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A Failed Cartel: When Collusive Firms Become Competitors

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Abstract

Collusive agreements may break down completely or leave behind a surviving coalition of firms that continues to coordinate its production. We study an exogenously specified partial-cartel structure in which one firm operates independently while the remaining firms maximize their joint profit. We use the term failed cartel to indicate the institutional origin of this market configuration, without claiming that the breakdown, the identity of the outsider, or the persistence of the surviving coalition is determined endogenously. For comparison, we also consider a Cournot–Stackelberg structure in which the former cartel members compete non-cooperatively while anticipating the quantity choice of a common follower. We therefore reformulate the stationary first-order conditions as a fixed-point problem in which the follower's implicitly defined stationary response $b_3(x, y)$, rather than the derivative-generated mapping $F_3^{\lambda_3}(x, y, z)$, forms the third component of the corresponding operator. Fixed points of this operator characterize stationary solutions and provide conditional criteria for uniqueness and numerical approximation on an invariant region. When feasibility and the relevant sufficient optimality conditions are satisfied, these stationary solutions determine market equilibria. The linear n-firm analysis compares equilibrium output, prices, profits, market shares, asymmetry, and concentration across the considered market structures. Consumer surplus and total surplus comparisons are provided separately for the illustrative linear quadratic triopoly and are not asserted for general nonlinear demand. A separate isoelastic triopoly with linear costs provides a globally optimal partial-cartel outcome and a verified scalar contraction algorithm, with full-cartel and Cournot benchmarks.

Keywords: collusion; market equilibrium; Stackelberg oligopoly; response functions; fixed-point theory

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1. Introduction

Understanding how firms interact strategically has been a central objective of industrial organization since the pioneering work of (Cournot, 1838/1960) and (von Stackelberg, 1934). The classical models of oligopoly describe two fundamental modes of competition. In the

Cournot framework, firms choose production quantities simultaneously, each treating the outputs of its rivals as fixed, leading to a Nash equilibrium in quantities (Cournot, 1838/1960). In contrast, the Stackelberg model introduces sequential decision-making, where leaders anticipate the reactions of followers and exploit their informational advantage to influence market outcomes (von Stackelberg, 1934). These two paradigms remain the cornerstones of oligopoly theory and have generated an extensive literature on market equilibrium, strategic behavior, and welfare analysis (Friedman, 1983; Tirole, 1988; Vives, 1999).

A related question concerns the consequences of coordination among only some of the firms in a market. A full cartel maximizes joint industry profit, whereas a partial cartel coordinates the quantities of its members while facing independent producers. The allocation of output and profit then depends on both coalition membership and the timing of quantity choices. Cartel stability and the incentives to depart from an agreement are separate questions, central to the literature on collusion and its enforcement (Green & Porter, 1984; Harrington, 2006; Motta, 2004; Porter, 1983; Stigler, 1964). In repeated-game models, continuation strategies and intertemporal incentive constraints determine whether cooperation can be sustained (Friedman, 1971; Green & Porter, 1984; Montero & Guzman, 2010; Rotemberg & Saloner, 1986; Salant, 1976).

The present analysis belongs to the established literature on partial cartels and cartel–fringe market structures. A quantity setting coalition facing independent producers is a classical object: Salant et al. (1983) study an exogenously specified coalition in a Cournot market, while d'Aspremont et al. (1983) and Donsimoni et al. (1986) investigate cartel-membership stability. The closest quantity-leadership framework is that of Shaffer (1995), in which a cartel leads a Cournot fringe; Konishi and Lin (1999) and Zu et al. (2012) study stable cartels under more general demand and cost conditions. The partial cartel considered here is a one-outsider specialization of this established architecture.

We take the partition into $n-1$ cooperating firms and one independent outsider as exogenously given. The coalition chooses its members' quantities jointly at the first stage, and the outsider observes these choices before selecting its own output. The term failed cartel identifies a possible institutional origin of this partition. The model analyzes the subsequent quantity setting game; it does not endogenize the initial breakdown, the identity of the outsider, or the persistence of the surviving coalition. Internal, external, and coalition-proof membership stability are therefore outside the results established here.

To isolate the effect of coordination under the same two-stage timing, we compare this partial cartel with a Cournot–Stackelberg configuration in which $n-1$ leaders choose quantities simultaneously and non-cooperatively before a common follower moves. The comparison is supplemented by simultaneous Cournot competition, a fully sequential canonical Stackelberg hierarchy, and full-cartel coordination. Hybrid and multiple-leader–follower oligopolies have an established literature (Anderson & Engers, 1992; Askar, 2018; Julien, 2017). Here, the Cournot–Stackelberg outcome serves as a comparative benchmark; no repeated-game punishment strategy is constructed.

This structure also belongs to the broader class of multiple-leader–follower oligopoly models. Early formulations of mixed Stackelberg–Nash–Cournot competition were developed by (Sherali et al., 1983), who studied the characterization and computation of equilibria with one Stackelberg leader and several Cournot followers. Sherali (1984) subsequently considered multiple leaders who incorporate the aggregate follower response into their individual production decisions. The allocation of firms between the leader and follower groups and its implications for output, concentration, profits, and welfare were further investigated by (Daughety, 1990).

More general non-cooperative multiple-leader–follower games can be formulated as equilibrium problems with equilibrium constraints. Pang and Fukushima (2005) related

such models to generalized Nash equilibrium problems and quasi-variational inequalities, emphasizing that equilibrium existence requires additional structural assumptions. EPEC formulations have also been applied to bilevel competition in electricity markets by Hu and Ralph (2007), while DeMiguel and Xu (2009) developed a stochastic multiple-leader Stackelberg model. The deterministic Cournot–Stackelberg model considered here is narrower: two or more leaders choose their quantities simultaneously while anticipating the response of a common follower. Related non-cooperative oligopoly models with multiple leaders and followers are studied in (Julien, 2017).

The principal results concern linear inverse demand $P(Q) = A - BQ$ and identical quadratic costs $C(x) = cx^2$. We derive explicit formulas for the Cournot, Cournot–Stackelberg, full-cartel, and one-outsider partial-cartel outcomes, together with a backward induction recursion for the canonical Stackelberg hierarchy. For the benchmark cost-to-demand-slope ratio $\delta = c/B = 1/2$, we prove strict aggregate output, price, and first- and last-firm output comparisons for every integer $n \geq 3$. We also examine asymptotic output, market shares, firm-level asymmetry, and concentration. The large-market limits keep the demand and cost parameters fixed and, in the failed-cartel case, maintain exactly one outsider for every n. Numerical profit comparisons are reported separately, and the consumer surplus and total surplus rankings are established only for the specified linear quadratic triopoly.

A complementary part of the paper concerns the stationary first-order conditions of nonlinear Cournot–Stackelberg models. The follower's implicitly defined stationary response $b_3(x, y)$ enters a product-space operator together with derivative-generated maps for the leaders. Fixed points characterize stationary solutions. Under the additional assumptions that a nonempty closed region is invariant, and the operator is contractive there, a standard fixed-point argument yields uniqueness in that region and convergence of the associated Picard iteration. The construction uses the Banach and Hardy–Rogers framework (Banach, 1922; Hardy & Rogers, 1973); writing it in terms of several coordinate maps does not constitute a new contraction principle.

An equilibrium interpretation requires further optimality conditions. Local second-order conditions give local maximizers, while global equilibrium requires optimality over the full admissible strategy sets and, for a subgame-perfect interpretation, optimal follower choices after every admissible leader profile. The derivative-generated iteration is a numerical stationary-point scheme, not an assumed best-response dynamic. For isoelastic demand $p(Q) = Q^{-1/2}$ and positive linear marginal cost, we instead construct a concrete partial-cartel example. Backward induction establishes global optimality, and a scalar fixed-point formulation provides an explicit invariant interval, a verified contraction bound, and a reproducible numerical iteration. Full-cartel and Cournot outcomes supply the nonlinear benchmarks.

The contribution is thus a common-framework comparison with explicit and recursive linear equilibrium calculations, self-contained output comparisons, and large-market characterizations under stated assumptions. The general nonlinear formulation identifies requirements for stationary-point computation, while the isoelastic partial-cartel example verifies a convergent scalar algorithm and global optimality. The economic conclusions concern the specified quantity setting games conditional on their timing and coalition partitions; they do not establish endogenous cartel formation or general welfare rankings across nonlinear models.

The paper first introduces the market games, then develops the linear n-firm comparisons and asymptotic results. The subsequent sections present the conditional Cournot–Stackelberg formulation and the verified isoelastic partial-cartel example, followed by the economic interpretation and limitations. Appendix A reports the numerical linear

quadratic triopoly, including profit and welfare comparisons. Appendix B collects the general fixed-point theory and the detailed isoelastic proof.

2. Materials and Methods

2.1. Oligopoly Market

Consider n firms $i = 1, 2, \ldots, n$ producing quantities $x_1, x_2, \ldots, x_n$ and competing in a single market with inverse demand $P(Z) = P(\sum_{i=1}^{n} x_i)$, where $Z = \sum_{i=1}^{n} x_i$ and P is continuous and, when necessary, differentiable. Firms have access to different production technologies reflected in their cost functions $c_i(x_i)$, $i = 1, 2, \ldots, n$. Firms' profits (payoff functions) are $\Pi_i(x_1, x_2, \ldots, x_n) = x_i P(\sum_{i=1}^{n} x_i) - c_i(x_i)$ for $i = 1, 2, \ldots, n$.

Standing Assumptions

The precise assumptions used in each result are stated locally. Unless otherwise specified, quantities are nonnegative, the inverse demand function is continuous on the admissible aggregate output set, and each cost function is continuous on the corresponding individual strategy set. Differentiability is imposed only when first-order or derivative-based fixed-point formulations are used. Whenever an interior stationary solution is interpreted as a global equilibrium, the relevant individual or reduced payoff is assumed to be concave in the firm's own decision, or global maximization is verified directly. Boundary solutions are characterized by the appropriate one-sided or Karush–Kuhn–Tucker conditions.

For abstract equilibrium-existence arguments, the individual strategy sets are assumed to be nonempty and compact unless coercivity or another explicit condition guarantees existence on an unbounded set. For contraction arguments, the operator is defined on a nonempty closed invariant subset of the product strategy space. Differentiability of an implicit continuation response is invoked only when the hypotheses of the implicit-function theorem, including a nonzero derivative with respect to the resolved variable, have been verified.

Each firm i chooses its quantity from a nonempty admissible strategy set $X_i \subseteq [0, \infty)$. Unless explicitly stated otherwise, the fixed-point reformulations developed below concern interior stationary solutions. For such solutions, the corresponding first-order equalities are necessary optimality conditions. A stationary solution represents a market equilibrium only when it is feasible and satisfies the relevant sufficient optimality conditions. If an optimum lies on the boundary of X_i, the corresponding first-order equality must be replaced by the appropriate boundary or Karush–Kuhn–Tucker optimality conditions (Nocedal & Wright, 2006).

The models considered below differ in the timing of quantity choices and in whether firms maximize individual or joint profits. We therefore specify separately the players, their admissible strategies, the order of decisions, the information available at each stage, and the corresponding equilibrium requirements. In the failed-cartel and Cournot–Stackelberg configurations, the identity of Firm n as the outsider or follower is imposed exogenously. The analysis begins after this market configuration has been established and does not model the preceding cartel-formation or breakdown process.

While much has been said about Stackelberg and Cournot models of competition, hybrid market structures that combine features of stylistic models of competitions have received less attention. A handful of papers have departed from the stylistic oligopolistic models and considered narrow extensions including (Anderson & Engers, 1992; Askar, 2018; Badev et al., 2024; Friedman, 1983; Ilchev et al., 2024; Julien, 2017).

Below we formulate the partial-cartel and Cournot–Stackelberg configurations used in the subsequent comparisons. The first represents an exogenously imposed coalition of $n - 1$ coordinating firms facing one independent producer. The second represents a non-

cooperative multiple-leader–one-follower market. The two configurations are compared as alternative post-cartel market structures; no claim is made that either structure constitutes an endogenous or sequentially rational punishment mechanism.

2.2. The Cournot Model

In the classical Cournot model, firms act simultaneously and non-cooperatively, i.e., each firm takes the output levels of its competitors as given, assuming they will not be changed over time when determining its own production (Cournot, 1838/1960; Friedman, 1971; Smith, 1982; Varian, 2014):

$$\begin{cases} \max\{\Pi_i(x_1, x_2, \dots x_n) : x_i \geq 0\}, \\ i = 1, 2, \dots, n. \end{cases}$$

Specifically, firms assume that rivals do not adjust their output in response to their decision. Under these assumptions, the first-order necessary conditions for optimality yield the following system:

$$\begin{cases} \dfrac{\partial \Pi_i}{\partial x_i}(x_1, x_2, \dots x_n) = P\left(\sum\limits_{i=1}^{n} x_i\right) + x_i P'\left(\sum\limits_{i=1}^{n} x_i\right) - c_i'(x_i) = 0, \\ i = 1, 2, \dots, n. \end{cases} \tag{1}$$

A Cournot–Nash equilibrium is a vector of feasible outputs such that each firm's quantity maximizes its payoff, given the quantities of its competitors. Every interior Cournot–Nash equilibrium satisfies system (1). The converse, however, does not hold without additional assumptions, because a solution of the first-order system may fail to be a profit-maximizing strategy. Accordingly, a solution of (1) represents a Cournot–Nash equilibrium when it is feasible and each firm's stationary quantity is a global maximizer of its payoff on the corresponding strategy set. This follows, for example, when each payoff function is concave in the firm's own quantity. If an equilibrium quantity lies on the boundary of the admissible set, the corresponding first-order equality must be replaced by the appropriate boundary or Karush–Kuhn–Tucker condition. See the discussions in (Badev et al., 2024; Dzhabarova et al., 2020; Friedman, 1971; Tron & Théra, 2026). For empirical applications to oligopoly markets based on real-world data, see (Badev et al., 2024) for a duopoly market and (Ali et al., 2025; Ilchev et al., 2024) for tripodal market models. The latter study constructs tripodal response functions directly from observed mobile-market data rather than deriving them from payoff maximization problems.

2.3. The Canonical Stackelberg Model

The canonical Stackelberg model, also referred to as the hierarchical Stackelberg model, assumes sequential decision making. For brevity, it will be denoted simply as the Stackelberg model throughout the tables, figures, and equations. Firm 1 is the initial leader and chooses its output first. Firm 2 observes the choice of Firm 1 and anticipates the subsequent firms' decisions. This procedure continues recursively, with each Firm i observing the quantities selected by Firms $1, \dots, i-1$ and anticipating the stationary continuation responses of Firms $i+1, \dots, n$. Firm n is the final follower.

Below we derive the stationary conditions for $n = 3$ to illustrate the backward induction construction. Using the notation of (Badev et al., 2026), we subsequently extend the analysis to the failed-cartel case.

For fixed (x_1, x_2), the first-order condition for Firm 3 is

$$\frac{\partial \Pi_3}{\partial x_3}(x_1, x_2, x_3) = P(Z) + x_3 P'(Z) - c_3'(x_3) = 0.$$

Assume that this equation has a unique solution for every admissible (x_1, x_2). We denote this solution by $b_3 = b_3(x_1, x_2)$ and call it the stationary continuation response of Firm 3. It is the unique best response of Firm 3 when the solution maximizes $\Pi_3(x_1, x_2, \cdot)$ on the follower's strategy set, for example under an appropriate strict-concavity assumption.

Given the stationary continuation response $b_3(x_1, x_2)$, Firm 2 anticipates the subsequent choice of Firm 3 and considers the reduced payoff $\Phi_2(x_1, x_2) = \Pi_2\big(x_1, x_2, b_3(x_1, x_2)\big)$. The interior stationary condition for Firm 2 is therefore $\frac{\partial \Phi_2}{\partial x_2}(x_1, x_2) = 0$

By the chain rule, we obtain

$$\frac{\partial \Phi_2}{\partial x_2}(x_1, x_2) = \frac{\partial \Pi_2}{\partial x_2}(x_1, x_2, b_3(x_1, x_2)) + \frac{\partial \Pi_2}{\partial x_3}(x_1, x_2, b_3(x_1, x_2))\frac{\partial b_3}{\partial x_2}(x_1, x_2).$$

Assume that, for every admissible x_1, the equation $\frac{\partial \Phi_2}{\partial x_2}(x_1, x_2) = 0$ has a unique solution. We denote this solution by $x_2 = b_2(x_1)$ and call it the stationary continuation response of Firm 2. It is the best response in the reduced problem when it maximizes $\Phi_2(x_1, \cdot)$ on the corresponding strategy set.

Finally, Firm 1 anticipates the stationary continuation responses of Firms 2 and 3 and considers the reduced payoff $\Phi_1(x_1) = \Pi_1\big(x_1, b_2(x_1), b_3(x_1, b_2(x_1))\big)$. The interior stationary condition for Firm 1 is $\frac{d\Phi_1}{dx_1}(x_1) = 0$.

Thus, the preceding backward induction construction initially determines a stationary Stackelberg triple. If the stationary quantities satisfy the relevant second-order sufficient conditions, the triple determines a strict local Stackelberg equilibrium. If the follower's payoff and the corresponding reduced payoff functions are concave on their admissible strategy sets, the stationary quantities are global maximizers and the triple determines a global Stackelberg equilibrium.

Equivalently,

$$\begin{aligned}\frac{d\Phi_1}{dx_1}(x_1) = \frac{\partial \Pi_1}{\partial x_1} + \frac{\partial \Pi_1}{\partial x_2}\frac{db_2}{dx_1}(x_1) \\ + \frac{\partial \Pi_1}{\partial x_3}\left(\frac{\partial b_3}{\partial x_1}(x_1, b_2(x_1)) + \frac{\partial b_3}{\partial x_2}(x_1, b_2(x_1))\frac{db_2}{dx_1}(x_1)\right) = 0,\end{aligned}$$

where all partial derivatives of Π_1 are evaluated at $(x_1, b_2(x_1), b_3(x_1, b_2(x_1)))$.

2.4. The Full-Cartel Model

Assume that all n firms have the same cost function c and form a full cartel. The cartel coordinates the firms' quantities to maximize total industry profit. Throughout this benchmark, equal allocation of aggregate output among the cartel members is imposed as a maintained assumption. Thus, if t denotes aggregate cartel output, then $x_i = t/n$ for every $i = 1, \ldots, n$. When c is strictly convex, this equal allocation also follows from cost minimization among identical firms for a given aggregate output.

The cartel's joint profit is $\Pi^{\text{Cartel}}(t) = tP(t) - nc\big(\frac{t}{n}\big)$.

This is the joint profit of the entire industry under the imposed equal allocation. In the absence of this restriction, the cartel would instead solve

$$\max_{(x_1,\ldots,x_n)} \left\{ QP(Q) - \sum_{i=1}^{n} c_i(x_i) \right\}, \qquad Q = \sum_{i=1}^{n} x_i,$$

subject to the firms' admissible quantity constraints.

When the economic interpretation requires a nonnegative market price, the admissible quantity profiles are restricted by $P(Q) \geq 0$. A stationary point outside this region is not interpreted as an economically feasible market outcome.

Let $X^{\text{Cartel}} \subseteq [0, \infty)$ denote the admissible set of aggregate cartel outputs. When nonnegative market price is required, this set is restricted further by $P(t) \geq 0$.

Consequently, an interior stationary cartel output satisfies

$$\frac{d\Pi^{\text{Cartel}}}{dt}(t) = P(t) + tP'(t) - c'\left(\frac{t}{n}\right) = 0.$$

A solution of this equation determines the profit-maximizing cartel output when the corresponding sufficient optimality condition holds. In particular, it is the global cartel optimum when Π^{Cartel} is concave on the admissible aggregate output set. If the optimum is attained at the boundary, the appropriate boundary optimality condition must be used.

Accordingly, the first-order equality is not used as a complete characterization when the cartel optimum is attained at $t = 0$, at a positive upper bound or at the boundary defined by $P(t) = 0$.

2.5. Hybrid Market Structures

2.5.1. A Failed Cartel

We now define the post-cartel game studied in this subsection. The initial cartel breakdown and the resulting partition $\mathcal{C} = \{1, \ldots, n-1\}$ and $\mathcal{O} = \{n\}$, are taken as exogenously given. The firms in $\mathcal{C}$ form the surviving coalition, whereas Firm n operates independently as the outsider. The model does not describe the decision to leave the initial cartel and does not require that departure to be profitable.

Unless explicitly stated otherwise, the failed-cartel model is considered for $n \geq 3$, so that the surviving coalition contains at least two firms. When $n = 2$, the surviving "coalition" contains only one firm, and the failed-cartel game coincides with the one-leader–one-follower Cournot–Stackelberg game.

The post-cartel game has two stages. At Stage 1, the surviving coalition acts as a single decision-making unit and selects the quantities of its $n-1$ members to maximize their joint profit. Under the maintained symmetry assumption, all coalition members choose the same quantity x. At Stage 2, Firm n observes the coalition quantities and selects $x_n \in X_n$ to maximize its own profit. Thus, the timing is

exogenously given partition $\rightarrow$ coalition chooses $x \rightarrow$ outsider observes x and chooses x_n.

The choice of timing is a substantive modeling assumption. In a model of tacit collusion, Escrihuela-Villar (2009) shows that the sequence of decisions by the cartel and the fringe affects cartel stability and that endogenous sequencing depends on the discount factor. Here, coalition leadership is imposed as part of the two-stage game; the analysis does not derive this timing from intertemporal incentives.

The surviving coalition anticipates the outsider's continuation response when selecting x. For notational convenience, write $x^{n-1} = \underbrace{(x, \ldots, x)}_{n-1 \text{ components}} \in \mathbb{R}^{n-1}$.

At Stage 2, for every admissible vector of coalition quantities $(u_1, \ldots, u_{n-1})$, the outsider solves $\max_{x_n \in X_n} \Pi_n(u_1, \ldots, u_{n-1}, x_n)$. Whenever the maximizer is unique, denote it by

$$b_n(u_1, \ldots, u_{n-1}) = \arg\max_{x_n \in X_n} \Pi_n(u_1, \ldots, u_{n-1}, x_n).$$

If only the first-order equation is known to possess a unique solution, b_n is initially interpreted as a stationary continuation response and becomes the best response only after the relevant global optimality condition has been verified.

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Assume, in addition, that b_n is continuously differentiable on the relevant domain of the first $n-1$ quantities. Along the symmetric path $x^{n-1} = (x, \ldots, x)$, the chain rule gives

$$\frac{d}{dx} b_n(x^{n-1}) = \sum_{i=1}^{n-1} \frac{\partial b_n}{\partial u_i}(x^{n-1}).$$

Along the symmetric path of the remaining cartel members, the outsider's stationary quantity is therefore $x_n = b_n(x^{n-1})$, where $x^{n-1} = (x, \ldots, x)$. The stationary response is the outsider's unique best response when it maximizes $\Pi_n(u_1, \ldots, u_{n-1}, \cdot)$ on X_n, for example under an appropriate strict-concavity assumption.

The remaining $n-1$ firms continue to coordinate and choose the common output x while anticipating the outsider's response $b_n(x^{n-1})$. Assuming that these firms have the same cost function c, their joint reduced payoff is

$$\Pi^{FC}(x) = \sum_{i=1}^{n-1} \Pi_i\big(x^{n-1}, b_n(x^{n-1})\big) = (n-1)\Big[xP\big((n-1)x + b_n(x^{n-1})\big) - c(x)\Big].$$

Since the factor $n-1$ is positive, maximizing Π^{FC} is equivalent to maximizing the reduced payoff

$$\Phi^{FC}(x) = xP\big((n-1)x + b_n(x^{n-1})\big) - c(x).$$

A failed-cartel equilibrium of the post-cartel game is a pair (x^*, x_n^*) such that $x_n^* = b_n((x^*)^{n-1})$ is a profit-maximizing continuation choice of the outsider after every admissible coalition quantity and

$$x^* \in \arg\max_{x \in X_{\mathcal{C}}} \Pi^{FC}(x),$$

where $X_{\mathcal{C}}$ is the admissible set of common coalition quantities. Under these requirements, the resulting outcome is subgame perfect within the specified two-stage post-cartel game. This conclusion does not imply that the initial cartel breakdown or the persistence and composition of the surviving coalition is itself an equilibrium outcome.

Stability with respect to cartel membership requires a separate coalition formation analysis. Thoron (1998) models firms' participation decisions non-cooperatively and uses coalition-proof Nash equilibrium to study cartel stability against self-enforcing coalitional deviations. The equilibrium defined above concerns quantity choices within the imposed partition and does not establish this stronger membership-stability property.

The first-order condition is

$$\frac{d\Phi^{FC}}{dx}(x) = P\big((n-1)x + b_n(x^{n-1})\big) + xP'\big((n-1)x + b_n(x^{n-1})\big)\left(n - 1 + \sum_{i=1}^{n-1} \frac{\partial b_n}{\partial u_i}(x^{n-1})\right) - c'(x) = 0.$$

A solution of this first-order equation is a stationary coalition quantity. It determines a failed-cartel equilibrium of the specified post-cartel game only when x globally maximizes Π^{FC} on $X_{\mathcal{C}}$ and b_n is the outsider's profit-maximizing continuation response after every admissible coalition choice. If the relevant objectives are strictly concave on convex strategy sets, these maximizers are unique.

2.5.2. Cournot–Stackelberg Model

We next define the Cournot–Stackelberg game. Firms $1, \ldots, n-1$ are leaders and Firm n is the common follower. At Stage 1, the leaders choose their quantities $x_1, \ldots, x_{n-1}$

simultaneously and non-cooperatively, each maximizing its individual reduced payoff while anticipating the follower's continuation response. At Stage 2, Firm n observes the vector $(x_1, \ldots, x_{n-1})$ and chooses $x_n \in X_n$ to maximize its own profit. The timing is

$$\begin{array}{c} \text{leaders choose } (x_1, \ldots, x_{n-1}) \text{ simultaneously} \\ \downarrow \\ \text{follower observes their choices and selects } x_n. \end{array}$$

The assignment of firms to the leader and follower groups is exogenous.

The slope of the follower's response is economically important because it determines how the follower adjusts its quantity when one of the leaders changes production. This observation is related to the classical distinction between strategic substitutes and strategic complements introduced by (Bulow et al., 1985) and subsequently developed in models of strategic investment and interdependent demand (Fudenberg & Tirole, 1984; Loomis, 1997). In the present model, however, the follower's response is initially defined through its stationary first-order condition. It receives a best-response interpretation only after the corresponding optimality conditions have been verified.

For fixed $(x_1, \ldots, x_{n-1})$, the follower's first-order condition is

$$\frac{\partial \Pi_n}{\partial x_n}(x_1, \ldots, x_n) = P(Z) + x_n P'(Z) - c_n'(x_n) = 0.$$

Assume that this equation has a unique solution for every admissible $(x_1, \ldots, x_{n-1})$. We denote this stationary response by $x_n = b_n(x_1, \ldots, x_{n-1})$.

The function b_n is initially the follower's stationary continuation response. It coincides with the follower's unique profit-maximizing best response when $b_n(x_1, \ldots, x_{n-1})$ is the unique global maximizer of $\Pi_n(x_1, \ldots, x_{n-1}, \cdot)$ on X_n. This conclusion holds, for example, when the follower's payoff is strictly concave in x_n on a convex strategy set. If the maximizing quantity lies on the boundary of X_n, it must instead satisfy the corresponding boundary or Karush–Kuhn–Tucker optimality condition.

Firms $1, \ldots, n-1$ anticipate the follower's stationary response and consider the reduced payoff functions

$$\begin{cases} \Phi_i^{CS}(x_1, x_2, \ldots, x_{n-1}) = \Pi_i(x_1, x_2, \ldots, x_{n-1}, b_n(x_1, x_2, \ldots, x_{n-1})), \\ i = 1, 2, \ldots n-1. \end{cases}$$

The interior stationary condition for each Firm i, $i = 1, \ldots, n-1$, is

$$\frac{\partial \Phi_i^{CS}}{\partial x_i}(x_1, \ldots, x_{n-1}) = 0.$$

A Cournot–Stackelberg equilibrium is a feasible profile $(x_1^*, \ldots, x_{n-1}^*, x_n^*)$ such that

$$x_n^* = b_n(x_1^*, \ldots, x_{n-1}^*)$$

and, for every leader $i = 1, \ldots, n-1$,

$$x_i^* \in \arg\max_{x_i \in X_i} \Phi_i^{CS}(x_1^*, \ldots, x_{i-1}^*, x_i, x_{i+1}^*, \ldots, x_{n-1}^*).$$

For a subgame-perfect interpretation, $b_n(x_1, \ldots, x_{n-1})$ must be a profit-maximizing continuation response after every admissible leader profile, not only at the equilibrium profile. Solutions of the displayed first-order equations are merely stationary leader profiles until these global maximization requirements have been verified.

These conditions determine profit-maximizing choices when the corresponding reduced payoff functions satisfy the relevant sufficient optimality conditions.

For brevity, write $\overline{x}_n = (x_1, \ldots, x_{n-1})$.

By the chain rule,

$$\begin{cases} \frac{\partial \Phi_i^{CS}}{\partial x_i}(\overline{x_n}) = \frac{\partial \Pi_i}{\partial x_i}(\overline{x_n}, b_n(\overline{x_n})) + \frac{\partial \Pi_i}{\partial x_n}(\overline{x_n}, b_n(\overline{x_n}))\frac{\partial b_n}{\partial x_i}(\overline{x_n}), \\ i = 1, 2, \ldots n-1. \end{cases}$$

Hence, a stationary Cournot–Stackelberg profile is characterized by

$$\begin{cases} \frac{\partial \Pi_n}{\partial x_n}(\overline{x}) = 0, \\ \frac{\partial \Phi_i^{CS}}{\partial x_i}(\overline{x_n}) = \frac{\partial \Pi_i}{\partial x_i}(\overline{x_n}, b_n(\overline{x_n})) + \frac{\partial \Pi_i}{\partial x_n}(\overline{x_n}, b_n(\overline{x_n}))\frac{\partial b_n}{\partial x_i}(\overline{x_n}) = 0, \\ \text{for } i = 1, 2, \ldots n-1. \end{cases}$$

This first-order system characterizes stationary profiles. Such a profile is a Cournot–Stackelberg equilibrium when the follower's stationary response and the leaders' stationary choices satisfy the corresponding sufficient optimality conditions. In particular, concavity of the follower's payoff with respect to x_n and concavity of each reduced payoff Φ_i^{CS} with respect to x_i ensure that the corresponding stationary choices are global maximizers.

3. Linear Demand: Large-Market Comparative Statics and Asymptotic Results

For an oligopoly with n firms, let Q_n, P_n, and x_i, $i = 1, 2, \ldots, n$, denote aggregate output, market price, and the output of Firm i, respectively. Superscripts identify the market structure: C denotes Cournot, S canonical Stackelberg, CS Cournot–Stackelberg, FC failed cartel, and $Cartel$ full cartel coordination. The same convention is used for aggregate output, prices, profits, and individual quantities.

3.1. Equilibrium Formulas for the n-Firm Models

Consider the inverse demand and cost functions $P(Q) = A - BQ$ and $C(x) = cx^2$, and let $M = \frac{A}{B}$ and $\delta = \frac{c}{B}$. In the appendix, we consider a numerical example with concrete choices for A, B, c, M, and δ.

Since $B > 0$ and $c > 0$, the individual and reduced optimization problems considered below have strictly concave quadratic objective functions in the relevant decision variables. Consequently, every feasible interior solution of the corresponding first-order conditions is the unique global maximizer of the relevant objective function. Thus, whenever the explicitly derived quantities are positive, the stationary responses are genuine best-response functions and the resulting stationary profiles are equilibrium profiles.

Proposition 1 (Cournot–Stackelberg equilibrium with $n-1$ leaders)**.** *Let Firms* $1, \ldots, n-1$ *be Cournot leaders and Firm n their common follower. The follower's best response is*

$$b_n(x_1, \ldots, x_{n-1}) = \max\left\{0, \frac{M - \sum_{i=1}^{n-1} x_i}{2(1+\delta)}\right\}.$$

At the equilibrium derived below, the response is interior, so the positive branch of this formula applies.

For $\delta = 1/2$*, the symmetric equilibrium quantities and aggregate output are*

$$x_1^{CS} = \cdots = x_{n-1}^{CS} = \frac{2M}{2n+3}, \qquad x_n^{CS} = \frac{5M}{3(2n+3)}, \qquad Q_n^{CS} = \frac{(6n-1)M}{3(2n+3)}.$$

The corresponding price is $P_n^{CS} = A - BQ_n^{CS}$.

Proof. Assume that firms $1, \ldots, n-1$ act simultaneously in a Cournot-type reduced game, while firm n is the follower. Denote $X = \sum_{i=1}^{n-1} x_i$. For fixed $x_1, \ldots, x_{n-1}$, the follower solves

$$\max_{z \geq 0} \Big\{ z(A - B(X+z)) - cz^2 \Big\}.$$

The first-order condition is $A - BX - 2(B+c)z = 0$. Since the follower's second derivative is

$$\frac{\partial^2 \Pi_n}{\partial z^2} = -2(B+c) < 0,$$

this condition determines the follower's unique interior stationary quantity. Taking the constraint $z \geq 0$ into account, the unique best response is

$$z = b_n(x_1, \ldots, x_{n-1}) = \max\left\{0, \frac{M - X}{2(1+\delta)}\right\}.$$

At the symmetric equilibrium obtained below,

$$X = (n-1)\frac{2M}{2n+3} < M,$$

and therefore the follower's equilibrium response is interior.

By symmetry among the $n-1$ leaders, we look for an equilibrium of the form $x_1 = \cdots = x_{n-1} = x$. Then $X = (n-1)x$. The reduced first-order condition for each leader gives

$$x = \frac{(1-\alpha)M}{(1-\alpha)n + 2\delta}, \qquad \alpha = \frac{1}{2(1+\delta)}.$$

In the particular case $\delta = \frac{1}{2}$, we have $\alpha = \frac{1}{3}$, and therefore

$$x_1 = \cdots = x_{n-1} = \frac{2M}{2n+3}.$$

The reduced payoff of each leader is a strictly concave quadratic function of its own output. Therefore, the reduced first-order condition determines the leader's unique best response, given the outputs of the other leaders and the anticipated response of the follower.

The follower's output is $x_n = z = \frac{5M}{3(2n+3)}$. Thus, the total output is

$$Q_n^{CS} = (n-1)\frac{2M}{2n+3} + \frac{5M}{3(2n+3)} = \frac{(6n-1)M}{3(2n+3)}.$$

The corresponding market price is $P_n^{CS} = A - BQ_n^{CS}$. For $n = 3$, this gives $x_1 = x_2 = 137760$ and $x_3 = 114800$, which coincides with the Cournot–Stackelberg values computed above. □

Proposition 2 (Cartel equilibrium with n firms)**.** *If all n firms form a cartel and divide the monopoly output equally, then*

$$Q_n^{Cartel} = \frac{M}{2(1+\delta/n)}, \qquad x_i^{Cartel} = \frac{Q_n^{Cartel}}{n}, \quad i = 1, \ldots, n.$$

Since $M > 0$, $\delta > 0$, *and* $n \geq 2$, *we have* $0 < Q_n^{Cartel} < M$.

Therefore, every firm produces a strictly positive quantity, and the aggregate cartel output lies in the economically admissible region $0 < Q_n^{Cartel} < A/B$. *Consequently, the corresponding market price* $P_n^{Cartel} = A - BQ_n^{Cartel}$ *is strictly positive.*

For $\delta = 1/2$, *these formulas reduce to*

$$Q_n^{Cartel} = \frac{Mn}{2n+1}, \qquad x_i^{Cartel} = \frac{M}{2n+1}.$$

The corresponding price is $P_n^{Cartel} = A - BQ_n^{Cartel}$.

Proof. Assume now that all n firms form a cartel and divide the monopoly output equally. If Q is the total cartel output, then each firm produces Q/n, and the cartel maximizes

$$\Pi^{Cartel}(Q) = QP(Q) - nC\left(\frac{Q}{n}\right).$$

That is,

$$\Pi^{Cartel}(Q) = Q(A - BQ) - nc\left(\frac{Q}{n}\right)^2 = AQ - BQ^2 - \frac{c}{n}Q^2.$$

The first-order condition is $A - 2\left(B + \frac{c}{n}\right)Q = 0$.

Moreover, $\frac{d^2\Pi^{Cartel}}{dQ^2} = -2\left(B + \frac{c}{n}\right) < 0$. Hence, the first-order condition determines the unique profit-maximizing aggregate cartel output.

Therefore,

$$Q_n^{Cartel} = \frac{A}{2\left(B + \frac{c}{n}\right)} = \frac{M}{2\left(1 + \frac{\delta}{n}\right)}.$$

For $\delta = \frac{1}{2}$, this becomes $Q_n^{Cartel} = \frac{Mn}{2n+1}$. Thus, each firm produces

$$x_i^{Cartel} = \frac{Q_n^{Cartel}}{n} = \frac{M}{2n+1}, \qquad i = 1, \ldots, n.$$

The cartel price is $P_n^{Cartel} = A - BQ_n^{Cartel}$. For $n = 3$, this gives $Q_3^{Cartel} = 265680$ and $x_1 = x_2 = x_3 = 88560$, which is exactly the cartel value obtained in the triopoly case. □

Proposition 3 (Failed-cartel equilibrium with one outsider)**.** *Let* $n \geq 3$. *Firms* $1, \ldots, n-1$ *coordinate their quantities as a surviving coalition, while Firm n operates independently as the outsider and selects its best response after observing the coalition quantities. For* $\delta = 1/2$, *the equilibrium quantities and aggregate output are*

$$x_1^{FC} = \cdots = x_{n-1}^{FC} = \frac{2M}{4n-1}, \qquad x_n^{FC} = \frac{(2n+1)M}{3(4n-1)},$$

and

$$Q_n^{FC} = \frac{(8n-5)M}{3(4n-1)}.$$

The corresponding price is $P_n^{FC} = A - BQ_n^{FC}$.

Proof. Finally, assume that firms $1, \ldots, n-1$ continue to behave as a cartel, while firm n deviates and chooses its best response. We again denote by x the common output of the $n-1$ cooperating firms: $x_1 = \cdots = x_{n-1} = x$. The deviating firm solves

$$\max_{z \geq 0} \left\{ z(A - B((n-1)x + z)) - cz^2 \right\}.$$

The deviating firm's first-order condition gives the interior stationary quantity

$$z = \frac{M-(n-1)x}{2(1+\delta)}.$$

Since $\frac{\partial^2 \Pi_n}{\partial z^2} = -2(B+c) < 0$, its unique best response on the nonnegative strategy set is

$$b_n(x^{n-1}) = \max\left\{0, \frac{M-(n-1)x}{2(1+\delta)}\right\}.$$

The cooperating firms maximize their joint profit

$$\Pi^{FC}(x) = (n-1)\left[xP((n-1)x+z) - cx^2\right].$$

For $\delta = \frac{1}{2}$, the outsider's response is $z = \frac{1}{3}(M-(n-1)x)$. We first consider the region $M-(n-1)x > 0$, in which the outsider's best response is interior.

The reduced joint profit of the cooperating firms is a strictly concave quadratic function of their common output x. Therefore, its first-order condition determines their unique profit-maximizing common output and gives

$$x_1 = \cdots = x_{n-1} = \frac{2M}{4n-1}.$$

At this quantity, the inequality holds $M-(n-1)x = \frac{(2n+1)M}{4n-1} > 0$. Hence, the outsider's response is indeed interior at the candidate equilibrium. Moreover, when $x \geq M/(n-1)$, the outsider's best response is zero and the market price is nonpositive, so the cooperating firms cannot obtain a higher joint profit in that region. Therefore, the interior solution above is the unique global optimum of the reduced cartel problem.

The deviating firm's output is $x_n = z = \frac{(2n+1)M}{3(4n-1)}$. Hence, the total output is

$$Q_n^{FC} = (n-1)\frac{2M}{4n-1} + \frac{(2n+1)M}{3(4n-1)}.$$

Equivalently, $Q_n^{FC} = \frac{(8n-5)M}{3(4n-1)}$.

All equilibrium quantities are positive and $0 < Q_n^{FC} = \frac{(8n-5)M}{3(4n-1)} < M$. Thus, the equilibrium is feasible, and the corresponding market price is positive.

The market price is $P_n^{FC} = A - BQ_n^{FC}$. In the context of the numerical example with $n = 3$, we obtain $x_1 = x_2 = \frac{2M}{11} \approx 112713$ and $x_3 = \frac{7M}{33} \approx 131498$. Therefore, $Q_3^{FC} \approx 356924$, which coincides with the failed-cartel values computed above. □

The canonical hierarchical Stackelberg benchmark is also considered in (Badev et al., 2026). For completeness, we derive the recursion used in the present comparisons directly from the firms' optimization problems.

Proposition 4 (Canonical Stackelberg recursion)**.** *Let* $P(Q) = A - BQ$, $C(x) = cx^2$, $A, B, c > 0$, $M = A/B$, *and* $\delta = c/B$. *Firms choose nonnegative quantities sequentially in the order* $1, \ldots, n$, *observing all preceding choices. Define*

$$r_0 = 1, \qquad r_k = \frac{r_{k-1}(r_{k-1}+2\delta)}{2(r_{k-1}+\delta)}, \qquad k \geq 1.$$

The backward induction equilibrium satisfies

$$Q_n^S = M(1-r_n), \qquad x_1^S = \frac{Mr_{n-1}}{2(r_{n-1}+\delta)}, \qquad x_n^S = \frac{Mr_n}{1+2\delta}.$$

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Moreover, $r_k > 0$, $r_k \downarrow 0$, and hence $Q_n^S \uparrow M$. Equivalently, $q_k = Q_k^S/M = 1 - r_k$ satisfies

$$q_0 = 0, \qquad q_k = q_{k-1} + \frac{(1-q_{k-1})^2}{2(1-q_{k-1}+\delta)}.$$

Proof. Suppose that the firms which have already moved have produced a total quantity H, and put $R = M - H$. We prove by induction on the number k of remaining firms that, for $R > 0$, the residual quantity $M - Q$ after their equilibrium continuation equals $r_k R$. For $k = 0$, this is immediate from $r_0 = 1$.

Assume the assertion holds for $k-1$ remaining firms. If the current firm chooses $x \in [0, R]$, its successors leave the final residual $r_{k-1}(R-x)$. Thus, the final price is $Br_{k-1}(R-x)$, and the current firm's reduced profit is

$$\Phi_k(x; R) = Br_{k-1}(R-x)x - cx^2 = B\left[r_{k-1}Rx - (r_{k-1}+\delta)x^2\right].$$

This function is strictly concave, with unique maximizer

$$x^*(R,k) = \frac{r_{k-1}R}{2(r_{k-1}+\delta)} \in (0, R).$$

For $x \geq R$, all successors optimally produce zero: the linear inverse demand is nonpositive and any positive output incurs positive cost. Such a choice gives the current firm a nonpositive profit, whereas $x^*(R,k)$ gives a positive profit. It therefore cannot improve upon the interior maximizer. If $R \leq 0$ at an off-equilibrium history, the unique optimal continuation consists of zero output by every remaining firm. If profiles with negative prices are excluded instead, the comparison over $x \in [0, R]$ already establishes optimality at every admissible history.

Substituting $x^*(R,k)$ gives the final residual

$$r_{k-1}\big(R - x^*(R,k)\big) = R\frac{r_{k-1}(r_{k-1}+2\delta)}{2(r_{k-1}+\delta)} = Rr_k,$$

which completes the induction. Initially $R = M$ and $k = n$, yielding the formulas for Q_n^S and x_1^S. For the last mover, the first-order condition is $A - BQ_n^S = (B+2c)x_n^S$. Since $A - BQ_n^S = BMr_n$, this gives $x_n^S = Mr_n/(1+2\delta)$.

Finally,

$$r_{k-1} - r_k = \frac{r_{k-1}^2}{2(r_{k-1}+\delta)} > 0.$$

The sequence is positive and decreasing. Its limit $\ell \geq 0$ satisfies $\ell^2/[2(\ell+\delta)] = 0$, so $\ell = 0$. The recursion for q_k follows by substituting $r_k = 1 - q_k$. □

Proposition 5 (Canonical Stackelberg versus Cournot–Stackelberg aggregate output)**.** *Let $\delta = 1/2$ and $M > 0$. Then $Q_2^S = Q_2^{CS}$, whereas $Q_n^S > Q_n^{CS}$ for every integer $n \geq 3$.*

Proof. Let $q_n = \frac{Q_n^S}{M}$ and $r_n = 1 - q_n$. For $\delta = 1/2$, the canonical Stackelberg recursion in Proposition 4 becomes

$$q_n = q_{n-1} + \frac{(1-q_{n-1})^2}{2(1-q_{n-1}+1/2)}.$$

Therefore, the residual sequence satisfies

$$r_n = r_{n-1} - \frac{r_{n-1}^2}{2r_{n-1}+1} = \frac{r_{n-1}(r_{n-1}+1)}{2r_{n-1}+1}.$$

Define $g(r) = \frac{r(r+1)}{2r+1}$ for $r \geq 0$. Since $g'(r) = \frac{2r^2+2r+1}{(2r+1)^2} > 0$, the function g is strictly increasing on $[0, \infty)$.

On the other hand, the Cournot–Stackelberg aggregate output is

$$\frac{Q_n^{CS}}{M} = \frac{6n-1}{3(2n+3)} = \frac{6n-1}{6n+9}.$$

Hence its residual from M is $a_n = 1 - \frac{Q_n^{CS}}{M} = \frac{10}{6n+9}$.

Starting from $q_0 = 0$, the canonical recursion gives $r_1 = \frac{2}{3}$ and $r_2 = \frac{10}{21}$. Since $a_2 = 10/21$, it follows that $Q_2^S = Q_2^{CS}$.

We now prove by induction that $r_n < a_n$ for every $n \geq 3$.

Let $n \geq 3$ and suppose that $r_{n-1} \leq a_{n-1}$. Since g is increasing, there holds

$$r_n = g(r_{n-1}) \leq g(a_{n-1}).$$

Putting $d = 6n + 3$, we have $a_{n-1} = 10/d$, and therefore $g(a_{n-1}) = \frac{10(d+10)}{d(d+20)}$.

Moreover, $\frac{10(d+10)}{d(d+20)} < \frac{10}{d+6} = a_n$ if and only if $(d+10)(d+6) < d(d+20)$, or equivalently $d > 15$.

Since $d = 6n + 3 > 15$ for every $n \geq 3$, we obtain $r_n < a_n$.

The equality $r_2 = a_2$ provides the initial induction step. Consequently, $r_n < a_n$ for every $n \geq 3$.

Finally, there holds the inequality

$$\frac{Q_n^S}{M} = 1 - r_n > 1 - a_n = \frac{Q_n^{CS}}{M}.$$

Since $M > 0$, this proves $Q_n^S > Q_n^{CS}$ for every integer $n \geq 3$. □

Ordering of Equilibrium Quantities

For the canonical hierarchical Stackelberg model, we use the backward induction formulas proved in Proposition 4. For $\delta = 1/2$, they give

$$Q_n^S = M(1 - r_n), \qquad x_1^S = \frac{Mr_{n-1}}{1 + 2r_{n-1}}, \qquad x_n^S = \frac{Mr_n}{2}.$$

The corresponding limits $\lim_{n\to\infty} nx_1^S = M$ and $\lim_{n\to\infty} nx_n^S = M/2$ are proved in Proposition 8 below.

Proposition 6 (Ordering of individual equilibrium quantities)**.** *Let $\delta = 1/2$, $M > 0$, and let $n \geq 3$ be an integer. For the first firm,*

$$x_1^{Cartel} < x_1^{FC} < x_1^{C} < x_1^{CS} < x_1^{S},$$

whereas for the last firm,

$$x_n^{Cartel} < x_n^{S} < x_n^{CS} < x_n^{C} < x_n^{FC}.$$

Proof. The individual output orderings used in the comparative discussion follow from the explicit formulas above.

For $\delta = \frac{1}{2}$, the cartel and Cournot quantities are

$$x_i^{Cartel} = M/(2n+1) \qquad \text{and} \qquad x_i^{Cournot} = M/(n+2).$$

In the Cournot–Stackelberg case,

$$x_1^{CS} = \cdots = x_{n-1}^{CS} = 2M/(2n+3) \qquad \text{and} \qquad x_n^{CS} = 5M/[3(2n+3)].$$

In the failed-cartel case,

$$x_1^{FC} = \cdots = x_{n-1}^{FC} = 2M/(4n-1) \qquad \text{and} \qquad x_n^{FC} = (2n+1)M/[3(4n-1)].$$

We first compare the first participant. Since $M > 0$ and $n > 2$, all denominators are positive. Moreover,

$$\frac{M}{2n+1} < \frac{2M}{4n-1} \iff 4n-1 < 4n+2,$$

$$\frac{2M}{4n-1} < \frac{M}{n+2} \iff 2n+4 < 4n-1 \iff n > \frac{5}{2},$$

and

$$\frac{M}{n+2} < \frac{2M}{2n+3} \iff 2n+3 < 2n+4.$$

Thus, for every integer $n \geq 3$, one obtains $x_1^{Cartel} < x_1^{FC} < x_1^{C} < x_1^{CS}$. To compare the first mover in the canonical model with a Cournot–Stackelberg leader, use Proposition 4. Since $r_1 = 2/3$ and

$$\frac{1}{r_k} - \frac{1}{r_{k-1}} = \frac{1}{1+r_{k-1}} < 1,$$

we have, for every integer $m \geq 2$,

$$\frac{1}{r_m} = \frac{3}{2} + \sum_{k=2}^{m} \frac{1}{1+r_{k-1}} < \frac{3}{2} + (m-1) = m + \frac{1}{2}.$$

Consequently, there holds the inequality

$$r_m > \frac{2}{2m+1}, \qquad m \geq 2. \tag{2}$$

Taking $m = n-1$, which is at least two, and using that $t \mapsto t/(1+2t)$ is strictly increasing for $t \geq 0$, we obtain

$$\frac{x_1^S}{M} = \frac{r_{n-1}}{1+2r_{n-1}} > \frac{2/(2n-1)}{1+4/(2n-1)} = \frac{2}{2n+3} = \frac{x_1^{CS}}{M}.$$

This proves the complete first-firm ordering $x_1^{Cartel} < x_1^{FC} < x_1^{C} < x_1^{CS} < x_1^{S}$.

We next compare the last participant. Again, since $M > 0$ and $n > 2$, all comparisons reduce to comparisons of positive fractions. We have

$$\frac{M}{2n+1} < \frac{5M}{3(2n+3)} \iff 3(2n+3) < 5(2n+1) \iff n > 1,$$

$$\frac{5M}{3(2n+3)} < \frac{M}{n+2} \iff 5(n+2) < 3(2n+3) \iff n > 1,$$

and

$$\frac{M}{n+2} < \frac{(2n+1)M}{3(4n-1)} \iff 3(4n-1) < (2n+1)(n+2) \iff 2n^2 - 7n + 5 > 0.$$

Since $2n^2 - 7n + 5 = (2n-5)(n-1)$, this polynomial is positive for every integer $n \geq 3$.

Therefore, for the non-canonical structures,

$$x_n^{Cartel} < x_n^{CS} < x_n^{C} < x_n^{FC}.$$

It remains to place the canonical last mover between the cartel and Cournot–Stackelberg quantities. Applying (2) with $m = n$ gives

$$x_n^{S} = \frac{Mr_n}{2} > \frac{M}{2n+1} = x_n^{Cartel}.$$

For the upper bound, Proposition 5 gives $Q_n^S > Q_n^{CS}$ for every $n \geq 3$. Thus

$$r_n = 1 - \frac{Q_n^S}{M} < 1 - \frac{Q_n^{CS}}{M} = \frac{10}{3(2n+3)},$$

and hence

$$x_n^{S} = \frac{Mr_n}{2} < \frac{5M}{3(2n+3)} = x_n^{CS}.$$

Combining these inequalities with the non-canonical comparisons proves

$$x_n^{Cartel} < x_n^{S} < x_n^{CS} < x_n^{C} < x_n^{FC}.$$

□

The numerical values in Table 1 and in the triopoly appendix use $A = 30996$, $B = 1/20$, and $c = 1/40$, so that $M = 619920$ and $\delta = 1/2$. All quantities, profits, surplus values, and percentage changes are computed from the exact equilibrium quantities before rounding. Quantities, prices, profits, and surplus values are displayed to the nearest integer; percentage changes are displayed to two decimal places. Totals are rounded independently of their components, so displayed rounded components need not sum exactly to a displayed total. The accompanying script games_linear_tables.py reproduces all numerical table entries using exact rational arithmetic.

Table 1. Total quantities, market prices, and aggregate profits in the Cournot–Stackelberg, cartel, and failed-cartel models.

	Cournot–Stackelberg			**Cartel**			**Failed Cartel**		
n	Q_n	P_n	**Total Profit**	Q_n	P_n	**Total Profit**	Q_n	P_n	**Total Profit**
2	324,720	14,760	3,463,935,840	247,968	18,598	3,843,008,064	324,720	14,760	3,463,935,840
3	390,320	11,480	3,202,506,720	265,680	17,712	4,117,508,640	356,924	13,150	3,625,978,683
4	432,065	9393	2,884,902,748	275,520	17,220	4,270,008,960	371,952	12,398	3,714,907,795
5	460,966	7948	2,596,114,915	28,1782	16,907	4,367,054,618	380,653	11,963	3,770,264,144
6	482,160	6888	2,348,504,928	286,117	16,690	4,434,240,074	386,327	11,680	3,807,895,514
7	498,367	6078	2,138,698,259	289,296	16,531	4,483,509,408	390,320	11,480	3,835,100,640
8	511,162	5438	1,960,537,355	291,727	16,410	4,521,185,958	393,283	11,332	3,855,671,462
9	521,520	4920	1,808,218,080	293,646	16,314	4,550,930,602	395,568	11,218	3,871,765,267
10	530,077	4492	1,676,926,959	295,200	16,236	4,575,009,600	397,385	11,127	3,884,697,501

3.2. Aggregate Output and Price Effects

The explicit equilibrium formulas also allow a systematic comparison of the five market structures as the number of firms increases. The main distinction is between cooperation and competition. Cartel coordination restricts production, whereas informational hierarchy expands output. The Cournot, Cournot–Stackelberg, and failed-cartel structures generate intermediate outcomes between these two benchmark forms of market organization.

Proposition 7 (Ordering of aggregate equilibrium outputs)**.** *Let $\delta = 1/2$, $M > 0$, and let $n \geq 3$ be an integer. Then*

$$Q_n^{Cartel} < Q_n^{FC} < Q_n^{C} < Q_n^{CS} < Q_n^{S}.$$

For $n = 2$, the failed-cartel and Cournot–Stackelberg structures coincide, so $Q_2^{FC} = Q_2^{CS}$.

Proof. For $\delta = 1/2$, the aggregate outputs in the first four structures are

$$Q_n^{Cartel} = \frac{nM}{2n+1}, \qquad Q_n^{FC} = \frac{(8n-5)M}{3(4n-1)},$$

$$Q_n^{C} = \frac{nM}{n+2}, \qquad Q_n^{CS} = \frac{(6n-1)M}{3(2n+3)}.$$

Since $M > 0$ and all denominators are positive, direct subtraction gives

$$Q_n^{FC} - Q_n^{Cartel} = \frac{(n-1)(4n+5)M}{3(4n-1)(2n+1)} > 0,$$

$$Q_n^{C} - Q_n^{FC} = \frac{2(n-1)(2n-5)M}{3(n+2)(4n-1)} > 0$$

and

$$Q_n^{CS} - Q_n^{C} = \frac{2(n-1)M}{3(2n+3)(n+2)} > 0$$

for every integer $n \geq 3$.

The preceding inequalities give $Q_n^{Cartel} < Q_n^{FC} < Q_n^{C} < Q_n^{CS}$.

By Proposition 5, $Q_n^{CS} < Q_n^{S}$ for every $n \geq 3$.

Consequently, $Q_n^{Cartel} < Q_n^{FC} < Q_n^{C} < Q_n^{CS} < Q_n^{S}$.

Moreover, the Cournot–Stackelberg structure contains only one follower and is less sequential than the canonical hierarchy. The corresponding aggregate output comparison established there yields $Q_n^{S} > Q_n^{CS}$. Combining these inequalities proves

$$Q_n^{Cartel} < Q_n^{FC} < Q_n^{C} < Q_n^{CS} < Q_n^{S}.$$

Finally, substituting $n = 2$ into the formulas gives $Q_2^{FC} = Q_2^{CS} = \frac{11M}{21}$. □

Figure 1 illustrates this ordering for $n \geq 3$, together with the equality $Q_2^{FC} = Q_2^{CS}$ in the two-firm case. The cartel consistently produces the smallest aggregate output, whereas the canonical Stackelberg model generates the largest. The failed-cartel and Cournot–Stackelberg structures remain intermediate cases, reflecting different degrees of competition following cartel breakdown.

For every integer $n \geq 3$, the comparison of individual outputs reveals two distinct patterns. For the first participant, $x_1^{Cartel} < x_1^{FC} < x_1^{C} < x_1^{CS} < x_1^{S}$, whereas for the last participant, $x_n^{Cartel} < x_n^{S} < x_n^{CS} < x_n^{C} < x_n^{FC}$.

These inequalities, established in Section Ordering of Equilibrium Quantities, show that the strategic role of a firm determines how it benefits from market organization. Informational hierarchy favors the leading firms, whereas the failed-cartel regime provides the greatest advantage to the deviating participant.

Since $P(Q) = A - BQ$ with $B > 0$, the inverse demand function is strictly decreasing. Therefore, the preceding proposition immediately implies that, for every integer $n \geq 3$,

$$P_n^{S} < P_n^{CS} < P_n^{C} < P_n^{FC} < P_n^{Cartel}.$$

For $n = 2$, the equality $Q_2^{FC} = Q_2^{CS}$ implies $P_2^{FC} = P_2^{CS}$.

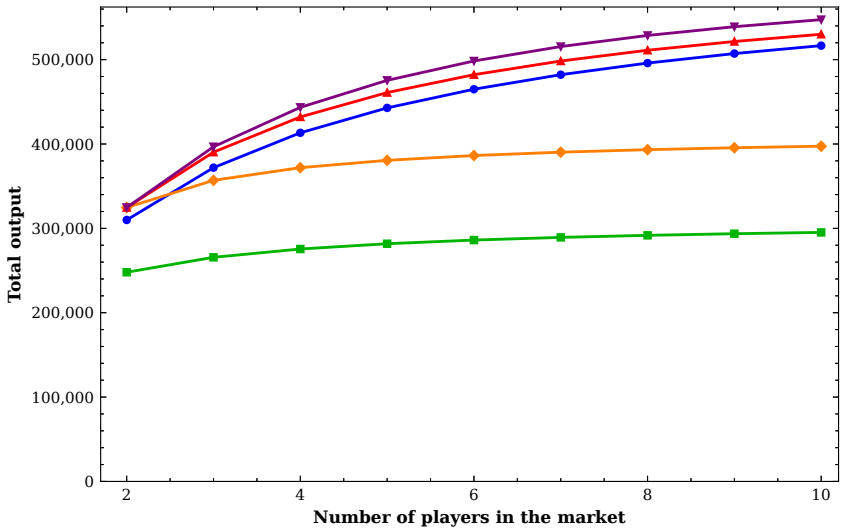


Figure 1. Total output as a function of the number of players in the market. Blue: Cournot. Green: cartel. Orange: failed cartel, where $(n-1)$ firms continue to cooperate and one firm deviates. Red: Cournot–Stackelberg. Purple: canonical Stackelberg.

Figure 2 therefore illustrates the same competitive mechanism from the consumers' perspective. Stronger competition reduces prices, whereas stronger coordination increases them through output restriction.

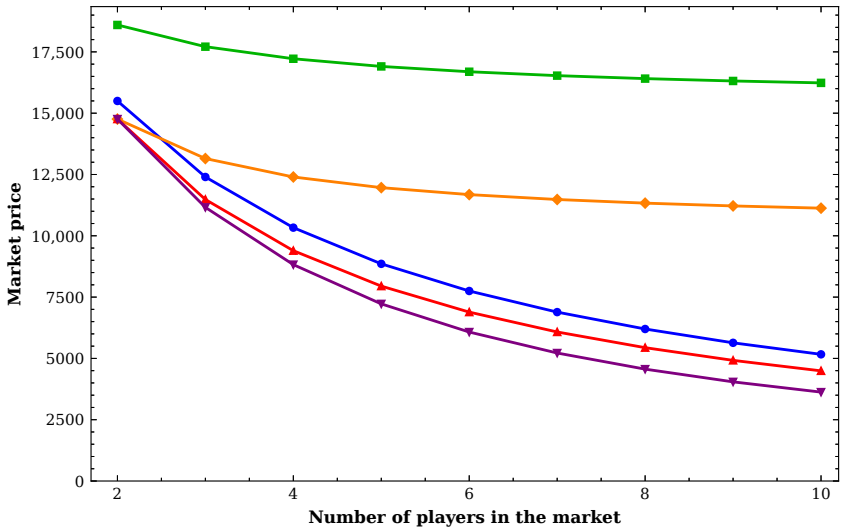


Figure 2. Market price as a function of the number of players in the market. Blue: Cournot. Green: cartel. Orange: failed cartel. Red: Cournot–Stackelberg. Purple: canonical Stackelberg.

For the values $n = 3, \ldots, 10$ reported in Table 1 and illustrated in Figure 3, the computed aggregate profits satisfy

$$\Pi_n^S < \Pi_n^{CS} < \Pi_n^C < \Pi_n^{FC} < \Pi_n^{Cartel}.$$

For $n = 2$, the Cournot–Stackelberg and failed-cartel structures coincide, and therefore $\Pi_2^{CS} = \Pi_2^{FC}$. These numerical comparisons illustrate the effect of coordination on aggregate producer profit for the selected parameter values. They do not, by themselves, establish the displayed ordering for every integer $n \geq 3$.

Figure 3 shows that, over the displayed range of n, aggregate producer profit increases with the degree of coordination. The full cartel generates the highest aggregate profit,

whereas the canonical Stackelberg model generates the lowest aggregate profit among the five reported structures. The failed-cartel structure again occupies an intermediate position, preserving part of the gains from cooperation despite the strategic deviation of one participant.

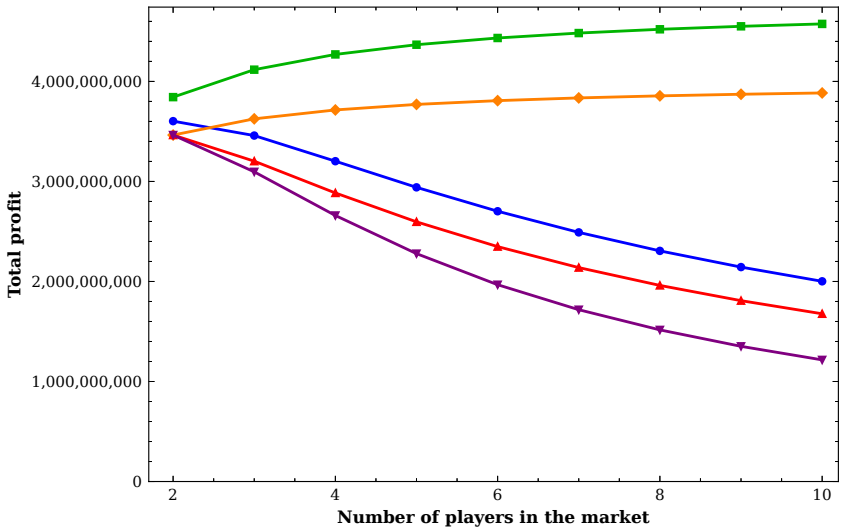


Figure 3. Total industry profit as a function of the number of players in the market. Blue: Cournot. Green: cartel. Orange: failed cartel. Red: Cournot–Stackelberg. Purple: canonical Stackelberg.

Overall, the five market structures illustrate the fundamental trade-off between competition and coordination. Informational hierarchy expands production, lowers prices, and improves consumer outcomes, while cooperation restricts output and increases producer profits. The failed-cartel model bridges these two mechanisms, providing a natural intermediate regime between full collusion and non-cooperative competition.

3.3. Asymptotic Behavior

The explicit formulas derived for the n-firm models make it possible to compare the large-market behavior of the five market structures as $n \to \infty$. Although the output of each firm converges to zero in the Cournot, Cournot–Stackelberg, canonical Stackelberg, and cartel models, the outsider in the failed-cartel model retains a positive limiting output. Aggregate production, prices, and market concentration approach model-dependent limiting values; some market structures have the same aggregate limit but differ in their firm-level output allocation and rates of convergence. These asymptotic outcomes reflect the persistent effects of informational hierarchy and cooperative behavior as the number of firms increases.

Figure 4 shows that the strict ordering of aggregate output remains unchanged for every integer $n \geq 3$. For $n = 2$, the failed-cartel and Cournot–Stackelberg outputs coincide. The canonical Stackelberg model consistently generates the highest production levels, whereas the cartel yields the lowest output. The Cournot, Cournot–Stackelberg, and failed-cartel regimes occupy intermediate positions. The Cournot and Cournot–Stackelberg aggregate outputs converge to the same limit, whereas the failed-cartel output converges to a lower limit.

Figure 5 illustrates how the sequences of equilibrium outcomes approach their corresponding large-market limits as the number of firms increases. The figure illustrates the rates at which the displayed sequences approach their respective asymptotic limits.

Since the normalized price satisfies $\frac{P_n}{A} = 1 - \frac{Q_n}{M}$, the comparison of aggregate output immediately determines the asymptotic price behavior.

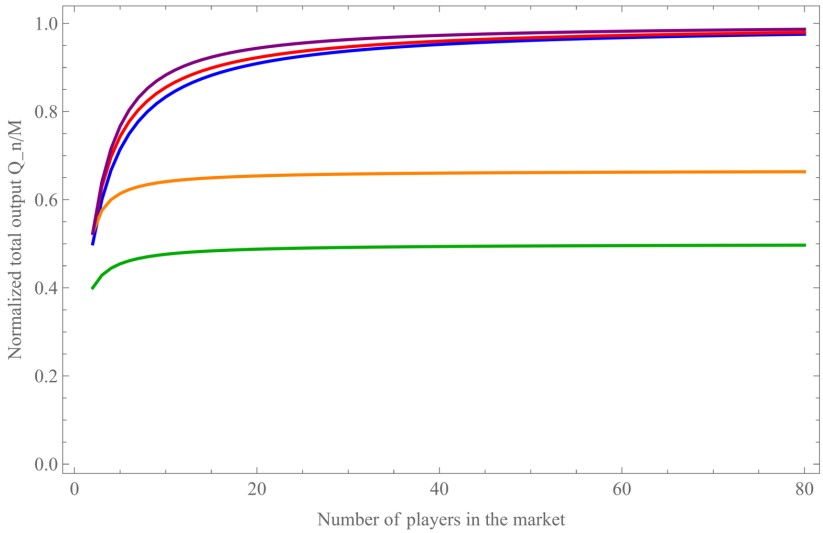


Figure 4. Normalized total output Q_n/M. Blue: Cournot. Green: cartel. Orange: failed cartel. Red: Cournot–Stackelberg. Purple: canonical Stackelberg.

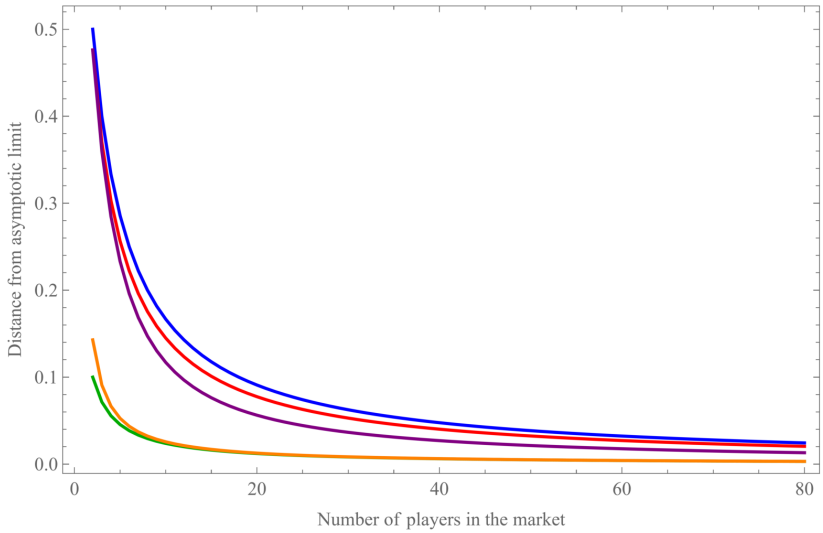


Figure 5. Absolute distance of normalized aggregate output from its corresponding limit, $|Q_n/M - \ell|$, where ℓ is 1, 1/2, or 2/3 according to the model. Blue: Cournot. Green: cartel. Orange: failed cartel. Red: Cournot–Stackelberg. Purple: canonical Stackelberg.

As illustrated in Figure 6, the price ordering is exactly the reverse of the output ordering. More competitive market structures generate lower equilibrium prices, whereas stronger coordination increases prices by restricting production.

The next comparison concerns the distribution of production among firms.

Figure 7 compares the asymmetry of individual equilibrium quantities. The Cournot and cartel models remain perfectly symmetric. The Cournot–Stackelberg model exhibits a moderate leader–follower asymmetry, while the canonical Stackelberg ratio approaches a finite value. By contrast, in the failed-cartel model the outsider's output has a positive limit whereas the output of each cooperating firm converges to zero. Consequently, the failed-cartel output-asymmetry ratio diverges as $n \to \infty$.

Finally, we examine market concentration using the Herfindahl–Hirschman index

$$HHI_n = \sum_{i=1}^{n} \left(\frac{x_i}{Q_n} \right)^2.$$

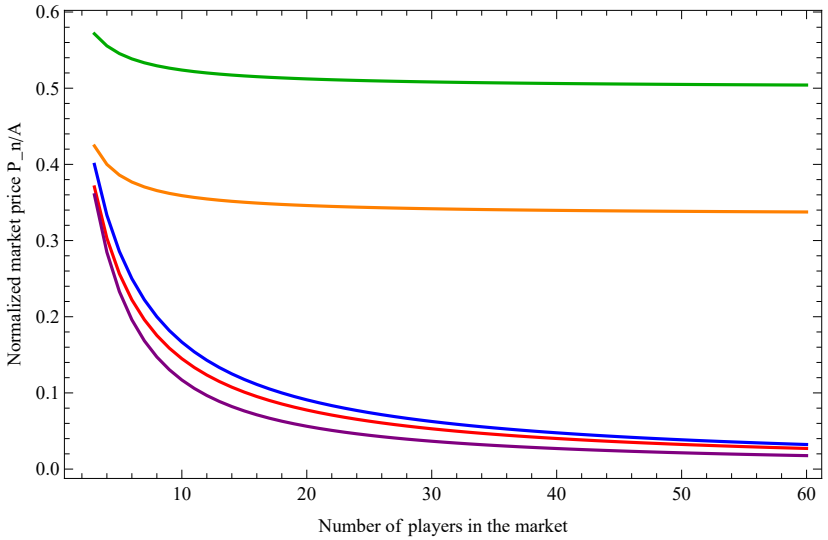


Figure 6. Normalized market price P_n/A. Blue: Cournot. Green: cartel. Orange: failed cartel. Red: Cournot–Stackelberg. Purple: canonical Stackelberg.

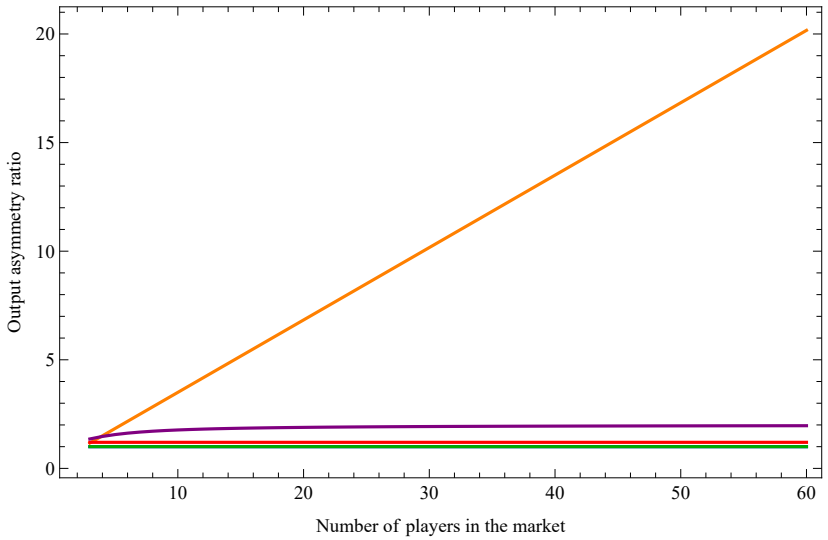


Figure 7. Output asymmetry ratio $\max_i x_i / \min_i x_i$. Green: cartel. Orange: failed cartel. Red: Cournot–Stackelberg. Purple: canonical Stackelberg.

Figure 8 shows that the symmetric Cournot and cartel structures become asymptotically unconcentrated, with $HHI_n \to 0$. The Cournot–Stackelberg and canonical Stackelberg models are asymmetric at the firm level, but their concentration indices also converge to zero because every individual market share vanishes. The failed-cartel model behaves differently: the outsider retains the limiting market share $1/4$, and consequently

$$\lim_{n\to\infty} HHI_n^{FC} = \frac{1}{16}.$$

Indeed, the market share of each coalition member is $\frac{x_i^{FC}}{Q_n^{FC}} = \frac{6}{8n-5}$ for $i = 1, \ldots, n-1$, whereas the outsider's market share is $\frac{x_n^{FC}}{Q_n^{FC}} = \frac{2n+1}{8n-5}$.

Therefore,

$$HHI_n^{FC} = (n-1)\left(\frac{6}{8n-5}\right)^2 + \left(\frac{2n+1}{8n-5}\right)^2 = \frac{4n^2+40n-35}{(8n-5)^2} \longrightarrow \frac{1}{16}.$$

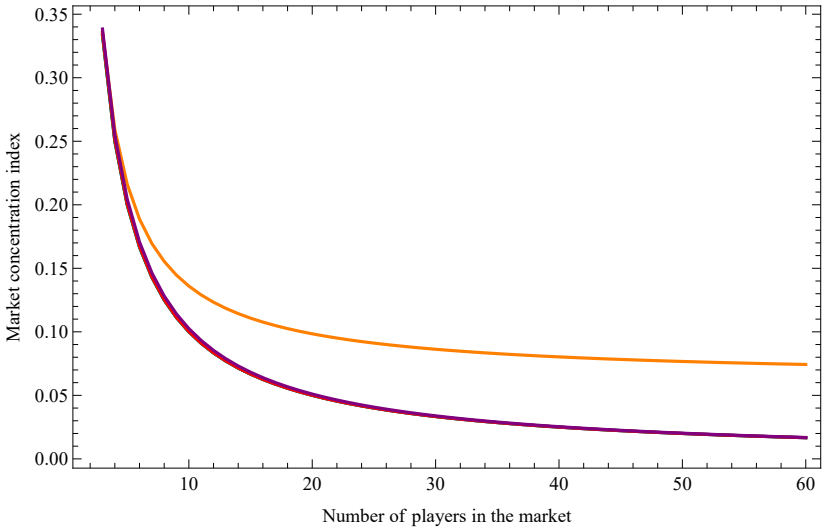


Figure 8. Market concentration index HHI_n. Green: cartel. Orange: failed cartel. Red: Cournot–Stackelberg. Purple: canonical Stackelberg.

Thus, only the failed-cartel structure exhibits persistent positive market concentration as the number of firms increases.

These failed-cartel limits are conditional on the maintained partition in which exactly $n-1$ firms remain coordinated and exactly one firm operates independently for every n. They do not describe the limiting size of an internally or externally stable cartel and should not be interpreted as a general prediction of endogenous cartel formation. Under the imposed partition, the outsider retains the positive limiting output $M/6$ and the market share $1/4$, while the output of each individual coalition member converges to zero.

3.4. Asymptotic Market Shares and Output Ratios

The explicit equilibrium formulas also permit an asymptotic comparison of the considered market structures. The following results summarize their large-market behavior as $n \to \infty$.

Proposition 8 (Canonical Stackelberg asymptotics)**.** *Let $\delta = 1/2$, and let (r_n) be the residual sequence from Proposition 4. Then*

$$\lim_{n\to\infty} nr_n = 1, \qquad \lim_{n\to\infty} Q_n^S = M, \qquad \lim_{n\to\infty} nx_1^S = M, \qquad \lim_{n\to\infty} nx_n^S = \frac{M}{2}, \qquad \lim_{n\to\infty} \frac{x_1^S}{x_n^S} = 2.$$

Proof. For $\delta = 1/2$, the recursion becomes $r_k = \frac{r_{k-1}(1+r_{k-1})}{1+2r_{k-1}}$ and $\frac{1}{r_k} - \frac{1}{r_{k-1}} = \frac{1}{1+r_{k-1}}$.

By Proposition 4, $r_k \to 0$. Telescoping and dividing by n yield

$$\frac{1}{nr_n} = \frac{1}{n} + \frac{1}{n}\sum_{k=1}^{n} \frac{1}{1+r_{k-1}} \longrightarrow 1$$

by Cesàro convergence. Hence, $nr_n \to 1$. Also, $nr_{n-1} \to 1$, because $(n-1)r_{n-1} \to 1$ and $n/(n-1) \to 1$. The equilibrium formulas now imply

$$Q_n^S = M(1-r_n) \longrightarrow M, \qquad nx_1^S = \frac{Mnr_{n-1}}{1+2r_{n-1}} \longrightarrow M, \qquad nx_n^S = \frac{Mnr_n}{2} \longrightarrow \frac{M}{2}.$$

Dividing the last two positive limits proves $x_1^S/x_n^S \to 2$. □

Theorem 1 (Asymptotic aggregate output)**.** *The aggregate equilibrium outputs satisfy*

$$\lim_{n\to\infty} Q_n^C = M,\ \lim_{n\to\infty} Q_n^{CS} = M,\ \lim_{n\to\infty} Q_n^S = M,\ \lim_{n\to\infty} Q_n^{FC} = \frac{2M}{3},\ \lim_{n\to\infty} Q_n^{Cartel} = \frac{M}{2}.$$

Proof. For $\delta = 1/2$, the explicit formulas are

$$Q_n^C = \frac{nM}{n+2}, \qquad Q_n^{CS} = \frac{(6n-1)M}{3(2n+3)}, \qquad Q_n^{FC} = \frac{(8n-5)M}{3(4n-1)}, \qquad Q_n^{Cartel} = \frac{nM}{2n+1}.$$

Dividing the numerator and denominator of each expression by n gives the stated limits for the Cournot, Cournot–Stackelberg, failed-cartel, and cartel models. By the preceding proposition, $Q_n^S = M(1 - r_n)$, where $r_n \to 0$. Consequently, $Q_n^S \to M$. □

Corollary 1. *The limiting aggregate outputs satisfy* $Q^{Cartel} < Q^{FC} < Q^C = Q^{CS} = Q^S$.

Proof. The preceding theorem gives the five limiting values $M/2$, $2M/3$, and M. Since $M > 0$, one has $M/2 < 2M/3 < M$, which proves the asserted ordering. □

Proposition 9 (Relative aggregate output ratios)**.** *The limiting aggregate output ratios satisfy*

$$\lim_{n\to\infty} \frac{Q_n^{CS}}{Q_n^C} = \lim_{n\to\infty} \frac{Q_n^{CS}}{Q_n^S} = 1, \qquad \lim_{n\to\infty} \frac{Q_n^{CS}}{Q_n^{FC}} = \frac{3}{2}, \qquad \lim_{n\to\infty} \frac{Q_n^{CS}}{Q_n^{Cartel}} = 2.$$

Proof. All denominators have strictly positive limits after normalization. Therefore, the quotient rule for limits and the preceding theorem yield $\frac{M}{M} = 1$, $\frac{M}{2M/3} = \frac{3}{2}$, and $\frac{M}{M/2} = 2$.

These are precisely the four stated ratios. □

Proposition 10 (Asymptotic first-firm outputs)**.** *The limiting leader-output ratios satisfy*

$$\lim_{n\to\infty} \frac{x_1^{CS}}{x_1^C} = \lim_{n\to\infty} \frac{x_1^{CS}}{x_1^S} = 1, \qquad \lim_{n\to\infty} \frac{x_1^{CS}}{x_1^{FC}} = 2, \qquad \lim_{n\to\infty} \frac{x_1^{CS}}{x_1^{Cartel}} = 2.$$

Proof. For the first firm, $\lim_{n\to\infty} x_1^C = \frac{M}{n+2}$, $\lim_{n\to\infty} x_1^{CS} = \frac{2M}{2n+3}$, $\lim_{n\to\infty} x_1^{FC} = \frac{2M}{4n-1}$, and $\lim_{n\to\infty} x_1^{Cartel} = \frac{M}{2n+1}$.

Thus $\lim_{n\to\infty} nx_1^C = M$, $\lim_{n\to\infty} nx_1^{CS} = M$, $\lim_{n\to\infty} nx_1^{FC} = M/2$, and $\lim_{n\to\infty} nx_1^{Cartel} = M/2$.

By the canonical Stackelberg recursion derived above, we get

$$\lim_{n\to\infty} nx_1^S = M, \qquad \lim_{n\to\infty} nx_n^S = \frac{M}{2}.$$

Dividing the corresponding normalized limits proves all four ratios. □

Proposition 11 (Asymptotic last-firm outputs)**.** *The limiting follower-output ratios satisfy*

$$\lim_{n\to\infty} \frac{x_n^{CS}}{x_n^C} = \frac{5}{6}, \qquad \lim_{n\to\infty} \frac{x_n^{CS}}{x_n^S} = \frac{5}{3}, \qquad \lim_{n\to\infty} \frac{x_n^{CS}}{x_n^{FC}} = 0, \qquad \lim_{n\to\infty} \frac{x_n^{CS}}{x_n^{Cartel}} = \frac{5}{3}.$$

Proof. The last-firm outputs satisfy

$$x_n^C = \frac{M}{n+2}, \quad x_n^{CS} = \frac{5M}{3(2n+3)}, \quad x_n^{FC} = \frac{(2n+1)M}{3(4n-1)}, \quad x_n^{Cartel} = \frac{M}{2n+1}.$$

Consequently,

$$\lim_{n\to\infty} nx_n^C = M, \quad \lim_{n\to\infty} nx_n^{CS} = \frac{5M}{6}, \quad \lim_{n\to\infty} x_n^{FC} = \frac{M}{6}, \quad \lim_{n\to\infty} nx_n^{Cartel} = \frac{M}{2}.$$

The canonical Stackelberg recursion gives $nx_n^S \to M/2$.

Therefore, $\lim_{n\to\infty} \frac{x_n^{CS}}{x_n^S} = \frac{5M/6}{M/2} = \frac{5}{3}$.

The first, second, and fourth ratios now follow by division. Since $\lim_{n\to\infty} x_n^{CS} = 0$, whereas $\lim_{n\to\infty} x_n^{FC} = M/6 > 0$, the third ratio tends to zero. □

Proposition 12 (Asymptotic first-to-last-firm output ratios)**.** *The first-to-last-firm output ratios satisfy* $\frac{x_1^C}{x_n^C} = \frac{x_1^{Cartel}}{x_n^{Cartel}} = 1$ *for every* n.

Moreover, $\frac{x_1^{CS}}{x_n^{CS}} = \frac{6}{5}$, $\lim_{n\to\infty} \frac{x_1^S}{x_n^S} = 2$, *and* $\lim_{n\to\infty} \frac{x_1^{FC}}{x_n^{FC}} = 0$.

Equivalently, $\lim_{n\to\infty} \frac{x_n^{FC}}{x_1^{FC}} = +\infty$.

Proof. In the Cournot and cartel models, symmetry gives $\frac{x_1^C}{x_n^C} = \frac{x_1^{Cartel}}{x_n^{Cartel}} = 1$. For the Cournot–Stackelberg model, the explicit formulas give

$$\frac{x_1^{CS}}{x_n^{CS}} = \frac{2M/(2n+3)}{5M/[3(2n+3)]} = \frac{6}{5}.$$

The canonical Stackelberg limits $\lim_{n\to\infty} nx_1^S = M$ and $\lim_{n\to\infty} nx_n^S = M/2$ imply

$$\lim_{n\to\infty} \frac{x_1^S}{x_n^S} = 2.$$

Finally, in the failed-cartel model,

$$\frac{x_n^{FC}}{x_1^{FC}} = \frac{(2n+1)M/[3(4n-1)]}{2M/(4n-1)} = \frac{2n+1}{6}.$$

Thus, the outsider-to-coalition-member output ratio grows linearly and there hold $\lim_{n\to\infty} \frac{x_n^{FC}}{x_1^{FC}} = +\infty$, $\lim_{n\to\infty} \frac{x_1^{FC}}{x_n^{FC}} = 0$. □

Theorem 2 (Asymptotic classification of the market structures)**.** *As* $n \to \infty$, *there holds*

$$Q^{Cartel} < Q^{FC} < Q^C = Q^{CS} = Q^S,$$

whereas

$$1 = \lim_{n\to\infty} \frac{x_1^C}{x_n^C} < \lim_{n\to\infty} \frac{x_1^{CS}}{x_n^{CS}} = \frac{6}{5} < \lim_{n\to\infty} \frac{x_1^S}{x_n^S} = 2.$$

Consequently, the Cournot, Cournot–Stackelberg and canonical Stackelberg models become asymptotically equivalent with respect to aggregate production, but remain fundamentally different in terms of the distribution of production among firms.

Proof. The aggregate output ordering is the corollary proved above. The three asymmetry limits are given by the preceding proposition, and $1 < 6/5 < 2$. Combining these two conclusions proves the classification. □

This bounded comparison does not include the failed-cartel asymmetry ratio. In the failed-cartel model, the outsider-to-coalition-member ratio equals $(2n+1)/6$ and therefore diverges. Hence, the failed-cartel structure, rather than the canonical Stackelberg structure, exhibits the greatest output inequality asymptotically.

https://doi.org/10.3390/g17050056

Theorem 3 (Asymptotic market shares). *In the Cournot–Stackelberg model, the aggregate market share of Firms* $1, \ldots, n-1$ *converges to one, i.e.,* $\lim_{n\to\infty} \frac{\sum_{i=1}^{n-1} x_i^{CS}}{Q_n^{CS}} = 1$.

In the failed-cartel model, the outsider's output satisfies $\lim_{n\to\infty} x_n^{FC} = \frac{M}{6}$, *and the limiting market shares of the cooperating firms and the outsider are* $\lim_{n\to\infty} \frac{\sum_{i=1}^{n-1} x_i^{FC}}{Q_n^{FC}} = \frac{3}{4}$ *and* $\lim_{n\to\infty} \frac{x_n^{FC}}{Q_n^{FC}} = \frac{1}{4}$, *respectively.*

Proof. For the Cournot–Stackelberg model, $\frac{\sum_{i=1}^{n-1} x_i^{CS}}{Q_n^{CS}} = 1 - \frac{x_n^{CS}}{Q_n^{CS}}$. Since $\lim_{n\to\infty} x_n^{CS} = 0$ and $\lim_{n\to\infty} Q_n^{CS} = M > 0$, the ratio converges to one.

In the failed-cartel model, $\lim_{n\to\infty} x_n^{FC} = \lim_{n\to\infty} \frac{(2n+1)M}{3(4n-1)} = \frac{M}{6}$, whereas

$$\lim_{n\to\infty} \sum_{i=1}^{n-1} x_i^{FC} = \lim_{n\to\infty} (n-1)\frac{2M}{4n-1} = \frac{M}{2}.$$

Since $\lim_{n\to\infty} Q_n^{FC} = 2M/3$, it follows that $\lim_{n\to\infty} \frac{\sum_{i=1}^{n-1} x_i^{FC}}{Q_n^{FC}} = \frac{M/2}{2M/3} = \frac{3}{4}$ and $\lim_{n\to\infty} \frac{x_n^{FC}}{Q_n^{FC}} = \frac{M/6}{2M/3} = \frac{1}{4}$. □

The previous results show that the limiting ratios are structural characteristics of the considered market organizations. For every fixed position in the hierarchy covered by the formulas below, the corresponding individual outputs vanish in the Cournot, Cournot–Stackelberg, canonical Stackelberg, and cartel models, whereas the outsider in the failed-cartel model retains the positive limiting output $M/6$. In particular, the hierarchical Stackelberg and Cournot–Stackelberg structures asymptotically attain the same aggregate production, whereas the failed-cartel and cartel models preserve only two thirds and one half of this level, respectively. At the firm level, informational hierarchy generates persistent leader advantages, while the failed-cartel model assigns one quarter of its limiting aggregate output to the outsider.

4. Fixed-Point Methods for Nonlinear Stationary Systems

Nonlinear demand may prevent an explicit solution of the stationary equations; examples include the isoelastic models in (Andaluz et al., 2020; Kabaivanov et al., 2022). Implicit stationary responses may nevertheless be available (Dzhabarova et al., 2020). The following formulation supplies conditional uniqueness and numerical convergence; the optimization conditions needed for an equilibrium are stated separately.

The general product-space theorem, its notation, and the optional symmetry condition are collected in Appendix B. Here we state the operator and the conditional convergence criterion for the economic stationary system. The general Cournot–Stackelberg criterion remains conditional. Section 5 separately verifies a scalar contraction for the partial-cartel game.

Equilibrium Characterization of the Cournot–Stackelberg Model

Firms 1 and 2 choose quantities simultaneously while anticipating Firm 3's response. We use a single-valued stationary-point formulation; broader set-valued implicit equilibrium problems are studied in (Zangenehmehr & Farajzadeh, 2022).

Let $X_i \subseteq [0, \infty)$ be nonempty closed strategy sets. Assume that the payoffs have the derivatives used below and that, for each admissible (x, y), the follower's stationary equation has a unique solution $z = b_3(x, y) \in X_3$, with b_3 differentiable. Define

$$\Phi_i(x, y) = \Pi_i\big(x, y, b_3(x, y)\big), \qquad i = 1, 2.$$

The stationary system is

$$\partial_x \Phi_1(x,y) = 0, \qquad \partial_y \Phi_2(x,y) = 0, \qquad \partial_z \Pi_3(x,y,z) = 0. \tag{3}$$

For fixed positive scaling coefficients λ_i, define

$$F_3^{\lambda_3}(x,y,z) = z + \lambda_3 \partial_z \Pi_3(x,y,z), \tag{4}$$

$$F_1^{\lambda_1}(x,y) = x + \lambda_1 \partial_x \Phi_1(x,y), \qquad F_2^{\lambda_2}(x,y) = y + \lambda_2 \partial_y \Phi_2(x,y).$$

The coefficients convert marginal profit into quantity. They affect contractivity, but not the stationary solutions. The maps $F_i^{\lambda_i}$ are numerical maps, not best-response functions. In particular, b_3 is a best response only when it maximizes the follower's payoff.

Set

$$\mathcal{G}_2(x,y,z) = \big(F_1^{\lambda_1}(x,y), F_2^{\lambda_2}(x,y), b_3(x,y)\big). \tag{5}$$

For $\mathbf{u} = (x,y,z)$ and $\mathbf{v} = (u,v,w)$, use the sum metric $M_1(\mathbf{u},\mathbf{v}) = |x-u| + |y-v| + |z-w|$. The relation to the general product-space construction is recorded in Appendix B.

Theorem 4 (Stationary characterization)**.** *A feasible triple solves* (3) *if and only if it is a fixed point of* $\mathcal{G}_2$*.*

Proof. Since $\lambda_1, \lambda_2 > 0$, the first two fixed-point equations are equivalent to the leaders' stationary equations. By the assumed uniqueness of the follower's stationary solution, the third equation is equivalent to $z = b_3(x,y)$. These equivalences hold in both directions. □

Theorem 5 (Invariant-region contraction)**.** *Let* $D \subseteq X_1 \times X_2 \times X_3$ *be nonempty and closed. Assume* $\mathcal{G}_2(D) \subseteq D$ *and, for some* $\alpha \in (0,1)$*,*

$$M_1(\mathcal{G}_2\mathbf{u}, \mathcal{G}_2\mathbf{v}) \le \alpha M_1(\mathbf{u},\mathbf{v}), \qquad \mathbf{u},\mathbf{v} \in D, \tag{6}$$

where M_1 *is the sum metric with* $d_i(a,b) = |a-b|$*. Then* D *contains a unique fixed point* $\mathbf{u}^* = (\xi,\eta,\theta)$*, equivalently a unique stationary triple. For every* $\mathbf{u}_0 \in D$*, the iteration* $\mathbf{u}_{m+1} = \mathcal{G}_2\mathbf{u}_m$ *converges to* $\mathbf{u}^*$*, with estimates* (A2) *for* $\kappa = \alpha$*.*

Proof. The space (D, M_1) is complete. Apply Theorem A1 to $T = \mathcal{G}_2|_D$, with $k_1 = \alpha$ and $k_2 = k_3 = 0$, and use Theorem 4. □

The iteration is a numerical procedure, not an economic adjustment process. An equilibrium interpretation additionally requires the optimality conditions below. Contractivity is sufficient for uniqueness within D and numerical convergence, but is not necessary for equilibrium existence; structural existence conditions for Cournot games are given in (Novshek, 1985).

Definition 1 (Strict local Cournot–Stackelberg equilibrium)**.** *Let*

$$\Phi_1(x,y) = \Pi_1\big(x,y,b_3(x,y)\big), \qquad \Phi_2(x,y) = \Pi_2\big(x,y,b_3(x,y)\big)$$

be the reduced payoff functions of Firms 1 and 2. A feasible triple (x^*,y^*,z^*)*, with* $z^* = b_3(x^*,y^*)$*, is called a strict local Cournot–Stackelberg equilibrium if there exist neighborhoods* U_1*,* U_2*, and* U_3 *of* x^**,* y^**, and* z^**, respectively, such that*

- $\Phi_1(x^*,y^*) > \Phi_1(x,y^*)$ *for every* $x \in U_1 \cap X_1$ *with* $x \neq x^*$*,*
- $\Phi_2(x^*,y^*) > \Phi_2(x^*,y)$ *for every* $y \in U_2 \cap X_2$ *with* $y \neq y^*$*,*
- $\Pi_3(x^*,y^*,z^*) > \Pi_3(x^*,y^*,z)$ *for every* $z \in U_3 \cap X_3$ *with* $z \neq z^*$*.*

The preceding definition concerns unilateral deviations in neighborhoods of the stationary quantities. It does not, by itself, imply global optimality or subgame perfection. A global Cournot–Stackelberg equilibrium requires the corresponding maximization inequalities on the entire strategy sets. In addition, a subgame-perfect interpretation requires the follower's continuation response to be optimal after every admissible pair of leader quantities, and not only at the equilibrium pair (x^*, y^*).

Proposition 13. *Let (ξ, η, θ) be an interior fixed point of $\mathcal{G}_2$. Assume that the relevant payoff functions are twice continuously differentiable in neighborhoods of the corresponding quantities and that*

$$\frac{\partial^2 \Phi_1}{\partial x^2}(\xi, \eta) < 0, \qquad \frac{\partial^2 \Phi_2}{\partial y^2}(\xi, \eta) < 0, \qquad \frac{\partial^2 \Pi_3}{\partial z^2}(\xi, \eta, \theta) < 0.$$

Then (ξ, η, θ) is a strict local Cournot–Stackelberg equilibrium in the sense of the preceding definition.

If, in addition, $\Pi_3(x, y, \cdot)$ is concave on X_3 for every admissible (x, y), and $\Phi_1(\cdot, y)$ and $\Phi_2(x, \cdot)$ are concave on X_1 and X_2, respectively, for every admissible choice of the other leader, then the stationary choices are global maximizers. In this case, the triple is a global Cournot–Stackelberg equilibrium.

Proof. By Theorem 4, the fixed point (ξ, η, θ) satisfies

$$\frac{\partial \Phi_1}{\partial x}(\xi, \eta) = 0, \qquad \frac{\partial \Phi_2}{\partial y}(\xi, \eta) = 0, \qquad \frac{\partial \Pi_3}{\partial z}(\xi, \eta, \theta) = 0.$$

The strict inequalities for the corresponding second derivatives and the one-dimensional second-derivative test show that ξ, η, and θ are strict local maximizers of their respective optimization problems. Hence, (ξ, η, θ) is a strict local Cournot–Stackelberg equilibrium.

Under the additional concavity assumptions, each stationary quantity is a global maximizer on the corresponding strategy set. Therefore, the triple is a global Cournot–Stackelberg equilibrium. □

5. A Verified Isoelastic Partial-Cartel Example

5.1. Game and Equilibrium

We retain the isoelastic demand and linear costs used in the original nonlinear specification, but choose the admissible elasticity $\mu = 2$:

$$P(Q) = Q^{-1/2}, \qquad C_i(q_i) = cq_i, \qquad c > 0.$$

Quantities are nonnegative. At the zero-output profile, profits are defined by continuous extension as zero; although price is singular there, each firm's revenue is bounded by $\sqrt{Q}$. The model is a special case of the isoelastic demand family discussed in (Andaluz et al., 2020; Kabaivanov et al., 2022). It is a nonlinear illustration with linear costs, separate from the linear-demand, quadratic-cost comparisons above.

Firms 1 and 2 form the coalition. They first choose aggregate output $s = x + y$, with the maintained equal allocation $x = y = s/2$; Firm 3 observes s and then chooses z. The follower maximizes $z(s + z)^{-1/2} - cz$, and the coalition maximizes $s(s + b(s))^{-1/2} - cs$, where $b(s)$ is the follower's global best response. Linear costs make allocation within the coalition indeterminate without the equal-sharing rule. Coalition membership is exogenous.

Proposition 14 (Isoelastic partial-cartel equilibrium)**.** *For every $c > 0$, the follower has a unique best response for every $s \geq 0$. The coalition has a unique optimal aggregate output. With the imposed equal split, the resulting subgame-perfect outcome is*

$$u^* = \frac{3+\sqrt{3}}{6}, \qquad Q^{FC} = \frac{(u^*)^2}{c^2}, \qquad P^{FC} = \frac{c}{u^*},$$
$$x^{FC} = y^{FC} = \frac{(u^*)^2(2u^*-1)}{2c^2}, \; z^{FC} = \frac{2(u^*)^2(1-u^*)}{c^2}. \tag{7}$$

Here, $u = c\sqrt{Q}$. These quantities are global optimizers in the specified sequential game, not merely solutions of its stationary equations.

The proof, including boundary choices and the benchmarks below, is given in Appendix B.3.

5.2. A Contractive Numerical Iteration

The coalition's stationary equation reduces to $6u^2 - 6u + 1 = 0$. Define

$$T(u) = 1 - \frac{1}{6u}, \qquad D = \left[\frac{3}{4}, \frac{5}{6}\right]. \tag{8}$$

The interval is nonempty and closed, and

$$T(D) = \left[\frac{7}{9}, \frac{4}{5}\right] \subset D, \qquad 0 < T'(u) = \frac{1}{6u^2} \le \frac{8}{27} =: \kappa < 1.$$

Consequently, Theorem A1, with $k_1 = \kappa$ and $k_2 = k_3 = 0$, proves that the Picard iteration

$$u_{m+1} = T(u_m) \tag{9}$$

converges to u^* from every $u_0 \in D$. In particular,

$$|u_m - u^*| \le \frac{8}{19}|u_m - u_{m-1}|, \qquad m \ge 1.$$

The corresponding quantities are recovered by

$$(x_m, y_m, z_m) = H(u_m) = \left(\frac{u_m^2(2u_m-1)}{2c^2}, \frac{u_m^2(2u_m-1)}{2c^2}, \frac{2u_m^2(1-u_m)}{c^2}\right).$$

For every $u \in D$, these quantities are positive, and the follower component equals its global best response to the coalition quantity. Thus, $H(D)$ is a compact feasible set preserved by the induced update $H \circ T \circ H^{-1}$; H is injective because its aggregate output is u^2/c^2. Contractivity is proved in the scalar coordinate u, and is not asserted in the sum metric on arbitrary quantity triples.

Indeed, there holds $\|H(u) - H(v)\|_1 \le \frac{10}{3c^2}|u - v|$ for every $u, v \in D$, so the scalar error estimate also controls the error in quantities.

The interval D is used only to certify numerical convergence; it is not a lower-output restriction on the economic optimization problem. Global optimality is established independently over all nonnegative coalition quantities. The algorithm is a one-dimensional numerical solver, not a best-response dynamic and not the three-dimensional operator $\mathcal{G}_2$. The closed form (7) provides an independent check of the computed limit; no claim is made that iteration is necessary to solve this special example.

5.3. Reproducible Numerical Comparison

For $c = 0.09$, Table 2 compares the partial cartel with simultaneous Cournot competition and a full cartel. Equal allocation is imposed for the full cartel, whose aggregate output is uniquely optimal. The benchmark outputs are

$$Q^{Cartel} = \frac{1}{4c^2}, \qquad Q^{C} = \frac{25}{36c^2}, \qquad Q^{Cartel} < Q^{FC} < Q^{C}.$$

The price ordering is reversed. Profits are $\pi_i = (P - c)q_i$.

Table 2. Verified isoelastic triopoly, $\mu = 2$, $c = 0.09$.

Model	$x = y$	z	Q	P	$\pi_1 = \pi_2$	π_3
Full cartel	10.288066	10.288066	30.864198	0.180000	0.925926	0.925926
Failed cartel	22.167701	32.455767	76.791169	0.114115	0.534584	0.782685
Cournot	28.577961	28.577961	85.733882	0.108000	0.514403	0.514403

Both coalition members earn more than under Cournot, while the outsider earns more than either member. Nevertheless, all three firms earn less than their equal full-cartel profit. These comparisons concern the specified continuation game and do not establish general membership stability or a nonlinear ranking across all five market structures.

The accompanying standard-library Python 3.12.14. script games_isoelastic_triopoly.py generates Tables 2 and 3 from 60-digit decimal calculations. It implements exactly (9), without projection or an implicit solver, and stops when $E_m \leq 10^{-12}$. Starting at $u_0 = 3/4$, this occurs at iteration 19. At $c = 0.09$, the resulting bound on the sum of the absolute quantity errors is below 4.12×10^{-10}. The script checks feasibility and the equilibrium identities and also checks convergence from the midpoint and the other endpoint of D; these checks supplement the analytical invariance and contraction proof. Figure 9 illustrates the convergence of the reconstructed quantities for $\mu = 2$, $c = 0.09$, and $u_0 = 3/4$.

Table 3. Scalar Picard iteration from $u_0 = 3/4$; $E_m = (8/19)|u_m - u_{m-1}|$.

m	u_m	E_m
0	0.750000000000	—
1	0.777777777778	1.170×10^{-2}
2	0.785714285714	3.342×10^{-3}
4	0.788461538462	2.454×10^{-4}
8	0.788674033149	1.267×10^{-6}
12	0.788675128917	6.531×10^{-9}
19	0.788675134594	6.477×10^{-13}

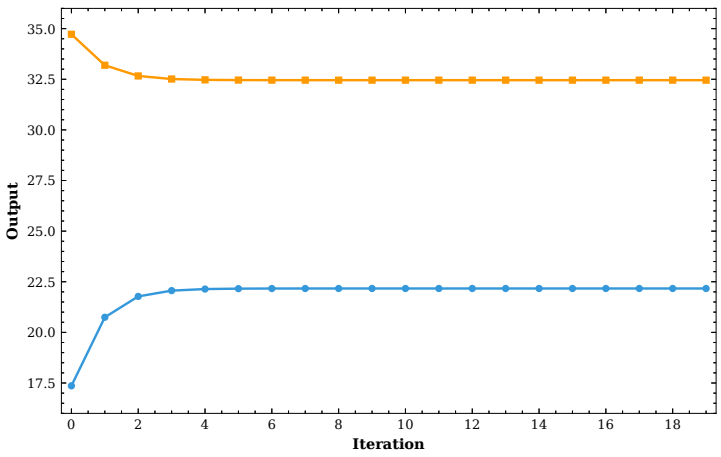


Figure 9. Verified isoelastic partial-cartel iteration with $\mu = 2$, $c = 0.09$, and $u_0 = 3/4$. Blue circles: each coalition member, $x_m = y_m$. Orange squares: the outsider, z_m. The quantities are reconstructed from the scalar Picard iteration; the paths are numerical approximations, not firms' adjustment behavior.

6. Discussion

The linear analysis separates two features of market organization: coordination of objectives and the order in which quantities are chosen. Under $P(Q) = A - BQ$, $C(x) = cx^2$, and $c/B = 1/2$, the aggregate output ordering for $n \geq 3$ is

$$Q_n^{Cartel} < Q_n^{FC} < Q_n^{C} < Q_n^{CS} < Q_n^{S},$$

with the reverse price ordering. The full cartel restricts output most strongly, while the fully sequential hierarchy generates the largest output. The comparison between the partial cartel and Cournot–Stackelberg is particularly informative because the coalition and the non-cooperative leaders face the same follower timing but optimize different objectives. The strict ordering is stated for $n \geq 3$: at $n = 2$, the canonical Stackelberg, Cournot–Stackelberg, and failed-cartel games all reduce to the same one-leader–one-follower game.

The partial-cartel outcome preserves some gains from coordination relative to Cournot in the reported numerical examples. The aggregate-profit comparisons for $n = 3, \ldots, 10$ are numerical results for the selected parameters, rather than a general profit-ordering theorem. Within the triopoly, the outsider earns more than either coalition member and more than a firm in the Cournot benchmark, but less than its equal share of full-cartel profit. These comparisons distinguish an advantage within the imposed post-cartel partition from an incentive to leave the original cartel. They do not establish that the partition would emerge from voluntary departure or remain stable when membership can change.

The asymptotic results also depend on the maintained market organization. Cournot, Cournot–Stackelberg, and canonical Stackelberg output converge to $M = A/B$, whereas full-cartel and failed-cartel output converge to $M/2$ and $2M/3$, respectively. In the failed-cartel sequence, exactly $n - 1$ firms remain coordinated for every n. The outsider retains output $M/6$ and a market share of $1/4$, while the coalition collectively supplies the remaining $3/4$. This is a limit conditional on an imposed partition, not a prediction about the size or composition of an endogenously stable cartel.

Aggregate convergence does not eliminate relative differences between firms. The canonical first-to-last-firm output ratio approaches 2, and the Cournot–Stackelberg leader-to-follower ratio equals $6/5$. In contrast, the failed-cartel outsider-to-member ratio equals $(2n + 1)/6$ and diverges. The firm-level Herfindahl–Hirschman index tends to $1/16$ in the failed-cartel model and to zero in the other four structures. These indices describe concentration of output across individual firms. They do not measure the independence of strategic decisions: a full cartel, for example, has equal firm shares and index $1/n$ while choosing output jointly.

The welfare calculations have a narrower scope. For the specified linear quadratic triopoly, both consumer surplus and total surplus follow the ordering canonical Stackelberg, Cournot–Stackelberg, Cournot, failed cartel, and full cartel, from highest to lowest. This agreement is verified for the example; it is not asserted for arbitrary numbers of firms, cost ratios, or nonlinear demand. In particular, an output comparison alone need not establish a total surplus comparison when production costs depend on the allocation of output across firms.

The general Cournot–Stackelberg fixed-point formulation remains conditional on invariance, contractivity, and optimality. The separate isoelastic partial-cartel example with $\mu = 2$ and linear costs verifies these requirements for a scalar numerical solver: the feasible invariant interval and contraction constant are explicit, and backward induction establishes global optimality. Its closed-form solution independently validates the iteration. This special example compares the partial cartel, full cartel, and Cournot; it does not establish nonlinear rankings for all five structures or validate the general three-dimensional operator.

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Heterogeneity also matters for the composition of a cartel. In an infinitely repeated price game with capacity constraints, Bos and Harrington (2010) endogenize cartel membership and show that, when firms are sufficiently patient, a stable cartel involving the largest firms exists, while firms with sufficiently small capacities remain outside every stable cartel. Their analysis provides a benchmark for extending the present imposed-partition comparisons to endogenous membership. Such an extension would require explicit participation and continuation incentives in addition to heterogeneous production technologies.

The main extensions are to endogenize coalition membership and post-breakdown timing, allow several outsiders and heterogeneous costs, and determine when the surviving coalition is stable against individual or coalitional deviations. Further nonlinear work should extend the verified scalar example to broader demand and cost specifications and to the remaining market structures. These extensions would connect the conditional quantity setting comparisons more closely to an economic account of cartel breakdown.

7. Conclusions

This paper compared five quantity setting market structures under common demand and cost assumptions. The linear analysis provides explicit equilibrium formulas and a backward induction recursion for the canonical hierarchy. For $c/B = 1/2$ and $n \geq 3$, it establishes aggregate output, price, and first- and last-firm output orderings. Profit and welfare calculations are reported separately for the stated numerical benchmarks, with welfare rankings confined to the illustrative triopoly.

The one-outsider partial cartel has a distinct large-market limit. Conditional on continued coordination by exactly $n - 1$ firms, aggregate output tends to $2M/3$, and the outsider retains output $M/6$ and a market share of $1/4$. The canonical hierarchy instead approaches aggregate output M, with scaled first- and last-mover quantities tending to M and $M/2$. These results describe outcomes under fixed timing and imposed coalition structures; they do not establish incentives for cartel departure or membership stability.

For nonlinear Cournot–Stackelberg models, the stationary conditions admit a conditional product-space formulation. Separately, the isoelastic triopoly with $\mu = 2$ and linear costs provides a verified partial-cartel example, with global optimization, an explicit invariant interval, a contraction factor $8/27$, and a reproducible scalar iteration. These results apply to the specified games and do not establish endogenous membership stability or general nonlinear welfare rankings.

Supplementary Materials: The following supporting information can be downloaded at https://www.mdpi.com/article/10.3390/g17050056/s1. The accompanying scripts games_linear_tables.py and games_isoelastic_triopoly.py reproduce the linear tables and the verified nonlinear example, respectively. The script generate_games_figures.py regenerates all ten figures.

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Appendix A. Linear Triopoly: Equilibrium and Welfare Comparisons

We will consider the illustrative example from (Badev et al., 2026). This example—although simplified—will present some ideas for market dynamics when initial agreements are not followed. We will illustrate first with just 3 players.

Example A1. *We consider a triopoly market with linear demand and quadratic costs. Let x, y, and z denote the output levels of the three firms. We use the linear inverse demand $P(Z) = A - BZ$ where $Z = x + y + z$, and the symmetric cost function $C(u) = Cu^2$.*

Consider the market in Example A1. Let $A = 30996$, $B = \frac{1}{20}$, and $C = \frac{1}{40}$. The payoff functions are

$$\begin{cases} \Pi_i = x_i\left(30996 - \frac{1}{20}\sum_{k=1}^{3} x_k\right) - \frac{1}{40}x_i^2 \\ \text{for } i = 1, 2, 3. \end{cases}$$

Appendix A.1. Cournot Benchmark

By Badev et al. (2026), the Cournot equilibrium is symmetric, with $x = y = z = 123984$. At this point, $\Pi_1 = \Pi_2 = \Pi_3 = 1152902419.20$, and the market price is $P(x, y, z) = 12398.4$.

Appendix A.2. Equilibrium in the Canonical Stackelberg Model

The equilibrium production levels are $x = 151200$, $y = 133920$, $z = 111600$.[1] The corresponding profits are $\Pi_1(x, y, z) = 1115856000$, $\Pi_2(x, y, z) = 1046183040$, and $\Pi_3(x, y, z) = 934092000$. The market price is $P(x, y, z) = 11160$.

Appendix A.3. Equilibrium in the Cournot–Stackelberg Model

In the Cournot–Stackelberg case, Firms 1 and 2 act in a Cournot-type reduced game while Firm 3 is the follower. For a fixed x and y, the follower solves

$$\frac{\partial \Pi_3}{\partial z}(x, y, z) = 30996 - \frac{1}{20}x - \frac{1}{20}y - \frac{3}{20}z = 0.$$

Hence the follower's response function is

$$z = b_3(x, y) = 206640 - \frac{1}{3}x - \frac{1}{3}y.$$

Firms 1 and 2 anticipate this response and maximize the reduced payoff functions $\Phi_i(x, y) = \Pi_i(x, y, b_3(x, y))$ for $i = 1, 2$.

Thus the first-order conditions of the reduced Cournot game are

$$\begin{cases} \frac{\partial \Phi_1}{\partial x}(x, y) = 20664 - \frac{7}{60}x - \frac{1}{30}y = 0, \\ \frac{\partial \Phi_2}{\partial y}(x, y) = 20664 - \frac{1}{30}x - \frac{7}{60}y = 0. \end{cases}$$

Solving this system gives $x = y = 137,760$. Substituting into the follower's response gives $z = 114,800$.

Therefore, the total output in the market is $Z = x + y + z = 390320$, market price is $P(x, y, z) = 11480$, the corresponding profits are $\Pi_1(x, y, z) = \Pi_2(x, y, z) = 1107039360$ and $\Pi_3(x, y, z) = 988428000$.

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Appendix A.4. Cartel Equilibrium

Suppose now that the three firms form a cartel and act as a monopolist. Since the firms are symmetric, the monopoly output is divided equally among them: $x = y = z = \frac{t}{3}$, where t denotes the total market production.

The cartel maximizes the total profit $\Pi(t) = tP(t) - 3C\left(\frac{t}{3}\right)$.

Using the data from Example A1, $P(t) = 30996 - \frac{1}{20}t$, $C(u) = \frac{1}{40}u^2$, we obtain

$$\Pi(t) = t\left(30996 - \frac{1}{20}t\right) - 3 \cdot \frac{1}{40}\left(\frac{t}{3}\right)^2.$$

Hence $\Pi(t) = 30996t - \frac{7}{120}t^2$ and the first-order condition is

$$\frac{d\Pi}{dt}(t) = 30996 - \frac{7}{60}t = 0,$$

which gives $t = 265680$.

Therefore, $x = y = z = 88560$. The market price is $P = 17712$. The profit of each firm is $\Pi_i = 1372502880$. The total market profit is $\Pi_1 + \Pi_2 + \Pi_3 = 4117508640$. Under the equal allocation of output in the full-cartel benchmark, Firm 3 earns $\Pi_3^{Cartel} = 1372502880$, whereas its profit in the failed-cartel regime is only $\Pi_3^{FC} \approx 1296882887$. Thus, Firm 3 is worse off after leaving the cartel in this example. The cartel breakdown is therefore not derived from an endogenous profitability condition but is taken as exogenously given.

Appendix A.5. Equilibrium in the Failed Cartel Case

Assume now that the cartel agreement collapses because player 3 deviates and chooses his output strategically, while firms 1 and 2 continue to cooperate and behave as a cartel.

For fixed (x, y), the third firm solves $\frac{\partial \Pi_3}{\partial z}(x, y, z) = 0$. Using the payoff functions from Example A1, we obtain

$$30996 - \frac{1}{20}x - \frac{1}{20}y - \frac{3}{20}z = 0.$$

Hence the follower's response function is $z = b_3(x, y) = 206640 - \frac{x}{3} - \frac{y}{3}$.

Since firms 1 and 2 remain in a cartel agreement, they choose equal outputs $x = y$. Substituting $y = x$ gives $z = b_3(x, x) = 206640 - \frac{2}{3}x$.

The joint profit of firms 1 and 2 is $\Pi(x) = 2\left(xP(2x + b_3(x, x)) - \frac{1}{40}x^2\right)$.

Since

$$2x + b_3(x, x) = 2x + 206640 - \frac{2}{3}x = 206640 + \frac{4}{3}x,$$

the market price becomes $P(x) = 20664 - \frac{x}{15}$.

Therefore, $\Pi(x) = 41328x - \frac{11}{60}x^2$. The first-order condition is

$$\frac{d\Pi}{dx}(x) = 41328 - \frac{11}{30}x = 0.$$

The exact quantities are $x = y = 1239840/11$ and $z = 1446480/11$. Hence

$$Q = \frac{3926160}{11} \approx 356923.636364, \qquad P(Q) = \frac{144648}{11} \approx 13149.818182.$$

The corresponding rounded firm quantities are $x = y \approx 112713$ and $z \approx 131498$. The calculations below use the exact fractions, not these rounded quantities.

The profits of firms 1 and 2 are $\Pi_1 = \Pi_2 \approx 1164547898$. The profit of firm 3 is $\Pi_3 \approx 1296882887$. The total market profit is $\Pi_1 + \Pi_2 + \Pi_3 \approx 3625978683$.

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The failed-cartel and Cournot–Stackelberg models describe two distinct market structures that may arise after the breakdown of an initial cartel agreement. In the failed-cartel model, Firms 1 and 2 continue to coordinate their production decisions and maximize their joint profit while anticipating the response of Firm 3. By contrast, in the Cournot–Stackelberg model, Firms 1 and 2 act non-cooperatively: they choose their quantities simultaneously and each maximizes its own reduced payoff while anticipating the response of the common follower, Firm 3. Consequently, the two models have different objective functions and generally produce different equilibrium outcomes, even when they are based on the same inverse demand and cost functions. The initial values used in a numerical fixed-point iteration affect only the computational procedure; they do not define the underlying market structure or the corresponding equilibrium concept.

Appendix A.6. Comparative Analysis of the Equilibrium Outcomes

Tables A1 and A2 summarize the equilibrium outcomes under the five market structures. Table A1 reports the equilibrium quantities, aggregate output, and market price, whereas Table A2 presents the corresponding individual and aggregate profits. Throughout this section, $Q = x + y + z$ denotes the total market output and P the corresponding equilibrium price.

Table A1. Equilibrium quantities and market prices in the triopoly market. All entries are rounded to the nearest integer.

Model	x	y	z	Q	P
Cournot	123,984	123,984	123,984	371,952	12,398
Canonical Stackelberg	151,200	133,920	111,600	396,720	11,160
Cournot–Stackelberg	137,760	137,760	114,800	390,320	11,480
Cartel	88,560	88,560	88,560	265,680	17,712
Failed cartel	112,713	112,713	131,498	356,924	13,150

The five market structures exhibit distinct economic characteristics. The cartel produces the lowest aggregate output and the highest market price, whereas the canonical Stackelberg model generates the highest output and the lowest price. The Cournot equilibrium lies between these extremes, while the Cournot–Stackelberg and failed-cartel structures represent intermediate regimes arising from informational hierarchy and partial cooperation, respectively.

Table A2. Individual and total profits in the triopoly market.

Model	Π_1	Π_2	Π_3	$\sum \Pi_i$
Cournot	1,152,902,419	1,152,902,419	1,152,902,419	3,458,707,258
Canonical Stackelberg	1,115,856,000	1,046,183,040	934,092,000	3,096,131,040
Cournot–Stackelberg	1,107,039,360	1,107,039,360	988,428,000	3,202,506,720
Cartel	1,372,502,880	1,372,502,880	1,372,502,880	4,117,508,640
Failed cartel	1,164,547,898	1,164,547,898	1,296,882,887	3,625,978,683

The profit distribution reflects the strategic organization of the market. The Cournot and cartel models remain symmetric, whereas the Stackelberg and failed-cartel structures introduce asymmetry through leadership and strategic deviation. In particular, the outsider in the failed-cartel regime captures the largest individual profit, while the cooperating firms still benefit from partial coordination.

Figure A1 illustrates the firm-level equilibrium quantities reported in Table A1.

Figure A1 visualizes the equilibrium production levels. It clearly illustrates the symmetry of the Cournot and cartel models and the asymmetric output allocation generated by the Stackelberg and failed-cartel structures.

The price–cost margins provide an additional economic interpretation. Since $MC(u) = C'(u) = \frac{u}{20}$, firms producing larger quantities incur higher marginal costs. Nevertheless, in the failed-cartel regime the outsider combines a relatively high output with a relatively high market price, explaining its superior individual profitability.

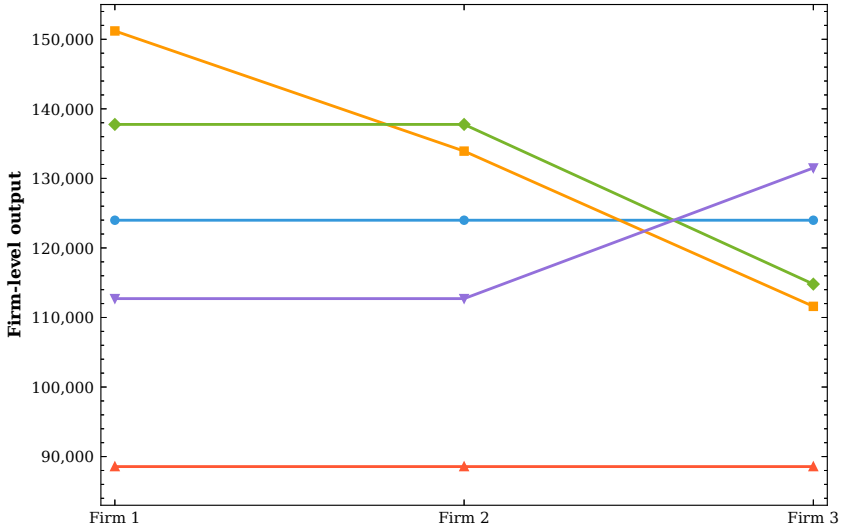


Figure A1. Firm-level outputs in the linear-demand example. Blue: Cournot. Orange: canonical Stackelberg. Green: Cournot–Stackelberg. Red: cartel. Purple: failed cartel.

The qualitative differences become more transparent when the equilibrium outcomes are expressed as percentage deviations from the symmetric Cournot equilibrium, which serves as the natural benchmark throughout the comparative analysis.

Table A3 highlights the different effects of informational hierarchy and partial cooperation. The canonical Stackelberg model strongly favors the leader, whereas the Cournot–Stackelberg model distributes the production increase more evenly between the leaders. In contrast, the failed-cartel regime shifts production toward the outsider while preserving part of the price-increasing effect of cartel coordination.

Table A3. Relative changes in equilibrium quantities and market price with respect to the Cournot equilibrium.

Market Structure	x	y	z	Q	P
Cournot (benchmark)	0.00%	0.00%	0.00%	0.00%	0.00%
Canonical Stackelberg	+21.95%	+8.01%	−9.99%	+6.66%	−9.99%
Cournot–Stackelberg	+11.11%	+11.11%	−7.41%	+4.94%	−7.41%
Cartel	−28.57%	−28.57%	−28.57%	−28.57%	+42.86%
Failed cartel	−9.09%	−9.09%	+6.06%	−4.04%	+6.06%

Table A4 illustrates the trade-off between competition and coordination. Informational hierarchy reduces aggregate producer profits, whereas full cooperation maximizes them. The failed-cartel regime remains particularly interesting because it preserves a higher aggregate industry profit than the Cournot benchmark while generating an asymmetric distribution of profits.

Table A4. Relative changes in individual and aggregate profits with respect to the Cournot equilibrium.

Market Structure	Π_1	Π_2	Π_3	$\sum \Pi_i$
Cournot (benchmark)	0.00%	0.00%	0.00%	0.00%
Canonical Stackelberg	−3.21%	−9.26%	−18.98%	−10.48%
Cournot–Stackelberg	−3.98%	−3.98%	−14.27%	−7.41%
Cartel	+19.05%	+19.05%	+19.05%	+19.05%
Failed cartel	+1.01%	+1.01%	+12.49%	+4.84%

In the linear quadratic triopoly, every firm earns 1,372,502,880 under the full cartel. Following one firm's departure, the outsider earns approximately 1,296,882,887, while each remaining cartel member earns approximately 1,164,547,898. Thus, all three firms strictly prefer the full-cartel allocation to the specified post-departure outcome. In particular, no firm benefits from leaving if departure leads to the continuation game studied here, in which the surviving coalition leads and the outsider follows.

Overall, the numerical comparisons reveal a consistent economic pattern. Stronger coordination restricts production, increases prices, and raises producer profits, whereas stronger informational hierarchy expands output and benefits consumers through lower prices. The Cournot–Stackelberg and failed-cartel models naturally bridge these two extremes by combining elements of competition and coordination.

Let us denote the model with superscript, i.e., x^{Model}, y^{Model}, and z^{Model} is the quantities for each player in $Model = \{Cartel, FC, C, CS, S\}$ where $FC = Failed - cartel$, $C = Cournot$, $CS = Cournot - -Stackelberg$, and $S = CanonicalStackelberg$.

The comparisons for the three players are as follows: for the leader, $x^{Cartel} < x^{FC} < x^{C} < x^{CS} < x^{S}$; for the middle players, $y^{Cartel} < y^{FC} < y_2^{C} < y^{S} < y^{CS}$; and for the followers, $z^{Cartel} < z^{S} < z^{CS} < z^{Cournot} < z^{FC}$.

Thus, the cartel outcome generates the lowest individual output for each participant. In contrast, the Stackelberg-type models increase the production of the leading firms, while the follower produces less than in the symmetric Cournot benchmark.

Appendix A.7. Consumer and Total Surplus

The welfare calculations in this subsection apply only to the linear quadratic triopoly specified in Example A1. No corresponding ordering is claimed for arbitrary n, general demand and cost functions, or the isoelastic specification.

Consumer surplus is computed as $CS = \int_0^Q P(q)\,dq - P(Q)Q$. For the linear demand function $P(Q) = 30996 - \frac{1}{20}Q$, this gives $CS = \frac{Q^2}{40}$. Total surplus is the sum of consumer surplus and total industry profit, i.e., $TS = CS +$ Total Profit.

In particular, the failed-cartel quantities give

$$CS^{FC} = \frac{385368308640}{121} \approx 3184862054.88, \qquad TS^{FC} = \frac{824111729280}{121} \approx 6810840737.85.$$

The rounded entries in Table A5 are obtained from these exact values.

Table A5. Consumer surplus and total surplus under the considered market structures.

Model	Consumer Surplus	Total Surplus
Cournot	3,458,707,258	6,917,414,515
Canonical Stackelberg	3,934,668,960	7,030,800,000
Cournot–Stackelberg	3,808,742,560	7,011,249,280
Cartel	1,764,646,560	5,882,155,200
Failed cartel	3,184,862,055	6,810,840,738

Table A6 confirms that market hierarchy benefits consumers, while cartel coordination benefits producers. The Cournot–Stackelberg model represents a compromise between these two extremes, whereas the failed-cartel regime remains much closer to the Cournot benchmark than to the full cartel in terms of both consumer and total welfare.

For the parameters of the linear quadratic triopoly, the market structures generating larger aggregate outputs also generate the largest consumer surplus and total surplus values. In this particular benchmark, full coordination produces the lowest consumer and total surplus, while the failed-cartel outcome mitigates, but does not eliminate, the welfare reduction relative to the Cournot equilibrium.

Table A6. Relative changes in consumer surplus and total surplus with respect to the Cournot equilibrium.

Market Structure	Consumer Surplus	Total Surplus
Cournot	0.00%	0.00%
Canonical Stackelberg	+13.76%	+1.64%
Cournot–Stackelberg	+10.12%	+1.36%
Cartel	−48.98%	−14.97%
Failed cartel	−7.92%	−1.54%

An additional noteworthy observation is that both welfare indicators induce exactly the same ranking of the considered market structures. More precisely,

$$CS^{S} > CS^{CS} > CS^{Cournot} > CS^{FC} > CS^{Cartel},$$

and

$$TS^{S} > TS^{CS} > TS^{Cournot} > TS^{FC} > TS^{Cartel}.$$

Thus, for this numerical linear quadratic triopoly, consumer surplus and total surplus induce the same ranking of the five market structures. This agreement is a property of the selected functional forms and parameter values. It should not be interpreted as a general welfare theorem for arbitrary demand functions, cost technologies, numbers of firms, or cartel compositions.

Appendix B. Fixed-Point Background and Technical Proofs

Appendix B.1. Mathematical Preliminaries: Notation and Known Results

We use ordinary fixed points on product spaces. For nonempty sets I_i and maps $F_i : \prod_{r=1}^{n} I_r \to I_i$, define $T(\mathbf{u}) = (F_1(\mathbf{u}), \ldots, F_n(\mathbf{u}))$. An n-tupled fixed point is exactly a fixed point of T, and its simultaneous iteration is $\mathbf{u}_{m+1} = T(\mathbf{u}_m)$; see (Ali et al., 2025; Ilchev et al., 2024; Petruşel, 2018; Petruşel et al., 2018; Zlatanov, 2021). Related coupled, cyclic, and tripled formulations appear in (Berinde & Borcut, 2011; Bhaskar and Lakshmikantham, 2006; Guo & Lakshmikantham, 1987; Kirk et al., 2003; Samet & Vetro, 2010; Sintunavarat & Kumam, 2012). No partial order is assumed here; the mixed-monotone approach of (Moradi et al., 2022) is a separate possible extension.

Definition A1 (Hardy–Rogers condition)**.** *A self-map T of a metric space (X, ρ) satisfies the symmetric Hardy–Rogers condition if, for all $u, v \in X$,*

$$\rho(Tu, Tv) \leq k_1\rho(u, v) + k_2\big[\rho(u, Tu) + \rho(v, Tv)\big] + k_3\big[\rho(u, Tv) + \rho(v, Tu)\big], \qquad \text{(A1)}$$

where $k_1, k_2, k_3 \geq 0$ and $k_1 + 2k_2 + 2k_3 < 1$. The usual five-coefficient form reduces to this form by interchanging u, v and averaging (Hardy & Rogers, 1973; Reich, 1971). The Banach, Kannan,

and Chatterjea conditions are obtained by retaining only k_1, k_2, or k_3, respectively (Banach, 1922; Chatterjea, 1972; Kannan, 1968).

Theorem A1 (Hardy–Rogers). *Let (X,ρ) be nonempty and complete, and let $T : X \to X$ satisfy (A1). Then T has a unique fixed point u^*, and $u_{m+1} = Tu_m$ converges to u^* for every $u_0 \in X$. With $\kappa = \frac{k_1+k_2+k_3}{1-k_2-k_3} < 1$, the following estimates hold for $m \geq 1$:*

$$\begin{aligned} \rho(u_m, u^*) &\leq \frac{\kappa^m}{1-\kappa}\rho(u_1, u_0), \\ \rho(u_m, u^*) &\leq \frac{\kappa}{1-\kappa}\rho(u_m, u_{m-1}), \\ \rho(u_m, u^*) &\leq \kappa\rho(u_{m-1}, u^*). \end{aligned} \tag{A2}$$

This is the standard fixed-point theorem of (Hardy & Rogers, 1973); we use it without repeating its proof. In the applications, n denotes the number of firms and m the iteration index.

Appendix B.2. Product-Space Formulation

Definition A2 (Update operators). *Let (X_i, d_i), $i = 1,2,3$, be nonempty metric spaces, $X = X_1 \times X_2 \times X_3$, and $\mathcal{F}_i : X \to X_i$. For $\mathbf{u} = (x,y,z)$, set $h = \mathcal{F}_3(x,y,z)$ and $g = \mathcal{F}_2(x,y,h)$. The two update operators used here are*

$$S_1(\mathbf{u}) = (\mathcal{F}_1(x,g,h), g, h), \qquad S_3(\mathbf{u}) = (\mathcal{F}_1(x,y,h), \mathcal{F}_2(x,y,h), h).$$

They correspond to the canonical and Cournot–Stackelberg update structures, respectively; the canonical construction was studied in (Badev et al., 2026). These compositions describe numerical updates, and do not by themselves establish sequential profit maximization.

Definition A3 (Metric and fixed points). *For $\mathbf{u} = (u_1,u_2,u_3)$, $\mathbf{v} = (v_1,v_2,v_3) \in X$, set*

$$M_1(\mathbf{u},\mathbf{v}) = \sum_{i=1}^{3} d_i(u_i, v_i).$$

For $j \in \{1,3\}$, write

$$\begin{aligned} M_2^{S_j}(\mathbf{u},\mathbf{v}) &= M_1(\mathbf{u}, S_j\mathbf{u}) + M_1(\mathbf{v}, S_j\mathbf{v}), \\ M_3^{S_j}(\mathbf{u},\mathbf{v}) &= M_1(\mathbf{u}, S_j\mathbf{v}) + M_1(\mathbf{v}, S_j\mathbf{u}). \end{aligned}$$

A mixed tripled fixed point for this update structure means simply a point $\mathbf{u}^* = S_j\mathbf{u}^*$.

Theorem A2 (Product-space consequence). *Assume that each (X_i, d_i) is complete. Fix $j \in \{1,3\}$, and suppose that, for all $\mathbf{u},\mathbf{v} \in X$,*

$$M_1(S_j\mathbf{u}, S_j\mathbf{v}) \leq k_1^{(j)} M_1(\mathbf{u},\mathbf{v}) + k_2^{(j)} M_2^{S_j}(\mathbf{u},\mathbf{v}) + k_3^{(j)} M_3^{S_j}(\mathbf{u},\mathbf{v}), \tag{A3}$$

where $k_r^{(j)} \geq 0$ and $k_1^{(j)} + 2k_2^{(j)} + 2k_3^{(j)} < 1$.

Then S_j has a unique fixed point $\mathbf{u}^*$, and $\mathbf{u}_{m+1} = S_j\mathbf{u}_m$ converges to it from every $\mathbf{u}_0 \in X$. The estimates (A2) apply with $\rho = M_1$ and $\kappa = (k_1^{(j)} + k_2^{(j)} + k_3^{(j)})/(1 - k_2^{(j)} - k_3^{(j)})$.

Proof. The product (X, M_1) is complete, and (A3) is precisely (A1) for $T = S_j$. Apply Theorem A1. □

If the component spaces coincide and $R(x,y,z) = (y,z,x)$, the additional condition $S_j(R\mathbf{u}^*) = R(S_j\mathbf{u}^*)$ makes $R\mathbf{u}^*$ another fixed point. Uniqueness then gives $x^* = y^* = z^*$. This local cyclic symmetry is an optional mathematical condition, not an assumption on firms with different strategic roles.

For the operator $\mathcal{G}_2$ in (5), choose $\mathcal{F}_1(x,y,z) = F_1^{\lambda_1}(x,y)$, $\mathcal{F}_2(x,y,z) = F_2^{\lambda_2}(x,y)$, and $\mathcal{F}_3(x,y,z) = b_3(x,y)$. Then $S_3 = \mathcal{G}_2$. The economically relevant contraction is imposed on the closed invariant set D, as stated in Theorem 5.

Appendix B.3. Proof of Proposition 14 and Its Benchmarks

Proof. For fixed $s \geq 0$, the follower's profit is $V(z;s) = z(s+z)^{-1/2} - cz$.

For $s + z > 0$, there hold

$$V_z = (s+z)^{-1/2} - \frac{z}{2}(s+z)^{-3/2} - c, \quad V_{zz} = -(s+z)^{-3/2} + \frac{3z}{4}(s+z)^{-5/2} < 0.$$

Since $V(z;s) \to -\infty$ as $z \to \infty$, a unique maximum exists.

For $s > 0$, the right derivative at zero is $s^{-1/2} - c$; hence $b(s) = 0$ precisely when $s \geq c^{-2}$.

At $s = 0$, direct maximization of $\sqrt{z} - cz$ gives $b(0) = 1/(4c^2)$.

For $0 \leq s < c^{-2}$, the positive best response satisfies, with $Q = s + b(s)$,

$$b(s) = 2Q(1 - c\sqrt{Q}), \qquad s = Q(2c\sqrt{Q} - 1).$$

Setting $u = c\sqrt{Q}$ parametrizes this entire branch by $1/2 \leq u < 1$, since

$$s(u) = \frac{u^2(2u-1)}{c^2}, \qquad s'(u) = \frac{2u(3u-1)}{c^2} > 0.$$

Let $s = x + y$ denote the coalition's aggregate output. Its joint profit is $s(P(Q) - c)$. Using $u = c\sqrt{Q}$, we have $P(Q) = c/u$ and $s(u) = u^2(2u-1)/c^2$. Thus, expressed as a function of u, the coalition's joint profit is

$$W(u) = s(u)\left(\frac{c}{u} - c\right) = \frac{u(1-u)(2u-1)}{c}.$$

Differentiating with respect to u gives $W'(u) = \frac{-6u^2+6u-1}{c}$ and $W''(u) = \frac{6-12u}{c}$.

The only stationary point in $[1/2, 1]$ is $u^* = (3+\sqrt{3})/6$. The derivative is positive before u^* and negative after it, so this is the unique global maximum on that interval, with positive profit. For $s \geq c^{-2}$, the follower chooses zero and the coalition obtains $\sqrt{s} - cs \leq 0$; at $s = 0$, its profit is zero. Thus u^* also determines the unique optimal coalition aggregate over all $s \geq 0$. The follower strategy $b(s)$ is optimal after every history. This proves the subgame-perfect outcome in (7) under the imposed equal split.

For the full cartel, maximize $\sqrt{Q} - cQ$ on $Q \geq 0$. Its unique maximum is $Q = 1/(4c^2)$; linear costs leave its allocation indeterminate, so equal shares are imposed. For Cournot, each payoff is strictly concave in its own positive output by the same derivative calculation as above. An equilibrium cannot have an inactive firm: any active firm's stationary condition implies $P(Q) > c$, giving an inactive firm a profitable small entry. The zero-output profile is not an equilibrium either. The interior equations yield equal outputs and

$$Q^{-1/2} - c - \frac{Q/3}{2Q^{3/2}} = 0,$$

hence the unique Cournot equilibrium has $Q = 25/(36c^2)$. Since $1/2 < u^* < 5/6$, the stated aggregate output ordering follows.

Finally, on $D = [3/4, 5/6]$, the two coalition quantities increase and the outsider quantity decreases as u increases. Direct differentiation of H gives

$$\sum_{i=1}^{3} |H_i'(u)| = \frac{6u(2u-1)}{c^2} \leq \frac{10}{3c^2},$$

which proves the claimed quantity-error bound. □

Note

[1] Badev et al. (2026) develops this example and here we describe the market outcome.

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