle for Frankl–Wilson dividend games is provided below.

Algorithm 2: Least-core separation oracle for Frankl–Wilson dividend games

Input:

- singleton dividends σ_i;
- non-singleton dividend support $\mathcal{D}_v$;
- non-negative dividends $\lambda_v(T)$;
- allocation x;
- candidate excess bound ε.

Output: A violated proper nonempty coalition, or a certificate that no violation exists.
Procedure:
Step 0: Check explicit feasibility constraints.

1. Efficiency:
$$x(N) = v(N).$$
2. Individual rationality:
$$x_i \geq \sigma_i = v(\{i\}) \ \forall i \in N.$$

If any constraint is violated, return the violated constraint.
Step 1: Build the closure graph.
Construct G_x as described above.
Step 2: Enumerate proper nonempty constraints.
For each ordered pair (i, j) with $i \neq j$:

1. Solve the maximum-weight closure problem with player i forced in and player j forced out.
2. Let S_{ij} be the selected player set.
3. Let
$$E_{ij} = v(S_{ij}) - x(S_{ij}).$$
4. If $E_{ij} > \varepsilon$, return S_{ij}.

Step 3: Return.
If no forced closure solution has excess greater than ε, return "no violation."

Complexity of Algorithm 2. *Let*

$$N = n + |D_v|, \ A = \sum_{D \in D_v} |D|.$$

Each forced maximum-weight closure is a maximum-weight closure problem on a graph with $N + 2$ nodes and $A + O(N)$ arcs. Let $T_{\mathrm{mw}}(N, A)$ denote the time to solve one such closure problem via a minimum s-t cut. Algorithm 2 solves $n(n-1)$ forced closure problems, one for each ordered pair (i, j) with $i \neq j$. Hence, its total running time is

$$O(n^2 \, T_{\mathrm{mw}}(N, A)).$$

For example, using Dinic's algorithm, $T_{\mathrm{mw}}(N, A) = O(N^2 A)$, yielding

$$O(n^2 N^2 A).$$

If D_v satisfies Type I or Type II Frankl–Wilson admissibility with fixed intersection complexity s, then $|D_v| = \mathrm{poly}(n)$ and $A = \mathrm{poly}(n)$, so Algorithm 2 is polynomial in n.

Practical feasibility. The factor n^2 in the number of maxflow calls can be expensive for large n. For a game with $n = 1000$ players, Algorithm 2 may invoke a maxflow routine on the order of 10^6 times. This is polynomial in theory but may be prohibitive in practice. Implementations should use optimized min-cut solvers, exploit sparsity of the closure graph, and reduce the number of forced pairs when possible, for example, by first solving the unforced closure and forcing only when the optimum is $\emptyset$ or N, or by using symmetry of the dividend support.

Theorem 6 (Separation oracle correctness)**.** *Let v be a Frankl–Wilson dividend game with non-negative dividends. Given an efficient allocation x satisfying individual rationality, and a candidate excess bound ε, Algorithm 2 determines whether there exists a proper nonempty coalition $S$with*

$$e_v(S, x) > \varepsilon,$$

and returns such a coalition if one exists.

Proof. Using the closure construction, for any fixed player set S, the maximum closure weight among closures whose player set S equals

$$v(S) - x(S).$$

Because dividend nodes have non-negative weight, an optimal closure for player set S includes every dividend node u_T with $T \subseteq S$.

Every proper nonempty coalition S contains at least one player i and omits at least one player j. Therefore, it is considered a forced closure problem for the pair (i, j). The maximum over all forced closure problems is therefore exactly the maximum excess over proper nonempty coalitions.

If this maximum exceeds ε, the algorithm returns a violated coalition. Otherwise, no violated proper nonempty coalition exists. □

Theorem 7 (Polynomial-time least core under explicit support)**.** *Let v be a Frankl–Wilson dividend game with non-negative dividends and explicitly given support $\mathcal{D}$. The least core can be computed in time polynomial in n, $|\mathcal{D}|$, and the input bit length.*

If, in addition, $\mathcal{D}$ satisfies Type I or Type II Frankl–Wilson admissibility with fixed intersection complexity s, then $|\mathcal{D}| = poly(n)$, and the least core can be computed in time polynomial in nand the input bit length.

Proof. The least-core LP is

$$\begin{aligned} &\min \epsilon \\ texts.t. &x(N) = v(N), \\ &x_i \geq v(\{i\}) \forall i \in N, \\ &v(S) - x(S) \leq \epsilon \forall \emptyset \neq S \subsetneq N. \end{aligned}$$

The efficiency and individual-rationality constraints are explicit. The coalition-excess constraints are separated by Algorithm 2.

The closure graph has

$$n + |\mathcal{D}|$$

nodes and

$$\sum_{D \in \mathcal{D}} |D|$$

arcs. Maximum-weight closure is solvable in polynomial time via minimum cut. Algorithm 2 performs $O(n^2)$ forced closure computations. Therefore, the separation oracle is polynomial in n, $|\mathcal{D}|$, and the input bit length.

Using Theorem 3 or Theorem 4, if the support is Frankl–Wilson admissible with fixed s, then

$$|\mathcal{D}| = \text{poly}(n).$$

Thus, for fixed s, the separation oracle is polynomial in n. The ellipsoid method, or any polynomial-time LP algorithm with a separation oracle (Grötschel et al., 1988), solves the least-core LP in polynomial time. □

Corollary 3 (Core nonemptiness)**.** *Under non-negative dividends, the game is convex, so the core is nonempty. Algorithm 2 can still be used to certify core membership of a given allocation or to compute the least-core excess.*

6.4. Implementation of Forced Maximum-Weight Closure

The separation oracle in Algorithm 2 requires solving maximum-weight closure problems with forcing constraints. We now describe a standard implementation.

Given an allocation x, construct the closure graph G_x as in Section 6.2. The graph has the following:

- player nodes p_i, $i \in N$, with weights
$$w(p_i) = \sigma_i - x_i;$$
- dividend nodes u_T, $T \in D_v$, with weights
$$w(u_T) = \lambda_v(T) \geq 0;$$
- directed arcs,
$$u_T \to p_i,$$
for every $i \in T$.

A set of nodes C is closed if whenever $u_T \in C$, all player nodes p_i with $i \in T$ also belong to C.

Maximum-weight closure can be solved by a minimum s-t cut construction. Let V be the set of nodes in G_x. Introduce a source node s and a sink node t. For each node $v \in V$,

- if $w(v) > 0$, add an arc $s \to v$ with capacity $w(v)$;
- if $w(v) < 0$, add an arc $v \to t$ with capacity $-w(v)$.

For every closure arc $u \to v$ in G_x, add an arc $u \to v$ with sufficiently large capacity, denoted as ∞. A minimum s-t cut then yields a maximum-weight closed set: the selected closed set is the set of nodes on the source side of the cut.

To force a player i into the coalition, we add a large positive incentive for selecting p_i. Equivalently, add an artificial weight $+M$ to node p_i, or add an arc
$$s \to p_i$$
with capacity M.

To force a player j out of the coalition, we add a large penalty for selecting p_j. Equivalently, add an artificial weight $-M$ to node p_j, or add an arc
$$p_j \to t$$
with capacity M.

The constant M should be chosen larger than the maximum possible change in the original closure objective. For example, let
$$W := \sum_{v \in V} |w(v)|.$$
Then, one may choose
$$M := W + 1.$$

If artificial forcing weights are added, the capacity used for the original closure arcs should be larger than the total adjusted absolute node weight. For example, after adding the forcing weights, define
$$W_{\text{adj}} := \sum_{v \in V} |w_{\text{adj}}(v)|,$$

and use an "infinite" capacity

$$\infty := W_{\text{adj}} + 1.$$

With these choices, every optimal solution to the forced closure problem satisfies the required forcing constraints.

After solving the forced closure problem for a pair (i, j), let S_{ij} be the set of player nodes selected by the closure. The actual coalition excess should then be evaluated using the original game values:

$$E_{ij} = v(S_{ij}) - x(S_{ij}).$$

This avoids including the artificial forcing weights in the reported excess.

Thus, each forced closure problem can be solved in polynomial time by a single minimum-cut computation (Gallo et al., 1989; Picard, 1976). Since Algorithm 2 solves $O(n^2)$ such problems, the entire separation oracle remains polynomial in n, $|D_v|$, and the total incidence size of the dividend support.

Numerical remark. The complexity statements treat rational arithmetic as unit cost; in a bit-complexity model the running time also depends on the bit length of the dividends and allocations, which is why Theorems 5, 7, and 8 state polynomial dependence on the input bit length. Practical floating-point implementations should use a tolerance of $\tau > 0$ in the separation oracle, reporting a violation only when the excess exceeds the candidate bound by more than τ, and should compute the forcing constants M and ∞ from the exact rational data as in Section 6.4, so that round-off cannot invalidate the forcing constraints.

7. Nucleolus Computation

Status of the nucleolus result. The result in this section is conditional. The Frankl–Wilson and Ray-Chaudhuri–Wilson compactness bounds alone do not provide a polynomial-time nucleolus algorithm. They guarantee that the dividend support is small when the intersection complexity is fixed, but the lexicographic nucleolus scheme also requires constrained separation: coalition constraints fixed at earlier stages must be excluded or enforced in later stages. We therefore assume that the fixed coalition constraints have a polynomial-size explicit representation (Assumption 7) and that a polynomial-time constrained K-best closure oracle is available (Assumption 8). Without these additional assumptions, polynomial-time nucleolus computation for general Frankl–Wilson dividend games remains open.

7.1. Lexicographic Scheme and Fixed Constraints

In a Maschler-style lexicographic scheme, one solves a sequence of linear programs. At each stage, some coalition constraints are fixed at their previously determined excess levels, while the remaining coalition constraints are bounded by the current excess variable.

Let F_t denote the set of proper nonempty coalitions whose excess constraints have been fixed before stage t. Let

$$q_t = |F_t|.$$

For a polynomial-time implementation, it is not sufficient merely to know that some linearly independent constraints define the current face. The separation procedure must be able to exclude or enforce the coalition constraints that are fixed by the lexicographic scheme. We therefore make the following explicit assumption.

Assumption 7 (Polynomial fixed-constraint representation)**.** *At every lexicographic stage t, the set F_t of fixed proper nonempty coalition constraints is explicitly represented and has size*

$$q_t \le \text{poly}(n, |\mathcal{D}|).$$

Moreover, the feasible region at stage t is correctly described by the efficiency constraint, the individual-rationality constraints, the fixed equalities for coalitions in F_t, and the current excess bound for the remaining proper nonempty coalitions.

This assumption should be viewed as part of the oracle model for nucleolus computation. It is not claimed to hold automatically for every Frankl–Wilson dividend game.

7.2. Constrained K-Best Closure Oracle

For a given allocation x, let G_x be the closure graph constructed in Section 6. For non-negative dividends, the excess of a coalition S is

$$e(S, x) = v(S) - x(S).$$

We require a K-best closure oracle that can enumerate high-excess coalitions while respecting forcing constraints.

Definition 1 (Constrained K-best closure oracle)**.** *Given the closure graph G_x, an integer K, a forced-in set $I \subseteq N$, and a forced-out set $O \subseteq N$, the oracle returns up to K distinct player sets $S \subseteq N$ with largest excess values*

$$e(S, x) = v(S) - x(S)$$

among all coalitions satisfying

$$I \subseteq S \text{ and } S \cap O = \emptyset.$$

The oracle returns the player sets in nonincreasing order of excess. If fewer than K such coalitions exist, it returns all of them. Ties may be broken arbitrarily, but the returned player sets must be distinct.

We make the following oracle assumption.

Assumption 8 (Polynomial-time constrained K-best closure oracle)**.** *For every allocation x arising in the lexicographic scheme and for every polynomially bounded K, the constrained K-best closure oracle runs in time polynomial in K, n, $|\mathcal{D}|$, and the input bit length.*

7.3. Constrained Separation Using K-Best Closures

Lemma 1 (Constrained K-best closure separation)**.** *Let F be a forbidden family of q proper nonempty coalitions. Suppose the constrained K-best closure oracle is available.*

Then, one can find a maximum-excess proper nonempty coalition not in F, or certify that none exists, using at most

$$n(n-1)(q+1)$$

oracle-generated candidate coalitions.

Proof. Let $S^\star$ be a maximum-excess proper nonempty coalition with $S^\star \notin F$. Since $S^\star$ is proper and nonempty, there exist players $i \in S^\star$ and $j \notin S^\star$.

Consider the forced closure problem with player i forced in and player j forced out. The coalition $S^\star$ is feasible for this forced problem. Among all feasible coalitions in this forced problem, at most q coalitions belong to the forbidden family F. Therefore, among the top $q+1$ distinct feasible coalitions returned by the oracle, at least one coalition is not in F. Because the oracle returns coalitions in nonincreasing order of excess, the best non-forbidden coalition returned for this pair has excess at least $e(S^\star, x)$. Since $S^\star$ is optimal among non-forbidden proper nonempty coalitions, this returned coalition is also optimal.

Enumerating all ordered pairs (i, j) with $i \neq j$ covers every proper nonempty coalition. There are $n(n-1)$ such pairs. Taking the best non-forbidden coalition over all pairs gives the desired constrained separator. □

7.4. Polynomial Nucleolus Computation Under the Oracle Assumptions

Theorem 8 (Conditional nucleolus tractability under oracle assumptions)**.** *This is a conditional result. It does not assert that the required oracle exists for general Frankl–Wilson dividend games, nor does it provide a construction for nontrivial instances.*

Let v be a Frankl–Wilson dividend game with non-negative dividends, explicitly given support $\mathcal{D}$, and the nonempty imputation set.

Suppose Assumptions 7 and 8 hold. Then, the nucleolus can be computed in time polynomial in n, $|\mathcal{D}|$, and the input bit length.

If, in addition, $\mathcal{D}$ satisfies Type I or Type II Frankl–Wilson admissibility with fixed intersection complexity s, then the nucleolus can be computed in time polynomial in n and the input bit length.

Proof. At each lexicographic stage, let F_t be the set of fixed coalition constraints. Using Assumption 7,

$$q_t = |F_t| \leq \text{poly}(n, |\mathcal{D}|).$$

To separate the remaining proper nonempty coalition constraints, apply Lemma 1 with $F = F_t$ and $K = q_t + 1$. For each ordered pair (i, j), the constrained K-best closure oracle returns enough candidate coalitions to identify the maximum-excess proper nonempty coalition not in F_t, or to certify that no such coalition improves the current excess bound.

With Assumption 8, each constrained K-best closure call is polynomial time. Since Lemma 1 uses only polynomially generated candidates, the constrained separation oracle at stage t is polynomial time.

Each stage therefore solves a linear program with a polynomial-time separation oracle. Using the ellipsoid method, each stage is solvable in polynomial time. Under the standard Maschler-style implementation, the number of stages is at most $n - 1$, because each stage increases the dimension of the fixed equality constraints by at least one. Hence, the total running time is polynomial.

If the support is Frankl–Wilson-admissible with fixed s, then $|\mathcal{D}| = \text{poly}(n)$, so the runtime is polynomial in n. □

Remark 3. *The Shapley and least-core results do not require the K-best closure oracle or Assumption 7. They rely only on explicit support, compactness, and maximum-weight closure separation.*

The nucleolus result should be read as a conditional theorem: if a polynomial-size fixed-constraint representation and a polynomial-time constrained K-best closure oracle are available, then the nucleolus is polynomial-time computable. For general Frankl–Wilson dividend games, constructing such an oracle or proving that it always exists remains an open problem.

7.5. A Concrete Constrained K-Best Closure Oracle for the Fano Plane

The Fano-plane instance has $n = 7$ players and non-singleton dividend support $D_v \subseteq \mathcal{L}$ with $|\mathcal{L}| = 7$. There are only

$$2^7 - 2 = 126$$

proper nonempty coalitions. We may therefore construct a constrained K-best closure oracle by brute-force enumeration:

Oracle O_{Fano}**.** Given K, a forced-in set $I \subseteq N$, and a forced-out set $O \subseteq N$, enumerate all proper nonempty coalitions S with $I \subseteq S$ and $S \cap O = \emptyset$; compute $e_v(S, x) = v(S) - x(S)$ for each; sort in nonincreasing order; return the top K distinct coalitions.

Complexity. There are at most 126 feasible coalitions. Each excess evaluation requires $O(|D_v|)$ arithmetic operations using the explicit dividend representation. Sorting takes $O(126 \log 126)$ comparisons. Hence, O_{Fano} runs in time

$$O(2^n\, |D_v| + 2^n \log 2^n)$$

with $n = 7$, which is a constant. For any fixed-n instance with explicit support, the same construction yields a polynomial-time oracle (constant time in n).

Consequence for the Fano-plane protocol. For the Fano-plane instance, Assumptions 7 and 8 hold: $q_t \leq 126$ at every stage, and O_{Fano} is a valid constrained K-best closure oracle. The nucleolus is therefore computable by Theorem 8 for this instance. In the symmetric case (all weights $w_\ell = 1$), the nucleolus is the equal division $(1, \dots, 1)$, as computed in Section 10.3.6.

What this example does and does not show. It shows that the conditional theorem is not vacuous: for small instances, the oracle exists and is constructible. It does not provide a polynomial-time oracle in n for growing n; the brute-force construction is exponential in n. Constructing a polynomial-time constrained K-best closure oracle for general Frankl–Wilson dividend games, or proving that none exists, remains open.

8. Approximation by L-Intersecting Substructures

The exact theory applies when the dividend support is exactly Frankl–Wilson admissible. In practice, a game may only be approximately L-intersecting. We can still obtain approximation guarantees.

Let v be the original game and let w be an approximating game obtained by retaining some dividends and omitting others. Let O be the set of omitted dividend coalitions.

Define the omitted absolute dividend mass

$$R = \sum_{S \in O} |\lambda_v(S)| .$$

If all omitted dividends are non-negative, this is simply the omitted dividend mass.

Proposition 2. *Shapley approximation bound*

$$\|\varphi(v) - \varphi(w)\|_1 \leq R.$$

Also,

$$\|\varphi(v) - \varphi(w)\|_\infty \leq R.$$

Proof. The difference game is

$$v - w = \sum_{S \in O} \lambda_v(S) u_S.$$

By linearity of the Shapley value,

$$\varphi(v) - \varphi(w) = \sum_{S \in O} \lambda_v(S) \varphi(u_S).$$

For an unanimity game u_S,

$$\|\varphi(u_S)\|_1 = 1.$$

Therefore,

$$\|\varphi(v) - \varphi(w)\|_1 \leq \sum_{S \in O} |\lambda_v(S)| = R.$$

For the ℓ_∞ bound, for each player i,

$$|\varphi_i(v) - \varphi_i(w)| \leq \sum_{S \in O : i \in S} |\lambda_v(S)| \frac{1}{|S|} \leq R.$$

□

Thus, if a high-mass L-intersecting substructure can be extracted, the exact Shapley value of the subgame gives a certified approximation to the original Shapley value.

Approximate Stability

Definition 2 (Approximate core)**.** *Let v be a transferable-utility game, and let $\varepsilon \geq 0$. An allocation $x \in \mathbb{R}^N$ is an ε-approximate core allocation if*

$$x(N) = v(N), e_v(S, x) = v(S) - x(S) \leq \varepsilon \ \forall \ \emptyset \neq S \subsetneq N,$$

and

$$x_i \geq v(\{i\}) - \varepsilon \ \forall i \in N.$$

The first condition is efficiency; the second is ε-stability against coalitional deviation; the third is ε-individual rationality. When $\varepsilon = 0$, this reduces to the standard core. The notion coincides with the standard ε-core for $\varepsilon \geq 0$ when efficiency is required exactly, and individual rationality is required up to ε.

Definition 3 (Approximate imputation)**.** *An allocation x is an ε-approximate imputation if $x(N) = v(N)$ and $x_i \geq v(\{i\}) - \varepsilon$ for all $i \in N$.*

Define the total absolute dividend error

$$E = E(v, w) = \sum_{S \subseteq N} |\lambda_v(S) - \lambda_w(S)| .$$

This includes omitted dividends and misestimated dividends.
For every coalition S,

$$|v(S) - w(S)| \leq E.$$

In particular,

$$|v(N) - w(N)| \leq E.$$

Theorem 9 (Preimputation least-core stability under dividend error)**.** *Let $\varepsilon^*_{\text{pre}}(v)$ denote the least-core excess over efficient allocations, without individual-rationality constraints. Then,*

$$| \varepsilon^*_{\text{pre}}(v) - \varepsilon^*_{\text{pre}}(w) | \leq 2E.$$

Proof. Let x be efficient for v. Define

$$\delta = w(N) - v(N),$$

and set

$$x'_i = x_i + \frac{\delta}{n} \ \forall i \in N.$$

Then, x' is efficient for w. For any coalition S,

$$x'(S) = x(S) + \delta \frac{|S|}{n}.$$

Thus,

$$e_w(S, x') - e_v(S, x) = (w(S) - v(S)) - \delta \frac{|S|}{n}.$$

Since $|w(S) - v(S)| \leq E$ and $|\delta| \leq E$,

$$e_w(S, x') \leq e_v(S, x) + 2E.$$

Taking maxima and minimizing over efficient allocations yields

$$\varepsilon^*_{\text{pre}}(w) \leq \varepsilon^*_{\text{pre}}(v) + 2E.$$

The reverse inequality follows symmetrically. □

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Theorem 10 (Approximate core allocation)**.** *Suppose whas a core allocation y. Define*

$$\delta = v(N) - w(N),$$

and set

$$x_i = y_i + \frac{\delta}{n} \ \forall i \in N.$$

Then x is efficient for v, and for every proper nonempty coalition S,

$$e_v(S, x) \leq 2E.$$

Moreover,

$$x_i \geq v(\{i\}) - 2E \ \forall i \in N.$$

Thus, in the sense of Definitions 2 and 3, x is a 2E-approximate core allocation and a 2E-approximate imputation of v.

Proof. Efficiency follows because

$$x(N) = y(N) + \delta = w(N) + v(N) - w(N) = v(N).$$

For any proper nonempty coalition S,

$$\begin{aligned} e_v(S, x) &= v(S) - x(S) \\ &= v(S) - y(S) - \delta\frac{|\,S\,|}{n} \\ &= e_w(S, y) + (v(S) - w(S)) - \delta\frac{|\,S\,|}{n}. \end{aligned}$$

Since $y \in \text{Core}(w)$,

$$e_w(S, y) \leq 0.$$

Also,

$$|v(S) - w(S)| \leq E$$

and

$$|\delta| \leq E.$$

Therefore,

$$e_v(S, x) \leq 0 + E + E = 2E.$$

For individual rationality,

$$x_i = y_i + \frac{\delta}{n} \geq w(\{i\}) + \frac{\delta}{n}.$$

Thus,

$$v(\{i\}) - x_i \leq v(\{i\}) - w(\{i\}) + \frac{|\,\delta\,|}{n} \leq E + E = 2E.$$

Hence,

$$x_i \geq v(\{i\}) - 2E.$$

□

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9. Extraction of L-Intersecting Substructures

The exact algorithms assume that the L-intersecting dividend support is given. In applications, one may need to extract a high-mass L-intersecting subfamily from a larger dividend representation.

The extraction problem is as follows:

$$\max_{\mathcal{F} \subseteq \mathcal{C}} \sum_{S \in \mathcal{F}} \lambda_v(S)$$

subject to $\mathcal{F}$ being L-intersecting or Frankl–Wilson admissible, where $\mathcal{C}$ is a candidate family of coalitions.

This problem is computationally difficult in general.

9.1. Hardness Example: $L = \{0\}$

For $L = \{0\}$, the extraction problem requires selecting a subfamily of pairwise disjoint coalitions to maximize the retained dividend mass.

Let $P_1, \ldots, P_m$ be the candidate coalitions with non-negative weights

$$w_j = |\lambda_v(P_j)|.$$

The problem is formulated as the following weighted set-packing integer program:

$$\max \sum m w_j z_j$$

subject to

$$z_i + z_j \leq 1 \text{ for all } i \neq j \text{ such that } P_i \cap P_j \neq \emptyset,$$

and $z_j \in \{0, 1\}$ for all $j \in \{1, \ldots, m\}$.

Because weighted set packing is NP-hard, optimal L-intersecting extraction should not be expected to be polynomial-time solvable in general without additional restrictions.

9.2. Fixed-Parameter and Heuristic Extraction

For small intersection complexity s, small coalition size, or special candidate families, extraction may be tractable. Possible approaches include the following:

1. exhaustive enumeration for small candidate families;
2. integer programming;
3. greedy selection by dividend density;
4. local search;
5. LP relaxations and rounding;
6. thresholding by dividend magnitude;
7. sampling-based dividend estimation.

The approximation guarantee provides a certificate: if the extracted subfamily retains high dividend mass, the exact Shapley value of the extracted subgame gives a certified approximation to the original Shapley value.

10. Illustrative Examples

10.1. Fano-Plane Dividend Games

Let

$$N = \{1, 2, 3, 4, 5, 6, 7\}.$$

Let $\mathcal{D}_v$ be the seven lines of the Fano plane:

$$\{1,2,3\},$$
$$\{1,4,5\},$$
$$\{1,6,7\},$$
$$\{2,4,6\},$$
$$\{2,5,7\},$$
$$\{3,4,7\},$$
$$\{3,5,6\}.$$

Every line has size 3, and any two distinct lines intersect in exactly one point. Therefore, $\mathcal{D}_v$ is a 3-uniform $\{1\}$-intersecting family. Here,

$$r = 3, L = \{1\}, s = 1.$$

The Ray-Chaudhuri–Wilson bound yields

$$|\mathcal{D}_v| \leq \binom{7}{1} = 7,$$

which is tight in this example.

Suppose all singleton dividends are zero and every line has dividend 1:

$$\sigma_i = 0 \ \forall i,$$

$$\lambda_v(T) = 1 \ \forall T \in \mathcal{D}_v.$$

Then,

$$v(S) = \#\{T \in \mathcal{D}_v : T \subseteq S\}.$$

The total value is

$$v(N) = 7.$$

10.1.1. Shapley Value

Each player belongs to exactly three lines. Each line dividend is split equally among its three members. Therefore, each player receives

$$3 \cdot \frac{1}{3} = 1.$$

Thus,

$$\varphi_i(v) = 1 \ \forall i \in N.$$

The Shapley vector is

$$\varphi(v) = (1, 1, 1, 1, 1, 1, 1).$$

The total is

$$7 = v(N).$$

10.1.2. Core

All dividends are non-negative, so the game is convex. The core is nonempty, and the Shapley value belongs to the core.

The equal allocation $x_i = 1$ satisfies efficiency. For any coalition S, the value $v(S)$ is the number of full lines contained in S. Since each line has three players and payoff is one per player, one expects

$$v(S) \leq |S|.$$

Indeed, for the Fano-plane dividend game, the equal allocation is in the core.

This example illustrates how finite-geometric L-intersecting families naturally define Frankl–Wilson dividend games with symmetric Shapley values and nonempty cores.

10.2. A Modular Type II Example

We now present a small modular Frankl–Wilson dividend game satisfying Type II admissibility.

Let

$$N = \{1, 2, 3, 4, 5\}.$$

Choose the prime

$$p = 3$$

and the allowed intersection residue set

$$A = \{1\} \subseteq \mathbb{Z}_3.$$

Define the non-singleton dividend support

$$D_v = \{\{1, 2\}, \{1, 3, 4\}, \{2, 3, 5\}\}.$$

Let

$$T_1 = \{1, 2\}, T_2 = \{1, 3, 4\}, T_3 = \{2, 3, 5\}.$$

The coalition sizes are

$$|T_1| = 2, |T_2| = 3, |T_3| = 3.$$

For Modulo 3, these sizes are

$$2, 0, 0,$$

and none of which belongs to $A = \{1\}$. Hence, the admissibility condition

$$|T| \quad p \notin A$$

is satisfied for every $T \in D_v$.

The pairwise intersections are

$$T_1 \cap T_2 = \{1\},$$

$$T_1 \cap T_3 = \{2\},$$

$$T_2 \cap T_3 = \{3\}.$$

Each intersection has size 1; therefore,

$$|T_i \cap T_j| \quad 3 = 1 \in A$$

for every pair of distinct coalitions T_i, $T_j \in D_v$. Thus, D_v is a modular Frankl–Wilson dividend support with

$$p = 3, A = \{1\}, s = |A| = 1.$$

The Frankl–Wilson bound yields

$$| D_v | \leq \sum_{i=0}^{1} \binom{5}{i} = 1 + 5 = 6.$$

The actual support size is 3, so the bound is satisfied.

This example is also non-uniform, since T_1 has size 2 while T_2 and T_3 have size 3. Therefore, it is not a Type I uniform ordinary L-intersecting support. It is nevertheless admissible under Type II modular Frankl–Wilson conditions.

Suppose all singleton dividends are zero and all non-singleton dividends are equal to one:

$$\sigma_i = 0 \ \forall i \in N,$$

$$\lambda_v(T) = 1 \ \forall T \in D_v.$$

Then, the total value is

$$v(N) = \sum_{T \in D_v} \lambda_v(T) = 3.$$

The Shapley value can be computed by dividend splitting. Each dividend is divided equally among the members of its coalition.

For player 1,

$$\varphi_1(v) = \frac{1}{|\,T_1\,|} + \frac{1}{|\,T_2\,|} = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}.$$

For player 2,

$$\varphi_2(v) = \frac{1}{|\,T_1\,|} + \frac{1}{|\,T_3\,|} = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}.$$

For player 3,

$$\varphi_3(v) = \frac{1}{|\,T_2\,|} + \frac{1}{|\,T_3\,|} = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}.$$

For player 4,

$$\varphi_4(v) = \frac{1}{|\,T_2\,|} = \frac{1}{3}.$$

For player 5,

$$\varphi_5(v) = \frac{1}{|\,T_3\,|} = \frac{1}{3}.$$

Thus,

$$\varphi(v) = \left(\frac{5}{6}, \frac{5}{6}, \frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right).$$

The components sum to

$$\frac{5}{6} + \frac{5}{6} + \frac{2}{3} + \frac{1}{3} + \frac{1}{3} = 3 = v(N).$$

Since all dividends are non-negative, the game is convex, the core is nonempty, and the Shapley value belongs to the core.

This example illustrates how modular Frankl–Wilson admissibility can apply to non-uniform dividend supports that would not satisfy the Type I uniformity requirement.

10.3. Cryptographic Instantiation: Block-Design Secret-Sharing Protocols with Frankl–Wilson Dividends

Important modeling caveat. The Frankl–Wilson dividend structure obtained below applies to the additive multi-secret value function

$$v(S) = \sum_{B \subseteq S} w_B.$$

It does not generally apply to the binary single-secret access game

$$v_{\text{binary}}(S) = 1\{\exists B{:}B \subseteq S\},$$

whose Harsanyi dividends are generally signed and may be supported by many additional coalitions.

The Fano-plane dividend game in Section 10.1 and the modular example in Section 10.2 are not merely abstract combinatorial constructions. They arise naturally from a well-known class of cryptographic multi-agent protocols based on block-design compartmented secret sharing. This subsection provides a concrete protocol instantiation, proves that its Harsanyi dividends are supported exactly on the blocks of a symmetric design, and verifies the Frankl–Wilson admissibility conditions. The construction connects the present paper to coding theory, threshold cryptography, and the combinatorial structures underlying quantum error-correcting codes.

10.3.1. Motivation from Coding Theory and Quantum Error Correction

The Fano plane is the projective plane of order two. Its incidence structure appears in several fundamental objects in coding theory and quantum information:

1. The $[7, 4, 3]$ binary Hamming code (MacWilliams & Sloane, 1977) has seven weight-three codewords whose supports are precisely the lines of the Fano plane.
2. The dual $[7, 3, 4]$ simplex code has nonzero codewords whose supports are the complements of Fano lines.
3. The $[7, 1, 3]$ Steane quantum error-correcting code (Steane, 1996) is constructed from the Hamming code and inherits the Fano-plane combinatorics in its stabilizer structure.

In distributed cryptographic protocols, these incidence structures naturally determine which coalitions of agents can reconstruct secret information. A coalition succeeds if and only if it contains the full support of a codeword or block. This motivates the following protocol family.

Remark on standard threshold games. A single t-out-of-n threshold secret-sharing scheme (Shamir, 1979) induces the simple game $v(S) = 1_{\{|S| \geq t\}}$. The Harsanyi dividends of this game are signed and supported on all coalitions of size at least t:

$$\lambda_v(T) = (-1)^{|T|-t}\binom{|\,T\,|-1}{t-1}, \; |\,T\,| \geq t.$$

This violates Assumption 5 (non-negative dividends) and does not yield a Frankl–Wilson-admissible support. The construction below avoids this difficulty by using a *multi-secret* additive protocol in which each compartment is independently shared.

10.3.2. The Fano-Plane Multi-Secret Sharing Protocol

Let $N = \{1, 2, 3, 4, 5, 6, 7\}$ be the seven points of the Fano plane, and let $\mathcal{L}$ be the seven lines:

$$\mathcal{L} = \{123, \; , 145, \; , 167, \; , 246 \;\; 257 \;\; 347 \;\; 356\}.$$

Fix a finite field $\mathbb{F}_q$ (for binary secrets, $q = 2$). For each line $\ell \in \mathcal{L}$, there is an independent secret $s_\ell \in \mathbb{F}_q$ with associated weight $w_\ell \geq 0$ representing the secret's length or value.

Protocol Π_{Fano}**.** For each line $\ell = \{a, b, c\} \in \mathcal{L}$:

1. Choose $r_{\ell,a}, r_{\ell,b} \in \mathbb{F}_q$ uniformly and independently at random.
2. Set $r_{\ell,c} := s_\ell - r_{\ell,a} - r_{\ell,b}$.
3. Assign the share component $r_{\ell,i}$ to each player $i \in \ell$.

Each player i receives the tuple of share components $(r_{\ell,i})_{\ell \in \mathcal{L},\, i \in \ell}$.

Correctness. If a coalition S contains all three players of a line $\ell = \{a, b, c\}$, then

$$r_{\ell,a} + r_{\ell,b} + r_{\ell,c} = s_\ell,$$

so S can reconstruct s_ℓ.

Perfect secrecy. If S does not contain all players of ℓ, then at least one share component $r_{\ell,i}$ is missing. Since the missing component is uniformly random and independent of s_ℓ, the view of S is statistically independent of s_ℓ. Hence, no unauthorized coalition gains any information about s_ℓ.

Coalition success. A coalition S recovers the secret s_ℓ if and only if $\ell \subseteq S$.

10.3.3. The Induced Cooperative Game

Define the cooperative game v_Π by

$$v_\Pi(S) := \sum_{\ell \in \mathcal{L}:\, \ell \subseteq S} w_\ell, S \subseteq N.$$

The value $v_{\Pi}(S)$ is the total weight of line secrets that coalition S can reconstruct. This is a natural multi-agent utility: the coalition's payoff equals the total amount of secret material it can recover.

Since each line secret is shared by a three-out-of-three additive threshold scheme (a degenerate threshold scheme requiring all participants), the protocol is a compartmented threshold secret-sharing scheme. The overall game is a non-negative linear combination of unanimity games:

$$v_{\Pi} = \sum_{\ell \in \mathcal{L}} w_{\ell}\, u_{\ell},$$

where u_{ℓ} is the unanimity game on ℓ.

10.3.4. Harsanyi Dividends of the Protocol

Theorem 11 (Dividends of the Fano-plane protocol)**.** *For the game v_{Π} induced by Protocol Π_{Fano}:*

1. *The singleton dividends satisfy $\sigma_i = \lambda_{v_{\Pi}}(\{i\}) = 0$ for all $i \in N$.*
2. *For each line $\ell \in \mathcal{L}$,*
$$\lambda_{v_{\Pi}}(\ell) = w_{\ell}.$$
3. *For every other coalition $T \subseteq N$ with $|T| \geq 2$ and $T \notin \mathcal{L}$,*
$$\lambda_{v_{\Pi}}(T) = 0.$$

Hence, the non-singleton dividend support is

$$D_{v_{\Pi}} = \{\ell \in \mathcal{L} : w_{\ell} > 0\}.$$

Proof of Theorem 11. By Möbius inversion,

$$\lambda_{v_{\Pi}}(T) = \sum_{U \subseteq T} (-1)^{|T|-|U|} v_{\Pi}(U).$$

Substitute

$$v_{\Pi}(U) = \sum_{\ell \in \mathcal{L}} w_{\ell}\, 1_{\{\ell \subseteq U\}}.$$

Then,

$$\lambda_{v_{\Pi}}(T) = \sum_{\ell \in \mathcal{L}} w_{\ell} \sum_{U \subseteq T} (-1)^{|T|-|U|} 1_{\{\ell \subseteq U\}}.$$

Fix a line $\ell \in \mathcal{L}$.

- If $\ell \not\subseteq T$, then no subset $U \subseteq T$ contains ℓ, so the inner sum is 0.
- If $\ell \subseteq T$, write $U = \ell \cup R$ with $R \subseteq T \setminus \ell$. Then the inner sum becomes

$$\sum_{R \subseteq T \setminus \ell} (-1)^{|T|-|\ell|-|R|} = (1-1)^{|T|-|\ell|}.$$

This equals 1 if $T = \ell$, and equals 0 if $|T| > |\ell|$.

Therefore,

$$\lambda_{v_{\Pi}}(T) = \begin{cases} w_T, & T \in \mathcal{L}, \\ 0, & T \notin \mathcal{L}, |T| \geq 2. \end{cases}$$

Since no line is a singleton, $\sigma_i = \lambda_{v_{\Pi}}(\{i\}) = 0$ for every $i \in N$. Hence,

$$D_{v_{\Pi}} = \{\ell \in \mathcal{L} : w_{\ell} > 0\}.$$

□

Corollary 4. *The game v_{Π} satisfies Assumptions 1–6 of the present paper:*

- *1 (rational data): choose rational weights* w_ℓ.
- *2 (reduced player set): every player belongs to at least one line.*
- *3 (Frankl–Wilson admissible support): verified below.*
- *4 (explicit support): the seven Fano lines are explicitly known.*
- *5 (non-negative dividends):* $w_\ell \geq 0$ *by construction.*
- *6 (nonempty imputation set):* $v_\Pi(N) = \sum_\ell w_\ell \geq 0 = \sum_i v_\Pi(\{i\})$.

10.3.5. Frankl–Wilson Admissibility of the Dividend Support

Theorem 12 (Type I admissibility)**.** *The dividend support* D_{v_Π} *is a three-uniform ordinary* $\{1\}$*-intersecting family. Hence, it is Type I admissible with*

$$r = 3, L = \{1\}, s = |L| = 1.$$

By the Ray-Chaudhuri–Wilson theorem (Theorem 2),

$$| D_{v_\Pi} | \leq \binom{7}{1} = 7.$$

If all weights are positive, $| D_{v_\Pi} | = 7$ *and the bound is tight.*

Proof. Every Fano line has size 3. Any two distinct Fano lines intersect in exactly one point. Therefore $D_{v_\Pi} \subseteq \mathcal{L}$ is 3-uniform and $\{1\}$-intersecting. The Ray-Chaudhuri–Wilson bound applies directly. □

Theorem 13 (Type II modular admissibility)**.** *The dividend support* D_{v_Π} *is a modular Frankl–Wilson admissible modulo* $p = 3$ *with residue set* $A = \{1\}$*. Hence, it is Type II admissible with*

$$p = 3, A = \{1\} \subseteq \mathbb{Z}_3, s = |A| = 1.$$

Using the Frankl–Wilson theorem (Theorem 1),

$$| D_{v_\Pi} | \leq \sum_{i=0}^{1} \binom{7}{i} = 1 + 7 = 8.$$

Proof. For every line $\ell \in \mathcal{L}$:

$$|\ell| = 3 \equiv 0 \pmod 3.$$

Since $0 \notin A = \{1\}$, the admissibility condition $|\ell| \quad p \notin A$ is satisfied.
For every pair of distinct lines $\ell, \ell' \in \mathcal{L}$:

$$|\ell \cap \ell'| = 1 \equiv 1 \pmod 3.$$

Since $1 \in A$, the intersection condition $|\ell \cap \ell'| \quad p \in A$ is satisfied.
Therefore, D_{v_Π} satisfies both conditions of Type II modular Frankl–Wilson admissibility.
□

Remark 4. *The Type I bound* $| D_{v_\Pi} | \leq 7$ *is tighter than the Type II bound* $| D_{v_\Pi} | \leq 8$ *in this example. Both certify polynomial compactness. The modular verification is included to demonstrate that the Fano-plane protocol satisfies the specific Frankl–Wilson modular conditions used throughout the present paper.*

10.3.6. Algorithmic Consequences

Since the Fano-plane protocol satisfies all assumptions of the present paper, the main algorithmic results apply directly.

Shapley value. By Theorem 5,

$$\varphi_i(v_\Pi) = \sum_{\ell \in \mathcal{L}:\, i \in \ell} \frac{w_\ell}{|\,\ell\,|} = \sum_{\ell \ni i} \frac{w_\ell}{3}.$$

If all weights are $w_\ell = 1$, each player belongs to exactly three lines, so

$$\varphi_i(v_\Pi) = 3 \cdot \frac{1}{3} = 1 \forall\, i \in N.$$

Core and least core. Since all dividends are non-negative, the game is convex (Proposition 1). The core is nonempty, and the Shapley value belongs to the core. The maximum-weight closure separation oracle (Algorithm 2) applies with a closure graph of $7 + 7 = 14$ nodes and 21 arcs.

Nucleolus (symmetric case). If all weights are equal, the automorphism group of the Fano plane acts transitively on the seven points. The nucleolus is unique and covariant under game automorphisms. Therefore, it assigns equal payoffs to all players. By efficiency,

$$\nu_i(v_\Pi) = \frac{v_\Pi(N)}{7} = 1 \forall\, i \in N.$$

Thus, in the symmetric Fano-plane protocol, the nucleolus coincides with the Shapley value and lies in the core.

10.3.7. Generalization to Finite Projective Planes

The Fano-plane construction generalizes to any finite projective plane of order q.

Let Π_q be a projective plane of order q. Then the following are true:

- The number of points (players) is $n = q^2 + q + 1$.
- The number of lines is also n.
- Every line contains $q + 1$ points.
- Any two distinct lines intersect in exactly one point.

Protocol $\Pi_{\mathrm{Proj}(q)}$**.** For each line ℓ, assign an independent secret s_ℓ of weight $w_\ell \geq 0$. Share s_ℓ by a $(q+1)$-out-of-$(q+1)$ additive threshold scheme among the players on ℓ. Define

$$v(S) = \sum_{\ell:\, \ell \subseteq S} w_\ell.$$

Theorem 14 (Projective-plane Frankl–Wilson dividends)**.** *The game induced by Protocol* $\Pi_{\mathrm{Proj}(q)}$ *has non-singleton dividend support equal to the set of positively weighted lines. This support satisfies the following:*

1. ***Type I admissibility.*** *The line family is* $(q+1)$*-uniform and* $\{1\}$*-intersecting with* $s = 1$*. Using the Ray-Chaudhuri–Wilson theorem,*

$$|\, D_v \,| \leq \binom{n}{1} = n.$$

This is tight since a projective plane has exactly n lines.

2. ***Type II modular admissibility.*** *Choose a prime p with* $p \nmid q$*. For example,*
   - *if q is odd, take* $p = 2$*;*
   - *if q is even, take* $p = 3$*.*

   Set $A = \{1\} \subseteq \mathbb{Z}_p$*. Then,*
   - *for every line* ℓ*:* $|\ell| = q + 1 \not\equiv 1 \pmod p$ *(since* $p \nmid q$*), so* $|\ell| \quad p \notin A$*.*
   - *for every pair of distinct lines:* $|\ell \cap \ell'| = 1 \equiv 1 \pmod p$*, so* $|\ell \cap \ell'| \quad p \in A$*.*

Hence, the support is Type II admissible with $s = 1$, *and*

$$| D_v | \leq \sum_{i=0}^{1} \binom{n}{i} = n + 1.$$

Proof. The dividend calculation is identical to Theorem 11. The intersection properties follow from the axioms of a projective plane. The modular admissibility follows because $p \nmid q$ implies $q + 1 \not\equiv 1 \pmod{p}$. □

Corollary 5. *For every prime-power order* q, *Protocol* $\Pi_{\mathrm{Proj}(q)}$ *induces a Frankl–Wilson dividend game with* $n = q^2 + q + 1$ *players, polynomial dividend support of size* n, *and intersection complexity* $s = 1$. *All algorithmic results of the present paper (Shapley computation, closure-based separation, conditional nucleolus) apply.*

10.3.8. Further Generalization: Symmetric Block Designs

The construction extends to any symmetric balanced incomplete block design (BIBD) (Beth et al., 1999).

Let $(N, \mathcal{B})$ be a symmetric 2-(v, k, λ) design. Thus,

- $|N| = v$ and $|\mathcal{B}| = v$;
- each block has size k;
- each point lies in k blocks;
- every pair of points lies in exactly λ blocks;
- any two distinct blocks intersect in exactly λ points.

Protocol Π_{BIBD}**.** For each block $B \in \mathcal{B}$, assign an independent secret of weight $w_B \geq 0$, shared by a k-out-of-k additive threshold scheme among the players of B. Define

$$v(S) = \sum_{B \in \mathcal{B}:\ B \subseteq S} w_B.$$

Theorem 15 (Symmetric-design Frankl–Wilson dividends)**.** *The game induced by Protocol* Π_{BIBD} *has non-singleton dividend support* $D_v \subseteq \mathcal{B}$ *satisfying the following:*

1. ***Type I admissibility.*** *The support is* k*-uniform with pairwise intersections equal to* λ. *Hence, it is Type I admissible with* $L = \{\lambda\}$, $s = 1$, *and*

$$| D_v | \leq \binom{v}{1} = v.$$

2. ***Type II modular admissibility.*** *Choose any prime* p *with* $k \not\equiv \lambda \pmod{p}$. *For example, any prime* $p > k$ *suffices. Set* $A = \{\lambda \quad p\}$. *Then,*
   - $|B| \quad p = k \quad p \neq \lambda \quad p$, *so* $|B| \quad p \notin A$;
   - $|B \cap B'| \quad p = \lambda \quad p \in A$ *for all distinct blocks* B, B'.

   Hence, the support is Type II admissible with $s = 1$, *and*

$$|D_v| \leq v + 1.$$

Proof. The dividend calculation follows from the unanimity expansion as in Theorem 11. The intersection property of symmetric designs (any two blocks meet in λ points) is a standard consequence of the incidence matrix identity $MM^{\top} = (k - \lambda)I + \lambda J$ combined with $b = v$. □

Examples of Symmetric Designs Yielding Frankl–Wilson Dividend Games:

Design	Parameters (v, k, λ)	Modular prime p
Fano plane	$(7, 3, 1)$	$p = 3$
Projective plane of order q	$(q^2, +, q+1\ \ q+1\ \ 1)$	$p = 2$ (if q odd) or $p = 3$ (if q even)
Paley biplane of order 11	$(11, 5, 2)$	$p = 7$
Hadamard 2-design	$(4t, 2t-1\ \ t-1)$	any prime $p > 4t-1$

Table 2A,B consolidates the verified instances discussed in Sections 10.1–10.3.8. Each row records an exact symbolic verification of the relevant Frankl–Wilson admissibility conditions and the resulting support bound; the table is not a runtime benchmark.

Table 2. (**A**) Admissibility verification for Frankl–Wilson dividend instances. All entries are exact symbolic verifications of admissibility; they are not runtime benchmarks. (**B**) Support bounds and consequences for Frankl–Wilson dividend instances. All entries are exact symbolic verifications of support bounds and consequences; they are not runtime benchmarks.

(**A**) Admissibility verification for Frankl–Wilson dividend instances

Instance (Section)	Combinatorial parameters	Type I check	Type II check
Fano-plane dividend game (Section 10.1)	$N = [7]$; seven Fano lines $\mathcal{L}$; every line has size 3; any two lines meet in 1 point	Yes: 3-uniform, $L = \{1\}$, $s = 1$	Yes: $p = 3$, $A = \{1\}$; line size $3 \equiv 0 \pmod 3$, and $0 \notin A$; intersection size $1 \in A$
Modular Type II example (Section 10.2)	$N = [5]$; $D_v = \{\{1,2\}, \{1,3,4\}, \{2,3,5\}\}$; $p = 3$; $A = \{1\}$	No: support sizes are 2, 3, 3, so it is non-uniform	Yes: sizes 2, 0, 0 (mod 3), none in A; every pairwise intersection has size $1 \in A$
Fano-plane multi-secret protocol Π_{Fano} (Section 10.3)	$N = [7]$; independent line secrets s_ℓ with weights $w_\ell \geq 0$; each line shared by a 3-out-of-3 additive scheme	Yes: 3-uniform, $L = \{1\}$, $s = 1$	Yes: $p = 3$, $A = \{1\}$; line size $3 \equiv 0 \pmod 3$, and $0 \notin A$; intersection size $1 \in A$
Projective plane $\Pi_{\text{Proj}(q)}$ (Section 10.3.7)	$n = q^2 + q + 1$ players and n lines; every line has size $q + 1$; any two lines meet in 1 point	Yes: $(q+1)$-uniform, $L = \{1\}$, $s = 1$	Yes: choose any prime $p \nmid q$, and set $A = \{1\}$; then $q + 1 \not\equiv 1 \pmod p$, so line size is not in A; intersections satisfy $1 \in A$
Symmetric 2-(v, k, λ) BIBD Π_{BIBD} (Section 10.3.8)	v players and v blocks; every block has size k; any two distinct blocks meet in λ points	Yes: k-uniform, $L = \{\lambda\}$, $s = 1$	Yes: choose a prime p with $k \not\equiv \lambda \pmod p$ (e.g., any $p > k$); set $A = \{\lambda p\}$; then $kp \notin A$, while $\lambda p \in A$

(**B**) Support bounds and consequences for Frankl–Wilson dividend instances

Instance (section)	Dividend support	Compactness bound	Verification and consequence
Fano-plane dividend game (Section 10.1)	$D_v \subseteq \mathcal{L}$, with equality in the unit-dividend case	Type I: $\binom{7}{1} = 7$; Type II: $\sum_{i=0}^{1} \binom{7}{i} = 8$	Ray-Chaudhuri–Wilson and Frankl–Wilson bounds; unit-dividend Shapley value $(1, \ldots, 1)$; convex game with nonempty core; symmetric nucleolus is equal division
Modular Type II example (Section 10.2)	$D_v = \{\{1,2\}, \{1,3,4\}, \{2,3,5\}\}$	$\lvert D_v \rvert = 3 \leq \sum_{i=0}^{1} \binom{5}{i} = 6.$	Direct Frankl–Wilson verification; Shapley value $(5/6, 5/6, 2/3, 1/3, 1/3)$ non-negative dividends give convexity and a nonempty core

Table 2. *Cont.*

(**B**) Support bounds and consequences for Frankl–Wilson dividend instances			
Instance (section)	**Dividend support**	**Compactness bound**	**Verification and consequence**
Fano-plane multi-secret protocol Π_{Fano} (Section 10.3)	$D_{\nu_\Pi} = \{\ell \in \mathcal{L} : w_\ell > 0\}$	Type I: 7; Type II: 8	Theorem 11 gives dividends supported exactly on lines; Theorems 12 and 13 verify Type I/II; if all $w_\ell > 0$, the closure graph has 14 nodes and 21 arcs; if all $w_\ell = 1$, the nucleolus equals the equal division $(1, \ldots, 1)$
Projective plane $\Pi_{\text{Proj}(q)}$ (Section 10.3.7)	Support is the set of positively weighted lines	Type I: $(\lvert D_v\rvert \le n)$; Type II: $(\lvert D_v\rvert \le n+1)$	Theorem 14; exact verification from projective-plane axioms; all algorithmic results apply with $s = 1$. Examples: $p = 2$ if q is odd; if q is even, choose any prime $p \nmid q$
Symmetric 2-(v, k, λ) BIBD Π_{BIBD} (Section 10.3.8)	Support is the set of positively weighted blocks	Type I: $(\lvert D_v\rvert \le n)$; Type II: $(\lvert D_v\rvert \le n+1)$	Theorem 15; includes the Paley biplane $(11, 5, 2)$ with $p = 7$, and Hadamard 2-$(4t, t, -12t - 1t - 1)$ designs with $p > 4t - 1$

10.3.9. Connection to Coding-Theoretic and Quantum Protocols

The Fano-plane protocol admits a coding-theoretic interpretation. The incidence matrix of the Fano plane over $\mathbb{F}_2$ is the parity-check matrix of the $[7, 4, 3]$ Hamming code. The seven lines correspond to the seven weight-three codewords. In the Steane $[7, 1, 3]$ quantum code, the same incidence structure governs the stabilizer generators.

In a distributed storage or secure network coding interpretation,

- Each line ℓ represents a stored encoded packet or a key compartment.
- A coalition of storage nodes (players) can decode the packet s_ℓ if and only if it controls all nodes in the support ℓ.
- The total recoverable information equals $v_\Pi(S)$.

The Frankl–Wilson admissibility of the dividend support is therefore not an artificial combinatorial condition but a consequence of the geometric intersection properties inherent in the code or design.

10.3.10. Important Modeling Caveat

The Frankl–Wilson dividend structure is obtained for the **additive multi-secret value function**

$$v(S) = \sum_{B \subseteq S} w_B.$$

It is **not** obtained for the binary single-secret access game

$$v_{\text{binary}}(S) = 1_{\{\exists\, B \in \mathcal{B}:\ B \subseteq S\}}.$$

The binary OR game is a monotone simple game whose Harsanyi dividends involve inclusion-exclusion over intersections of blocks and are generally signed and supported on many additional coalitions. The clean Frankl–Wilson dividend property relies on the direct-sum multi-secret representation, where each compartment contributes independently to the coalition value.

This distinction is important for applications. In cryptographic protocols with multiple independent secrets, keys, or compartments, the additive multi-secret model is natural: the coalition's utility is the total amount of information recovered. In protocols with a single binary access decision, the dividend structure may not satisfy Frankl–Wilson admissibility.

11. Discussion

Frankl–Wilson dividend games provide a broad extension of extremal dividend structures.

11.1. Compared with Sunflower-Dividend Games

Sunflower-dividend games have a very specific structure: a common kernel and disjoint petals. This structure yields strong symmetry aggregation and a simple $O(2^k + m)$ separation oracle.

Frankl–Wilson dividend games allow much more general intersection patterns. The price is that they generally lack petal symmetry. Their tractability comes from support compactness and maximum-weight closure rather than from petal aggregation.

Feature	**Sunflower-dividend games**	**Frankl–Wilson dividend games**
Intersection pattern	common kernel, disjoint petals	pairwise intersections in L or modulo p in A
Special case relationship	active coalitions are $\{\kappa\}$-intersecting	includes more general ordinary and modular intersection patterns
Support compactness	follows from explicit petal structure	follows from Frankl–Wilson or Ray-Chaudhuri–Wilson bounds
Shapley computation	closed form with petal symmetry	dividend summation over compact explicit support
Core separation	specialized sunflower oracle	maximum-weight closure oracle
Nucleolus computation	FPT via petal aggregation	polynomial under constrained K-best closure oracle
Extraction	weighted set packing/ sunflower extraction	weighted L-intersecting or modular selection

The cryptographic instantiation in Section 10.3 addresses a natural question: which multi-agent environments produce Frankl–Wilson dividend structures? The answer is compartmented secret-sharing protocols based on symmetric block designs. In these protocols, each block carries an independent secret shared among its members by an all-or-nothing threshold scheme. The coalition value is the total weight of recovered secrets, and the Harsanyi dividends are supported exactly on the blocks. Since symmetric designs have constant pairwise block intersections, the dividend support automatically satisfies L-intersecting and modular Frankl–Wilson admissibility. This provides a concrete bridge between extremal set theory, coding theory, and cooperative game theory.

11.2. Compared with Laminar and VC-Dividend Games

Laminar-dividend games use nested/disjoint support families and admit tree dynamic programming. VC-dividend games use bounded VC dimension to obtain polynomial

support size and PAC learnability. Frankl–Wilson dividend games use modular or uniform intersection constraints. These classes overlap but are not identical.

For example,

- laminar families have VC dimension at most 2 but are not generally L-intersecting;
- sunflower-active coalitions are L-intersecting but not necessarily laminar;
- block designs are L-intersecting but not laminar.

Thus, Frankl–Wilson dividend games form a distinct and useful branch of the extremal dividend research program.

Frankl–Wilson dividend games also differ from other tractability paradigms. Games of bounded treewidth exploit decomposable graphical structure; anonymous games exploit player symmetry; sparse marginal-contribution networks restrict the syntactic representation of marginal contributions; VC-dividend games use statistical dimension. Frankl–Wilson dividend games instead restrict the pairwise intersection pattern of the dividend support, a condition that is combinatorial rather than graphical, syntactic, or statistical. These classes overlap but are not nested.

11.3. Potential Applications Beyond Secret Sharing

The exact Frankl–Wilson dividend structure arises when a coalition value decomposes additively over independent tasks, compartments, or blocks, and when those blocks satisfy controlled intersection constraints. In addition to compartmented secret sharing, this can occur in project-based team production, distributed storage, cost-sharing games with independent joint facilities, and high-level multi-agent task allocation.

In multi-agent formation control, for example, one may define local subformations or task groups whose successful completion generates a reward. If the agent sets corresponding to these subformations are designed as block-design or finite-geometric families, then the induced high-level reward-allocation game can have a Frankl–Wilson dividend structure. The cooperative game then serves as a fair allocation or credit-assignment layer on top of the control system, rather than as a model of the continuous control dynamics itself.

11.4. Robustness to Imperfect Dividend Information

The exact algorithms assume that the Harsanyi dividends are known accurately. In networked multi-agent systems, communication disruptions, packet loss, sensor faults, or false-data-injection attacks may corrupt the estimated coalition values or dividends. Let $\hat{v}$ be an estimated game and let

$$E = \sum_{S \subseteq N} | \lambda_v(S) - \lambda_{\hat{v}}(S) |.$$

The approximation results of Section 8 show that the Shapley error is bounded by E, and that a core allocation of $\hat{v}$ yields a $2E$-approximate core allocation of v. Thus, if a resilient estimator provides a bound on the dividend error, the framework can still provide certified approximate fair allocations. Without such a bound, the exact tractability guarantees do not apply.

11.5. Application Outlook: Networked Multi-Agent Systems and Reliable Dividend Information

The framework in this paper is structural: it identifies when the dividend support is compact, not which physical systems produce such support. Two application-facing questions deserve discussion.

Formation and networked multi-agent control. In formation maneuvering of air-ground swarms, coordinated subgroups of unmanned aerial and ground vehicles jointly enable capabilities that individual agents cannot provide, and safety-critical constraints

couple the subgroups (Yan et al., 2026). Such settings are natural candidates for coalitional modeling. For the Frankl–Wilson framework to apply, however, the coalition value would need to decompose additively over a family of overlapping subgroups whose pairwise intersections are themselves structured, for example, subgroups sharing a common core of agents, or subgroups whose intersection sizes are constrained modulo a prime. The block-design protocols of Section 10.3 are one clean instance of such a decomposition, where each compartment is shared by an all-or-nothing threshold scheme. Whether value functions arising in formation control admit a similar decomposition is an application-specific modeling question. We do not claim that the present framework covers formation control as stated; we flag it as a direction for application-driven work.

Reliability of dividend information. The algorithmic results assume that dividends are observable, credible, and unaltered (Assumption 4). In networked multi-agent systems, this assumption can fail: event-triggered communication introduces delays and packet loss (Wang et al., 2025), and FDI attacks can corrupt the value signals used to estimate dividends (Liu et al., 2025; Zhang et al., 2026). Section 12.8 discusses the consequences and the graceful-degradation guarantees that follow from the approximation bounds of Section 8.

Cost allocation and weighted voting. Beyond formation control, cost-sharing and weighted-voting games are standard cooperative-game domains. Whether their value functions admit L-intersecting or modular dividend support is problem-specific, we do not develop these instantiations here, but they are natural targets for the framework.

Multi-compartment voting. In multi-compartment or multi-issue voting, the payoff may be the weighted number of independently passed compartments. If the voter sets required to pass the compartments form an L-intersecting or Frankl–Wilson-admissible family, then the additive voting game is a Frankl–Wilson dividend game. This is distinct from a single binary threshold voting rule, whose dividends are generally signed and therefore do not satisfy the non-negative-dividend closure construction.

Network alliance game. In network alliance games, the coalition value may be additive over independently secured routes, links, or key compartments. If the squads required to secure those routes form a block-design or modular intersection family, the induced allocation game has Frankl–Wilson dividend structure.

12. Limitations

Several limitations should be noted.

12.1. Frankl–Wilson Admissibility and Fixed Intersection Complexity Are Required

The polynomial support bounds rely on Frankl–Wilson admissibility: either uniform ordinary L-intersection or modular proper A-intersection. Arbitrary L-intersecting families without these conditions may not admit the same bounds. Moreover, the polynomial bounds are valid only when the intersection complexity $s = |L|$ or $s = |A|$ is fixed. If s grows with n, the Frankl–Wilson and Ray-Chaudhuri–Wilson bounds may be exponential. Thus, the exact tractability results are fixed-parameter results with respect to s, not general polynomial-time results for arbitrary dividend support.

12.2. Algorithms Depend on Explicit Support

The exact algorithms assume that the dividend support is explicitly given or can be enumerated. If the support is given only implicitly, enumeration may be nontrivial.

12.3. Expanded Limitation: Signed Dividends

The closure-based separation oracle relies on the assumption that all non-singleton dividends are non-negative:

$$\lambda_v(T) \geq 0 \ \forall T \in D_v.$$

This assumption is essential for the maximum-weight closure construction. If some dividend $\lambda_v(T)$ is negative, then a closed set containing the players in T may still exclude the dividend node u_T, because including u_T would decrease the closure weight. However, the coalition value $v(S)$ requires that every dividend $T \subseteq S$ be included, regardless of sign. Therefore, with signed dividends, the maximum-weight closure objective no longer equals

$$\max_{S \subseteq N} \{v(S) - x(S)\}.$$

Thus, the polynomial-time separation oracle does not automatically extend to signed dividends.

There are several possible directions for future work. If a game with signed dividends is nevertheless convex, then the excess function

$$S \mapsto v(S) - x(S)$$

is supermodular, and maximizing it is equivalent to minimizing a submodular function. In that case, general submodular-function minimization may provide a separation oracle. However, this approach is more general and typically less combinatorially transparent than the maximum-weight closure method. It also does not follow from Frankl–Wilson compactness alone.

For arbitrary signed dividends, the game may fail to be convex, and coalition-excess maximization can become computationally difficult. Developing structured separation methods for signed-dividend games is therefore an important open problem.

12.4. Nucleolus Tractability Is Conditional

The nucleolus result assumes a polynomial-time constrained K-best maximum-closure oracle. Without such an oracle, nucleolus computation for general Frankl–Wilson dividend games remains open.

12.5. Expanded Limitation: Extraction Hardness

The exact algorithms assume that an L-intersecting or Frankl–Wilson-admissible dividend support is already provided. In applications, one may instead start with a dense dividend representation and wish to extract a high-mass admissible subfamily.

The extraction problem is as follows:

$$\max_{\mathcal{F} \subseteq \mathcal{C}} \sum_{T \in \mathcal{F}} \lambda_v(T)$$

subject to $\mathcal{F}$ being L-intersecting or Frankl–Wilson admissible, where $\mathcal{C}$ is a candidate family of coalitions.

This problem is computationally difficult in general. Already for $L = \{0\}$, the problem requires selecting a maximum-weight family of pairwise disjoint coalitions. This is a weighted set-packing problem, which is NP-hard for general candidate coalitions.

Some special cases may be easier. For example,

- if all candidate coalitions have size two and $L = \{0\}$, the problem reduces to maximum-weight matching;
- if the candidate family is laminar, dynamic programming may be possible;

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- if coalition sizes are bounded or the candidate family has bounded intersection width, fixed-parameter or branch-and-bound methods may be practical;
- integer programming, LP relaxation, greedy density selection, and local search can provide useful heuristics.

The approximation results in this paper provide a natural certification mechanism. If an extraction procedure returns a subgame w whose omitted absolute dividend mass is

$$R = \sum_{T \in O} |\lambda_v(T)|,$$

then Proposition 2 yields

$$\|\varphi(v) - \varphi(w)\|_1 \leq R, \|\varphi(v) - \varphi(w)\|_\infty \leq R.$$

Thus, even when optimal extraction is intractable, heuristic extraction can still yield certified approximate Shapley values, provided the omitted dividend mass is explicitly computed.

12.6. Stability Approximation Is Approximate

The approximation results give additive error bounds. They do not imply exact core nonemptiness of the original game from core nonemptiness of the approximating subgame, except when the dividend error is zero.

12.7. New Limitation: Implicit Supports and Verification

The Frankl–Wilson and Ray-Chaudhuri–Wilson theorems provide support-size bounds. They do not, by themselves, construct or enumerate the dividend support.

The exact algorithms in this paper rely on Assumption 4: the dividend support D_v and the dividends $\lambda_v(T)$ are explicitly given or can be enumerated efficiently.

If the support is explicitly given and the parameters are known, admissibility can be verified directly.

For Type I, one checks that

1. every coalition in D_v has the same size r;
2. for every pair of distinct coalitions $T, T' \in D_v$,

$$|T \cap T'| \in L.$$

For Type II, one checks that

1. there is a prime p and a residue set $A \subseteq \mathbb{Z}_p$;
2. for every $T \in D_v$,

$$|T| \quad p \notin A;$$

3. for every pair of distinct coalitions $T, T' \in D_v$,

$$|T \cap T'| \quad p \in A.$$

Given D_v explicitly, these checks require examining all pairs of coalitions. A naive implementation takes $O(|D_v|^2)$ intersection checks. With ordinary set representations, each check may take $O(n)$ time, giving $O(n|D_v|^2)$ total time. With bit-vector representations, the pairwise checks can be substantially faster for moderate support sizes.

However, if the parameters r, L, p, A are not known, finding an admissible representation with small intersection complexity s is a separate combinatorial inference problem. The paper does not claim that this inference problem is polynomial-time solvable.

Similarly, if the game is given only by a value oracle or by an implicit rule, computing the Harsanyi dividends may require Möbius inversion over exponentially many coalitions. In

such cases, the Frankl–Wilson support-size bound does not automatically yield an efficient algorithm. The exact tractability results therefore depend crucially on explicit or efficiently enumerable dividend supports.

12.8. Robustness to Communication Disruptions and Adversarial Attacks

Assumption 4 treats the dividend support and the dividends as explicitly given and reliable. In networked multi-agent settings this assumption may be violated. Event-triggered communication can delay or drop the information needed to evaluate coalition values (Wang et al., 2025), and stochastic or unknown FDI attacks can corrupt the value signals from which dividends are computed (Liu et al., 2025; Zhang et al., 2026).

The structural results of Sections 4 and 5 concern the true dividend structure and are unaffected by corruption of the input data. The algorithmic results are affected: if the dividends used by Algorithm 1 or Algorithm 2 differ from the true dividends, the computed Shapley value and least-core allocation are those of the corrupted game, not of the true game. The approximation results of Section 8 bound this discrepancy.

If the total dividend error between the true dividends λ_v and the corrupted dividends λ_w satisfies

$$E(v, w) = \sum_{S \subseteq N} |\lambda_v(S) - \lambda_w(S)| \leq E,$$

then Proposition 2 yields $\|\phi(v) - \phi(w)\|_1 \leq E$, and Theorems 9 and 10 yield a $2E$-approximate core allocation of the true game (Definition 2).

Thus, when the corruption has bounded total dividend mass, the framework degrades gracefully: applying the algorithms of this paper to the corrupted game yields certified approximate solutions of the true game.

When corruption is unbounded, adversarial, or structurally targeted, these bounds become vacuous. Robust or adversarially resilient dividend estimation, for example, estimation under event-triggered communication, filtering under FDI attacks, or learning of dividend supports from noisy coalition-value observations, is not addressed in this paper and is listed as future work in Section 13.

13. Future Work

The Frankl–Wilson dividend framework suggests several research directions.

a. **Signed dividend extensions.** Develop separation techniques for games with positive and negative dividends.
b. **Unconditional nucleolus algorithms.** Prove polynomial-time nucleolus computation without relying on K-best closure oracles, perhaps by exploiting additional design structure.
c. **Block-design dividend games.** Study games whose dividend supports are balanced incomplete block designs, finite geometries, or coding-theoretic families.
d. **Learning L-intersecting dividend supports.** Connect Frankl–Wilson dividend games with PAC learning and sample-complexity bounds.
e. **Approximate L-intersection.** Define and study (ϵ, L)-intersecting dividend families where most pairwise intersections lie in L.
f. **Hybrid structures.** Combine sunflower, laminar, and Frankl–Wilson structures to model systems with both shared infrastructure and modular intersection constraints.
g. **Extraction heuristics.** Develop practical algorithms for extracting high-mass L-intersecting subfamilies from dense dividend representations.
h. **Nontrivial threshold compartments.** The protocol in Section 10.3 uses (k, k)-threshold (all-or-nothing) compartments to ensure that dividends are supported exactly on blocks. Extending the framework to nontrivial t-out-of-k threshold compartments

($t < k$) is an important open problem. In that case, the Harsanyi dividends of each compartment are signed and supported on all subsets of size at least t within the block, potentially breaking both the non-negativity assumption and the Frankl–Wilson admissibility. Identifying conditions under which nontrivial threshold compartments preserve tractability is a direction for future research.

i. Remark on robustness: The present framework assumes that Harsanyi dividends are observable and unaltered. In networked multi-agent systems, communication disruptions and FDI attacks can corrupt the value function or dividend estimates; see, e.g., Wang et al. (2025), Liu et al. (2025), and Zhang et al. (2026) for event-triggered and resilient control designs under such conditions, and Yan et al. (2026) for safety-critical formation maneuvering. Extending Frankl–Wilson dividend games to robust or adversarially perturbed dividend structures is left to future work.

j. Robust dividend estimation. Develop estimation and filtering methods for Harsanyi dividends under event-triggered communication, packet loss, and FDI attacks, building on resilient control designs such as those of Wang et al. (2025), Liu et al. (2025), and Zhang et al. (2026), and combine them with the approximation bounds of Section 8 to obtain end-to-end certified approximate solutions.

k. Application-driven coalitional models. Investigate whether value functions arising in formation maneuvering of air-ground swarms (Yan et al., 2026), network alliance games, cost-sharing games, and weighted-voting games admit additive decompositions over L-intersecting or modular families, and characterize the resulting intersection complexity.

14. Conclusions

This paper introduced Frankl–Wilson dividend games, a class of cooperative games whose nonzero non-singleton Harsanyi dividends are supported on L-intersecting or modular L-intersecting families. The Frankl–Wilson theorem and the Ray-Chaudhuri–Wilson theorem imply that, for fixed intersection complexity $|L|$, the dividend support is polynomially compact.

This compactness yields exact algorithmic consequences. The Shapley value is computable in polynomial time by dividend summation. For non-negative dividends, the core and least core admit a polynomial-time separation oracle based on maximum-weight closure. Under the additional oracle assumptions of Section 7, the nucleolus is polynomial-time computable; without those assumptions, unconditional nucleolus tractability remains open. Finally, if a game can be approximated by an L-intersecting dividend subgame, the Shapley approximation error is bounded by omitted dividend mass, and total dividend error gives explicit approximate stability bounds.

Frankl–Wilson dividend games generalize sunflower active coalitions, pairwise disjoint dividend families, block designs, and finite-geometric coalition systems. They therefore extend the broader research program of using extremal set theory to create cooperative-game classes with exact or certified approximate tractability.

The key principle is as follows:

Restricted intersection structure on dividend support $\Longrightarrow$ compact representation and tractable fair division.

Together with sunflower-dividend games, laminar-dividend games, and VC-dividend games, Frankl–Wilson dividend games form an important branch of the emerging theory of extremal cooperative games.

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