

Article

Power Indices with Threats in Precoalitions

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Abstract

We investigate power indices for simple games with precoalitions which distribute power among players in an external and an internal step. We extend an existing approach which uses the Public Good index both on the external level in the quotient game as well as on the internal level for measuring the leverage of players to threaten their peers through departing the precoalition. We replace the Public Good index in that model by five other efficient power indices, i.e., the Shapley–Shubik index, the Deegan–Packel index, the Johnston index and two indices based on null player free winning coalitions. Axiomatizations of the novel power indices with threat partitions are presented. We also propose a slight modification to the existing framework for threat power indices which guarantees that null players are always assigned zero power. Numerical results for all power indices combined with different threat partitions are presented and discussed.

Keywords: cooperative game theory; simple games; weighted voting games; coalition structure; precoalitions; power indices

JEL Classification: C71

1. Introduction

Power indices quantify influence in simple games, i.e., cooperative games in which there are only winning and losing coalitions. Commonly used power indices are based on different axiomatic assumptions and include the Shapley–Shubik index (Shapley & Shubik, 1954), the Banzhaf index (Banzhaf, 1964; Penrose, 1946), the Deegan–Packel index (Deegan & Packel, 1978), the Public Good index (Holler, 1982; Holler & Packel, 1983), the Johnston index (Johnston, 1978) as the well as two more recent power indices based on null player free winning coalitions (Álvarez-Mozos et al., 2015).

This work investigates power indices with coalition structures, i.e., we assume a partition of the player set into disjoint precoalitions (also known as a priori unions and sometimes simply referred to as unions) is given implying that either all or none of the members of a precoalition join a certain coalition. In voting situations, precoalitions may capture the effects of strict party discipline in legislative bodies, where voting behavior is constrained by formal party affiliations. Alternatively, precoalitions can represent a mechanism to make negotiations more efficient, with each precoalition choosing one person to represent them, thus reducing transaction costs in multilateral negotiations.

In his seminal work, Owen (1977) proposed a generalization of the Shapley value (Shapley, 1953) for cooperative games with precoalitions. Owen’s solution concept operates through a two-stage allocation mechanism. On the external level, the total value is first distributed among precoalitions according to their Shapley values in the induced quotient



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game. On the internal level, each precoalition's allocation is afterwards divided among its members via a second application of the Shapley value, accounting for potential mobility between precoalitions.

In this manuscript, we study a different approach going back to [Holler and Nohn \(2009\)](#) and [Alonso-Meijide et al. \(2010\)](#) based on so-called threat partitions for simple games. On the internal level, threat games are used to reflect individual players' threat potential through possible withdrawal from their precoalition.

Imagine a voting situation with four players such that a motion is only passed if both Player 1 agrees and at least two out of the three Players 2, 3 and 4 are in favor of the motion as well. Assuming Player 1 forms a precoalition of their own and Players 2, 3 and 4 form the second precoalition, one would conclude that both of these two a priori unions are equally powerful. What happens if one of the Players 2, 3 or 4 threatens his colleagues with their departure from the precoalition in the sense of forming their own separate precoalition? As long as the other two players stay together in a precoalition, the one departing player will never be able to cast the decisive vote, i.e., Player 1 needs the precoalition with the two remaining players to win. How can we reconcile the fact that an individual player separating into their individual precoalition appears to have no threat power, whereas the original precoalition consisting of Players 2, 3, and 4 still own half the power in the game? To take our example one step further, let us enter Player 5 who is never needed to cast the decisive vote in the simple game without precoalitions, i.e., in the terminology of simple games Player 5 is a null player. Let us continue with two precoalitions and let Player 5 join Players 2, 3, and 4 in their precoalition. As long as the other three players stay together in a precoalition, Players 2, 3, 4, and 5 all possess a threat power of zero despite that fact that Players 2, 3, and 4 can turn losing coalitions into winning coalitions in the simple game without precoalitions whereas Player 5 cannot. Obviously, we owe this conundrum to our assumption that the rest of the players keep together in a precoalition when one of their colleagues breaks away. Assuming that the precoalition of Players 2, 3, 4, and 5 splits into singletons as soon as one player breaks away, we attribute a threat power of one-sixth (in the sense of one-third of half the power in the game) to Players 2, 3, and 4 and zero threat power to player 5. Actually, a threat partition ([Alonso-Meijide et al., 2010](#)) can be any partition structure in which an individual player breaks away from his precoalition forming a separate precoalition of their own. It is even thinkable that a threat partition breaks symmetries between otherwise symmetric players. Let us return to our original situation with two precoalitions consisting of Players 1 and the three equally influential Players 2, 3, and 4, respectively. We assume that whenever Player 3 breaks away, then Players 2 and 4 remain in a precoalition, whereas whenever either Player 2 or Player 4 leaves, then the precoalition breaks into singletons. In that situation, Player 3 has zero threat power whereas we attribute a threat power of one fourth (in the sense of one half of half the power in the game) to Players 2 and 4. We revisit, formalize, and discuss these motivating examples in detail in Section 4.

[Holler and Nohn \(2009\)](#) suggest to initially apportion power among precoalitions according to the Public Good index (PGI) applied to the quotient game. For any internal threat games, the PGI is employed as well. [Holler and Nohn \(2009\)](#) elaborate three canonical cases concerning how the precoalition structure changes upon defection of a player from their union. In [Alonso-Meijide et al. \(2010\)](#), these so-called Threat PGIs are axiomatized and a more general notion of a threat partition is introduced. In [Staudacher \(2023\)](#), efficient algorithms for computing the three Threat PGIs pertaining to the three canonical cases from [Holler and Nohn \(2009\)](#) for weighted voting games are presented.

We take up the concept of Threat PGIs from [Alonso-Meijide et al. \(2010\)](#) and [Holler and Nohn \(2009\)](#) and substitute the PGI with five alternative efficient power indices: the

Shapley–Shubik index (Shapley & Shubik, 1954), the Deegan–Packel index (Deegan & Packel, 1978), the Johnston index (Johnston, 1978), and the two power indices f^{np} and g^{np} based on null player free winning coalitions (Álvarez-Mozos et al., 2015).

In Section 2, we provide some background information on simple games, power indices, and their axiomatizations. In Section 3, we discuss simple games with a coalition structure and provide general notions of Shapley–Shubik, Deegan–Packel, Johnston, and the f^{np} and g^{np} indices in the presence of precoalitions. Section 4 contains the core theoretical and methodological contributions of this paper. We present axiomatic characterizations of the introduced power indices incorporating threat partitions. Furthermore, we propose a minimal but crucial modification to the existing framework for power indices with threats that formally enforces the null player property. In Section 5, we present numerical results for the various threat power indices, focusing on an example with three equally strong precoalitions of different sizes from Holler and Nohn (2009) and a real-world example from Alonso-Mejide et al. (2010). The paper concludes with a brief discussion of key findings and identifies a few open research questions in Section 6.

2. Simple Games, Power Indices, and Their Axiomatizations

In this section, we briefly summarize some terminology for cooperative games and simple games along the lines of Bertini et al. (2013). We introduce six power indices and discuss their axiomatizations.

2.1. Cooperative Game Theory and Simple Games

Cooperative game theory (Chakravarty et al., 2015) studies the potential outcomes and gains that players can realize through forming coalitions.

We let $N = 1, \dots, n$ be a finite set of n players. Any subset $S \subseteq N$ is called a *coalition*, and 2^N represents the power set of N . The sets $\emptyset$ and N are called the *empty coalition* and *grand coalition*, respectively. For any coalition S , its cardinality is denoted by $|S|$, with $|N| = n$. An n -person cooperative game is defined as a pair (N, v) , where $v : 2^N \rightarrow \mathbb{R}$ is the *characteristic function* satisfying the normalization $v(\emptyset) = 0$. The cooperative game is *monotone* if $v(S) \leq v(T)$ whenever $S \subseteq T$ for any coalitions $S, T \in 2^N$.

A cooperative game is called *simple* if it is monotone with $v(N) = 1$ and $v(S) \in \{0, 1\}$ for all $S \subseteq N$. In simple games:

- Coalitions with $v(S) = 1$ are called *winning coalitions*;
- Coalitions with $v(S) = 0$ are called *losing coalitions*;
- A player i is *critical* in a winning coalition S if $v(S \setminus \{i\}) = 0$;
- A player i is called a *null player* if $v(S \cup i) - v(S) = 0$ for any coalition $S \in 2^N$;
- A winning coalition S is a *minimal winning coalition* if it consists entirely of critical players;
- A winning coalition S is a *vulnerable coalition* if it contains at least one critical player;
- A winning coalition S is a *null player free winning coalition* if it contains no null players;
- For a winning coalition S , its *set of critical players* is denoted by $Cr(S)$;
- We let $SI(N)$ denote the set of all simple games on the player set N .

Weighted voting games (also known as weighted majority games or weighted games) are an important subclass of simple games. An n -player weighted voting game is given by n non-negative real weights $w_i, i = 1, \dots, n$, and a non-negative real quota q . Its characteristic function $v : 2^N \rightarrow \{0, 1\}$ takes the value $v(S) = 1$ if and only if the weights the players in coalition S reach or exceed the quota, i.e., $w(S) = \sum_{i \in S} w_i \geq q$, and there holds $v(S) = 0$ otherwise, i.e., when coalition S is losing.

Another important subclass of simple games are *unanimity games*. For any coalition $S \subseteq N$, the unanimity game u_S of that coalition is defined as $u_S(T) = 1$ if $S \subseteq T$ and $u_S(T) = 0$ otherwise.

2.2. Power Indices and Their Axiomatic Characterizations

In general, a *power index* f is a point-valued solution concept for a simple game, i.e., the power index f outputs a unique vector $f(N, v) = (f_1(N, v), \dots, f_n(N, v))$ provided with a given simple game specified by its player set N and its characteristic function v as its input. In the following, we define those six power indices investigated in the rest of the article and supply their axiomatic characterizations from the literature. For a deeper discussion on power indices and their properties, we refer to the overview article by Bertini et al. (2013).

We let (N, v) be a simple game with n players and let $W^{np}(v)$ and $W^m(v)$ stand for the sets of null player free winning coalitions and minimal winning coalitions, respectively, and $W_i^{np}(v)$ and $W_i^m(v)$ for the associated subsets containing player i . Further, we let $VC(v)$ signify the set of vulnerable coalitions and $\eta_i(v, c)$ the number of coalitions of cardinality c for which i is a critical player.

- (a) The *Shapley–Shubik index* (Shapley & Shubik, 1954) of player i is defined as

$$SS_i(N, v) = \sum_{c=1}^n \frac{\eta_i(v, c)}{c \binom{n}{c}}. \quad (1)$$

The Shapley–Shubik index is derived from the Shapley value, which was originally defined by Shapley (1953) and is one of the most important point-valued solution concepts in cooperative game theory.

- (b) The *Public Good index* (also known as Holler index) (Holler, 1982; Holler & Packel, 1983) of player i is defined as

$$PG_i(N, v) = \frac{|W_i^m(v)|}{\sum_{k=1}^n |W_k^m(v)|}. \quad (2)$$

- (c) The *Deegan–Packel index* (Deegan & Packel, 1978) of player i is defined as

$$DP_i(N, v) = \frac{1}{|W^m(v)|} \sum_{S \in W_i^m(v)} \frac{1}{|S|}. \quad (3)$$

- (d) The *Johnston index* (Johnston, 1978) of player i is defined as

$$J_i(N, v) = \frac{\sum_{S \in VC(v), i \in Cr(S)} \frac{1}{|Cr(S)|}}{\sum_{k=1}^n \sum_{S \in VC(v), k \in Cr(S)} \frac{1}{|Cr(S)|}} \quad (4)$$

if i is not a null player and $J_i(N, v) = 0$ otherwise.

- (e) The *null player free index* f^{np} (Álvarez-Mozos et al., 2015) of player i is defined as

$$f_i^{np}(N, v) = \frac{1}{|W^{np}(v)|} \sum_{S \in W_i^{np}(v)} \frac{1}{|S|}. \quad (5)$$

- (f) The *null player free index* g^{np} (Álvarez-Mozos et al., 2015) of player i is defined as

$$g_i^{np}(N, v) = \frac{|W_i^{np}(v)|}{\sum_{k=1}^n |W_k^{np}(v)|}. \quad (6)$$

A power index f meets the *efficiency property* when, for every simple game (N, v) (except the null game), the equation $\sum_{i=1}^n f_i(N, v) = v(N) = 1$ holds.

We say f has the *null player property* if it assigns $f_i(N, v) = 0$ to every null player $i \in N$ in all simple games (N, v) .

The *symmetry property* (also known as equal treatment property and distinct from the anonymity property as shown in Algaba et al. (2019) and Malawski (2020)) is satisfied when, for any simple game (N, v) and any two players $i, j \in N$ with $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \in 2^{N \setminus \{i, j\}}$, we obtain $f_i(N, v) = f_j(N, v)$.

The *non-negativity property* guarantees $f_i(N, v) \geq 0$ for all players $i \in N$ and all simple games (N, v) on $SI(N)$.

We let $(N \setminus \{i\}, v^{N \setminus \{i\}})$ denote the game (N, v) without player i . Lastly, f satisfies the *null player removable property* (also known as the null player out property) if for each null player $i \in N$ in a simple game $(N, v) \in SI(N)$ there holds $f_j(N \setminus \{i\}, v^{N \setminus \{i\}}) = f_j(N, v)$ for all players $j \in N \setminus \{i\}$. The null player removable property implies that removing any null players from a simple game (N, v) does not change the power indices of the remaining players. Note that the null player property and the null player removable property are not equivalent; see, e.g., Staudacher et al. (2021) for a discussion of a power index used to measure indirect control in corporate structures which exhibits null player but lacks null player removability.

Properties such as the five introduced above can be employed in order to characterize a power index by a set of axioms. Concretely, it has been pointed out that all six power indices introduced in this subsection can be shown to be the unique power index satisfying efficiency, symmetry, and null player property combined with a fourth axiom. Before discussing further details, we introduce a result from Van den Brink and Van der Laan (1998) which points out that whenever a power index is efficient and shares the null player removable property, then the null player property is automatically fulfilled. We later use this fact to replace the null player axiom by the null player removable axiom for our six efficient power indices.

Lemma 1 (Van den Brink & Van der Laan, 1998). *Let f be a power index on $SI(N)$ satisfying both the efficiency and the null player removable property. Then f is guaranteed to also possess the null player property.*

The following six results characterize the introduced power indices via efficiency, symmetry, null player removability, and a fourth axiom. We note in passing that the fourth axiom need not be unique, e.g., one can find alternatives to the fourth axioms presented below for the Shapley–Shubik, Public Good, and Deegan–Packel indices in the literature.

The following axiomatization of the Shapley–Shubik index was first established in Dubey (1975).

Lemma 2 (Dubey, 1975). *The Shapley–Shubik index defined in Equation (1) is the unique power index f defined on $SI(N)$ that satisfies efficiency, symmetry, null player removability, and the transfer property, i.e., for two simple games (N, v) and (N, w) there holds*

$$f(N, v) + f(N, w) = f(N, v \vee w) + f(N, v \wedge w)$$

with $(N, v \vee w)$ defined by $(v \vee w)(S) = \max\{v(S), w(S)\}$ and $(N, v \wedge w)$ defined by $(v \wedge w)(S) = \min\{v(S), w(S)\}$ for all $S \subseteq N$.

The following axiomatization is due to Alonso-Meijide et al. (2008).

Lemma 3 (Alonso-Meijide et al., 2008; Álvarez-Mozos et al., 2015). *The Public Good index defined in Equation (2) is the unique power index f defined on $SI(N)$ that satisfies efficiency,*

symmetry, null player removability, and the PG-minimal monotonicity property, i.e., for two simple games (N, v) and (N, w) and all players $i \in N$ with $W_i^m(v) \subseteq W_i^m(w)$, the following inequality is guaranteed:

$$f_i(N, v) \sum_{j \in N} |W_j^m(v)| \leq f_i(N, w) \sum_{j \in N} |W_j^m(w)|.$$

The following two axiomatizations appeared first in [Lorenzo-Freire et al. \(2007\)](#).

Lemma 4 ([Álvarez-Mozos et al., 2015](#); [Lorenzo-Freire et al., 2007](#)). The Deegan–Packel index defined in Equation (3) is the unique power index f defined on $SI(N)$ that satisfies efficiency, symmetry, null player removability, and the DP-minimal monotonicity property, i.e., for two simple games (N, v) and (N, w) and all players $i \in N$ with $W_i^m(v) \subseteq W_i^m(w)$, the following inequality is guaranteed:

$$f_i(N, v) |W^m(v)| \leq f_i(N, w) |W^m(w)|.$$

Lemma 5 ([Lorenzo-Freire et al., 2007](#)). The Johnston index defined in Equation (4) is the unique power index f defined on $SI(N)$ that satisfies efficiency, symmetry, null player removability, and the critical mergeability property, i.e., for any simple game (N, v) with k minimal winning coalitions $M(v) = \{S_1, \dots, S_k\}$, the following equality is guaranteed for all players $i \in N$:

$$f_i(N, v) = \sum_{S \in \mathcal{F}} \frac{|VC(S, v)|}{|VC(v)|} f_i(N, u_S).$$

In the above equation, $VC(S, v)$ denotes the set of vulnerable coalitions such that the set of players $S \subseteq N$ are critical, $VC(v)$ the set of vulnerable coalitions of the game (N, v) , u_S the unanimity game defined by a coalition S , $\mathcal{F} = \{\cap_{j \in R} S_j : \cap_{j \in R} S_j \neq \emptyset, R \subseteq K\}$ and $K = \{1, \dots, k\}$.

The final two axiomatizations are due to [Álvarez-Mozos et al. \(2015\)](#).

Lemma 6 ([Álvarez-Mozos et al., 2015](#)). The null player free index f^{np} defined in Equation (5) is the unique power index f defined on $SI(N)$ that satisfies efficiency, symmetry, null player removability, and the f^{np} -minimal monotonicity property, i.e., for two simple games (N, v) and (N, w) and all players $i \in N$ with $W_i^m(v) \subseteq W_i^m(w)$, the following inequality is guaranteed:

$$f_i(N, v) |W^{np}(v)| \leq f_i(N, w) |W^{np}(w)|.$$

Lemma 7 ([Álvarez-Mozos et al., 2015](#)). The null player free index g^{np} defined in Equation (6) is the unique power index f defined on $SI(N)$ that satisfies efficiency, symmetry, null player removability, and the g^{np} -minimal monotonicity property, i.e., for two simple games (N, v) and (N, w) and all players $i \in N$ with $W_i^m(v) \subseteq W_i^m(w)$, the following inequality is guaranteed:

$$f_i(N, v) \sum_{j \in N} |W_j^{np}(v)| \leq f_i(N, w) \sum_{j \in N} |W_j^{np}(w)|.$$

3. Simple Games with Precoalitions and Coalitional Power Indices

In this section, we briefly introduce some terminology for cooperative games with a priori unions and coalitional power indices along the lines of [Alonso-Meijide et al. \(2010\)](#). We extend a definition from [Alonso-Meijide et al. \(2010\)](#) to all six power indices from Section 2.2 followed by a short discussion of their properties.

3.1. Simple Games with Precoalitions

We are looking at an external division of our player set $N = \{1, \dots, n\}$ into precoalitions (also known as a priori unions), where members of each precoalition are supposed to join a coalition either collectively or not at all. We let $\mathcal{P}(N)$ denote the set of all partitions of N , where a partition is a collection of non-empty subsets of N such that N is the disjoint union of these subsets. We refer to an element $P \in \mathcal{P}(N)$ as a coalition structure (also known as a system of unions) of the set N . A simple game with a coalition structure can be represented as a triplet (N, v, P) . We express the coalition structure in the form $P = \{P_1, \dots, P_l\}$, meaning there are l precoalitions $P_1, \dots, P_l$, and the set $L = \{1, \dots, l\}$ serves as the index set for the partition P . For an individual player $i \in N$, the notation $P(i)$ refers to the precoalition i which belongs to, i.e., $P(i) = P_k$ with $i \in P_k$ and $k \in L$.

The simple game played entirely between the precoalitions (P, v^P) is called the *quotient game*. In the following subsection, we measure the power of a precoalition $Q \in P = \{P_1, \dots, P_l\}$ in terms of the six power indices from Section 2.2. Let us write M^P for the set of minimal winning coalitions in the quotient game (P, v^P) and M_Q^P for the set of those minimal winning coalitions in the quotient game (P, v^P) containing precoalition $Q \in P$. Following Alonso-Meijide et al. (2010), we refer to a subset $S \subseteq Q$ of a precoalition Q as an *essential part* with respect to a minimal winning coalition $R \in M_Q^P$ (in the quotient game (P, v^P)) if $S \cup \bigcup_{Q' \in R \setminus \{Q\}} Q'$ is a winning coalition in (N, v) and $T \cup \bigcup_{Q' \in R \setminus \{Q\}} Q'$ is a losing coalition in (N, v) for all true subsets $T \subset S$.

There are two extreme coalition structures. In $N^0 = P^n = \{\{1\}, \{2\}, \dots, \{n\}\}$, each player forms their own precoalition as a singleton whereas for $P^N = \{N\}$ there is merely one union and that is the grand coalition.

3.2. Coalitional Power Indices and Their Properties

A *coalitional power index* g is a function which obtains an n -person simple game with a coalition structure specified by its player set N , its characteristic function v and a partition P as its input and yields a unique vector $g(N, v, P) = (g_1(N, v, P), \dots, g_n(N, v, P))$ as its output.

Given some power index f on (N, v) , we call a coalitional power index g a *generalized coalitional f -index* if $g(N, v, N^0) = f(N, v)$ for all simple games (N, v) on $SI(N)$, implying that g equals f for all simple games (N, v) and singleton precoalitions N_0 . Setting f in the previous definition equal to the Shapley–Shubik index SS defined in Equation (1), the Public Good index PG defined in Equation (2), the Deegan–Packel index DP defined in Equation (3), the Johnston index J defined in Equation (4), the null player free index f^{np} defined in Equation (5), and the null player free index g^{np} defined in Equation (6), we obtain *generalized coalitional Shapley–Shubik indices*, *generalized coalitional Public Good indices*, *generalized coalitional Deegan–Packel indices*, *generalized coalitional Johnston indices*, *generalized coalitional f^{np} –Packel indices*, and *generalized coalitional g^{np} –Packel indices*. Note that our terminology differs slightly from Alonso-Meijide et al. (2010), where what we refer to as a generalized coalitional Public Good index would simply be referred to as a coalitional Public Good index. Given that in Alonso-Meijide et al. (2011) an extension of the Deegan–Packel index following the approach by Owen (1977) is called *the* coalitional Deegan–Packel index and that in some parts of the literature the Owen value is called *the* coalitional Shapley value, we aim to rule out any confusion by speaking of generalized coalitional f -indices. In terms of notation, we from now on reserve f for power indices on (N, v) and use g for coalitional power indices on (N, v, P) .

Let us summarize some properties of our six families of generalized coalitional indices:

Generalized coalitional Shapley–Shubik indices, generalized coalitional Public Good indices, generalized coalitional Deegan–Packel indices, generalized coalitional Johnston

indices, generalized coalitional f^{np} -Packel indice, and generalized coalitional g^{np} -Packel indices possess the following properties:

- *singleton efficiency*, i.e., for all simple games (N, v) , there holds $\sum_{i=1}^n g_i(N, v, N^0) = 1$;
- *singleton symmetry*, i.e., for all simple games and all players i and j which are symmetric players in the simple game (N, v) , there holds $g_i(N, v, N^0) = g_j(N, v, N^0)$;
- *singleton null player removability*, i.e., removing any null players from the simple game (N, v) does not change the power index of any remaining player in the game (N, v, N^0) ;
- Lemma 1 tells us that singleton efficiency and singleton null player removability together imply *singleton null player*, i.e., $g_i(N, v, N^0) = 0$ is guaranteed for all null players in (N, v) .

We can now generalize a result from [Alonso-Meijide et al. \(2010\)](#).

Theorem 1. *Let g be a coalitional power index satisfying singleton efficiency, singleton symmetry and singleton null player removability.*

g is a generalized coalitional Shapley–Shubik index if and only if it furthermore possesses the singleton transfer property, i.e., the transfer property from Lemma 2 is guaranteed for $SS(N, v, N^0)$.

g is a generalized coalitional Public Good index if and only if it furthermore possesses the singleton PG-minimal monotonicity property, i.e., the PG-minimal monotonicity property from Lemma 3 is guaranteed for $PG(N, v, N^0)$.

g is a generalized coalitional Deegan Packel index if and only if it furthermore possesses the singleton DP-minimal monotonicity property, i.e., the DP-minimal monotonicity property from Lemma 4 is guaranteed for $DP(N, v, N^0)$.

g is a generalized coalitional Johnston index if and only if it furthermore possesses the singleton critical mergeability property, i.e., the critical mergeability property from Lemma 5 is guaranteed for $J(N, v, N^0)$.

g is a generalized coalitional f^{np} -index if and only if it furthermore possesses the singleton f^{np} -minimal monotonicity property, i.e., the f^{np} -minimal monotonicity property from Lemma 6 is guaranteed for $f^{np}(N, v, N^0)$.

g is a generalized coalitional g^{np} -index if and only if it furthermore possesses the singleton g^{np} -minimal monotonicity property, i.e., the g^{np} -minimal monotonicity property from Lemma 7 is guaranteed for $g^{np}(N, v, N^0)$.

Proof. All statements follow immediately from the definition of a generalized coalitional f -index based upon a power index f defined on $SI(N)$. $\square$

Let us return to the quotient game (P, v^P) on the external level of the precoalitions. A generalized coalitional f -index g on (N, v, P) derived from a power index f on (N, v) satisfies the *quotient game property* if for all precoalitions

$$\sum_{i \in Q} g_i(N, v, P) = g_Q(P, v^P, P^0) = f_Q(P, v^P) \quad (7)$$

is guaranteed; see [Alonso-Meijide et al. \(2010\)](#).

The quotient game property (7) enables us to generalize another result from [Alonso-Meijide et al. \(2010\)](#).

Theorem 2. *Generalized coalitional Shapley–Shubik indices, generalized coalitional Public Good indices, generalized coalitional Deegan–Packel indices, generalized coalitional Johnston indices, generalized coalitional f^{np} -indices, and generalized coalitional g^{np} -indices satisfying the quotient game property (7) also exhibit the following two properties:*

(a) the null union property, i.e., for all simple games with precoalitions (N, v, P) and any precoalition Q being a null player in the quotient game (P, v^P) , there holds

$$\sum_{i \in Q} g_i(N, v, P) = 0.$$

(b) the symmetry among unions property, i.e., for all simple games with precoalitions (N, v, P) and all precoalitions Q and Q' being symmetric players in the quotient game (P, v^P) , there holds

$$\sum_{i \in Q} g_i(N, v, P) = \sum_{i \in Q'} g_i(N, v, P).$$

In both (a) and (b), g stands for a generalized coalitional f -index based upon any of the six power indices SS, PG, DP, J, f^{np} or g^{np} .

Proof. All statements follow immediately from the properties of the six generalized coalitional f -indices and the quotient game property (7). $\square$

Following [Alonso-Meijide et al. \(2010\)](#), we finally note that combining our six generalized coalitional f -indices with the quotient game property does *not* guarantee

- the *null player property* in the sense that for all simple games with precoalitions (N, v, P) and for any null player i in (N, v) there also holds $g_i(N, v, P) = 0$;
- the *null player removable property* in the sense that for any null player $i \in N$ in any simple game (N, v) there holds $g_j(N \setminus \{i\}, v^{P \setminus \{P(i)\} \cup \{P(i) \setminus \{i\}\}}, P \setminus \{P(i)\} \cup \{P(i) \setminus \{i\}\}) = g_j(N, v, P)$ for all players $j \in N \setminus \{i\}$. The null player removable property implies that removing any null players from the game (N, v, P) does not change the coalitional power indices of the remaining players;
- the *symmetry within unions property* in the sense that for all simple games with precoalitions (N, v, P) and for all players $i, j \in Q$, i.e., belonging to the identical precoalition Q , which are also symmetric players in (N, v) there also holds $g_i(N, v, P) = g_j(N, v, P)$.

We study the null player property, the null player removable property, and symmetry within unions for coalitional power indices in more detail in the next section.

4. Threat Power Indices and Their Axiomatic Characterization

We introduce the concepts of threat partitions and threat power indices and emphasize upon the generality of the concept and point out how the Public Good index can be replaced smoothly by five other efficient power indices in the framework established in [Alonso-Meijide et al. \(2010\)](#). We revisit three practically relevant approaches for reflecting an individual player's threat power to leave his precoalition proposed by [Holler and Nohn \(2009\)](#). We discuss a slight modification of the framework to ensure the null player property in every instance. Finally, we present an axiomatic characterization of any discussed threat power indices.

4.1. Power Indices and Threat Partitions

In [Alonso-Meijide et al. \(2010\)](#), a general concept of a threat partition is introduced, with threat power indices based upon the Public Good index and the assumption to distribute power within precoalitions proportional to each player's power in the corresponding threat game. A threat power index is a coalitional power index computed in two steps. In the first step, power among precoalitions is distributed according to one of the six efficient power indices f on $SI(N)$ introduced in Section 2.2 for the quotient game. In the second step, that power is distributed internally among individual members of precoalitions proportional to their power in threat games, i.e., the threat power of each

individual player i is determined by applying the same power index f in a threat game in which player i acts on his own.

Formally, a mapping TP is defined as a threat partition if for any given pair (P, i) —where P represents a partition of the player set N and $i \in N$ is a player—it assigns another partition $TP(P, i)$ of N satisfying the condition that $\{i\} \in TP(P, i)$. Any partition $TP(P, i)$ can be interpreted as the result of player i departing his precoalition $P(i)$ preferring to play entirely on their own instead. Given a threat partition TP , the *Threat Power index* $T_i^{TP}(f)$ of player $i \in P(i)$ (based upon an efficient power index f on $SI(N)$ as introduced in Section 2.2) is defined by

$$T_i^{TP}(f)(N, v, P) = f_{P(i)}(P, v^P) \frac{f_{\{i\}}(TP(P, i), v^{TP(P, i)})}{\sum_{j \in P(i)} f_{\{j\}}(TP(P, i), v^{TP(P, i)})}. \quad (8)$$

whenever $\sum_{j \in P(i)} f_{\{j\}}(TP(P, i), v^{TP(P, i)}) > 0$ and $T_i^{TP}(f)(N, v, P) = \frac{f_{P(i)}(P, v^P)}{|P(i)|}$ otherwise.

Holler and Nohn (2009) study three canonical procedures for measuring an individual player's threat power when that player considers to leave his precoalition. We review these three canonical cases and confirm that the Public Good index PG can be replaced by any of the five efficient indices SS , DP , J , f^{np} , and g^{np} without any problems.

For *Threat Partition 1* (Holler & Nohn, 2009), the precoalition structure P is assumed to exhibit merely a minimal degree of stability. Once an individual member i leaves their union $P(i)$, the complete precoalition structure P breaks apart in the very same way the precoalition $P(i)$ does. Regarding intra-union power allocation, this model suggests that subsets of a union can cooperate not only with other precoalitions, but also with subsets within those precoalitions. The *Threat Power index* $T_i^1(f)$ of player $i \in P(i)$ (based upon an efficient power index f on $SI(N)$ as introduced in Section 2.2) is given by

$$T_i^1(f)(N, v, P) = f_{P(i)}(P, v^P) \frac{f_i(N, v)}{\sum_{j \in P(i)} f_j(N, v)}. \quad (9)$$

whenever $\sum_{j \in P(i)} f_j(N, v) > 0$ and $T_i^1(f)(N, v, P) = 0$ otherwise.

Note that $T^1(f)$ was introduced not only by Holler and Nohn (2009) for the Public Good index but also in the preprint Alonso-Mejide and Carreras (2009) for the Shapley–Shubik index.

Threat Partition 2 (Holler & Nohn, 2009) assumes a more stable precoalition structure P . Once a single player i leaves their union $P(i)$, then merely that precoalition $P(i)$ splits into singletons, whereas the rest of the precoalition structure stays unchanged. Regarding intra-union power allocation, this model suggests that subsets of a precoalition only enjoy the possibility to cooperate with other unions, but not with any subsets of these other unions. As introduced in Holler and Nohn (2009), for precoalition $Q \in P$, we let $P/Q = P \setminus \{Q\} \cup \{\{i\} | i \in Q\}$ stand for the new precoalition structure after Q splits into singletons $\{i\}$, $i \in Q$. The *Threat Power index* $T_i^2(f)$ of player $i \in P(i)$ (based upon an efficient power index f on $SI(N)$ as introduced in Section 2.2) is given by

$$T_i^2(f)(N, v, P) = f_{P(i)}(P, v^P) \frac{f_{\{i\}}(P/P(i), v^{P/P(i)})}{\sum_{j \in P(i)} f_{\{j\}}(P/P(i), v^{P/P(i)})}. \quad (10)$$

whenever $\sum_{j \in P(i)} f_{\{j\}}(P/P(i), v^{P/P(i)}) > 0$ and $T_i^2(f)(N, v, P) = 0$ otherwise.

Threat Partition 3 (Holler & Nohn, 2009) is based on the assumption of maximal stability for the union structure P . If a single member i leaves their precoalition $P(i)$, then the rest of that union $P(i)$ stays unchanged and so do all the other precoalitions. Following Holler and Nohn (2009), we let $P/i = P \setminus \{P(i)\} \cup \{\{i\}, P(i) \setminus \{i\}\}$ stand for the precoalition

structure after player i depart from their union $P(i)$ and act entirely on their own. The Threat Power index $T_i^3(f)$ of player $i \in P(i)$ (based upon an efficient power index f on $SI(N)$ as introduced in Section 2.2) is given by

$$T_i^3(f)(N, v, P) = f_{P(i)}(P, v^P) \frac{f_{\{i\}}(P/i, v^{P/i})}{\sum_{j \in P(i)} f_{\{j\}}(P/j, v^{P/j})}. \quad (11)$$

whenever $\sum_{j \in P(i)} f_{\{j\}}(P/j, v^{P/j}) > 0$ and $T_i^3(f)(N, v, P) = \frac{f_{P(i)}(P, v^P)}{|P(i)|}$ otherwise.

While we consider the above three canonical cases to be the most practically relevant threat partitions and are not aware of any examples or calculations for different threat partitions in the literature, we strive to emphasize upon the generality of the framework. Threat partitions different from the three canonical cases are thinkable. Let us therefore introduce a threat partition EO (“even–odd”). If a single member i leaves their precoalition $P(i)$, then all the other precoalitions remain intact whereas the rest of $P(i)$ splits into two precoalitions consisting of the remaining players with even or odd indices, respectively. Following the notation introduced at the beginning of this subsection, we could write the precoalition structure after player i departs from their union $P(i)$ in the form $EO(P, i) = P \setminus \{P(i)\} \cup \{\{i\}, A, B\}$ with $A \cup B = P(i) \setminus \{i\}$ and the two precoalitions A and B containing the players from $P(i) \setminus \{i\}$ with even or odd indices, respectively. As a specific case of the general Formula (8), we study the following threat power index of player $i \in P(i)$ (based upon an efficient power index f on $SI(N)$ as introduced in Section 2.2) given by

$$T_i^{EO}(f)(N, v, P) = f_{P(i)}(P, v^P) \frac{f_{\{i\}}(EO(P, i), v^{EO(P, i)})}{\sum_{j \in P(i)} f_{\{j\}}(EO(P, i), v^{EO(P, i)})}. \quad (12)$$

whenever $\sum_{j \in P(i)} f_{\{j\}}(EO(P, i), v^{EO(P, i)}) > 0$ and $T_i^{EO}(f)(N, v, P) = \frac{f_{P(i)}(P, v^P)}{|P(i)|}$ otherwise.

Let us take this opportunity to comment on the notation $T_i^{TP}(f)$ for threat power indices introduced in Equation (8). The lower index i stands for a specific player whose threat power is evaluated. The upper index stands TP for the threat partition at hand, i.e., it can be a specific threat partition like EO or one of the three canonical cases specified by Shorthands 1, 2 or 3. f is the efficient power index used in the quotient game and the internal threat games. That way, we are keeping the analogy with existing literature in the sense that $T_i^{TP}(PG)$ in this work coincides with T_i^{TP} in Holler and Nohn (2009) and Alonso-Mejide et al. (2010).

Let us look at some examples and focus on cases of a precoalition which is not a null union in the quotient game and whose individual members are null unions in the threat games. As for the three canonical threat partitions, this situation can only happen for Threat Partition 3, but can be ruled out for Threat Partitions 1 and 2 since in these two cases a precoalition must be a null union in the quotient game whenever all its members form singleton null unions in their threat games. The following examples appeared first in the third and fourth footnote of Alonso-Mejide et al. (2010). We specify them as weighted voting games.

Example 1. Let us investigate the four-player weighted voting game with weights $w_1 = 4$, $w_2 = w_3 = w_4 = 2$, quota $q = 8$ and the precoalition structure $P = \{\{1\}, \{2, 3, 4\}\}$ with $f_{\{2, 3, 4\}}(P, v^P) = 0.5$ in the quotient game and f being any of the six power indices SS , PG , DP , J , f^{np} or g^{np} .

(a) All members of the precoalition $\{2, 3, 4\}$ are null unions $\{i\}$, $i = 2, 3, 4$, attributed zero power in the threat games $(P/i, v^{P/i})$ corresponding to Threat Partition 3. We obtain

$$T_i^3(f)(N, v, P) = \frac{f_{\{2,3,4\}}(P, v^P)}{|P(i)|} = \frac{0.5}{3} = \frac{1}{6}$$

for $i = 2, 3, 4$.

(b) As for Threat Partitions 1 and 2, the individual members of the precoalition $\{2, 3, 4\}$ are attributed positive power in their threat games and hence $T_i^1(f) = T_i^2(f) = \frac{1}{6}$ for $i = 2, 3, 4$.

(c) For the threat partition EO from (12), we observe that Player 3 forms a null union $\{3\}$ in their threat game as the two Players $\{2, 4\}$ with even indices remain united in a precoalition. On the other hand, Players 2 and 4 both possess positive threat power as they face two singleton coalitions, i.e., one singleton coalition consisting of Player 3 and another consisting of the other even-indexed player, in their threat games. Hence $T_3^{EO}(f) = 0$ and $T_i^{EO}(f) = \frac{1}{4}$ for $i = 2, 4$.

Example 2. Let us introduce a null player in the game from Example 1 and turn it into a five-player weighted voting game. We use the weights $w_1 = 4$, $w_2 = w_3 = w_4 = 2$, $w_5 = 1$, quota $q = 8$ and the precoalition structure $P = \{\{1\}, \{2, 3, 4, 5\}\}$ with $f_{\{2,3,4,5\}}(P, v^P) = 0.5$ in the quotient game and f standing for any of the six power indices SS, PG, DP, J, f^{np} or g^{np} . Clearly, Player 5 is a null player in (N, v) .

(a) Again, all members of the precoalition $\{2, 3, 4, 5\}$ are null unions $\{i\}$, $i = 2, 3, 4, 5$, attributed zero power in the threat games $(P/i, v^{P/i})$ corresponding to Threat Partition 3. We obtain

$$T_i^3(f)(N, v, P) = \frac{f_{\{2,3,4,5\}}(P, v^P)}{|P(i)|} = \frac{0.5}{4} = \frac{1}{8}$$

for $i = 2, 3, 4, 5$. Thus, $T_5^3(f) = \frac{1}{8} > 0$ assigns power greater zero to Player 5 despite the fact that they are a null player in (N, v) .

(b) As for Threat Partitions 1 and 2, Players 2, 3, and 4 are attributed positive power in their threat games, whereas Player 5 is attributed zero power in their threat game. Hence $T_5^1(f) = T_5^2(f) = 0$ and $T_i^1(f) = T_i^2(f) = \frac{1}{6}$ for $i = 2, 3, 4$.

(c) For threat partition EO from (12), we observe that Player 3 faces the null union $\{5\}$ and $\{2, 4\}$ in their threat game and hence forms a null union $\{3\}$ themselves. Clearly, the null player 5 also forms a null union $\{5\}$ in their threat game. On the other hand, Players 2 and 4 both possess positive threat power as they face $\{3, 5\}$ and the singleton coalition consisting the other even-indexed player in their threat games. Hence $T_i^{EO}(f) = \frac{1}{4}$ for $i = 2, 4$ and $T_i^{EO}(f) = 0$ for $i = 3, 5$.

In the following, we suggest a slight modification that guarantees the null player property for general threat partitions. In case $\sum_{j \in P(i)} f_{\{j\}}(TP(P, i), v^{TP(P, i)}) = 0$, we suggest to divide the power $f_{P(i)}(P, v^P)$ of precoalition $P(i)$ in the quotient game by the number of non-null players of $P(i)$ in (N, v) and assign zero power to null players in (N, v) . We let $np(v, N^0, P_k)$ denote the number of null players in (N, v) belonging to union P_k . Thus, our alternative suggestion for general threat partitions specified as a mapping TP reads as follows. Given a threat partition TP , the null player free Threat Power index $T_i^{TP-np}(f)$ of player $i \in P(i)$ (based upon an efficient power index f on $SI(N)$ as introduced in Section 2.2) is defined by

$$T_i^{TP-np}(f)(N, v, P) = f_{P(i)}(P, v^P) \frac{f_{\{i\}}(TP(P, i), v^{TP(P, i)})}{\sum_{j \in P(i)} f_{\{j\}}(TP(P, i), v^{TP(P, i)})}. \quad (13)$$

whenever $\sum_{j \in P(i)} f_{\{j\}}(TP(P, i), v^{TP(P, i)}) > 0$. In case $\sum_{j \in P(i)} f_{\{j\}}(TP(P, i), v^{TP(P, i)}) = 0$, we assign

$$T_i^{TP-np}(f)(N, v, P) = \frac{f_{P(i)}(P, v^P)}{np(v, N^0, P(i))} \quad (14)$$

whenever $f_i(N, v) > 0$ and $T_i^{TP-np}(f)(N, v, P) = 0$ otherwise.

We keep the notation for our null player free threat power indices introduced in (13) and (14) analogous to (8) by simply adding the suffix “-np” following the threat partition in the upper index. In terms of the three canonical threat partitions, there is no need to study threat indices T^{1-np} or T^{2-np} separately, because the situation of a precoalition Q which is not a null union in the quotient game, i.e., $f_Q(P, v^P) > 0$, and whose individual members are null unions in the threat games can not occur for Threat Partitions 1 and 2. The alternative definition for Threat Partition 3 looks as follows. The *null player free Threat Power index* $T_i^{3-np}(f)$ of player $i \in P(i)$ (based upon an efficient power index f on $SI(N)$) as introduced in Section 2.2) is given by

$$T_i^{3-np}(f)(N, v, P) = f_{P(i)}(P, v^P) \frac{f_{\{i\}}(P/i, v^{P/i})}{\sum_{j \in P(i)} f_{\{j\}}(P/j, v^{P/j})}. \quad (15)$$

whenever $\sum_{j \in P(i)} f_{\{j\}}(P/j, v^{P/j}) > 0$. In case $\sum_{j \in P(i)} f_{\{j\}}(P/j, v^{P/j}) = 0$, we assign

$$T_i^{3-np}(f)(N, v, P) = \frac{f_{P(i)}(P, v^P)}{np(v, N^0, P(i))} \quad (16)$$

whenever $f_i(N, v) > 0$ and $T_i^{3-np}(f)(N, v, P) = 0$ otherwise.

Example 3. Let us revisit our five-player weighted voting game with weights $w_1 = 4$, $w_2 = w_3 = w_4 = 2$, $w_5 = 1$, quota $q = 8$, and the precoalition structure $P = \{\{1\}, \{2, 3, 4, 5\}\}$ from Example 2. For our alternative definition of $T^{3-np}(f)$, we obtain

$$T_i^{3-np}(f)(N, v, P) = \frac{f_{\{2,3,4,5\}}(P, v^P)}{np(v, N^0, P(i))} = \frac{0.5}{3} = \frac{1}{6}$$

for $i = 2, 3, 4$ and $T_5^{3-np}(f) = 0$, i.e., the null player is assigned zero power and does not affect the power indices of the non-null Players 2, 3, and 4. Obviously, the null player is now removable from the game (N, v, P) .

The following theorem generalizes some observations on threat power indices based on the Public Good index from Holler and Nohn (2009) and Alonso-Mejide et al. (2010) for the three canonical cases.

Theorem 3. We let f stand for one of the following six power indices Shapley–Shubik SS, Public Good PG, Deegan–Packel DP, Johnston J, the null player free index f^{np} , or the the null player free index g^{np} . Further, we let the threat power indices $T^1(f)$, $T^2(f)$, $T^3(f)$ and $T^{3-np}(f)$ be defined according to (9)–(11) and (15). Then, we observe:

- (a) If at most one precoalition contains more than one player, then the threat indices $T^1(f)$ and $T^2(f)$ yield the same result (depending only on f);
- (b) If no precoalition consists of more than two players, then the threat indices $T^2(f)$, $T^3(f)$ and $T^{3-np}(f)$ yield the same result (depending only on f);
- (c) $T^1(f)$ attributes non-zero power to any Player i who is not a null player in (N, v) and whose precoalition $P(i)$ is not a null union in (P, v^P) ;

(d) For any of the six choices for f , there holds $T_i^2(f)(N, v, P) = 0$ if and only if Player i does not belong to any essential part;

(e) $T_i^2(f)(N, v, P) = 0$ implies $T_i^3(f)(N, v, P) = T_i^{3-np}(f)(N, v, P) = 0$ for any choice of f ;

(f) Even in case Player i is not a null player in (N, v) and their precoalition $P(i)$ is not assigned non-zero power in the quotient game (P, v^P) , player i might be assigned zero power by $T_i^2(f)(N, v, P)$, $T_i^3(f)(N, v, P)$ and $T_i^{3-np}(f)(N, v, P)$ depending only upon the structure of P and irrespective of the choice of f .

Proof. (a) is obvious from the definitions of $T^1(f)$ and $T^2(f)$;

(b) is obvious from the definitions of $T^2(f)$, $T^3(f)$ and $T^{3-np}(f)$;

(c) follows from the definitions of $T^1(f)$ and the quotient game;

(d) is due to the fact that player i forms a null union $\{i\}$ in the threat game $(N, P/P(i), v^{P/P(i)})$ if and only if player i does not belong to any essential part (which was defined towards the end of Section 3.1). In other words, $\{i\}$ is a critical union within the threat game $(N, P/P(i), v^{P/P(i)})$ if and only if player i is a member of at least one essential part. Hence the latter condition impacts all six choices for f equally;

(e) whenever player i forms a null union $\{i\}$ in the threat game $(N, P/P(i), v^{P/P(i)})$, that implies player i also forms a null union $\{i\}$ in the threat game $(N, P/i, v^{P/i})$;

(f) is discussed below in Example 4. $\square$

Example 4. Let us investigate the five-player weighted voting game with weights $w_1 = 4, w_2 = w_3 = 2, w_4 = 4, w_5 = 1$, quota $q = 10$, and the precoalition structure $P = \{\{1, 2, 3\}, \{4, 5\}\}$ with $f_{\{1, 2, 3\}}(P, v^P) = f_{\{4, 5\}}(P, v^P) = 0.5$ in the quotient game and f being any of the six power indices SS, PG, DP, J, f^{np} or g^{np} . In (N, v) , none of the five players are null players. Still, in the threat game $(N, P/i, v^{P/i})$ corresponding to Threat Partition 3, there holds $T_i^3(f)(N, v, P) = T_i^{3-np}(f)(N, v, P) = 0$ for $i = 2, 3, 5$. With $T_i^3(f)(N, v, P) = T_i^{3-np}(f)(N, v, P) = 0.5$ for $i = 1, 4$ all power is attributed to Players 1 and 4. However, the picture looks differently for $T^2(f)$ with $T_4^2(f)(N, v, P) = 0.5$ and $T_5^2(f)(N, v, P) = 0$, whereas $T_i^2(f)(N, v, P) > 0$ for $i = 1, 2, 3$.

We look at two very similar examples in detail in Section 5.

4.2. Axiomatic Characterizations of Threat Power Indices

In Alonso-Meijide et al. (2010), axiomatic characterizations of threat power indices based upon the Public Good index are discussed for general threat partitions TP and two new properties of coalitional power indices are introduced. We list these two properties below and formulate our analogy to the second property taking into account our suggestions for incorporating the null player property reflecting Equation (16) and our model from Section 4.1.

Given a threat partition TP , a coalitional power index g is said to exhibit

- the property *TP proportionality within unions* whenever for all simple games with precoalitions (N, v, P) the equation

$$g_i(N, v, P)g_j(N, v, TP(P, j)) = g_j(N, v, P)g_i(N, v, TP(P, i)) \quad (17)$$

is satisfied for all players i and j participating in the same precoalition $Q \in P$;

- the property *TP empty threats* whenever for all simple games with precoalitions (N, v, P) the equation

$$g_i(N, v, P) = g_j(N, v, P) \quad (18)$$

is satisfied for all players i and j participating in the same precoalition $Q \in P$ in the case $\sum_{k \in Q} g_k(N, v, TP(P, k)) = 0$;

- the property *TP empty threats among non-null players* whenever for all simple games with precoalitions (N, v, P) the equation

$$g_i(N, v, P) = g_j(N, v, P) \quad (19)$$

is satisfied for all players i and j participating in the same precoalition $Q \in P$ with neither i nor j being a null player in (N, v) in the case $\sum_{k \in Q} g_k(N, v, TP(P, k)) = 0$.

TP proportionality within unions is designed to guarantee that power within precoalitions is divided proportionally to the players' power in their corresponding threat games. Note that the property is specific to the threat partition TP at hand, i.e., the axiom incorporates the threat partition TP .

As for the remainder of this subsection, we first axiomatize all threat power indices discussed in Section 4.1. Afterwards, we argue the necessity of all axioms and present two corollaries on threat power indices.

The following theorem generalizes and extends a proposition from [Alonso-Meijide et al. \(2010\)](#).

Theorem 4. *Let f stand for one of the following six power indices: Shapley–Shubik SS, Public Good PG, Deegan–Packel DP, Johnston J, the null player free index f^{np} , or the the null player free index g^{np} . Further, let the threat power indices $T^{TP}(f)$ be defined according to (8) and the null player free threat power indices $T^{TP-np}(f)$ be defined according to (13) and (14).*

These threat power indices are the unique generalized coalitional f -indices satisfying the quotient game property, TP proportionality within unions and

- (a) *TP empty threats in case of $T^{TP}(f)$;*
- (b) *TP empty threats among non-null players and null player removability in case of $T^{TP-np}(f)$.*

Proof. The proof follows the corresponding proof in [Alonso-Meijide et al. \(2010\)](#). Nevertheless, it appears worthwhile to appreciate its individual steps once again.

1. *Existence:* We first prove that $T^{TP}(f)$ and $T^{TP-np}(f)$ are generalized coalitional f -indices satisfying the quotient game property, TP proportionality within unions. In a slight abuse of notation, we write $T(f)$ whenever it is clear that arguments and calculations are identical for $T^{TP}(f)$ and $T^{TP-np}(f)$, i.e., it is implied that $T(f)$ comes with a specific threat partition TP .

Generalized coalitional f -index: There holds

$$T_i(f)(N, v, N^0) = f_{\{i\}}(N^0, v^{N^0}) = f_i(N, v)$$

for all players i and all simple games (N, v) .

Quotient game property: For all simple games with precoalitions (N, v, P) and any precoalition $Q \in P$, we obtain

$$\sum_{i \in Q} T_i(f)(N, v, P) = f_Q(P, v^P) = T_Q(f)(P, v^P, P^0).$$

TP proportionality within unions: We let i and j be players belonging to the identical precoalition $Q \in P$. TP proportionality within unions is trivially satisfied in the case $\sum_{k \in Q} f_{\{k\}}(TP(P, k), v^{TP(P, k)}) = 0$. Otherwise, there holds

$$\begin{aligned} & T_i(f)(N, v, P) T_j(f)(N, v, TP(P, j)) \\ &= f_Q(P, v^P) \frac{f_{\{i\}}(TP(P, i), v^{TP(P, i)})}{\sum_{k \in Q} f_{\{k\}}(TP(P, k), v^{TP(P, k)})} f_{\{j\}}(TP(P, j), v^{TP(P, j)}) \\ &= f_Q(P, v^P) \frac{f_{\{j\}}(TP(P, j), v^{TP(P, j)})}{\sum_{k \in Q} f_{\{k\}}(TP(P, k), v^{TP(P, k)})} f_{\{i\}}(TP(P, i), v^{TP(P, i)}) \\ &= T_j(f)(N, v, P) T_i(f)(N, v, TP(P, i)). \end{aligned}$$

(a) *TP empty threats* follows immediately from the definition of $T^{TP}(f)$.

(b) *TP empty threats among non-null players and null player removability* follow immediately from the definition of $T^{TP-np}(f)$.

2. *Uniqueness:* We point out that any coalitional power index g fulfilling the itemized properties coincides with $T^{TP}(f)$ or $T^{TP-np}(f)$, respectively, reflecting the axioms and threat partition employed.

Given that g is a generalized coalitional f -index and that g retains the quotient game property, we know that for all simple games with precoalitions (N, v, P) and all precoalitions $Q \in P$

$$\sum_{i \in Q} g_i(N, v, P) = g_Q(P, v^P, P^0) = f_Q(P, v^P)$$

is guaranteed. For the sake of clarity, we point out that in the above equation the first equality is due to the quotient game property and the second due to g being a generalized coalitional f -index.

The latter also implies $g_i(N, v^{TP(P, i)}, TP(P, i)) = f_{\{i\}}(TP(P, i), v^{TP(P, i)})$ for each player $i \in Q$. Because g obeys TP proportionality within unions as well, for each $i \in Q$, we obtain

$$\begin{aligned} g_i(N, v, P) \sum_{j \in Q} f_{\{j\}}(TP(P, j), v^{TP(P, j)}) &= g_i(N, v, P) \sum_{j \in Q} g_j(N, v^{TP(P, j)}, TP(P, j)) \\ &= \sum_{j \in Q} g_j(N, v, P) g_i(N, v^{TP(P, i)}, TP(P, i)) = f_Q(P, v^P) f_{\{i\}}(TP(P, i), v^{TP(P, i)}). \end{aligned}$$

If $\sum_{j \in Q} f_{\{j\}}(TP(P, j), v^{TP(P, j)}) > 0$, it follows that

$$g_i(N, v, P) = f_Q(P, v^P) \frac{f_{\{i\}}(TP(P, i), v^{TP(P, i)})}{\sum_{j \in Q} f_{\{j\}}(TP(P, j), v^{TP(P, j)})} = T_i^{TP}(f)(N, v, P) = T_i^{TP-np}(f)(N, v, P).$$

If $\sum_{j \in Q} f_{\{j\}}(TP(P, j), v^{TP(P, j)}) = 0$, then

(a) TP empty threats ensures $g_i(N, v, P) = \frac{f_Q(P, v^P)}{|P(i)|} = T_i^{TP}(f)(N, v, P)$.

(b) TP empty threats among non-null players and null player removability guarantee

$$g_i(N, v, P) = \frac{f_Q(P, v^P)}{np(v, N^0, P(i))} = T_i^{TP-np}(f)(N, v, P)$$

if i is not a null player in (N, v) and $g_i(N, v, P) = 0 = T_i^{TP-np}(f)(N, v, P)$ otherwise. $\square$

Let us recapitulate the proof of Theorem 4 and appreciate the necessity of all axioms. g being a generalized coalitional f -index ensures that the efficient power index f is applied both in the quotient game and in the threat games. While the quotient game property guarantees that the members of each precoalition divide the power of their precoalition

in the quotient game amongst each other, TP proportionality within unions makes certain that this internal power division occurs proportionally to the players' power in their corresponding threat games. Thus, TP proportionality within unions is the one axiom carrying the specific information on the threat partition TP . Clearly, these three properties are indispensable.

The case $\sum_{j \in Q} f_{\{j\}}(TP(P, j), v^{TP(P, j)}) = 0$, i.e., all members j of the precoalition Q have zero power in their threat games, deserves a closer look. For the approach T^{TP} , the axiom TP empty threats ensures that all players $i \in Q$ obtain $\frac{f_Q(P, v^P)}{|P(i)|} = T_i^{TP}(f)(N, v, P)$, whereas without the axiom any power distribution among the players in Q satisfying $\sum_{j \in Q} g_j(N, v, P) = f_Q(P, v^P)$ would be possible. For the null player free threat power indices T^{TP-np} , the axiom TP empty threats among non-null players and null player removability together ensure that all non-null players $i \in Q$ obtain equal threat power $\frac{f_Q(P, v^P)}{np(v, N^0, P(i))} = T_i^{TP-np}(f)(N, v, P)$ and that any null players in (N, v) are attributed zero threat power. Clearly, the combination of the two axioms TP empty threats among non-null players and null player removability is needed. The former axiom alone only guarantees that all non-null players $i \in Q$ receive the identical threat power, but not that the sum of their threat powers equals $f_Q(P, v^P)$ as it does not specify the share of the null players to be zero. On the other hand, null player removability alone would only guarantee that the threat powers of non-null players total to $f_Q(P, v^P)$, but not that they are equal.

It is obvious that we could have replaced the null player removability axiom for T^{TP-np} in Theorem 4 with the null player axiom. In order to keep our axiomatization in sync with Section 2.2 and Theorem 1, we opted for null player removability.

As emphasized before, the case of a precoalition Q with $\sum_{j \in Q} f_{\{j\}}(TP(P, j), v^{TP(P, j)}) = 0$, i.e., all members j of the precoalition Q have zero power in their threat games, and $f_Q(P, v^P) > 0$ cannot occur for the two canonical Threat Partitions 1 and 2. For these threat partitions, a precoalition has zero power in the quotient game whenever all its members have zero threat power. As indicated in Alonso-Meijide et al. (2010), for these two special cases, we can replace the axioms listed in (a) and (b) in Theorem 4 by the weaker and simpler nonnegativity axiom.

Corollary 1. *Let the threat power indices $T^1(f)$ and $T^2(f)$ be defined according to (9) and (10) for the canonical Threat Partitions 1 and 2, respectively, with f as in Theorem 4. These threat power indices are the unique generalized coalitional f -indices satisfying the quotient game property, TP proportionality within unions and nonnegativity.*

We remind the reader that in Corollary 1 the distinction between $T^1(f)$ and $T^2(f)$ and their underlying threat partitions is elegantly incorporated in the property TP proportionality within unions underlining the versatility of that axiom.

Finally, let us revisit symmetry within unions introduced at the end of Section 3.2 and discuss this property for threat power indices. We saw in Examples 1 and 2 for the threat power indices (12) pertaining to our “even–odd” threat partition EO that in general, threat power indices do not exhibit symmetry within unions. However, for the special cases of the three canonical Threat Partitions 1, 2, and 3 the picture looks differently.

Corollary 2. *Let the threat power indices $T^1(f)$, $T^2(f)$ and $T^3(f)$ be defined according to (9)–(11) for the canonical Threat Partitions 1, 2, and 3, respectively, and the null player free threat power index $T^{3-np}(f)$ be defined according to (15) and (16) for the canonical Threat Partition 3, with f as in Theorem 4. These threat power indices satisfy symmetry within unions, i.e., for two players $i, j \in Q$ who are both members of precoalition Q and are also symmetric players in (N, v) the*

four equalities $T_i^1(f) = T_j^1(f)$, $T_i^2(f) = T_j^2(f)$, $T_i^3(f) = T_j^3(f)$, and $T_i^{3-np}(f) = T_j^{3-np}(f)$ are guaranteed.

5. Numerical Results

We analyze the values of threat power indices for both an example from [Holler and Nohn \(2009\)](#) with three equally strong precoalitions of different sizes and a real-world scenario originally investigated in [Alonso-Meijide et al. \(2010\)](#). In all tables in this section, values of power indices are rounded to five digits after the decimal point.

5.1. An Example with Three Equally Strong Precoalitions

We revisit the following example from [Holler and Nohn \(2009\)](#) writing it as a weighted voting game with small integers.

Example 5 ([Holler & Nohn, 2009](#)). We investigate the six-player weighted voting game with weights $w_1 = 7$, $w_2 = w_3 = 4$, $w_4 = 3$, $w_5 = w_6 = 1$, quota $q = 11$, and the precoalition structure $P = \{\{1, 5, 6\}, \{2, 3\}, \{4\}\}$.

For Example 5, all three precoalitions $P_1 = \{1, 5, 6\}$, $P_2 = \{2, 3\}$ and $P_3 = \{4\}$ are equally strong as there are three minimal winning coalitions on the external level, i.e., any pair of two precoalitions is minimal winning. Thus, for the quotient game, there holds $f_{P_k}(P, v^p) = \frac{1}{3}$ for $k = 1, 2, 3$ and f standing for any of the six power indices SS , PG , DP , J , f^{np} , or g^{np} . Given that we have three equally powerful precoalitions of three different sizes, this example is well suited to illustrate the influence of the six efficient power indices from Section 2.2 combined with the three threat partitions introduced in Section 4.1.

Studying the power indices on (N, v) , i.e., ignoring the precoalition structure, in Table 1, we recognize that the game has no null players and the Public Good index and the Deegan–Packel index are the two indices lacking monotonicity with respect to the weights. As for Player 1, i.e., the player with the largest voting weight, the Johnston index attributes the largest number, followed by Shapley–Shubik, Deegan–Packel, Public Good, f^{np} , and g^{np} . We observe that the latter two indices which are based on null player free winning coalitions distribute power most evenly among the six players.

Table 1. Power indices on (N, v) for Example 5.

Player Number	SS	PG	DP	J	f^{np}	g^{np}
1	0.46667	0.30769	0.33333	0.59877	0.24731	0.22881
2	0.18333	0.15385	0.16667	0.14198	0.17204	0.16949
3	0.18333	0.15385	0.16667	0.14198	0.17204	0.16949
4	0.13333	0.23077	0.20000	0.09259	0.15591	0.16102
5	0.01667	0.07692	0.06667	0.01235	0.12634	0.13559
6	0.01667	0.07692	0.06667	0.01235	0.12634	0.13559

We list the results for Threat Partitions 1 and 2 in Tables 2 and 3, respectively. As for Threat Partition 3, one simply obtains $T_1^3(f) = T_4^3(f) = \frac{1}{3}$, $T_2^3(f) = T_3^3(f) = \frac{1}{6}$ and $T_5^3(f) = T_6^3(f) = 0$ irrespective of the choice of f . As from Tables 2 and 3, we confirm that the two symmetric Players 2 and 3 forming precoalition P_2 are treated symmetrically within their union. As for the internal threat games of Players 1, 5, and 6 forming precoalition P_1 , it is worthwhile comparing the indices of Player 1 (as the strongest player) in Table 1. Again, the Johnston index attributes the largest number, followed by Shapley–Shubik, Deegan–Packel, Public Good, f^{np} , and g^{np} indices for Threat Partition 1. In Table 2, we see the same ranking for the power indices of Player 1, but for Threat Partition 2 the effect is

less pronounced in the sense that all numbers are smaller with a little more power being attributed to the weaker Players 5 and 6 instead.

Table 2. Power indices on (N, v, P) for Threat Partition 1 and Example 5.

Player Number	$T^1(SS)$	$T^1(PG)$	$T^1(DP)$	$T^1(J)$	$T^1(f^{np})$	$T^1(g^{np})$
1	0.31111	0.22222	0.23810	0.32013	0.16487	0.15254
2	0.16667	0.16667	0.16667	0.16667	0.16667	0.16667
3	0.16667	0.16667	0.16667	0.16667	0.16667	0.16667
4	0.33333	0.33333	0.33333	0.33333	0.33333	0.33333
5	0.01111	0.05556	0.04762	0.00660	0.08423	0.09040
6	0.01111	0.05556	0.04762	0.00660	0.08423	0.09040

Table 3. Power indices on (N, v) for Threat Partition 2 and Example 5.

Player Number	$T^2(SS)$	$T^2(PG)$	$T^2(DP)$	$T^2(J)$	$T^2(f^{np})$	$T^2(g^{np})$
1	0.26984	0.20000	0.21212	0.27536	0.14449	0.13580
2	0.16667	0.16667	0.16667	0.16667	0.16667	0.16667
3	0.16667	0.16667	0.16667	0.16667	0.16667	0.16667
4	0.33333	0.33333	0.33333	0.33333	0.33343	0.33333
5	0.03175	0.06667	0.06061	0.02899	0.09442	0.09877
6	0.03175	0.06667	0.06061	0.02899	0.09442	0.09877

As we pointed out before, the situation for the two small Players 5 and 6 within union P_1 is assessed most drastically in Threat Partition 3. The two minor players constitute null unions in their respective threat games, i.e., in that model, their power derives solely from Player 1, who thus obtains the entirety of the power share irrespective of the power index f . Thus, we obtained another illustration of statement f) from Theorem 3. Needless to add, $T^{3-np}(f)$ and $T^3(f)$ coincide as there are no null players in (N, v) .

5.2. A Real World Example

We revisit the following real world example from [Alonso-Meijide et al. \(2010\)](#) and compare our novel threat power indices to the existing results based on the Public Good index.

Example 6 (Elections to the Parliament of the Basque Country held on 14 April 2005 [Alonso-Meijide et al., 2010](#)). We look at the Parliament of the Basque Country with its 75 seats apportioned following to the elections held on 14 April 2005. The allocation of seats was as follows: 22 seats for the Basque nationalist conservative party EAJ/PNV (Party 1); 18 seats for the Spanish socialist party PSE-EE/PSOE (Party 2); 15 seats for the Spanish conservative party PP (Party 3); 9 seats for the Basque left-wing party EHAK/PCTV (Party 4); 7 seats for the Basque nationalist social democrat party EA (Party 5); 3 seats for the Spanish left-wing party EB/IU (Party 6); 1 seat for the Basque nationalist left-wing party Aralar (Party 7). We can model the scenario as a weighted voting game with quota $q = 38$ and the weights $w_1 = 22$, $w_2 = 18$, $w_3 = 15$, $w_4 = 9$, $w_5 = 7$, $w_6 = 3$, and $w_7 = 1$. Given that EAJ/PNV (1), EA (5), and EB/IU (6) formed the government before the elections in April 2005, it stands to reason that these three parties continue to act as a precoalition $P_1 = \{1, 5, 6\}$. Due to both PSE-EE/PSOE (2) and PP (3) possessing no formal links to any other party in that parliament, they can both be modelled as singleton precoalitions $P_2 = \{2\}$ and $P_3 = \{3\}$, respectively. The two left-wing Basque parties EHAK/PCTV (4) and Aralar (7) may be modelled as a fourth precoalition $P_4 = \{4, 7\}$.

Table 4 summarizes the structure of the Parliament from Example 6 and lists the values of our six efficient power indices on (N, v) , i.e., ignoring the precoalition structure. None of the seven parties is a null player. As in the previous Example 5, the Public Good and

Deegan–Packel indices both display non-monotonicity of power, i.e., even though Party 2 gained more seats than Party 3, the latter party belongs to two more minimal winning coalitions and hence Public Good and Deegan–Packel indices attribute more power to it; see [Alonso-Mejide et al. \(2010\)](#). As for the power measurement of Party 1, the player with largest voting weight, again the Johnston index comes out to be most favorable, with the Shapley–Shubik index coming second.

Table 4. Power indices on simple game (N, v) for Example 6.

Party	Seats	SS	PG	DP	J	f^{np}	g^{np}
EAJ/PNV (1)	22	0.35238	0.21429	0.24074	0.44444	0.19494	0.18280
PSE-EE/ PSOE (2)	18	0.25238	0.14286	0.15185	0.23268	0.16760	0.16129
PP (3)	15	0.18571	0.21429	0.22222	0.17647	0.15666	0.15412
EHAK/PCTV (4)	9	0.08571	0.14286	0.13333	0.06275	0.12932	0.13262
EA (5)	7	0.08571	0.14286	0.13333	0.06275	0.12932	0.13262
EB/IU (6)	3	0.01905	0.07143	0.05926	0.01046	0.11109	0.11828
Aralar (7)	1	0.01905	0.07143	0.05926	0.01046	0.11109	0.11828

Table 5 summarizes the quotient game and lists the values of our six efficient power indices on (N, v^P, P) , i.e., on the external level. As observed in [Alonso-Mejide et al. \(2010\)](#), the largest precoalition P_1 constitutes a minimal winning coalition with any of the three smaller precoalitions P_2 , P_3 , and P_4 , with the latter three being able to forge a fourth minimal winning coalition on their own (without P_1). According to both the Johnston and Shapley–Shubik indices, the largest precoalition P_1 gains more power than the total power of its Members 1, 5, and 6 in the simple game (N, v) analyzed in Table 4. For Public Good, Deegan–Packel, and the two null player free indices, the observation for P_1 reverses.

Table 5. Power indices on quotient game (N, v, P) for Example 6.

Precoalition	Seats	SS	PG	DP	J	f^{np}	g^{np}
$P_1 = \{1, 5, 6\}$	32	0.50000	0.33333	0.37500	0.64286	0.34375	0.31818
$P_2 = \{2\}$	18	0.16667	0.22222	0.20833	0.11905	0.21875	0.22727
$P_3 = \{3\}$	15	0.16667	0.22222	0.20833	0.11905	0.21875	0.22727
$P_4 = \{4, 7\}$	10	0.16667	0.22222	0.20833	0.11905	0.21875	0.22727

In Tables 6–8 we display the threat power indices for Threat Partitions 1, 2, and 3, respectively. Interpreting the internal threat games of precoalition P_1 in Table 6, we confirm that the Johnston index favors the major player EAJ/PNV (party 1) very strongly. We observe a similar effect for the Shapley–Shubik index, although slightly mitigated.

Table 6. Power indices on (N, v, P) for Threat Partition 1 and Example 6.

Party	$T^1(SS)$	$T^1(PG)$	$T^1(DP)$	$T^1(J)$	$T^1(f^{np})$	$T^1(g^{np})$
EAJ/PNV (1)	0.38542	0.16667	0.20833	0.55195	0.15393	0.13411
PSE-EE/PSOE (2)	0.16667	0.22222	0.20833	0.11905	0.21875	0.22727
PP (3)	0.16667	0.22222	0.20833	0.11905	0.21875	0.22727
EHAK/PCTV (4)	0.13636	0.14815	0.14423	0.10204	0.11767	0.12013
EA (5)	0.09375	0.11111	0.11538	0.07792	0.10211	0.09730
EB/IU (6)	0.02083	0.05556	0.05128	0.01299	0.08771	0.08678
Aralar (7)	0.03030	0.07407	0.06410	0.01701	0.10108	0.10714

For Threat Partition 2, the two weaker Parties 5 and 6 within union P_1 fare slightly better than with respect to all six power indices than under Threat Partition 1; see Table 7.

Most strikingly, Aralar (Party 7) never belongs to any essential part in (N, v, P) and hence is assigned zero power, i.e., $T_7^2(f) = 0$, illustrating statement d) from Theorem 3.

Table 7. Power indices on (N, v, P) for Threat Partition 2 and Example 6.

Party	$T^2(SS)$	$T^2(PG)$	$T^2(DP)$	$T^2(J)$	$T^2(f^{np})$	$T^2(g^{np})$
EAJ/PNV (1)	0.35714	0.15152	0.18750	0.51598	0.14993	0.13040
PSE-EE/PSOE (2)	0.16667	0.22222	0.20833	0.11905	0.21875	0.22727
PP (3)	0.16667	0.22222	0.20833	0.11905	0.21875	0.22727
EHAK/PCTV (4)	0.16667	0.22222	0.20833	0.11905	0.21875	0.22727
EA (5)	0.10714	0.12121	0.12784	0.09727	0.10422	0.09911
EB/IU (6)	0.03571	0.06061	0.05966	0.02961	0.08959	0.08867
Aralar (7)	0	0	0	0	0	0

Table 8 pertaining to Threat Partition 3 illustrates both statement (e) from Theorem 3 (in the sense of $T_7^2(f) = 0$ implying $T_7^3(f) = 0$) and statement (f) from Theorem 3 as it is the only scenario in which EB/IU (Party 6) becomes a null union in its respective threat game. The strong dependency of EB/IU upon its precoalition P_1 is revealed.

Table 8. Power indices on (N, v, P) for Threat Partition 3 and Example 6.

Party	$T^3(SS)$	$T^3(PG)$	$T^3(DP)$	$T^3(J)$	$T^3(f^{np})$	$T^3(g^{np})$
EAJ/PNV (1)	0.40909	0.18841	0.23116	0.55714	0.20404	0.18036
PSE-EE/PSOE (2)	0.16667	0.22222	0.20833	0.11905	0.21875	0.22727
PP (3)	0.16667	0.22222	0.20833	0.11905	0.21875	0.22727
EHAK/PCTV (4)	0.16667	0.22222	0.20833	0.11905	0.21875	0.22727
EA (5)	0.09091	0.14493	0.14384	0.08571	0.13971	0.13782
EB/IU (6)	0	0	0	0	0	0
Aralar (7)	0	0	0	0	0	0

Again, $T^{3-np}(f)$ and $T^3(f)$ coincide, because there are no null players in (N, v) .

6. Conclusions

In this paper, we studied power indices for simple cooperative games with precoalitions which distribute power among players in an external step in the shape of the quotient game and an internal step in the shape of threat games. Our main purpose was to expand existing work (Alonso-Mejide et al., 2010; Holler & Nohn, 2009) which relies entirely upon the Public Good index both on the external level and in the internal threat games. We substituted the Public Good index PG in that model by five alternative efficient power indices, i.e., the Shapley–Shubik index SS , the Deegan–Packel index DP , the Johnston index J , and the two indices f^{np} and g^{np} based on null player free winning coalitions. The various new power indices with threat partitions are axiomatically characterized along the lines of Alonso-Mejide et al. (2010). Furthermore, this article pointed out the generality of the concept of threat partitions and showed that while the three canonical threat partitions introduced in Holler and Nohn (2009) exhibit symmetry within unions, there also exist threat power indices which assign different values to symmetric players belonging to the same precoalition. In addition, we suggested a small but crucial modification to the existing framework for threat power indices which guarantees that null players in the original game without precoalitions are always assigned zero power. Along the way, a new property for coalitional power indices called TP empty threats among non-null players was introduced. Our numerical experiments in Section 5 point out how the underlying power index and the choice of threat partition influence the results of the derived threat power indices.

Our mechanism for constructing threat power indices on the basis of a threat partition and a power index f on $SI(N)$ is very clearly not confined to the six power indices SS , PG , DP , J , f^{np} , and g^{np} . Any power index holding the symmetry, efficiency, and null player removability properties (and hence the null player property) can be supplanted. One might, for example, also test power indices with the aforementioned properties which are defined only for a subclass of simple games like the power index based on the minimum sum representation (MSR) of a weighted voting game (Freixas & Kaniovski, 2014). The Colomer–Martinez (Colomer & Martinez, 1995) and Holler–Colomer–Martinez (Armijos-Toro et al., 2024) indices, which are both defined merely for weighted voting games and were both recently characterized in Armijos-Toro et al. (2024), could be further suitable candidates.

Finally, we stress that our modification of the existing paradigm for threat power indices enforcing the null player property should be regarded as an alternative rather than an imperative. We are aware of research on power indices not exhibiting the null player property in the context of solidarity and egalitarianism; see recent works by Bertini and Stach on solidarity measures (Stach & Bertini, 2022) and subcoalitional values as power indices (Stach & Bertini, 2025).

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