

Article

The Power of Passivity in the Hirshleifer Contest Under Small Noise

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Abstract

Hirshleifer's difference-form contest technology is a useful tool in the study of a class of conflict, especially military combats. We aim to highlight an important feature that the Hirshleifer contest model distinctively has, namely passivity (bidding zero effort) may stand as an effective choice in conflict even when the contest is highly deterministic (i.e., with small noise). For that purpose, we establish two propositions on the contest with $n \geq 2$ risk-neutral contestants under small noise. The first proposition states that every contestant bids arbitrarily close to zero (if not bidding zero with positive probability at all) under sufficiently small noise. The second proposition, more strikingly, states that every contestant either bids arbitrarily close to the second-highest valuation (among all the contestants' valuations), or simply remains passive with certainty under any sufficiently small noise. We further show that the first proposition holds for the contest between risk-averse individuals endowed with constant absolute risk aversion as well, and illustrate by an example how quickly polarization in bidding among contestants, as is predicted by the propositions, may emerge as the noise of the contest abates. These results help pave the way toward a complete characterization of the difference-form contest.

Keywords: difference-form contest; passivity; Nash equilibrium; small noise; all-pay auction



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1. Introduction

Contest theory has proved to be useful in the study of conflict, be it political rent-seeking, military combats, sports games, lawsuits, labor union's strikes, R&D competition, etc. (refer to, e.g., [Konrad \(2009\)](#) for an introduction, especially on dynamic contests; [Beviá and Corchón \(2024\)](#) for a postgraduate-level textbook; [Corchón and Serena \(2018\)](#) and [Kimbrough et al. \(2020\)](#) for two recent literature reviews). Among the contest technologies that have been extensively employed in applied analysis stand two canonical models, respectively represented by [Gordon Tullock's \(1980\)](#) ratio-form and [Jack Hirshleifer's \(1988, 1989\)](#) difference-form contest success functions (CSFs). Both forms of the contest can be derived axiomatically (refer to, e.g., [Skaperdas, 1996](#)), or stochastically (see chapter 10 in [Hirshleifer and Riley \(1992\)](#) on the special case of lottery and [Jia's \(2008\)](#) extension that allows for any power functional form of the Tullock contest, and [McFadden's \(1974\)](#) estimation of the logit model that can serve as a stochastic derivation of the Hirshleifer CSF).

Compared to Tullock's ratio-form contest technology that has drawn a great deal of attention, the Hirshleifer-type contest technology, in part due to analytical challenges,¹ was rigorously analyzed only recently ([Baik, 1998](#); [Che & Gale, 2000](#); [Alcalde & Dahm, 2007](#); [Beviá & Corchón, 2015](#); [Cubel & Sanchez-Pages, 2016](#); [Ewerhart & Sun, 2018, 2024](#); [Levine & Mattozzi, 2022](#)). It is worth recalling that Hirshleifer deliberately developed his

very difference-form contest model to address two important drawbacks of the Tullockian technology. First and foremost, in the Tullock ratio-form contest, any contestant who bids zero effort would lose the game with certainty when any opponent commits an extremely small amount of resources. This stands in striking contrast with many actual conflicts. One strong case in point is a widely known observation in military history, which is perhaps best summarized by Sun Tzu's famous remark that the supreme art of war is to subdue the enemy without fighting, or von Clausewitz's widely quoted advice that defense is the stronger form of fighting than attack. Indeed, episodes that align with this insight abound. A good number of eras over the history of humanity are actually defense-dominated in military conflicts, ranging from medieval Europe, part of the civil war of the United States, to World War I (Hirshleifer, 2000, p. 786). Moreover, the power of inaction continued to be demonstrated in the modern wars with tanks and airplanes as well (see, e.g., Lawrence, 2017, Chapters 2 and 3). At times, it is reasonable to surrender rather than resist in military conflict between nations (Hirshleifer, 1989, p. 103).² Secondly, the Tullock CSF fails to display increasing returns to effort up to the inflection point at equal resource commitments, another important observation often found in conflict. In particular, when one side is slightly inferior to its opponent in terms of committed resources, the gain of bidding more to be slightly superior would be quite remarkable. But the Tullock ratio-form contest technology is silent on this salient phenomenon, which is well known to researchers of military conflict. A good example is Dupuy's (1987, Chapter 11) meticulous research of the combats between the German and Allied forces in World War II, which, as Hirshleifer (1989, p. 111; 2000, p. 778) points out, is equivalent to a game-theoretical analysis based on the difference-form contest.

Among the existing literature on difference-form contests, Ewerhart and Sun (2024), hereafter referred to as ES 2024, investigate the equilibrium set of the multiplayer Hirshleifer contest, obtaining a number of novel results. But the analysis made in the said paper on the effectiveness of passivity is far from complete. The present piece is therefore motivated to explore how far we may go along this Hirshleiferian line of inquiry of conflict. We establish two new results on the equilibria of the multiplayer Hirshleifer contest under small noise, and thereby further illuminate the power of passivity in the contest. The first proposition says that every contestant bids arbitrarily close to zero (if not bidding zero at all) with positive probability. The second proposition, perhaps a much more interesting result, says that each contestant either bids arbitrarily close to the second-highest valuation (among all the contestants' valuations) or simply remains passive (bidding zero) with certainty when the noise of the contest is small enough (it does not need to vanish). It is worth emphasizing that these two propositions simultaneously apply to any equilibrium under small noise, a point that has significant implications (refer to Corollary 1 in Section 3, below).

As is noted in several existing studies on contests (e.g., Che & Gale, 2000; Ewerhart, 2017; Levine & Mattozzi, 2022), the all-pay auction turns out to be robust in the sense that, under certain circumstances, the equilibrium set or/and the equilibrium payoffs in some contests under vanishing noise converge to those in the all-pay auction. Che and Gale (2000) develop a two-player difference-form contest, which can hardly be extended to a contest with more than two contestants, and prove the uniform convergence of equilibrium bids as the noise vanishes, while Levine and Mattozzi (2022) characterize the limiting behavior of the two-player Tullock contest. The equilibrium set in the all-pay auction with multiple players, however, is strikingly complex (Baye et al., 1996), much more complicated compared to the auction with only two bidders (Hillman & Riley, 1989). It remains unclear which part of the equilibrium set under the all-pay auction stands as the limit for any difference- or ratio-form contest with multiple contestants under vanishing noise. Relatedly, virtually nothing is known yet about the convergence process of the equilibrium set when

there are multiple constantans. The results we obtain in the present study help pave the way for exploring these problems.

We shall describe the model and make some technical preparations in the next section. We then state and briefly elaborate on our main results in Section 3, relegating rigorous proof of these results to Appendix A. In Section 4, we offer two illustrative examples. The first one demonstrates how fast the strategy of a contestant, even if she is as capable as anyone of bidding aggressively, may converge to passivity (bidding zero with certainty) as the noise of the contest becomes small enough. The second one shows that part of our main results, namely Proposition 1, is robust with respect to some variation in risk attitude of the contestants, while whether Proposition 2 remains valid as well is unknown. Section 5 concludes with remarks.

2. Set-Up and Preparations

We now describe the multiplayer Hirshleifer contest. Denote by $N \equiv \{1, \dots, n\}$ where $n \geq 2$, the set of all the contestants with valuations $V_1 \geq V_2 \geq \dots \geq V_n > 0$, and by x_i , the effort made by any $i \in N$. Any $i \in N$ wins the prize with probability $p_i(x_i, x_{-i}) \equiv p_i(x_i, x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) = \frac{1}{1 + \sum_{k \neq i} \exp(\alpha(x_k - x_i))}$, where $\alpha \geq 0$ is the decisiveness parameter. The greater the value of α , the less noisy the contest is, and in the limit of $\alpha \rightarrow \infty$, the contest becomes an all-pay auction in which the highest bidder wins with certainty (Baye et al., 1996). The cost of bidding is identical to the effort exerted. Every contestant is risk-neutral and therefore receives a payoff $\pi_i(x_i, x_{-i}) = V_i p_i(x_i, x_{-i}) - x_i$, under effort profile (x_i, x_{-i}) , $\forall i \in N$.

At any $\alpha > 0$, a standard argument invoking Glicksberg's (1952) fixed point theorem suffices to prove the existence of an equilibrium in mixed strategies, denoted as $\mu^* = (\mu_1^*, \dots, \mu_n^*)$ (refer to, for instance, Lemma 1 in Ewerhart and Sun (2024)). To ease exposition, we shall restate two lemmas that have already been stated in the existing studies as the starting point of the present analysis.

Lemma 1. *At any $\alpha > 0$,*

- (i) *The equilibrium strategy of any contestant $i \in N$ randomizes over a finite number of bids, denoted as $y_i^{(1)}(\alpha) > \dots > y_i^{(L_i)}(\alpha) \geq 0$, where $L_i = L_i(\alpha) < \infty$;*
- (ii) *Zero is contained in the support of the equilibrium strategy for at least $(n - 1)$ contestants, and each of them receives a payoff no greater than $\frac{n}{(n-1)\alpha}$.*

Part (i) of the lemma follows from analyticity of the payoff functions on an open neighborhood and compactness of the set of possible effort (Ewerhart & Sun, 2018, Lemma 1). A technique that invokes Cauchy's theorem on one-variable complex functions to establish discreteness of the support of the equilibrium strategy for analytic payoff functions was first introduced into the literature of contest theory in Ewerhart (2015, Theorem 3.2), and was then used in Levine and Mattozzi (2022) and Ewerhart and Sun (2024), among a few other studies. An alternative real-analysis approach to proving that the equilibrium a symmetric Hirshleifer contest admits has a finite support is found in Sun (2017). Part (ii) combines Lemmas 2 and A.1 in Ewerhart and Sun (2024).

For the Hirshleifer contest under vanishing noise, it has been shown that the payoff for all but the single highest-valuation contestant (in the case that $V_1 > V_2$) approaches zero, and that everyone bids up to V_2 or *ultimately* becomes inactive. It is formally stated as,

Lemma 2 (Ewerhart & Sun, 2024, Proposition 7). *Let ϵ be a small positive number. For sufficiently large α ,*

- (i) At least two constants, including contestant one if $V_1 > V_2$, bid more than $V_2 - \varepsilon$ with positive probability;
- (ii) $|\Pi_1^* - (V_1 - V_2)| < \varepsilon$, $\forall i \in N \setminus \{1\}$, $\Pi_i^* < \varepsilon$;
- (iii) No contestant i such that $y_i^{(1)}(\alpha) \in [\varepsilon, V_2 - \varepsilon]$;
- (iv) If $V_1 = V_2$, $\mu_i^*(\{0\}) < \varepsilon$ for at least two contestants $i \in N$.

We now introduce a new and technically useful lemma. For any contestant i , let $E_{\mu_{-i}^*}[p_i(x, x_{-i})]$ be the probability with which i wins when bidding x against any possible x_{-i} under equilibrium $\mu^* = \mu^*(\alpha)$ (we routinely suppress dependence on α wherever convenient). As usual, let $O(x)$ represent a (positive) infinitesimal that approaches zero at the same order as does x when x approaches zero.

Lemma 3. $\forall i \in N$, for any $y_i \in \text{supp}\{\mu_i^*(\alpha)\}$, the following holds when α is sufficiently large.³

- (i) $E_{\mu_{-i}^*}\left[p_i\left(y_i + \frac{1}{\sqrt{\alpha}}, x_{-i}\right)\right] \geq \left[1 - O\left(\exp\left(-\frac{1}{2}\sqrt{\alpha}\right)\right)\right] \prod_{k \neq i} \mu_k^*\left(\left[0, y_i + \frac{1}{2\sqrt{\alpha}}\right]\right)$;
- (ii) If there exists some $b > 0$, $E_{\mu_{-i}^*}\left[p_i\left(y_i + \frac{1}{\sqrt{\alpha}}, x_{-i}\right)\right] \geq b$ for any large α , then

$$E_{\mu_{-i}^*}\left[p_i\left(y_i + \frac{1}{\sqrt{\alpha}}, x_{-i}\right)\right] < [1 + O(\exp(-\sqrt{\alpha}))] \prod_{k \neq i} \mu_k^*\left(\left[0, y_i + \frac{2}{\sqrt{\alpha}}\right]\right).$$

Proof. (i) By Lemma 1(i), at any $\alpha > 0$ the equilibrium support of each contestant is finite. Therefore,

$$\begin{aligned} & E_{\mu_{-i}^*}\left[p_i\left(y_i + \frac{1}{\sqrt{\alpha}}, x_{-i}\right)\right] \\ & \geq \sum_{\forall k \neq i, x_k \leq y_i + \frac{1}{2\sqrt{\alpha}}} \frac{1}{1 + \sum_{k \neq i} \exp\left(\alpha\left(x_k - y_i - \frac{1}{\sqrt{\alpha}}\right)\right)} \prod_{k \neq i} \mu_k^*(\{x_k\}) \\ & \geq \sum_{\forall k \neq i, x_k \leq y_i + \frac{1}{2\sqrt{\alpha}}} \frac{1}{1 + (n-1)\exp\left(-\frac{1}{2}\sqrt{\alpha}\right)} \prod_{k \neq i} \mu_k^*(\{x_k\}) \\ & = \sum_{\forall k \neq i, x_k \leq y_i + \frac{1}{2\sqrt{\alpha}}} \frac{1}{\left[1 - \frac{(n-1)\exp\left(-\frac{1}{2}\sqrt{\alpha}\right)}{1 + (n-1)\exp\left(-\frac{1}{2}\sqrt{\alpha}\right)}\right]} \prod_{k \neq i} \mu_k^*(\{x_k\}) \\ & = \left[1 - O\left(\exp\left(-\frac{1}{2}\sqrt{\alpha}\right)\right)\right] \prod_{k \neq i} \mu_k^*\left(\left[0, y_i + \frac{1}{2\sqrt{\alpha}}\right]\right). \end{aligned}$$

(ii) First note

$$\begin{aligned} & E_{\mu_{-i}^*}\left[p_i\left(y_i + \frac{1}{\sqrt{\alpha}}, x_{-i}\right)\right] \\ & = \sum_{\forall k \neq i, x_k \leq y_i + \frac{1}{\sqrt{\alpha}}} \frac{1}{1 + \sum_{k \neq i} \exp\left(\alpha\left(x_k - y_i - \frac{1}{\sqrt{\alpha}}\right)\right)} \prod_{k \neq i} \mu_k^*(\{x_k\}) \\ & \quad + \sum_{\exists k \neq i, x_k > y_i + \frac{1}{\sqrt{\alpha}}} \frac{1}{1 + \sum_{k \neq i} \exp\left(\alpha\left(x_k - y_i - \frac{1}{\sqrt{\alpha}}\right)\right)} \prod_{k \neq i} \mu_k^*(\{x_k\}) \\ & < \sum_{\forall k \neq i, x_k \leq y_i + \frac{1}{\sqrt{\alpha}}} \frac{2}{\sqrt{\alpha}} \prod_{k \neq i} \mu_k^*(\{x_k\}) + O(\exp(-\sqrt{\alpha})). \end{aligned}$$

In light of the fact that $E_{\mu_{-i}^*} \left[p_i \left(y_i + \frac{1}{\sqrt{\alpha}}, x_{-i} \right) \right] \geq b > 0$, we then obtain

$$E_{\mu_{-i}^*} \left[p_i \left(y_i + \frac{1}{\sqrt{\alpha}}, x_{-i} \right) \right] < [1 + O(\exp(-\sqrt{\alpha}))] \prod_{k \neq i} \mu_k^* \left(\left[0, y_i + \frac{2}{\sqrt{\alpha}} \right] \right).$$

□

3. The Main Results

At any $\alpha > 0$, by Lemma 1(i), the support of any contestant's strategy has a finite number of elements. We first establish a result on the lower bound of the support of any contestant.

Proposition 1. *In the Hirshleifer contest between $n \geq 2$ risk-neutral contestants with linear effort-cost functions and heterogenous valuations of the prize, for any contestant i , $\liminf_{\alpha \rightarrow \infty} \{ \text{supp} \{ \mu_i^*(\alpha) \} \} = 0$.*

In the light of Lemma 1(ii), the message conveyed by the proposition is that at least $n - 1$ contestants bid zero with positive probability at any $\alpha > 0$, and that the other one, if any, who does not bid arbitrarily close to zero for sufficiently large α . A rigorous proof of the proposition is found in Appendix A. The proof proceeds by negation, starting from the assumption, without loss of generality, that there exists contestant j who always bids at minimum $\underline{y}_j(\alpha) \equiv \min \{ \text{supp} \{ \mu_j^*(\alpha) \} \} = y_j^{(L_j)}(\alpha)$ that is above zero and approaches some $b > 0$ as $\alpha \rightarrow \infty$ in each of the two possible scenarios, namely $V_1 = V_2$ and $V_1 > V_2$. The intuition of the argument for the case of $V_1 = V_2$ is as follows. There then exists contestant $i \neq j$ who bids zero with an arbitrarily small probability as $\alpha \rightarrow \infty$, by Lemma 2(iv). But no contestant bids positive weakly below $\underline{y}_j(\alpha)$, by the second-order condition at any interior bid. The reason that contestant j always bids close to $b > 0$ for any α large enough is that each contestant $i \neq j$ necessarily bids around $\underline{y}_j(\alpha)$ with a substantially large probability. That in turn implies that contestant j can then overwhelmingly subdue all the other contestants with a much higher probability to win the prize and marginally higher effort-cost by bidding $\underline{y}_j(\alpha) + \frac{1}{\sqrt{\alpha}}$ than bidding $\underline{y}_j(\alpha)$ at any large α , contradicting $\underline{y}_j(\alpha) \in \text{supp} \{ \mu_j^*(\alpha) \}$. The argument for the case of $V_1 > V_2$ can be likewise understood intuitively as follows. By Lemmas 1(ii) and 2(ii), the always active contestant (i.e., anyone who bids zero effort with probability zero) must be contestant one. In this case, any contestant $i \neq 1$ bids zero with a probability above a certain positive number. Otherwise, each of them necessarily bids around $\underline{y}_1(\alpha) \equiv \min \{ \text{supp} \{ \mu_1^*(\alpha) \} \}$ with a substantial probability so that bidding $\underline{y}_1(\alpha)$ guarantees contestant one a payoff approaching $V_1 - V_2$ (by Lemma 2(ii) again), which in turn implies that, by bidding slightly above $\underline{y}_1(\alpha)$, contestant one would receive a strictly greater payoff than bidding $\underline{y}_1(\alpha)$, an impossibility. Note that $\underline{y}_1(\alpha)$ is close to b for large α . It then follows that bidding $b/2$ would be a strictly better option for contestant one, in conflict with $\underline{y}_1(\alpha) \in \text{supp} \{ \mu_1^*(\alpha) \}$. It is worth emphasizing that both the above explanation and the rigorous proof offered in Appendix A apply to any equilibrium of the contest.

Proposition 1 establishes that the lower bound of the support of every contestant is zero when the noise of the Hirshleifer contest vanishes, consistent with Lemma 2 in Baye et al. (1996, p. 299) on the equilibria of the all-pay auction. We now sharply strengthen claim (iii) in Lemma 2 on the upper bound of any contestant's support under small noise.

Proposition 2. *In the Hirshleifer contest between $n \geq 2$ risk-neutral contestants with linear effort-cost functions and heterogeneous valuations of the prize, for any contestant i , either $y_i^{(1)}(\alpha) \rightarrow V_2$ as $\alpha \rightarrow \infty$, or $y_i^{(1)}(\alpha) = 0$ at any sufficiently large α .*

That is, polarization even among homogenous contestants emerges once α reaches some *finite* threshold: any contestant either bids aggressively or drops out. When $n = 2$, the proposition directly follows from Lemma 2(i). The proof of the proposition for any $n \geq 3$, relegated to Appendix A, proceeds by negation. Starting with the assumption that there exists some contestant j whose highest possible bid $y_j^{(1)}$ is arbitrarily small but always remains positive as $\alpha \rightarrow \infty$, we demonstrate that this assumption necessarily results in an absurdity. The reasoning is more involved than what is used to establish Proposition 1, but the roadmap and intuition of the argument can be summarized as follows.

We first take care of the case of $V_1 = V_2$. That arbitrarily small $y_j^{(1)}(\alpha) > 0$ holds true at any α for contestant j implies the existence of some contestant $i \neq j$ who bids zero with positive probability but bids around or below $y_j^{(1)}$ with a vanishing probability. By bidding $y_j^{(1)}$ against μ_{-i}^* , therefore, contestant i weakly subdues contestant j , while the latter, when bidding $y_j^{(1)}$ against μ_{-j}^* , is subdued by the former almost surely (with a probability approaching one as $\alpha \rightarrow \infty$). As such, by bidding $y_j^{(1)}$ against μ_{-i}^* , contestant i would win with a probability that is enormously larger than the probability with which contestant j wins when bidding $y_j^{(1)}$ against μ_{-j}^* . As a consequence, bidding $y_j^{(1)}$, relative to bidding zero, against μ_{-i}^* by contestant i entails a vanishing increase in effort-cost, but brings about a substantial increase in gross benefit. That is, $y_j^{(1)}$ strictly dominates zero for contestant i , hence in conflict with the equilibrium property of the zero bid for the contestant.

We then come to grips with the more challenging case $V_1 > V_2$. Clearly, in the light of Lemmas 1(ii) and 2(ii), contestant one never bids zero for α large; that is, $j \neq 1$. We shall argue that there exists a contestant whose valuation of the prize equals V_2 and who bids zero with positive probability, on the one hand, and bidding $y_j^{(1)}(\alpha) > 0$ strictly dominates bidding zero for α large enough, on the other hand, hence a contradiction. The argument proceeds in a few steps. We first establish a preparatory lemma, Lemma A1, in step 1. We show that, in spite of the fact that contestant one's minimal possible bid $y_1(\alpha) \equiv \min\{\text{supp}\{\mu_1^*(\alpha)\}\} > 0$ is below $y_j^{(1)}(\alpha)$, $\alpha y_1(\alpha)$ is unbounded as α grows, while everyone else bids zero with a probability that is bounded away from zero but virtually does not bid any positive effort around or below $y_j^{(1)}(\alpha)$. We also observe that there exists a contestant $i \neq j$ such that $V_i = V_2$ who bids aggressively (say above $V_2/2$) with a probability bounded away from zero. We are then enabled to demonstrate in steps 2 and 3 that, by bidding $y_j^{(1)}$ against μ_{-i}^* , contestant i wins with a probability that is not only substantially larger than the probability with which contestant j wins when bidding $y_j^{(1)}$ against μ_{-j}^* (refer to inequality (A21) in Appendix A), but incomparably larger than the probability with which contestant i wins when bidding zero against μ_{-i}^* (refer to inequality (A22)). To understand the latter observation, one needs to notice the following technically useful point: against any possible x_{-i} in equilibrium, the probability with which contestant i wins when bidding zero is $\frac{1}{1 + \sum_{k \neq i} \exp(\alpha x_k)}$, while the probability of winning if contestant i bids $y_j^{(1)}$ instead is $\frac{1}{1 + \sum_{k \neq i} \exp(\alpha(x_k - y_j^{(1)}))}$. Noting that $\sum_{k \neq i} \exp(\alpha x_k) > \exp(\alpha y_1)$ for any α , therefore, the ratio of the former probability to the latter probability equals $\frac{1}{1 + \sum_{k \neq i} \exp(\alpha x_k)} \cdot \frac{\exp(\alpha y_j^{(1)}) + \sum_{k \neq i} \exp(\alpha x_k)}{\exp(\alpha y_j^{(1)})} < \frac{1}{\sum_{k \neq i} \exp(\alpha x_k)}$.

$$\frac{\exp(\alpha y_j^{(1)}) + \sum_{k \neq i} \exp(\alpha x_k)}{\exp(\alpha y_j^{(1)})} = \frac{1}{\sum_{k \neq i} \exp(\alpha x_k)} + \frac{1}{\exp(\alpha y_j^{(1)})} < \frac{2}{\exp(\alpha y_1)} \rightarrow 0 \text{ as } \alpha \rightarrow \infty.$$

Using the equilibrium property of bid $y_j^{(1)}$ by contestant j , we can then finally show in step 4 that bidding $y_j^{(1)}$, relative to bidding zero, against μ_{-i}^* by contestant i entails slightly more effort-cost but results in a larger gain in gross benefit. Therefore, $y_j^{(1)}$ strictly dominates zero for contestant i , contradicting $0 \in \text{supp}\{\mu_i^*\}$.

As consequences of Propositions 1 and 2, we have the following observation (a rigorous proof is found in Appendix A).

Corollary 1. *In the Hirshleifer contest between $n \geq 2$ risk-neutral contestants with the same linear effort-cost function, and, respectively, with valuations $V_1 \geq V_2 = \dots = V_m > V_{m+1} \geq \dots \geq V_n > 0$ (let $V_{m+1} = 0$ if $m = n$),*

- (i) *For any $b \in (0, 1)$, there exists $\alpha^\# = \alpha^\#(b, V_2, n, m)$ such that at any $\alpha > \alpha^\#$, no less than two contestants $i \in N$ among the m top ones bid $y_i^{(1)}(\alpha) > bV_2$, and all the others, including each contestant $j > m$ if any, bid zero with certainty;*
- (ii) *When $V_1 = V_2$, at least $m - 1$ contestants among the top m ones bid zero with positive probability at any $\alpha > 0$, and the only one, if any, who does not do so bids arbitrarily close to zero as α grows unboundedly;*
- (iii) *When $V_1 > V_2$, contestant one's highest bid is arbitrarily close to V_2 and the lowest bid, which remains positive at any finite α , is arbitrarily close to zero as α grows unboundedly, while any contestant $i \neq 1$ bids zero with $\mu_i^*(\{0\})$ such that $\prod_{k \neq 1} \mu_k^*(\{0\})$ is arbitrarily close to $\frac{V_1 - V_2}{V_1}$ as α grows unboundedly.*

In the case of $V_1 = V_2 = V$, by part (i) of the corollary, every contestant either remains passive (bidding zero with probability one) or is very active (bidding arbitrarily close to V with some positive probability) as the noise of the contest becomes small enough. Such polarization in bidding among the contestants with equal and top-level valuation under small noise is, in and of itself, of interest. For instance, the expected payoffs could differ among ex ante identical contestants. In the case of $V_1 > V_2$, while any contestant whose valuation is strictly below V_2 completely drops out when $\alpha > \alpha^\#$, each of those of valuation V_2 bids zero with a significantly large probability that is at least around $\frac{V_1 - V_2}{V_1}$ as α grows unbounded, by part (ii) of the corollary. This aligns with one well-known observation on the all-pay auction with $V_1 > V_2$ (refer to part A of Theorem 2 in Baye et al., 1996).

4. Illustrative Examples: Quick Convergence to Passivity and Passivity Under Risk Aversion

We offer two illustrative examples, as described in the above section title.

Example 1. *How quickly is the convergence to polarized bidding among contestants?*

Consider the Hirshleifer contest between three homogeneous constants, all with the same valuation of the prize, normalized to one. Since at least two contestants, say contestants 1 and 2, become aggressive in bidding as the noise of the contest abates to a certain extent (refer to Lemma 2(i) and Corollary 1(i)), it is of interest to explore how quickly passivity emerges as the equilibrium strategy for some contestants over the course. For that purpose, we focus on such possible semi-symmetric equilibria in which contestants 1 and 2 play the same strategy, while contestant 3 behaves differently. It goes without saying that such a symmetric contest often admits multiple equilibria, mostly being asymmetric (refer to Section 4.1 in Ewerhart and Sun (2024)). What is of particular interest is the possible

scenario in which both contestants 1 and 2 aggressively bid in the sense that each possibly bids more than a half of the prize and therefore ex post rent overdissipation (the total bid by all exceeds the value of the prize) occurs with positive probability (Baye et al., 1999), while contestant 3 is passive.

It is already shown via numerical simulation in Ewerhart and Sun (2024, Section 4.2, Example 2) that when $\alpha \in (4.12, 7.02)$, for the three-player contest described above, there exists such an equilibrium that two constants randomize over zero and a positive bid while the other one plays a zero-bid pure strategy. As an extension, we used Mathematica 14.0 to make simulation allowing for a larger range of the parameter. We found that, when $\alpha \in (7.02, 9.20)$, at equilibrium the strategy of contestant 3, against two aggressive opponents both of whom individually randomize over zero and two positive bids, is being passive. For instance, when $\alpha = 8.12$, each of constants 1 and 2 bids 0.628, 0.313 and zero respectively with probability 0.416, 0.203 and 0.381, and contestant 3 bids zero with probability one. That is, when $\alpha = 8.12$, the chance of rent overdissipation is already around 17%, while passivity remains the equilibrium strategy for the other individual. In Figure 1, we illustrate the expected payoff of contestant 3 against two aggressive opponents for any α over the interval $(7.02, 9.20)$. Passivity clearly stands as the best reply for the contestant. When α is even larger, passivity appears to be the equilibrium strategy for one contestant against two aggressive opponents with a larger support that contains zero and 3 or more positive bids.

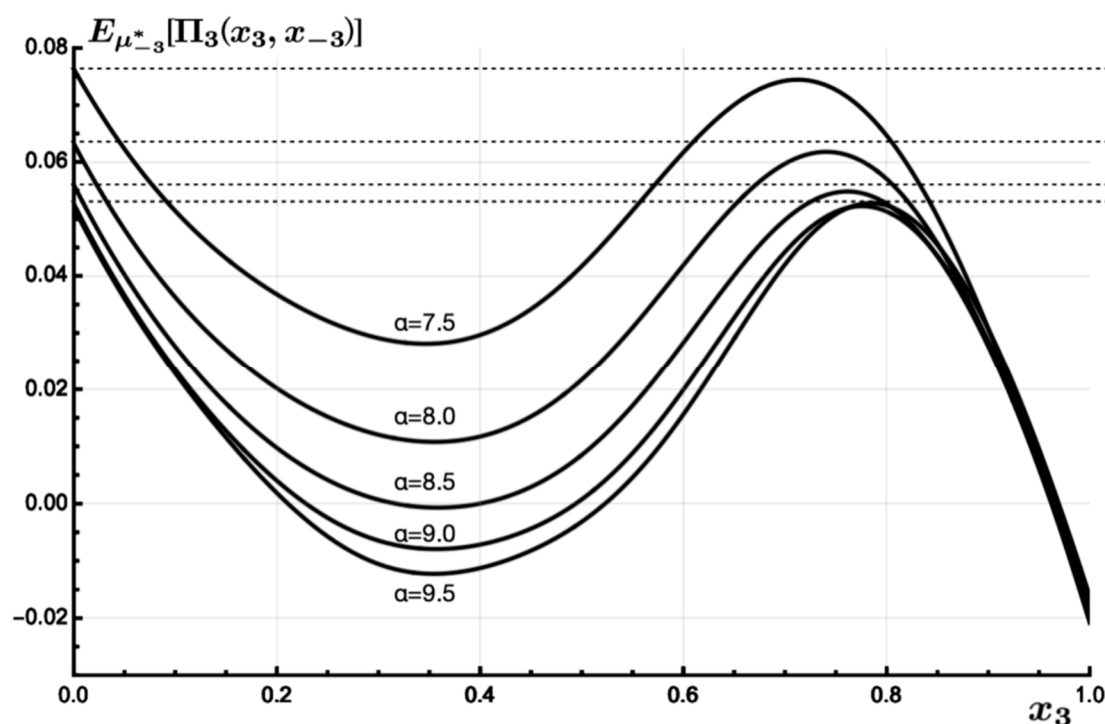


Figure 1. The expected payoff of contestant 3 against two aggressive opponents.

What can we learn from this simple example? Perhaps not much, except that polarization in bidding among contestants (even if ex ante identical) can emerge surprisingly quickly. In other words, the strong qualification in the statement of Proposition 2 on the noise, viz. “sufficiently large α ”, might better be loosely interpreted as a condition of “(for) not too small α ”, to make the insight that passivity could be a powerful choice even sharper. That said, the example only serves an illustrative purpose, and therefore we cannot go too far here to risk overinterpretation.

Example 2. *Passivity in the Hirshleifer contest with risk averse contestants.*

Our analysis has thus far been confined to the Hirshleifer contest with risk-neutral contestants. We now consider a Hirshleifer contest, which is the same as that described in Section 2 except the contestants' risk attitude. We assume that all contestants are risk averse, and everyone has a constant measure of absolute risk aversion γ_i , $i \in N$. Therefore, any contestant i 's preference can be described by a CARA utility function $u_i(c) = 1 - \exp(-\gamma_i c)$, where c is consumption, and parameter $\gamma_i > 0$ is the so-called Arrow-Pratt measure that could differ across the contestants. Denote by W_i the endowment of any player i . The expected utility of any contestant i then equals $p_i(x_i, x_{-i})u_i(V_i + W_i - x_i) + (1 - p_i(x_i, x_{-i}))u_i(W_i - x_i)$ when bidding x_i against a bid profile x_{-i} by the other contestants. At any $\alpha > 0$, a standard argument would do for establishing the existence of an equilibrium in mixed strategies, still denoted as $\mu^* = (\mu_1^*, \dots, \mu_n^*)$. At equilibrium, the expected payoff of contestant i when bidding $x_i \in \text{supp}\{\mu_i^*\}$ against μ_{-i}^* equals $E_{\mu_{-i}^*}[\Pi_i(x_i, x_{-i})] = E_{\mu_{-i}^*}[p_i(x_i, x_{-i})u_i(V_i + W_i - x_i) + (1 - p_i(x_i, x_{-i}))u_i(W_i - x_i)]$. By the analytical techniques described in Section 2, one finds that the support of any contestant's equilibrium strategy is a finite set. The equilibrium payoff, therefore, assumes a polylinear form in probabilities with a finite number of terms.

Based on the above observations, we show that the first half of Lemma 1(ii) holds still, i.e., zero is contained in the support of at least $n - 1$ contestants. The argument runs as follows. Take any positive element $x_i \in \text{supp}\{\mu_i^*(\alpha)\}$, and let $\Delta u_i(x_i) \equiv u_i(V_i + W_i - x_i) - u_i(W_i - x_i)$. Recall that for the Hirshleifer CSF, $\frac{\partial}{\partial x_i} p_i(x_i, x_{-i}) = \alpha p_i(1 - p_i)$ for any bid x_i against a profile x_{-i} by the other contestants. In the light of the above observation that $E_{\mu_{-i}^*}[\Pi_i(x_i, x_{-i})]$ assumes a polylinear form in probabilities with a finite number of terms under μ_{-i}^* , we can exchange differentiation and integration to compute $\frac{\partial}{\partial x_i} E_{\mu_{-i}^*}[\pi_i(x_i, x_{-i})]$. The first-order condition at $x_i > 0$ for contestant i turns out to be $E_{\mu_{-i}^*}[\alpha p_i(x_i, x_{-i})(1 - p_i(x_i, x_{-i}))\Delta u_i(x_i) - [p_i(x_i, x_{-i})u'_i(V_i + W_i - x_i) + (1 - p_i(x_i, x_{-i}))u'_i(W_i - x_i)]] = 0$, i.e.,

$$\begin{aligned} & \alpha \Delta u_i(x_i) E_{\mu_{-i}^*}[p_i(x_i, x_{-i})(1 - p_i(x_i, x_{-i}))] \\ & = u'_i(V_i + W_i - x_i) E_{\mu_{-i}^*}[p_i(x_i, x_{-i})] + u'_i(W_i - x_i) E_{\mu_{-i}^*}[1 - p_i(x_i, x_{-i})]. \end{aligned} \quad (1)$$

Noting $\frac{\partial \Delta u_i}{\partial x} = u'_i(W_i - x_i) - u'_i(V_i + W_i - x_i)$, $u''_i(c) = -\gamma_i u'_i(c)$ and $u'_i(c) = -\gamma_i [u_i(c) - 1]$ for any consumption c , we likewise compute the second-order condition,

$$\begin{aligned} & \frac{\partial^2}{\partial x_i^2} E_{\mu_{-i}^*}[\pi_i(x_i, x_{-i})] \\ & = \alpha^2 \Delta u_i(x_i) E_{\mu_{-i}^*}[(1 - 2p_i(x_i, x_{-i}))p_i(x_i, x_{-i})(1 - p_i(x_i, x_{-i}))] \\ & \quad + 2\alpha \gamma_i \Delta u_i(x_i) E_{\mu_{-i}^*}[p_i(x_i, x_{-i})(1 - p_i(x_i, x_{-i}))] \\ & \quad - \gamma_i u'_i(V_i + W_i - x_i) E_{\mu_{-i}^*}[p_i(x_i, x_{-i})] - \gamma_i u'_i(W_i - x_i) E_{\mu_{-i}^*}[(1 - p_i(x_i, x_{-i}))] \\ & = \alpha^2 \Delta u_i(x_i) E_{\mu_{-i}^*}[(1 - 2p_i(x_i, x_{-i}))p_i(x_i, x_{-i})(1 - p_i(x_i, x_{-i}))] \\ & \quad + \gamma_i \left\{ 2\alpha \Delta u_i(x_i) E_{\mu_{-i}^*}[p_i(x_i, x_{-i})(1 - p_i(x_i, x_{-i}))] \right. \\ & \quad \left. - u'_i(V_i + W_i - x_i) E_{\mu_{-i}^*}[p_i(x_i, x_{-i})] - u'_i(W_i - x_i) E_{\mu_{-i}^*}[(1 - p_i(x_i, x_{-i}))] \right\} \\ & < 0. \end{aligned}$$

But what is in braces in the second last term, by FOC (Equation (1)), equals $\alpha \Delta u_i(x_i) E_{\mu_{-i}^*}[p_i(x_i, x_{-i})(1 - p_i(x_i, x_{-i}))] > 0$. Therefore,

$$E_{\mu_{-i}^*}[(1 - 2p_i(x_i, x_{-i}))p_i(x_i, x_{-i})(1 - p_i(x_i, x_{-i}))] < 0.$$

Should there be two contestants i and j who never bid zero at some $\alpha > 0$, we may assume, without loss of generality, that the smallest element of contestant i 's support, denoted as $\underline{y}_i$, is the minimum among all the positive elements of $\bigcup_{k \in N} \text{supp}\{\mu_k^*(\alpha)\}$. Hence, $\underline{y}_i \leq \underline{y}_j$, where $\underline{y}_j$ is the smallest number among $\text{supp}\{\mu_j^*(\alpha)\}$. Then, against any possible bid profile x_{-i} by the other contestants, the probability of winning the prize by contestant i when bidding $\underline{y}_i$ is at most a half. The probability equals a half if and only if contestant j plays a pure strategy that equals $\underline{y}_i$ and $n = 2$. But that is impossible, since the SOC, by the above reasoning, then turns out to be (noting $p_j(\underline{y}_i, \underline{y}_j) = 1/2$),

$$\frac{\partial^2}{\partial y_i^2} \pi_i(\underline{y}_i, \underline{y}_j) = \frac{\gamma \alpha \Delta u(\underline{y}_i)}{2} > 0.$$

We are ready to show that at least $n - 1$ contestants bid zero with positive probability at any $\alpha > 0$. Suppose otherwise. For the contestant who is always active (never bids zero) and whose minimal element among his support is no greater than the minimal bid of any other always active contestant, by the above analysis, SOC at his minimal bid does not hold. That is, the first half of part (ii) in Lemma 1 holds in the case that each contestant has a constant Arrow–Pratt measure of absolute risk aversion (it is perhaps worth emphasizing again that the claim allows for Arrow–Pratt measures that are different across contestants).

We now outline, without providing algebraic details (available upon request from the author) to save space, an argument for that Proposition 1 holds true for the Hirshleifer contest between risk-averse contestants with CARA preferences.⁴ We only need to appropriately modify the proof by negation for the contest with risk-neutral contestants. Suppose that there exists contestant i who always bids at minimum positive $\underline{y}_i(\alpha) \equiv \min\{\text{supp}\{\mu_i^*(\alpha)\}\}$ that approaches some $b > 0$ as $\alpha \rightarrow \infty$. No contestant bids positive weakly below $\underline{y}_i(\alpha)$, by the second-order condition at any interior bid as is shown in the reasoning above. Therefore, the reason that contestant j always bids close to $b > 0$ for any α large enough is that each contestant $i \neq j$ necessarily bids zero or slightly above $\underline{y}_i(\alpha)$ with a substantially large probability. That in turn implies that contestant j can then substantially raise the probability of winning when against some x_{-j} under μ_{-j}^* , thereby raising the value of $p_j(x_j, x_{-j}) \Delta u_j(x_j)$, at the expense of a marginal reduction in $u_j(W_j - x_j)$, by bidding $\underline{y}_j(\alpha) + \frac{1}{\sqrt{\alpha}}$ rather than bidding $\underline{y}_j(\alpha)$ at any large α .⁵ It then results in an increase in $E_{\mu_{-j}^*} [p_j(x_j, x_{-j}) u_j(V_j + W_j - x_j) + (1 - p_j(x_j, x_{-j})) u_j(W_j - x_j)]$, contradicting $\underline{y}_j(\alpha) \in \text{supp}\{\mu_j^*(\alpha)\}$. As far as Proposition 2 is concerned, for the contest with two contestants, it is not difficult to see that both bid at most close to V_2 if $W_i > V_i$ for each contest $i = 1, 2$, as α grows unboundedly. When $n \geq 3$, however, we have not been able to rigorously prove, or falsify by a counterexample, Proposition 2 for risk-averse contestants with CARA preferences.

5. Concluding Remarks

Our analysis depicts a polarized picture of the equilibrium strategy profiles in the Hirshleifer contest when the noise of the contest is sufficiently small: any contestant either bids aggressively with positive probability, or simply drops out. Indeed, the power of passivity figures prominently in a class of conflict, a theme that Hirshleifer powerfully expounded over decades of inquiry into what he refers to as “the dark side of the force” (Hirshleifer, 2001). We further formalize the idea, showing that ex post polarization in bidding among (possibly ex ante identical) contestants emerges as the noise of the contest abates: some contestants bid widely, ranging from zero (or very close to zero) to some level of effort close to the second highest valuation, while the rest play a pure strategy of passivity as the value parameter α reaches some finite number.

We speculate that Proposition 1 and, more significantly, Proposition 2, hold not only for Hirshleifer contests but for quite some other difference-form contests as well. To rigorously prove the propositions, especially Proposition 2, for a class of difference-form contests, however, appears challenging. Recall that the most difficult part in our proof by negation of Proposition 2 is the argument used to deal with the case $V_1 > V_2$ (refer to the Appendix A). Indeed, much effort is made there in establishing inequalities (A21) and (A22) in order to demonstrate that $y_j^{(1)}$ strictly dominates zero for some contestant i who bids zero at equilibrium, hence a contradiction. In deriving these inequalities, we invoke several technically useful results on the Hirshleifer contest technology, either borrowed from other work (Lemmas 1 and 2) or established in the present study (Lemmas 3 and A1). To follow more or less the same argument to prove Proposition 2 for a class of difference-form contests, therefore, necessitates verifying a number of similar claims, a task that turns out to be difficult. Alternatively, one may develop an argument radically differing from the one used in the present analysis. Moreover, it is as tempting to explore whether the proposition holds for multiplayer contests with multiplicative noise, such as the Tullock contest. Due to a remarkable lack of understanding of the equilibrium set of such contests under vanishing noise in the literature, however, this open question appears to be even harder.

Another valuable avenue of further exploration is how risk attitude shapes the equilibrium in difference-form contests, an entirely uncharted territory in contest theory as far as we know. We demonstrate in the preceding section that, should all contestants be risk averse and endowed with CARA utility functions, our Proposition 1 holds still. However, whether Proposition 2 holds is an open question. Indeed, for the Hirshleifer contest between any number of risk-averse contestants (not necessarily with some CARA preference), it remains unknown whether zero is contained in the equilibrium supports of at least $n - 1$ contestants at any $\alpha > 0$, a crucial step for establishing Propositions 1 and 2 for risk-neutral Hirshleifer contestants. To the best of our knowledge, no serious work has been done on implications of risk attitude for the equilibrium set in difference-form contests, a topic that appears significantly rewarding, especially in the light of the explanatory power of such CSF in the study of war. We, therefore, leave for future research the problem of how risk attitude shapes the equilibrium set at any level of the noise of the contest and convergence of the set as the noise abates and ultimately vanishes.

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Appendix A

Proof of Proposition 1. We first deal with the case that $V_1 = V_2$. By Lemma 1(ii), at any α at most one contestant doesn't bid zero with positive probability. If some contestant j never bids zero for a sequence of α that goes to infinity, we show that $\liminf_{\alpha \rightarrow \infty} \{ \text{supp} \{ \mu_j^*(\alpha) \} \} = 0$. Suppose otherwise. To ease argument, we assume without loss of generality that there exists $b > 0$ such that $\underline{y}_j(\alpha) \equiv \min \{ \text{supp} \{ \mu_j^*(\alpha) \} \} \rightarrow b$ as $\alpha \rightarrow \infty$. We now show that such an assumption necessarily leads to an absurdity.

By Lemma 2(iv),

$$\exists i \neq j, \mu_i^*(\{0\}) \rightarrow 0 \text{ as } \alpha \rightarrow \infty. \quad (\text{A1})$$

We now invoke (A1) to show that $E_{\mu_{-j}^*} \left[\Pi_j \left(\underline{y}_j + \frac{1}{\sqrt{\alpha}}, x_{-j} \right) \right] > \frac{b}{4}$ for large α . Note

$$\begin{aligned} 0 &< E_{\mu_{-j}^*} \left[\Pi_j \left(\underline{y}_j, x_{-j} \right) \right] \\ &= V_j E_{\mu_{-j}^*} \left[p_j \left(\underline{y}_j, x_{-j} \mid \forall k \neq j, x_k \leq \underline{y}_j \right) \right] \\ &+ V_j E_{\mu_{-j}^*} \left[p_j \left(\underline{y}_j, x_{-j} \mid \exists k \neq j, x_k > \underline{y}_j; \forall k \neq j, x_k \leq \underline{y}_j + \frac{1}{2\sqrt{\alpha}} \right) \right] \\ &+ V_j E_{\mu_{-j}^*} \left[p_j \left(\underline{y}_j, x_{-j} \mid \exists k \neq j, x_k > \underline{y}_j + \frac{1}{2\sqrt{\alpha}} \right) \right] - \underline{y}_j. \end{aligned} \quad (\text{A2})$$

By the second-order condition at the smallest positive mass point (if any) of any $k \neq j$, contestant k has only one single mass point (zero) below $\underline{y}_j$. Consequently, the first term of right-hand side (RHS) of Equation (A2) equals $\frac{V_j}{1+(n-1)\exp(-\alpha\underline{y}_j)} \Pi_{k \neq j} \mu_k^*(\{0\})$, which is smaller than $V_j \Pi_{k \neq j} \mu_k^*(\{0\}) \rightarrow 0$ by claim (A1), and the second term is no greater than $\frac{V_j}{2} \Pi_{k \neq j} \mu_k^* \left(\left[0, \underline{y}_j + \frac{1}{2\sqrt{\alpha}} \right] \right)$, while the third term is at most an infinitesimal $O\left(\exp\left(-\frac{1}{2}\sqrt{\alpha}\right)\right)$. In the light of that $V_j E_{\mu_{-j}^*} \left[p_j \left(\underline{y}_j, x_{-j} \right) \right] > \underline{y}_j \geq b > 0$, we then obtain,

$$0 < \frac{3}{5} V_j \Pi_{k \neq j} \mu_k^* \left(\left[0, \underline{y}_j + \frac{1}{2\sqrt{\alpha}} \right] \right) - \underline{y}_j \text{ for any large } \alpha. \quad (\text{A3})$$

By Lemma 3(i), we observe that $E_{\mu_{-j}^*} \left[\Pi_j \left(\underline{y}_j + \frac{1}{\sqrt{\alpha}}, x_{-j} \right) \right]$ is no less than $[1 - O(\exp(-\frac{1}{2}\sqrt{\alpha}))] \times V_j \Pi_{k \neq j} \mu_k^* \left(\left[0, \underline{y}_j + \frac{1}{2\sqrt{\alpha}} \right] \right) - \left(\underline{y}_j + \frac{1}{\sqrt{\alpha}} \right)$. In view of inequality (A3), one then observes that for α large enough,

$$\begin{aligned} &E_{\mu_{-j}^*} \left[\Pi_j \left(\underline{y}_j + \frac{1}{\sqrt{\alpha}}, x_{-j} \right) \right] \\ &> \frac{9}{10} V_j \Pi_{k \neq j} \mu_k^* \left(\left[0, \underline{y}_j + \frac{1}{2\sqrt{\alpha}} \right] \right) - \left(\underline{y}_j + \frac{1}{\sqrt{\alpha}} \right) \\ &> \frac{9}{10} \times \frac{5}{3} \underline{y}_j - \left(\underline{y}_j + \frac{1}{\sqrt{\alpha}} \right) \\ &= \frac{1}{2} \underline{y}_j - \frac{1}{\sqrt{\alpha}} \\ &> \frac{1}{4} b, \end{aligned}$$

contradicting Lemma 2(ii).

When $V_1 > V_2$, by Lemma 2(i), $\Pi_1^* > V_1 - V_2 - \varepsilon$ holds for any sufficiently large α , where ε is a small positive number. Then, by Lemma 1(ii), $0 \notin \text{supp}\{\mu_1^*(\alpha)\}$ and $0 \in \text{supp}\{\mu_i^*(\alpha)\}$, $\forall i \geq 2$ for any α large. Suppose that there exists some $b > 0$ such that $\underline{y}_1(\alpha) \rightarrow b$ as $\alpha \rightarrow \infty$. If $\Pi_{k \neq 1} \mu_k^*(\{0\}) \rightarrow 0$ as $\alpha \rightarrow \infty$, we can slightly modify the argument as above for the case of $V_1 = V_2$ and use Lemma 3(i) to obtain $E_{\mu_{-1}^*} \left[\Pi_1 \left(\underline{y}_1 + \frac{1}{\sqrt{\alpha}}, x_{-1} \right) \right] > \frac{3}{2}(V_1 - V_2) + \frac{1}{4}b$, in conflict with Lemma 2(ii). We can therefore conclude, without loss of generality, that,

$$\exists \delta > 0, \Pi_{k \neq 1} \mu_k^*(\{0\}) \rightarrow \delta, \text{ as } \alpha \rightarrow \infty. \quad (\text{A4})$$

We now show that,

$$\prod_{k \neq 1} \mu_k^* \left(\left[0, \underline{y}_1 + \frac{1}{\sqrt{\alpha}} \right] \right) - \prod_{k \neq 1} \mu_k^* \left(\left[0, \underline{y}_1 \right] \right) \rightarrow 0 \text{ as } \alpha \rightarrow \infty. \quad (\text{A5})$$

Suppose not. That is, without loss of generality, there exists some $\delta' > 0$,

$$\prod_{k \neq 1} \mu_k^* \left(\left[0, \underline{y}_1 + \frac{1}{\sqrt{\alpha}} \right] \right) - \prod_{k \neq 1} \mu_k^* \left(\left[0, \underline{y}_1 \right] \right) \geq \delta' \text{ for any } \alpha \text{ large.} \quad (\text{A6})$$

Then by bidding $\underline{y}_1 + \frac{2}{\sqrt{\alpha}}$, by Lemma 3(i), contestant one receives a payoff

$$\begin{aligned} & E_{\mu_{-1}^*} \left[\Pi_1 \left(\underline{y}_1 + \frac{2}{\sqrt{\alpha}}, x_{-1} \right) \right] \\ &= V_1 E_{\mu_{-1}^*} \left[p_1 \left(\underline{y}_1 + \frac{2}{\sqrt{\alpha}}, x_{-1} \right) \right] - \left(\underline{y}_1 + \frac{2}{\sqrt{\alpha}} \right) \\ &\geq V_1 \left[1 - O(e^{-\sqrt{\alpha}}) \right] \prod_{k \neq 1} \mu_k^* \left(\left[0, \underline{y}_1 + \frac{1}{\sqrt{\alpha}} \right] \right) - \left(\underline{y}_1 + \frac{2}{\sqrt{\alpha}} \right). \end{aligned} \quad (\text{A7})$$

But

$$\begin{aligned} & E_{\mu_{-1}^*} \left[\Pi_1 \left(\underline{y}_1, x_{-1} \right) \right] \\ &= V_1 E_{\mu_{-1}^*} \left[p_1 \left(\underline{y}_1, x_{-1} \mid \forall k \neq 1, x_k \leq \underline{y}_1 \right) \right] \\ &+ V_1 E_{\mu_{-1}^*} \left[p_1 \left(\underline{y}_1, x_{-1} \mid \exists k \neq 1, x_k > \underline{y}_1; \forall k \neq 1, x_k \leq \underline{y}_1 + \frac{1}{\sqrt{\alpha}} \right) \right] \\ &+ V_1 E_{\mu_{-1}^*} \left[p_1 \left(\underline{y}_1, x_{-1} \mid \exists k \neq 1, x_k > \underline{y}_1 + \frac{1}{\sqrt{\alpha}} \right) \right] - \underline{y}_1. \end{aligned} \quad (\text{A8})$$

The first term of RHS of Equation (A8) is less than $V_1 \prod_{k \neq 1} \mu_k^* \left(\left[0, \underline{y}_1 \right] \right)$, the second term is no greater than $\frac{V_1}{2} \left[\prod_{k \neq 1} \mu_k^* \left(\left[0, \underline{y}_1 + \frac{1}{\sqrt{\alpha}} \right] \right) - \prod_{k \neq 1} \mu_k^* \left(\left[0, \underline{y}_1 \right] \right) \right]$, while the third term is an infinitesimal $O(\exp(-\sqrt{\alpha}))$. Thus, for α large enough, by Formulas (A7) and (A8), $E_{\mu_{-1}^*} \left[\Pi_1 \left(\underline{y}_1 + \frac{2}{\sqrt{\alpha}}, x_{-1} \right) \right] - E_{\mu_{-1}^*} \left[\Pi_1 \left(\underline{y}_1, x_{-1} \right) \right]$ is approaching at least $\frac{V_1}{2} \left[\prod_{k \neq 1} \mu_k^* \left(\left[0, \underline{y}_1 + \frac{1}{\sqrt{\alpha}} \right] \right) - \prod_{k \neq 1} \mu_k^* \left(\left[0, \underline{y}_1 \right] \right) \right] - \frac{2}{\sqrt{\alpha}}$,⁶ which, by claim (A6), exceeds $\frac{\delta' V_1}{4} > 0$ for α large enough, contradicting $\underline{y}_1 \in \text{supp}\{\mu_1^*\}$. This proves inequality (A5).

Since no contestant $k \neq 1$ possibly bids any positive effort below $\underline{y}_1$, $\mu_k^* \left(\left[0, \underline{y}_1 \right] \right) = \mu_k^* (\{0\})$, $\forall k \neq 1$. Thus, by Formulas (A4) and (A5), $\prod_{k \neq 1} \mu_k^* \left(\left[0, \underline{y}_1 + \frac{1}{\sqrt{\alpha}} \right] \right) \rightarrow \delta$ as $\alpha \rightarrow \infty$. Then, by Lemma 3(ii), $E_{\mu_{-1}^*} \left[\Pi_1 \left(\underline{y}_1, x_{-1} \right) \right] \leq V_1 [1 + O(\exp(-\sqrt{\alpha}))] \prod_{k \neq 1} \mu_k^* \left(\left[0, \underline{y}_1 + \frac{1}{\sqrt{\alpha}} \right] \right) - \underline{y}_1$, which in turn approaches $\delta V_1 - b$ as $\alpha \rightarrow \infty$. Apparently, $E_{\mu_{-1}^*} \left[\Pi_1 \left(\frac{b}{2}, x_{-1} \right) \right]$ approaches $\delta V_1 - \frac{b}{2}$. Thus, $E_{\mu_{-1}^*} \left[\Pi_1 \left(\frac{b}{2}, x_{-1} \right) \right] > E_{\mu_{-1}^*} \left[\Pi_1 \left(\underline{y}_1, x_{-1} \right) \right] + \frac{b}{4}$ for large α , contradicting $\underline{y}_1 \in \text{supp}\{\mu_1^*\}$.

This completes the proof of Proposition 1. $\square$

Proof of Proposition 2. The statement obviously holds when $n = 2$ since, by Lemma 2(i), both contestants bid arbitrarily close to V_2 . In what follows we assume $n \geq 3$.

By Lemma 2(iii), $\forall i \in N$, either $y_i^{(1)}(\alpha) > V_2 - \varepsilon$ or $y_i^{(1)}(\alpha) < \varepsilon$ for sufficiently large α . We now show that for any contestant j such that $\limsup_{\alpha \rightarrow \infty} \{y_j^{(1)}(\alpha)\} = 0$, $y_j^{(1)}(\alpha) = 0$ holds

for α large enough. Suppose not; that is, without loss of generality, for some contestant j , $y_j^{(1)}(\alpha) \rightarrow 0$ but $y_j^{(1)}(\alpha) > 0$ for any $\alpha > 0$. We shall show that this would lead to an absurdity.

We first consider the case that $V_1 = V_2$. In this case, by the second-order condition at $y_j^{(1)}(\alpha)$, namely $\alpha^2 V_j E_{\mu_{-j}^*} \left\{ p_j(y_j^{(1)}, x_{-j}) \left[1 - p_j(y_j^{(1)}, x_{-j}) \right] \left[1 - 2p_j(y_j^{(1)}, x_{-j}) \right] \right\} < 0$, at least for some profile x_{-j} , $p_j(y_j^{(1)}, x_{-j}) > 1/2$. Note that $p_j(y_j^{(1)}, x_{-j}) \leq p_j(y_j^{(1)}, 0_{n-1}) = \frac{\exp(\alpha y_j^{(1)})}{\exp(\alpha y_j^{(1)}) + (n-1)}$. We thus have,

$$y_j^{(1)}(\alpha) > \frac{\ln(n-1)}{\alpha}. \quad (\text{A9})$$

The rest of our argument will proceed in three steps.

Step 1. We first observe that there exists a set of contestants that contains at least two contestants $i \in N$ such that $\mu_i^*([0, \varepsilon(\alpha)]) \rightarrow 0$ as $\alpha \rightarrow \infty$, where $\varepsilon(\alpha)$ is an appropriately defined infinitesimal such that $y_j^{(1)} < \varepsilon(\alpha) \rightarrow 0$ as $\alpha \rightarrow \infty$. Otherwise, there exists a contestant that, by bidding slightly above $\varepsilon(\alpha)$ at any large α , would receive a positive payoff bounded away from zero, in conflict with Lemma 2(ii). Apparently, contestant j does not belong to this set. By Lemma 1(ii), therefore, there exists contestants i belonging to the set such that $0 \in \text{supp}\{\mu_i^*\}$. We shall choose $\varepsilon(\alpha) = y_j^{(1)} + \frac{1}{\sqrt{\alpha}}$, for algebraic simplicity in steps 2 and 3, below.

Step 2. We show that for any α large, by bidding $y_j^{(1)}$, contestant i would win with a much greater probability than that for contestant j when bidding $y_j^{(1)}$; that is, $E_{\mu_{-i}^*} [p_i(y_j^{(1)}, x_{-i})] \gg E_{\mu_{-j}^*} [p_j(y_j^{(1)}, x_{-j})]$. To facilitate analysis in Step 3, however, it would do by establishing Formula (A11), below. Note,

$$\begin{aligned} & E_{\mu_{-j}^*} [p_j(y_j^{(1)}, x_{-j})] \\ & \leq \mu_i^* \left(\left[0, y_j^{(1)} + \frac{1}{\sqrt{\alpha}} \right] \right) E_{\mu_{-i,j}^*} [p_j(y_j^{(1)}, 0, x_{-i,j})] \\ & + \left[1 - \mu_i^* \left(\left[0, y_j^{(1)} + \frac{1}{\sqrt{\alpha}} \right] \right) \right] E_{\mu_{-i,j}^*} \left[p_j \left(y_j^{(1)}, y_j^{(1)} + \frac{1}{\sqrt{\alpha}}, x_{-i,j} \right) \right]. \end{aligned} \quad (\text{A10})$$

Further notice that

$$\begin{aligned} & p_j(y_j^{(1)}, 0, x_{-i,j}) \\ & = \frac{2\exp(\alpha y_j^{(1)})}{2 \left[\exp(\alpha y_j^{(1)}) + 1 + \sum_{k \neq i,j} \exp(\alpha x_k) \right]} \\ & < \frac{2\exp(\alpha y_j^{(1)})}{2\exp(\alpha y_j^{(1)}) + \sum_{k \neq i,j} \exp(\alpha x_k)} \\ & = 2p_i(y_j^{(1)}, y_j^{(1)}, x_{-i,j}). \end{aligned}$$

When α is sufficiently large, the value of $\mu_i^* \left(\left[0, y_j^{(1)} + \frac{1}{\sqrt{\alpha}} \right] \right)$ is very small, below $\frac{V_i}{8V_j}$ of course, and therefore the first term of RHS of Equation (A10) must be smaller than $\frac{V_i}{4V_j} E_{\mu_{-i,j}^*} [p_i(y_j^{(1)}, y_j^{(1)}, x_{-i,j})] \leq \frac{V_i}{4V_j} E_{\mu_{-i,j}^*} [p_i(y_j^{(1)}, x_j, x_{-i,j})] \equiv \frac{V_i}{4V_j} E_{\mu_{-i}^*} [p_i(y_j^{(1)}, x_{-i})]$. The second term is no greater than $E_{\mu_{-i,j}^*} \left[p_j \left(y_j^{(1)}, y_j^{(1)} + \frac{1}{\sqrt{\alpha}}, x_{-i,j} \right) \right] < \exp(-\sqrt{\alpha})$. Invoking inequality (A9), we obtain $V_j E_{\mu_{-j}^*} [p_j(y_j^{(1)}, x_{-j})] > y_j^{(1)} > \frac{\ln(n-1)}{\alpha}$. Therefore, the

second term on RHS of (A10) must be smaller than $\frac{1}{4}E_{\mu_{-j}^*}[p_j(y_j^{(1)}, x_{-j})]$ when α is large enough to render $\alpha \exp(-\sqrt{\alpha}) < \frac{\ln(n-1)}{4V_j}$. As a consequence, when α is sufficiently large, $E_{\mu_{-j}^*}[p_j(y_j^{(1)}, x_{-j})] < \frac{V_i}{4V_j}E_{\mu_{-i}^*}[p_i(y_j^{(1)}, x_{-i})] + \frac{1}{4}E_{\mu_{-j}^*}[p_j(y_j^{(1)}, x_{-j})]$. Hence,

$$V_i E_{\mu_{-i}^*}[p_i(y_j^{(1)}, x_{-i})] > 3V_j E_{\mu_{-j}^*}[p_j(y_j^{(1)}, x_{-j})]. \quad (\text{A11})$$

Step 3. We invoke inequality (A11) to show $E_{\mu_{-i}^*}[\Pi_i(y_j^{(1)}, x_{-i})] > E_{\mu_{-i}^*}[\Pi_i(0, x_{-i})]$ for α large. By Lemma 1(i), the support of each contestant is finite. We start with,

$$E_{\mu_{-i}^*}[p_i(0, x_{-i})] = \sum_{k \neq i} \frac{1}{1 + \sum_{k \neq i} \exp(\alpha x_k)} \prod_{k \neq i} \mu_k^*(\{x_k\}), \quad (\text{A12})$$

$$E_{\mu_{-i}^*}[p_i(y_j^{(1)}, x_{-i})] = \sum_{k \neq i} \frac{1}{1 + \sum_{k \neq i} \exp(\alpha(x_k - y_j^{(1)}))} \prod_{k \neq i} \mu_k^*(\{x_k\}). \quad (\text{A13})$$

Note that for any profile of $x_k, k \neq i$,

$$\begin{aligned} & \frac{1}{1 + \sum_{k \neq i} \exp(\alpha(x_k - y_j^{(1)}))} - \frac{1}{1 + \sum_{k \neq i} \exp(\alpha x_k)} \\ &= \frac{1 - \exp(-\alpha y_j^{(1)})}{1 + \sum_{k \neq i} \exp(\alpha(x_k - y_j^{(1)}))} \cdot \frac{\sum_{k \neq i} \exp(\alpha x_k)}{1 + \sum_{k \neq i} \exp(\alpha x_k)} \\ &\geq \frac{1}{1 + \sum_{k \neq i} \exp(\alpha(x_k - y_j^{(1)}))} \cdot (1 - \exp(-\alpha y_j^{(1)})) \cdot \frac{n-1}{n}. \end{aligned}$$

Also note that by inequality (A9), $1 - \exp(-\alpha y_j^{(1)}) > \frac{n-2}{n-1}$. As a consequence, subtracting (A12) from (A13) yields, $E_{\mu_{-i}^*}[p_i(y_j^{(1)}, x_{-i})] - E_{\mu_{-i}^*}[p_i(0, x_{-i})] > \frac{n-2}{n} E_{\mu_{-i}^*}[p_i(y_j^{(1)}, x_{-i})]$. By inequality (A11), in light of $n \geq 3$,

$$\begin{aligned} & V_i E_{\mu_{-i}^*}[p_i(y_j^{(1)}, x_{-i})] - V_i E_{\mu_{-i}^*}[p_i(0, x_{-i})] \\ &> \frac{3(n-2)}{n} V_j E_{\mu_{-j}^*}[p_j(y_j^{(1)}, x_{-j})] \\ &\geq V_j E_{\mu_{-j}^*}[p_j(y_j^{(1)}, x_{-j})] \\ &> y_j^{(1)}. \end{aligned}$$

Hence, $E_{\mu_{-i}^*}[\Pi_i(y_j^{(1)}, x_{-i})] > E_{\mu_{-i}^*}[\Pi_i(0, x_{-i})]$, contradicting $0 \in \text{supp}\{\mu_i^*\}$.

We now consider the case that $V_1 > V_2$. Suppose $V_1 > V_2 = \dots = V_m > V_{m+1} \geq \dots \geq V_n > 0$ (let $V_{m+1} = 0$ if $m = n$). In this case, as is analyzed in the proof of Proposition 1, $0 \in \text{supp}\{\mu_i^*\}$, $\forall i \neq 1$ and $0 \notin \text{supp}\{\mu_1^*\}$. Then, by invoking the same argument used to derive inequality (A9) we likewise obtain a lower bound of the bid by contestant one,

$$\underline{y}_1 \equiv \min\{\text{supp}\{\mu_1^*(\alpha)\}\} > \frac{\ln(n-1)}{\alpha}. \quad (\text{A14})$$

By Lemma 2(i), contestant one bids arbitrarily close to V_2 as α becomes sufficiently large. Hence, $j \neq 1$. By the second-order condition at $y_j^{(1)}(\alpha) \equiv \max\{\text{supp}\{\mu_j^*(\alpha)\}\}$ for contestant j , we obtain,

$$y_j^{(1)} > \frac{\ln(\exp(\alpha \underline{y}_1) + n - 2)}{\alpha} > \underline{y}_1. \quad (\text{A15})$$

The rest of the argument is organized into three parts. In the first and preparatory part, step 1, we introduce a lemma that is to be used in the second part (steps 2 and 3), and the proof of Corollary 1 as well.

Step 1. We claim,

Lemma A1. *In the Hirshleifer contest between $n \geq 3$ risk-neutral contestants with the same linear effort-cost function and respectively with valuations $V_1 > V_2 \geq \dots \geq V_n > 0$,*

- (i) $\prod_{k \neq 1} \mu_k^*(\{0\})$ approaches at least $\frac{V_1 - V_2}{V_1}$ as $\alpha \rightarrow \infty$;
- (ii) $\forall k \neq 1, \mu_k^*\left(\left[0, y_j^{(1)} + \frac{1}{\sqrt{\alpha}}\right]\right) - \mu_k^*\left(\left[0, \underline{y}_1\right]\right) \rightarrow 0$ as $\alpha \rightarrow \infty$;
- (iii) The value of $\alpha \underline{y}_1$ is unboundedly large as α grows unboundedly;
- (iv) $\exists \delta > 0, \exists i \neq 1, j, V_i = V_2$ and $\mu_i^*\left(\left[0, \frac{V_2}{2}\right]\right) < 1 - 2\delta$ at any sufficiently large α .⁷

Proof of Lemma A1.

(i) By Proposition 1, $\underline{y}_1 \rightarrow 0$ as $\alpha \rightarrow \infty$. By Lemma 2(ii), $E_{\mu_{-1}^*}[\Pi_1(\underline{y}_1, x_{-1})] = V_1 E_{\mu_{-1}^*}[p_1(\underline{y}_1, x_{-1})] - \underline{y}_1$ approaches $V_1 - V_2$ as $\alpha \rightarrow \infty$. Hence, $E_{\mu_{-1}^*}[p_1(\underline{y}_1, x_{-1})] > (V_1 - V_2)/(2V_1)$ for α large enough. Also note that no contestant $i \neq 1$ bids any positive effort weakly below $\underline{y}_1$ by the second-order condition at any interior bid. Then, by Lemma 3(ii) and Formula (A5), $E_{\mu_{-1}^*}[\Pi_1(\underline{y}_1, x_{-1})] \leq V_1[1 + O(\exp(-\sqrt{\alpha}))]$. $\prod_{k \neq 1} \mu_k^*\left(\left[0, \underline{y}_1 + \frac{1}{\sqrt{\alpha}}\right]\right) - \underline{y}_1 \rightarrow V_1 \prod_{k \neq 1} \mu_k^*(\{0\})$ as $\alpha \rightarrow \infty$. In light of Lemma 2(ii), we then establish the claim.

(ii) It follows from part (i) and Formula (A5).

(iii) Suppose otherwise, there then exists some finite $M < \infty$, such that $\exp(\alpha \underline{y}_1) \leq M, \forall \alpha$. Consider the first-order condition at $\underline{y}_1$ for contestant one,

$$\frac{1}{\alpha} = V_1 E_{\mu_{-1}^*}[p_1(\underline{y}_1, x_{-1}) (1 - p_1(\underline{y}_1, x_{-1}))]. \quad (\text{A16})$$

The left-hand side of Equation (A16) is arbitrarily close to zero as α grows unboundedly, while the RHS is greater than $V_1 p_1(\underline{y}_1, 0_{n-1}) (1 - p_1(\underline{y}_1, 0_{n-1})) \prod_{k \neq 1} \mu_k^*(\{0\}) = \frac{V_1(n-1)\exp(\alpha \underline{y}_1)}{[\exp(\alpha \underline{y}_1) + (n-1)]^2} \prod_{k \neq 1} \mu_k^*(\{0\}) > \frac{V_1(n-1)^2}{[\exp(\alpha \underline{y}_1) + (n-1)]^2} \prod_{k \neq 1} \mu_k^*(\{0\})$ as a consequence of inequality (A14). Hence, RHS of (A16) must be greater than

$$\prod_{k \neq 1} \mu_k^*(\{0\}) \rightarrow \frac{(V_1 - V_2)(n-1)^2}{[M + (n-1)]^2}, \text{ a constant positive number. We are thus led to an absurdity.}$$

(iv) By Lemma 2(iii), each contestant $k > m$ if any bids arbitrarily around zero with certainty when α is large enough. If each contestant $i \in \{2, \dots, m\}$ bids below $\tilde{V} \equiv \max\left\{V_{m+1}, \frac{V_2}{2}\right\}$ with a probability arbitrarily close to one, then contestant one, by bidding slightly above $\tilde{V}$ will always receive a payoff close to $V_1 - \tilde{V} > V_1 - V_2$, contradicting Lemma 2(ii). This establishes the claim. $\square$

We now turn to the second part of the proof, organized into two steps. The aim of this part is to establish two inequalities, namely (A21) and (A22). In step 2, we use parts (ii) and (iv) of Lemma A1 to show that by bidding $y_j^{(1)}$ against μ_{-i}^* , contestant i wins with a

probability that is substantially larger than the probability with which contestant j wins when bidding $y_j^{(1)}$ against μ_{-j}^* , formulated by inequality (A21). In step 3, we invoke part (iii) of Lemma A1 to derive the other inequality, i.e., inequality (A22), which says that the probability with which contestant i wins if bidding $y_j^{(1)}$ against μ_{-i}^* is incomparably greater than the probability with which the contestant wins when bidding zero against μ_{-i}^* .

Step 2. For contestant i identified in Lemma A1(iv), consider,

$$\begin{aligned} & E_{\mu_{-j}^*} \left[p_j \left(y_j^{(1)}, x_{-j} \right) \right] \\ &= Pr \left(x_i \leq y_j^{(1)} + \frac{1}{\sqrt{\alpha}} \right) \cdot E_{\mu_{-i}^*} \left[p_j \left(y_j^{(1)}, x_i, x_{-ij} \mid x_i \leq y_j^{(1)} + \frac{1}{\sqrt{\alpha}} \right) \right] \\ &+ Pr \left(x_i > y_j^{(1)} + \frac{1}{\sqrt{\alpha}} \right) \cdot E_{\mu_{-i}^*} \left[p_j \left(y_j^{(1)}, x_i, x_{-ij} \mid x_i > y_j^{(1)} + \frac{1}{\sqrt{\alpha}} \right) \right]. \end{aligned} \quad (A17)$$

The second term of RHS of (A17) is smaller than $\exp(-\sqrt{\alpha})$. Invoking inequalities (A14) and (A15), we obtain, $E_{\mu_{-j}^*} \left[p_j \left(y_j^{(1)}, x_{-j} \right) \right] > \frac{y_j^{(1)}}{V_j} > \frac{y_1}{V_j} > \frac{\ln(n-1)}{\alpha V_j}$. That is, the second term on RHS of (A17) is only an infinitesimal relative to $E_{\mu_{-j}^*} \left[p_j \left(y_j^{(1)}, x_{-j} \right) \right]$ when α is large enough. By Lemma A1(iv), $\mu_i^* \left(\left[0, \frac{V_2}{2} \right] \right) < 1 - 2\delta$ for α large. Consequently, when α is sufficiently large, the first term of RHS of (A17) must be smaller than $(1 - \frac{7}{4}\delta) E_{\mu_{-ij}^*} \left[p_j \left(y_j^{(1)}, 0, x_{-ij} \right) \right]$. We thus obtain,

$$E_{\mu_{-j}^*} \left[p_j \left(y_j^{(1)}, x_{-j} \right) \right] < \left(1 - \frac{3}{2}\delta \right) E_{\mu_{-ij}^*} \left[p_j \left(y_j^{(1)}, 0, x_{-ij} \right) \right] \text{ for } \alpha \text{ large.} \quad (A18)$$

Note that contestant i does not bid any positive effort weakly below $\underline{y}_1$. Hence,

$$\begin{aligned} & E_{\mu_{-i}^*} \left[p_i \left(y_j^{(1)}, x_{-i} \right) \right] \\ &= Pr \left(x_j \leq \underline{y}_1 \right) \cdot E_{\mu_{-i}^*} \left[p_i \left(y_j^{(1)}, x_j, x_{-ij} \mid x_j \leq \underline{y}_1 \right) \right] \\ &+ Pr \left(\underline{y}_1 < x_j \leq y_j^{(1)} \right) \cdot E_{\mu_{-i}^*} \left[p_i \left(y_j^{(1)}, x_j, x_{-ij} \mid \underline{y}_1 < x_j \leq y_j^{(1)} \right) \right] \\ &\geq \mu_j^* (\{0\}) \cdot E_{\mu_{-ij}^*} \left[p_i \left(y_j^{(1)}, 0, x_{-ij} \right) \right] \\ &+ \left[\mu_j^* \left(\left[0, y_j^{(1)} \right] \right) - \mu_j^* \left(\left[0, \underline{y}_1 \right] \right) \right] \cdot E_{\mu_{-ij}^*} \left[p_i \left(y_j^{(1)}, y_j^{(1)}, x_{-ij} \right) \right]. \end{aligned} \quad (A19)$$

By Lemma A1(ii), $\mu_j^* \left(\left[0, y_j^{(1)} \right] \right) - \mu_j^* \left(\left[0, \underline{y}_1 \right] \right) = 1 - \mu_j^* \left(\left[0, \underline{y}_1 \right] \right) \rightarrow 0$ as $\alpha \rightarrow \infty$. But contestant j does not bid any positive effort weakly below $\underline{y}_1$. Thus, $\mu_j^* (\{0\}) \rightarrow 1$ as $\alpha \rightarrow \infty$. Also note that $E_{\mu_{-ij}^*} \left[p_i \left(y_j^{(1)}, y_j^{(1)}, x_{-ij} \right) \right] < E_{\mu_{-ij}^*} \left[p_i \left(y_j^{(1)}, 0, x_{-ij} \right) \right]$. Therefore, the second term on RHS of (A19) is only an infinitesimal relative to the first term when α is very large. We are thus led to,

$$E_{\mu_{-i}^*} \left[p_i \left(y_j^{(1)}, x_{-i} \right) \right] > (1 - \delta) E_{\mu_{-ij}^*} \left[p_i \left(y_j^{(1)}, 0, x_{-ij} \right) \right] \text{ for } \alpha \text{ large.} \quad (A20)$$

Combing inequalities (A18) and (A20) yields (noting that $\frac{1-\delta}{1-\frac{3}{2}\delta} > 1 + \frac{\delta}{2}$),

$$E_{\mu_{-i}^*} \left[p_i \left(y_j^{(1)}, x_{-i} \right) \right] > \left(1 + \frac{\delta}{2} \right) E_{\mu_{-j}^*} \left[p_j \left(y_j^{(1)}, x_{-j} \right) \right] \text{ for } \alpha \text{ large.} \quad (A21)$$

Step 3. For any effort profile $\{x_k, k \neq i\}$, $\frac{1}{1 + \sum_{k \neq i} \exp(\alpha(x_k - y_j^{(1)}))} - \frac{1}{1 + \sum_{k \neq i} \exp(\alpha x_k)} = \frac{1}{1 + \sum_{k \neq i} \exp(\alpha(x_k - y_j^{(1)}))} \left[1 - \frac{1 + \sum_{k \neq i} \exp(\alpha(x_k - y_j^{(1)}))}{1 + \sum_{k \neq i} \exp(\alpha x_k)} \right]$. Noting $y_j^{(1)} > y_1$ by inequality (A15), one obtains, $\frac{1 + \sum_{k \neq i} \exp(\alpha(x_k - y_j^{(1)}))}{1 + \sum_{k \neq i} \exp(\alpha x_k)} < \frac{1 + \sum_{k \neq i} \exp(\alpha(x_k - y_j^{(1)}))}{\sum_{k \neq i} \exp(\alpha x_k)} = \frac{1}{\sum_{k \neq i} \exp(\alpha x_k)} + \frac{1}{\exp(\alpha y_j^{(1)})} < \frac{2}{\exp(\alpha y_1)}$. But $\frac{2}{\exp(\alpha y_1)}$ is arbitrarily small when α grows unboundedly, by Lemma A1(iii). Subtraction of (A12) from (A13), therefore, must be arbitrarily close to $E_{\mu_{-i}^*} [p_i(y_j^{(1)}, x_{-i})]$. Thus,

$$E_{\mu_{-i}^*} [p_i(y_j^{(1)}, x_{-i})] - E_{\mu_{-i}^*} [p_i(0, x_{-i})] > \left(1 + \frac{\delta}{2}\right)^{-1} E_{\mu_{-i}^*} [p_i(y_j^{(1)}, x_{-i})] \text{ for } \alpha \text{ large. (A22)}$$

As the final part of the proof, the following step 4 combines the above inequalities (A21) and (A22) and uses the equilibrium property of bid $y_j^{(1)}$ by contestant j , showing that for contestant i , bidding $y_j^{(1)}$ instead of zero against μ_{-i}^* entails slightly more effort-cost but results in a strictly greater increase in gross benefit.

Step 4. Combining inequalities (A21) and (A22) and invoking the observation $V_i = V_2$ in Lemma A1(iv) results in $V_i E_{\mu_{-i}^*} [p_i(y_j^{(1)}, x_{-i})] - V_i E_{\mu_{-i}^*} [p_i(0, x_{-i})] > V_2 E_{\mu_{-j}^*} [p_j(y_j^{(1)}, x_{-j})] \geq V_j E_{\mu_{-j}^*} [p_j(y_j^{(1)}, x_{-j})] > y_j^{(1)}$, for any α large enough. That is, $E_{\mu_{-i}^*} [\Pi_i(y_j^{(1)}, x_{-i})] > E_{\mu_{-i}^*} [\Pi_i(0, x_{-i})]$ for α large, contradicting $0 \in \text{supp}\{\mu_i^*\}$.

We have thus proved Proposition 2. $\square$

Proof of Corollary 1. (i) It follows from the Proposition 2. (ii) It follows from Lemma 1(ii) and the proof of Proposition 1 for the case that $V_1 = V_2$. (iii) The claim on the highest bid by contestant one follows from Lemma 2(i). By Lemmas 1(ii) and 2(ii), his lowest bid $y_1^{(L_1)}(\alpha)$ is positive at any $\alpha > 0$ and approaches zero as $\alpha \rightarrow \infty$. Regarding the claim on the other contests, by Lemma A1(i) that is introduced in the proof of Proposition 2, it suffices to show that $\prod_{k \neq 1} \mu_k^*(\{0\})$ approaches at most $\frac{V_1 - V_2}{V_1}$ as $\alpha \rightarrow \infty$. Suppose otherwise; i.e., without loss of generality, $\prod_{k \neq 1} \mu_k^*(\{0\}) > \frac{V_1 - V_2}{V_1} + \delta$ holds for some $\delta > 0$ at any large α . Then, by bidding $\frac{V_1 \delta}{2}$, contestant one receives a payoff $E_{\mu_{-1}^*} \left[\Pi_1 \left(\frac{V_1 \delta}{2}, x_{-1} \right) \right] = V_1 E_{\mu_{-1}^*} \left[p_1 \left(\frac{V_1 \delta}{2}, x_{-1} \right) \right] - \frac{V_1 \delta}{2}$ that approaches at least $V_1 \prod_{k \neq 1} \mu_k^*(\{0\}) - \frac{V_1 \delta}{2}$ as $\alpha \rightarrow \infty$. But $V_1 \prod_{k \neq 1} \mu_k^*(\{0\}) - \frac{V_1 \delta}{2} > V_1 \left(\frac{V_1 - V_2}{V_1} + \delta \right) - \frac{3V_1 \delta}{4} = V_1 - V_2 + \frac{V_1 \delta}{4}$ for any sufficiently large α , in conflict with Lemma 2(ii). $\square$

Notes

- ¹ Another hurdle faced by the Hirshleifer contest technology, especially in applied analysis, is that it does not admit any interior pure equilibrium (refer to, for instance, [Hirshleifer, 1989](#), p. 107). In contrast, interesting enough, the Tullock contest does not admit any corner pure equilibrium (one or more contestants bid zero effort).
- ² That said, it is not to say that defense, or even surrender, is necessarily costless. It could be costly, even very costly indeed, especially because defense sometimes serves a temporary measure to enable the player to be on the offensive side later. Alternatively put, the nature of defense or passivity as a strategy in a multi-stage game may remarkably differ from that in a

static one. Nevertheless, in general, it is much more effort demanding and costly to be offensive than defensive. Admittedly, the present analysis focuses on a single one-shot contest only.

3 The term $\frac{1}{\sqrt{\alpha}}$ used in the following claims (i) and (ii) can be replaced by any infinitesimal $\varepsilon(\alpha)$ such that $\alpha\varepsilon(\alpha) \rightarrow \infty$ as $\alpha \rightarrow \infty$. We choose the specific $\frac{1}{\sqrt{\alpha}}$ for ease of exposition.

4 The following argument, once carefully refined, can be developed into a rigorous proof of Proposition 1 for the contest between risk-neutral contestants with CARA utility functions. But the proof offered in the appendix, which separately deals with the cases of $V_1 = V_2$ and $V_1 > V_2$, is of value, not least because it plays an important part in the proof of both Proposition 2 and Corollary 1.

5 It is worth noting that, technically, Lemma 3 is independent of the risk attitude of the contestants, and that such a small increase in bid from $y_j(\alpha)$ also possibly causes a *marginal* change of the value of some other terms (when contestant j happens to stand against other bid profile rather than the said x_{-j}) in the expression of $E_{\mu_{-j}^*}[\Pi_j(x_j, x_{-j})]$.

6 By saying that term $A(\alpha)$ approaches term $B(\alpha)$, we mean that the gap between them is an infinitesimal as $\alpha \rightarrow \infty$.

7 This estimation of the upper bound of bid by some contestant i such that $V_i = V_2$ is a restatement of Lemma A.4 in Ewerhart and Sun (2024), yet deliberately put in a slightly different format to facilitate analysis below.

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