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Simple Mediation in a Cheap-Talk Game †

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Abstract: In the Crawford–Sobel (uniform, quadratic utility) cheap-talk model, we consider a *simple* mediation scheme (a communication device) in which the informed agent reports one of the N possible elements of a partition to the mediator and then the mediator suggests one of the N actions to the uninformed decision-maker according to the probability distribution of the device. We show that no such *simple* mediated equilibrium can improve upon the unmediated N-partition Crawford–Sobel equilibrium when the preference divergence parameter (bias) is small.

Keywords: cheap talk; mediated equilibrium

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1. Introduction

We consider the (uniform, quadratic utility) cheap-talk model of [1] (hereafter referred to as the CS model) to study the effect of mediation in comparison to unmediated cheap-talk communication. In this game, the strategic interaction and information transmission between an uninformed decision-maker (called the receiver) and an informed agent (called the sender) has been studied, and what can be achieved by unmediated cheap-talk communication is well established. CS have proved that any (Bayesian–Nash) equilibrium in this cheap-talk game is equivalent to a partition equilibrium where the informed agent reveals one of the finitely many elements of the partition in which the true state of nature lies. The number of elements, N, in the most informative partition equilibrium in the CS model depends on the value of the preference divergence parameter, b.

Ref. [2] introduced an unmediated communication protocol which improved the welfare of two players relative to the CS equilibrium. Krishna and Morgan also constructed an example of mediated communication in the CS model, demonstrating the possibility of Pareto improvement.

Ref. [3] consider the effect of adding noise to the sender's message in the Crawford–Sobel model. The noise can be interpreted as communication error. In addition, one can think of their communication scheme as a special kind of a mediator who passes on messages from the sender to the receiver with some exogenous noise added on. Specifically, with some probability, the mediator passes on the sender's message to the receiver unchanged; otherwise, independent of the message sent by the sender, the mediator passes on a random message from some fixed error distribution. They show that for a sufficiently small amount of noise, it is possible to improve upon the N-partition CS equilibrium as long as $b \neq \frac{1}{2N^2}$ for any integer N and $b < \frac{1}{2}$.

Ref. [4] subsequently analysed the general form of mediation in the cheap-talk framework of CS. They derive the optimal unconstrained mediated mechanism and an upper bound on the receiver's payoff that can be achieved by any mediated equilibrium. This upper bound is achieved by the construction of [3] when the level of noise is chosen appropriately.[1] This bound is also the same as the value of the expected payoff to the receiver that is achieved by the equilibrium of the modified CS game constructed by Krishna and Morgan when $b < \frac{1}{8}$.

Here, we should point out that in all the above three models, welfare-improving mediation or noise involves a larger number of messages compared to the minimal number of messages required in the most informative CS equilibrium. None of the papers mentioned above addresses this issue of whether welfare-improving (non-strategic) mediation must *necessarily* involve additional messages and more actions induced in equilibrium.

We, in this paper, consider mediation schemes[2] in which the informed agent reports one possible element of a partition to a mediator (a communication device) and then the mediator suggests an action to the uninformed decision-maker according to the probability distribution of the device. We ask the question whether it is sufficient to use *simple* schemes (involving the same number of messages, N, as in the most informative CS equilibrium) for mediation to be Pareto superior to the CS equilibrium.[3]

In particular, we concentrate on a specific form of mediated equilibria, that we call *N-simple mediated equilibria*, in which the mediator is restricted to use the same number (N) of inputs and outputs as the number of elements of the N-partition CS equilibrium. This clearly imposes a restriction on the mediated mechanisms that one may consider; however, this restriction is made in order to enable us to answer the above question. The mediator, associated with a specific probability distribution, can be interpreted as a communication scheme that the players mutually agree to use. In this scheme, the possible number of elements the sender is allowed to use and the receiver is expected to receive via the mediator is restricted to N, as in the most informative CS equilibrium. Note that b, and hence N, is commonly known to the players.

The terminology "simple mechanism" also arises elsewhere in the context of mechanism design. An interesting strand of this literature studies "robust mechanisms" which relax some of the common-knowledge assumptions about beliefs, priors or functional forms of preferences (see [10]). The objective is to identify "simple" mechanisms which are reasonably detail-free compared to traditional Bayesian mechanisms and hence robust to structural uncertainty about the environment. Simple mechanisms and rules that give simple predictions are also appealing as they are more realistic and easily understandable. Although the motivation of this literature is somewhat different from our paper, it helps to justify our focus on the "simplicity" of the language or message space used by a mediator in our model.[4]

Our setup of cheap talk and mediation can be used to understand real-world scenarios, for example, one involving a politician (an uninformed decision-maker), a civil servant (a mediator), and an expert scientist (an informed agent). We should point out that we are not formally modelling such a situation here, in particular, the role of a civil servant. However, one can certainly consider a situation in which the expert meets the civil servant and reports (stochastically) the true state of the nature, and the civil servant in turn suggests an action (again, probabilistically) to the politician. In such a setup, it is natural to presume that the civil servant would use a message space that is not richer or more complex than what would be used in direct communication between the expert scientist and the politician. The main question of this paper can be interpreted as asking whether such a "simple" civil servant is worth having as a mediating channel in a conversation between an uninformed politician and an informed expert.

The main result of our paper (Theorem 1) is that the N-partition CS equilibrium cannot be improved upon by the corresponding N-simple mediated equilibrium when the preference divergence parameter b is small (less than $\frac{1}{2N^2}$). In other words, when b is small, mediation needs to use more messages (relative to the minimal number of messages that can be used in the best CS equilibrium) in order to improve upon the N-partition CS equilibrium. If mediation or noise is simply a randomisation over the messages that the sender would have reported in an original unmediated CS equilibrium, then it cannot improve information transmission when the degree of preference misalignment is small.

Mediation can usually improve on the cheap talk outcome because the mediator introduces noise in the communication between the informed agent and the decision-maker (see [11]). By randomising over the actions recommended to the receiver, mediation makes it easier to satisfy the incentive compatibility constraints. However, in our setup where the message space is restricted, there is no such additional benefit from mediation when the interests of the sender and the receiver are sufficiently closely aligned.

Although [4] provided a necessary and sufficient condition for an incentive compatible mediation rule to be optimal and showed that two specific mediation rules proposed in the literature (that of [3] and by [2]) are indeed optimal for certain values of b, we do not know what the structure of other optimal mediation rules might be.

One might ask if there exists an optimal mediation rule which is also an N-simple mediated equilibrium. If the answer is yes, this could imply that a suitable randomisation over the messages used in the original unmediated CS equilibrium could improve information transmission and there would be no need to use additional messages. However, we cannot deduce the minimal number of messages required in all such optimal mediation rules from [4].

Theorem 1, in this paper, thus advances our understanding of the effect of mediation in the CS model. It partially answers the question of whether the minimal size of the message space needs to be larger for mediation to Pareto dominate the CS equilibrium. Although we do not answer the question of whether or not an N-simple mediated equilibrium can always improve on the CS equilibrium for large b, we provide a suggestive example.

2. The Model

2.1. Crawford–Sobel Game

Our setup is identical to the uniform-quadratic utility CS model, as presented in the literature. Informed readers may wish to skip this subsection.

There are *two* agents. The informed agent, called the sender (S), precisely knows the state of the world, θ, where $\theta \sim U[0,1]$, and they can send a message at no cost, based on his private information, to the other agent, called the receiver (R). The receiver, however, does not know θ but must choose some decision y based on the information contained in the signal. The receiver's payoff is $U^R(y,\theta) = -(y-\theta)^2$, and the sender's payoff is $U^S(y,\theta,b) = -(y-(\theta+b))^2$, where $b > 0$ is a parameter that measures the 'bias' in their preferences.

CS have shown that any equilibrium of this game is essentially equivalent to a partition equilibrium where only a finite number of actions are chosen in equilibrium and each action corresponds to an element of the partition. For $b < \frac{1}{2N(N-1)}$, where $N \geq 2$ is an integer, there is a partition equilibrium[5] in which the state space is partitioned into N elements, characterised by $0 = a_0 < a_1 < a_2 < \ldots\ldots < a_{N-1} < a_N = 1$, where $a_k = \frac{k}{N} + 2bk(k-N)$; in this equilibrium, S sends a message for each element $[a_{k-1}, a_k)$, and given this message, R takes the optimal action $y_k = \frac{a_{k-1}+a_k}{2}$. We call this the N-partition CS equilibrium. For $\frac{1}{2N(N+1)} \leq b < \frac{1}{2N(N-1)}$, the "best" equilibrium (the one that maximises the receiver's expected payoff, EU^R) is the N-partition CS equilibrium[6]. For such an equilibrium, the receiver's expected payoff is $EU^R = -\frac{1}{12N^2} - \frac{b^2(N^2-1)}{3}$, while the sender's expected payoff is $EU^S = EU^R - b^2$.

2.2. Mediated Equilibrium

Within the CS framework, we now consider mediation, a possible structure of which could be as follows: S sends a message based on his private information to the mediator; the mediator then chooses an action according to a commonly known probability distribution and recommends it to R. We here consider a specific form of direct mediation (in the spirit of canonical mechanisms, as initiated and analysed extensively by [13–15]) and formally define such a mediated talk below.

Definition 1. *An $N \times M$ mediated talk is $[\{x_k\}_{k=0}^{N}, \{y_j\}_{j=1}^{M}, \left\{p_{kj}\right\}_{k=1,\ldots N; j=1,\ldots M}]$ where $0 = x_0 < x_1 < x_2 < \ldots\ldots < x_{N-1} < x_N = 1$, each $y_j \in [0,1]$ for $j \in \{1,2,\ldots M\}$, each $p_{kj} \in [0,1]$ for $k \in \{1,2,\ldots N\}$, $j \in \{1,2,\ldots M\}$ with $\sum_{j=1}^{M} p_{kj} = 1$.*

In an $N \times M$ mediated talk, S reports one of the N possible elements, $[x_{k-1}, x_k)$, in which the true state θ may lie, to the mediator and given the report $\theta \in [x_{k-1}, x_k)$, the mediator then recommends to R one action, y_j, out of the M possible actions with probability p_{kj}.

Our mechanism[7] (an $N \times M$ mediated talk) is said to be in equilibrium if it is incentive compatible for both players, that is, if (i) S has the incentive to be truthful to the mediator given the probabilities p_{kj}, and (ii) R has the incentive to obey the mediator's recommendation y_j, given the posterior probabilities on the state of nature.

In what follows, we will focus only on $N \times N$ direct[8] mediated equilibria. We will call such an equilibrium an N-simple mediated equilibrium.

Let $g_k(\theta)$ denote the difference in expected utility from inducing the distributions $\left\{p_{k+1j}\right\}_{j=1}^{N}$ and $\left\{p_{kj}\right\}_{j=1}^{N}$ that a type θ sender would obtain. Formally, $g_k(\theta) = \sum_{j=1}^{N}\left(p_{kj} - p_{k+1j}\right)[y_j - (\theta + b)]^2$.

Definition 2. *For any specific value of b, an $N \times N$ (or, N-simple) mediated equilibrium is an $N \times M$ (with $M = N$) mediated talk that satisfies*

(i) incentive compatibility for S: for all $k \in \{1,2,\ldots,N-1\}$, $g_k(x_k) = 0$ and $g_k'(\theta) \geq 0\ \forall\theta$.

(ii) incentive compatibility for R: $y_j = \underset{y}{\operatorname{argmax}} - \sum_{k=1}^{N} q_{kj} \frac{1}{(x_k - x_{k-1})} \int_{x_{k-1}}^{x_k} (y-\theta)^2 d\theta$ for all $j \in \{1,2,\ldots,N\}$, where q_{kj}, the posterior probability that $\theta \in [x_{k-1}, x_k)$, is given by $q_{kj} = \frac{(x_k - x_{k-1})p_{kj}}{\sum_{k=1}^{N}(x_k - x_{k-1})p_{kj}}$.

In (i), $g_k(x_k) = 0$ essentially corresponds to the "arbitrage condition" in the CS equilibrium. The incentive compatibility for the sender requires that the x_k type sender is indifferent between inducing $\left\{p_{k+1j}\right\}_{j=1}^{N}$ and $\left\{p_{kj}\right\}_{j=1}^{N}$. In addition, a sender with type $\theta \in (x_k, x_{k+1})$ should weakly prefer the distribution $\left\{p_{k+1j}\right\}_{j=1}^{N}$ over $\left\{p_{kj}\right\}_{j=1}^{N}$and a sender with type $\theta \in (x_{k-1}, x_k)$ should weakly prefer the distribution $\left\{p_{kj}\right\}_{j=1}^{N}$ over $\left\{p_{k+1j}\right\}_{j=1}^{N}$, implying that $g_k'(\theta) \geq 0$. Note that $g_k'(\theta) \geq 0$ at $\theta = x_k$ and since $g_k'(\theta)$ is constant, it has to be non-negative everywhere.

In (ii), the incentive compatibility for R requires that when y_j has been recommended, R indeed chooses the action y_j because it maximises his expected utility given his posterior beliefs.

2.3. Characterisation

An N-simple mediated equilibrium can be characterised easily. Incentive compatibility for R, as in Definition 2(ii), requires for all $j = 1, \ldots, N$, $\sum_{k=1}^{N} p_{kj}\left[(y_j - x_{k-1})^2 - (y_j - x_k)^2\right] = 0$. This implies:

$$y_j = \frac{1}{2}\left[\frac{(1 - \sum_{k\neq j} p_{jk})(x_j^2 - x_{j-1}^2) + \sum_{k\neq j} p_{kj}(x_k^2 - x_{k-1}^2)}{(1 - \sum_{k\neq j} p_{jk})(x_j - x_{j-1}) + \sum_{k\neq j} p_{kj}(x_k - x_{k-1})}\right] \tag{1}$$

The incentive compatibility for S, as in Definition 2(i), requires $\sum_{j=1}^{N-1}(p_{k+1j} - p_{kj})(y_j - y_N) \geq 0$ and $\sum_{j=1}^{N}(p_{kj} - p_{k+1j})\left[y_j - (x_k + b)\right]^2 = 0$ for all $k = 1, \ldots, N-1$.

The latter implies for all $k = 1, \ldots, N-1$,

$$\begin{aligned} 2(x_k + b) &= \frac{\sum_{j=1}^{N-1}(p_{kj} - p_{k+1j})(y_j^2 - y_N^2)}{\sum_{j=1}^{N-1}(p_{kj} - p_{k+1j})(y_j - y_N)} \\ &= \frac{(1 - \sum_{j\neq k} p_{kj} - p_{k+1k})(y_k^2 - y_N^2) + (p_{kk+1} - 1 + \sum_{j\neq k+1} p_{k+1j})(y_{k+1}^2 - y_N^2) + \sum_{j\neq k,k+1}(p_{kj} - p_{k+1j})(y_j^2 - y_N^2)}{(1 - \sum_{j\neq k} p_{kj} - p_{k+1k})(y_k - y_N) + (p_{kk+1} - 1 + \sum_{j\neq k+1} p_{k+1j})(y_{k+1} - y_N) + \sum_{j\neq k,k+1}(p_{kj} - p_{k+1j})(y_j - y_N)} \end{aligned} \tag{2}$$

Thus, an N-simple mediated equilibrium is characterised by $[\{x_k\}_{k=0}^{N}, \{y_j\}_{j=1}^{N}, \left\{p_{kj}\right\}_{k=1,\ldots,N;j=1,..,N}]$ satisfying Equations (1) and (2) with the constraints that $\sum_{j=1}^{N-1}(p_{k+1j} - p_{kj})(y_j - y_N) \geq 0$ for all $k = 1, \ldots, N-1$.

3. Results

To state and prove our results, we take $b < \frac{1}{2N(N-1)}$, for which the N-partition CS equilibrium exists. We first observe that the N-partition CS equilibrium is actually equivalent to a particular N-simple mediated equilibrium, namely, one with $x_k = a_k = \frac{k}{N} + 2bk(k - N)$ for all $k \in \{1, \ldots, N\}$; $y_j = \frac{a_{j-1} + a_j}{2}$ for all $j \in \{1, \ldots, N\}$ and $p_{kj} = 0$ for all $k, j \in \{1, \ldots, N\}$, $k \neq j$.

Note that in the class of simple mediated equilibria, $EU^S = EU^R - b^2$, that is, ex ante, the sender's and receiver's welfare ranking agree. For any simple mediated equilibrium, it is also clear that $EU^R = -\sum_{k=1}^{N}[(1 - \sum_{j\neq k} p_{kj})\int_{x_{k-1}}^{x_k}(y_k - \theta)^2 d\theta + \sum_{\substack{j=1 \\ j\neq k}}^{N} p_{kj}\int_{x_{k-1}}^{x_k}(y_j - \theta)^2 d\theta]$, which implies

$$EU^R = -\frac{1}{3}\sum_{k=1}^{N}[(1 - \sum_{j\neq k} p_{kj})\{(y_k - x_{k-1})^3 - (y_k - x_k)^3\} + \sum_{\substack{j=1 \\ j\neq k}}^{N} p_{kj}\{(y_j - x_{k-1})^3 - (y_j - x_k)^3\}].$$

We are interested in the question of when can the N-partition CS equilibrium not be improved upon by an N-simple mediated equilibrium, that is, when is the N-partition CS equilibrium is indeed the best among the set of N-simple mediated equilibria. Our main result answers the above question by solving the following constrained maximisation problem that we call the *final problem*.

Final problem: Maximise EU^R among the set of N-simple mediated equilibria (as characterised in the previous section).

Before we state and prove our main result, as a first step, we consider the following constrained maximisation problem that we call the *initial problem*:

Initial problem: $\underset{\{x_k\}_{k=1}^{N-1}, \{p_{kj}\}_{k\neq j}}{\text{Maximise}}$ EU^R subject to Equations (1) and (2).

The difference between the *final problem* and the *initial problem* is simply the set of restrictions that $\sum_{j=1}^{N-1}(p_{k+1j} - p_{kj})(y_j - y_N) \geq 0$, for all $k = 1, \ldots, N-1$.

Our first lemma below proves that for $b < \frac{1}{2N^2}$, a "corner" solution satisfies the necessary first-order conditions of the *initial problem*.

Lemma 1. *For $b < \frac{1}{2N^2}$, the necessary conditions for a solution of the initial problem are satisfied at $p_{kj} = 0$, for all $k \neq j$; $k, j \in \{1, \ldots, N\}$.*

Lemma 1 also characterises the values of $\{x_k\}_{k=1}^{N-1}$ at this solution of the *initial problem*. They are indeed the CS values given by $x_k = x_k^{CS} = \frac{k}{N} + 2bk(k-N)$, for all $k \in \{1, \ldots, N-1\}$. The formal proof of Lemma 1 has been relegated to the Appendix A.

The following result shows that the N-partition CS equilibrium satisfies the first-order conditions for a (local) optimiser of EU^R among the set of N-simple mediated equilibria. The result is an immediate consequence of Lemma 1, and thus, the formal proof is postponed to the Appendix A.

Corollary 1. *For $b < \frac{1}{2N^2}$, the necessary conditions for a local maximum of the final problem are satisfied at $p_{kj} = 0$ for all $k \neq j$; $k, j \in \{1, \ldots, N\}$ and $x_k = x_k^{CS} = \frac{k}{N} + 2bk(k-N)$ for all $k \in \{1, \ldots, N-1\}$.*

This corollary shows that the CS equilibrium values constitute a candidate solution to the *final problem*.

We are now ready to state our main result.

Theorem 1. *Consider $b < \frac{1}{2N(N-1)}$, for which the N-partition CS equilibrium exists. If $b < \frac{1}{2N^2}$ $(< \frac{1}{2N(N-1)})$, then no N-simple mediated equilibrium can improve upon the N-partition CS equilibrium.*

Our theorem above proves that for $b < \frac{1}{2N^2}$, the N-partition CS equilibrium is actually a global maximum among the set of N-simple mediated equilibria. Furthermore, it suggests that there may exist N-simple mediated equilibria that can improve upon the N-partition CS equilibrium when $b \in [\frac{1}{2N^2}, \frac{1}{2N(N-1)})$, as we demonstrate using an example in Section 4.1 below.

We first note that the global maximum exists by appealing to the Weierstrass Theorem, since the objective function is continuous and is defined over a compact set.

To prove the theorem, we first reconsider EU^R of any N-simple mediated talk and write the following expression:

$$-3EU^R = \sum_{k=1}^{N}[(1 - \sum_{j \neq k} p_{kj})\{(y_k - x_{k-1})^3 - (y_k - x_k)^3\} + \sum_{\substack{j=1 \\ j \neq k}}^{N} p_{kj}\{(y_j - x_{k-1})^3 - (y_j - x_k)^3\}].$$

We work with the above expression, involving the variables $[\{x_k\}_{k=0}^{N}, \{y_j\}_{j=1}^{N}, \left\{p_{kj}\right\}_{k=1,\ldots N; j=1,\ldots N}]$, satisfying Equation (1) from the previous section, $y_j = \frac{1}{2}\left[\frac{(1-\sum_{k\neq j} p_{jk})(x_j^2 - x_{j-1}^2) + \sum_{k \neq j} p_{kj}(x_k^2 - x_{k-1}^2)}{(1-\sum_{k\neq j} p_{jk})(x_j - x_{j-1}) + \sum_{k \neq j} p_{kj}(x_k - x_{k-1})}\right]$. We redefine the above expression as a function, f, explicitly as a function of the variables $\{x_k\}_{k=1}^{N-1}$ and $\{p_{kj}\}_{k \neq j}$ such that the domain satisfies Equation (1). Let us thus write

$$\begin{aligned}
&f(x_1, \ldots, x_{N-1}; p_{12}, p_{13}, \ldots, p_{1N}; p_{21}, p_{23} \ldots, p_{2N}; \ldots\ldots\ldots; p_{N1}, \ldots, p_{NN-1}) \\
&= \sum_{k=1}^{N}[(1 - \sum_{j\neq k} p_{kj})\{(y_k - x_{k-1})^3 - (y_k - x_k)^3\} + \sum_{\substack{j=1 \\ j \neq k}}^{N} p_{kj}\{(y_j - x_{k-1})^3 - (y_j - x_k)^3\}].
\end{aligned}$$

The function, f, is therefore the negative of the original objective function with only the constraint (1) incorporated but without taking into account the incentive compatibility constraint for S.

Abusing notation, we treat f as a function of $\{p_{kj}\}_{j \neq k}$, for any fixed level of $(x_1, \ldots, x_{N-1})$, and also, as a function of $\{x_k\}_{k=1}^{N-1}$, for fixed values of p_{kj}, for all $k \neq j$. We first observe the following result which characterises a property of the function, f.

Lemma 2. *For any* $(x_1, \ldots, x_{N-1})$, $\frac{\partial f}{\partial p_{kj}} > 0$, *at* $p_{kj} = 0$, *for all* $k \neq j$; $k, j \in \{1, \ldots, N\}$.

This means that at such a corner value of the probabilities where $p_{kj} = 0$, for all $k \neq j$, $k, j \in \{1, \ldots, N\}$, there is locally an open neighbourhood around this corner value where the value of the function f is larger than at the corner for any $(x_1, \ldots, x_{N-1})$. So, Lemma 2 shows that f, as a function of $\{p_{kj}\}_{j \neq k}$, attains a local minimum at $p_{kj} = 0$, for all $k \neq j$; k, $j \in \{1, \ldots, N\}$.

We then consider the global minimisation of f with respect to p_{kj} for all $k \neq j$, and we prove the following lemma.

Lemma 3. *For any* $(x_1, \ldots, x_{N-1})$, f, *as a function of* $\{p_{kj}\}_{j \neq k}$, *attains a (global) minimum at* $p_{kj} = 0$ *for all* $k \neq j$; $k, j \in \{1, \ldots, N\}$.

The next lemma indicates an important property of f as a function of $(x_1, \ldots, x_{N-1})$, for a particular value of $\{p_{kj}\}_{j \neq k}$.

Lemma 4. *At* $p_{kj} = 0$, *for all* $k \neq j$, f, *as a function of* $\{x_k\}_{k=1}^{N-1}$, *is strictly convex.*

The proof of Theorem 1 is now a direct consequence of the above lemmata. We provide the formal argument in the Appendix. Theorem 1 proves that for $b < \frac{1}{2N^2}$, a global maximum among the set of N-simple mediated equilibria must coincide with the N-partition CS equilibrium.

4. Remarks

4.1. Example

We provide here an example for $N = 2$ where *simple* mediation improves on the corresponding CS equilibrium when the bias b is appropriately larger than the bound mentioned in Theorem 1. Recall that a 2-partition CS equilibrium exists for $b < \frac{1}{4}$ and for $\frac{1}{12} \leq b < \frac{1}{4}$, the best CS partition equilibrium involves two elements. Theorem 1 confirms that the 2-partition CS equilibrium cannot be improved upon by any 2-simple mediated equilibrium when $b < \frac{1}{8}$. Our result also suggests that a 2-simple mediated equilibrium may improve upon the 2-partition CS equilibrium when b is large enough, that is, for $\frac{1}{8} \leq b < \frac{1}{4}$. We illustrate this comment for $b = \frac{1}{6}$. Here, the 2-partition CS equilibrium is characterised by $a = \frac{1}{6}$, $y_1 = \frac{1}{12}$, and $y_2 = \frac{7}{12}$ with utilities $EU^R = -\frac{7}{144} \simeq -0.0486$ and $EU^S = -\frac{11}{144} \simeq -0.0764$.

From the characterisation presented in Section 2.3, a 2-simple mediated equilibrium is given by[9] $(x, y_1, y_2, p_{11}, p_{12}, p_{21}, p_{22})$, where $x, y_1, y_2 \in (0, 1)$, $p_{11}, p_{12}, p_{21}, p_{22} \in [0, 1]$ and $p_{11} + p_{12} = 1$, $p_{21} + p_{22} = 1$. The incentive compatibility constraints for S and for R can all be combined into one equation given by $\frac{(1-p_{12})x^2 + p_{21}(1-x^2)}{4[(1-p_{12})x + p_{21}(1-x)]} + \frac{p_{12}x^2 + (1-p_{21})(1-x^2)}{4[p_{12}x + (1-p_{21})(1-x)]} - x = b$.

Thus, a 2-simple mediated equilibrium in this setup can be characterised by three variables (p_{12}, p_{21}, x), where $x \in (0, 1)$ and $p_{12}, p_{21} \in [0, 1]$, satisfying the above equation.

It is now easy to check that for $b = \frac{1}{6}$, $x = 0.2245201023$, $y_1 = 0.1745967377$, $y_2 = 0.6077768002$, $p_{12} = 0.03$, and $p_{21} = 0.04$ constitute a 2-simple mediated equilibrium with utilities $EU^R \simeq -0.0483$ and $EU^S \simeq -0.0760$ and can improve upon the corresponding 2-partition CS equilibrium.

The interpretation of the above example is as follows. One can see that the partitioning point x and the two decisions y_1 and y_2 of the 2-simple mediated equilibrium are all larger than the corresponding values of the 2-partition CS equilibrium. The fact that the lower interval is bigger and the higher element of the partition is smaller in size means that more information is being transmitted. This is possible because the mediator is allowed to randomise between y_1 and y_2.

4.2. Relationship with Goltsman et al. (2009)

One might be interested in knowing how our theorem compares with the corresponding results in [4] and in particular, if there is a connection between our Theorems 1 and 2 of [4]. In Theorem 2 of [4], an optimal mediation rule is provided. One interesting corollary of this theorem is that for $b = \frac{1}{2N^2}$, this particular optimal mediation rule cannot improve upon the N-partition CS equilibrium. This implies that the N-partition CS equilibrium is optimal, irrespective of the number of messages that the players are allowed to use, when $b = \frac{1}{2N^2}$. However, since Theorem 2 of [4] is about a specific optimal mediation rule which uses more messages than our N-simple mediated equilibrium and in general, there might be a continuum of optimal mediation rules. We thus feel that this theorem is not useful in answering the question posed in our paper.

A more meaningful approach might be to ask if the technique used by [4] to prove their Theorem 2 can provide any insight or an alternative way of proving our result. Ref. [4] introduced a lemma (Lemma 2 in their paper) to derive a necessary and sufficient condition for an incentive compatible mediation rule to be optimal and to provide an upper bound on the objective function using an incentive compatible mediation rule. One might try to identify such a condition and an upper bound in the more constrained setting of *N-simple* mediation rules. If the characterisation of optimal mediation rules in Lemma 2 of [4] could be appropriately modified to derive a characterisation of optimal N-simple mediated equilibria, then this would provide another method of proof of our Theorem 1. We would like to point out that we do not derive such a characterisation in this paper and that such an alternative proof is not straightforward, either.

It is also worth mentioning two recent papers on the connection between communication equilibrium and correlated equilibrium [5,18] that are relevant to our work. Ref. [18] proved that (essentially) every communication equilibrium of any finite Bayesian game with two players can be implemented as a strategic form correlated equilibrium of a game extended by a cheap-talk phase before the original Bayesian game. On the other hand, specific to the CS model, [5] constructed a strategy-correlated equilibrium, that sends messages to both players before the sender sends any message to the receiver, to achieve the best possible payoff from the mediated equilibrium of the CS model. Importantly, in his construction, unlike our work, neither player needs to send messages to the device. Following these new results, one may be interested to know whether our *N-simple* mediated equilibrium can be obtained as a correlated equilibrium in the sense of [18], or as a strategy-correlated equilibria, as in [5]. In particular, one may further ask whether or not these new constructible correlated or strategy-correlated equilibria will involve only a few (N many) messages. Clearly, these are important questions for future research.

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Appendix A

All the proofs of the results are collected in the appendix.

Proof of Lemma 1. It suffices to show that for $b < \frac{1}{2N^2}$, the N-simple mediated equilibrium corresponding to $p_{kj} = 0$, for all $k \neq j$, $k, j \in \{1, \ldots, N\}$ provides a candidate solution for the initial problem.

Let us first consider the Lagrangian:

$$\mathcal{L} = -\sum_{k=1}^{N}[(1-\sum_{j\neq k} p_{kj})\{(y_k - x_{k-1})^3 - (y_k - x_k)^3\} + \sum_{j=1}^{N} p_{kj}\{(y_j - x_{k-1})^3 - (y_j - x_k)^3\}]$$
$$-\sum_{k=1}^{N-1}\lambda_k[\frac{(1-\sum_{j\neq k} p_{kj} - p_{k+1k})(y_k^2 - y_N^2) + (p_{kk+1} - 1 + \sum_{j\neq k+1} p_{k+1j})(y_{k+1}^2 - y_N^2) + \sum_{j\neq k,k+1}(p_{kj} - p_{k+1j})(y_j^2 - y_N^2)}{(1-\sum_{j\neq k} p_{kj} - p_{k+1k})(y_k - y_N) + (p_{kk+1} - 1 + \sum_{j\neq k+1} p_{k+1j})(y_{k+1} - y_N) + \sum_{j\neq k,k+1}(p_{kj} - p_{k+1j})(y_j - y_N)}]$$
$$+2\sum_{k=1}^{N-1}\lambda_k x_k + 2b\sum_{k=1}^{N-1}\lambda_k.$$

To prove the result, we just need to show that at the proposed candidate solution, there exists $\{\lambda_k\}_{k=1}^{N-1}$ such that $\frac{\partial\mathcal{L}}{\partial x_k} = 0$ for all $k = 1, \ldots, N-1$ and $\frac{\partial\mathcal{L}}{\partial p_{kj}} < 0$ for all $k \neq j$ when $b < \frac{1}{2N^2}$.

First, at $p_{kj} = 0$, for all $k \neq j$; $k, j \in \{1, \ldots N\}$, it is easy to check that $\frac{\partial y_k}{\partial x_k} = \frac{\partial y_k}{\partial x_{k-1}} = \frac{1}{2}$, and $\frac{\partial y_k}{\partial x_j} = 0$ for all $j \neq k, k-1$. In addition, $\frac{\partial y_j}{\partial p_{kj}} = \frac{(x_k - x_{k-1})(x_k + x_{k-1} - x_j - x_{j-1})}{2(x_j - x_{j-1})}$ for all $k \neq j$ and $\frac{\partial y_j}{\partial p_{kl}} = 0$ for all $l \neq j$ for all k.

Subsequently, it can be shown that $\frac{\partial\mathcal{L}}{\partial x_k} = -3[(y_k - x_k)^2 - (y_{k+1} - x_k)^2] + \lambda_k - \frac{\lambda_{k+1}}{2} - \frac{\lambda_{k-1}}{2}$ for all $k = 1, \ldots, N-1$ (since λ_0 and λ_N are not defined, define them to be equal to zero).

Now, $\frac{\partial\mathcal{L}}{\partial x_k} = 0$ implies $12b(y_{k+1} - y_k) + 2\lambda_k - \lambda_{k+1} - \lambda_{k-1} = 0$ for all $k = 1, \ldots, N-1$.

This gives us a system of $(N-1)$ equations in $(N-1)$ variables, $\lambda_1, \ldots \lambda_{N-1}$, which can be succinctly written in matrix form as

$$\underbrace{\begin{bmatrix} 2 & -1 & 0 & 0 & . & . & 0 \\ -1 & 2 & -1 & 0 & . & . & . \\ 0 & -1 & 2 & -1 & 0 & . & . \\ 0 & 0 & -1 & . & . & . & 0 \\ . & . & . & . & . & . & 0 \\ . & . & . & 0 & -1 & 2 & -1 \\ 0 & . & . & 0 & 0 & -1 & 2 \end{bmatrix}}_{(N-1)\times(N-1)} \underbrace{\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ . \\ . \\ . \\ \lambda_{N-1} \end{bmatrix}}_{(N-1)\times 1} = \underbrace{\begin{bmatrix} 12b(y_1 - y_2) \\ 12b(y_2 - y_3) \\ 12b(y_3 - y_4) \\ . \\ . \\ . \\ 12b(y_{N-1} - y_N) \end{bmatrix}}_{(N-1)\times 1}$$

The $(N-1) \times (N-1)$ matrix above is symmetric and tridiagonal, the ij-th element of the inverse of which is given by $\frac{1}{4N}(i + j- \mid j - i \mid)(2N- \mid j - i \mid -i - j)$ (using results by [19,20]).

Thus, solving the equations, we obtain $\lambda_k = -\frac{2bk(N-k)}{N}[3 - 2bN^2 + 4bkN]$ (which is < 0 for all $k = 1, \ldots, N-1$).

We are now ready to show that when $b < \frac{1}{2N^2}$, $\frac{\partial\mathcal{L}}{\partial p_{kj}} < 0$ for all $k \neq j$, at the proposed candidate solution (the CS equilibrium values of x_ks and y_ks) and with the above values λ_k, for all $k = 1, \ldots, N-1$.

For all $k \neq j$, we have

$$\frac{\partial\mathcal{L}}{\partial p_{kj}} = [(y_k - x_{k-1})^3 - (y_k - x_k)^3] - [(y_j - x_{k-1})^3 - (y_j - x_k)^3] - \lambda_k[\frac{(y_j - y_k)(y_j - y_{k+1})}{(y_k - y_{k+1})}] - \lambda_{k-1}[\frac{(y_k - y_j)(y_j - y_{k-1})}{(y_{k-1} - y_k)}] - (\lambda_j + \lambda_{j-1})[\frac{\partial y_j}{\partial p_{kj}}].$$

We first prove that $\frac{\partial\mathcal{L}}{\partial p_{kj}} < 0$, for $k = 1$, when $b < \frac{1}{2N^2}$. Here,

$$\frac{\partial\mathcal{L}}{\partial p_{1j}} = \frac{(1-j)^2(2bN^2+1)(2bN^2 - 2bjN - 1)(12b^2N^4 - 12b^2N^3 - 12jb^2N^3 - 4b^2jN^2 + 8b^2j^2N^2 - 12bN^2 + 6bN + 6bjN + 3)}{N^3(2bN^2 + 2bN - 4bjN - 1)(2bN^2 - 4bN - 1)}.$$

Clearly, (i) $(1-j)^2 > 0$; (ii) $(2bN^2 + 1) > 0$; and (iii) $N^3 > 0$. Note also that as $b < \frac{1}{2N(N-1)}$, we have $2bN^2 - 2bN - 1 < 0$. It is now easy to check that (iv) $2bN^2 - 2bjN - 1 < 0$; (v) $2bN^2 + 2bN - 4bjN - 1 < 0$, and (vi) $2bN^2 - 4bN - 1 < 0$.

Finally, the factor,
(vii) $12b^2N^4 - 12b^2N^3 - 12jb^2N^3 - 4b^2jN^2 + 8b^2j^2N^2 - 12bN^2 + 6bN + 6bjN + 3$
$= 3(2N^2b - 1)^2 + 6Nb(1+j)(2Nb + 1 - 2N^2b) + 4N^2b^2[(j-2)^2 - 7 + j^2]$,

which we need to show is > 0 for all $j \geq 2$. Clearly, it is so for all $j \geq 3$. For $j = 2$, the factor is equal to $3(2N^2b-1)^2 + 18Nb(2Nb+1-2N^2b) - 12N^2b^2$, which can be shown to be > 0 whenever $b < \frac{1}{2N^2 - \frac{4N}{3}}$. Since $\frac{1}{2N^2-\frac{4N}{3}} > \frac{1}{2N^2}$, we have that the factor is for > 0, for all $j \geq 2$, when $b < \frac{1}{2N^2}$. Hence, $\frac{\partial \mathcal{L}}{\partial p_{1j}} < 0$, for all $j \geq 2$, when $b < \frac{1}{2N^2}$.

We now show that $\frac{\partial \mathcal{L}}{\partial p_{kj}} < 0$ for all $k > 1$ when $b < \frac{1}{2N^2}$. Substituting the values for the x_ks, y_ks and λ_ks, we have

$$\frac{\partial \mathcal{L}}{\partial p_{kj}} = \frac{(k-j)^2(2bN^2-1)(2bN^2+1)(2bN^2+2bN-2bjN-2bkN-1)}{N^3(2bN^2+2bN-4bjN-1)(2bN^2+4bN-4bkN-1)(2bN^2-4bkN-1)}A,$$

where

$$\begin{aligned}A &= 12b^2N^4 - 36b^2kN^3 - 12jb^2N^3 + 24b^2N^3 + 32b^2k^2N^2 + 8jb^2N^2k + 4b^2N^2 - 36b^2N^2k - 12b^2jN^2 \\ &+8b^2j^2N^2 - 12bN^2 + 18bkN + 6bjN - 12bN + 3.\end{aligned}$$

Here, clearly, (i) $(k-j)^2 > 0$; (ii) $(2bN^2+1) > 0$; and (iii) $N^3 > 0$. Once again, as $b < \frac{1}{2N(N-1)}$, we have $2bN^2 - 2bN - 1 < 0$. Thus, one can verify that (iv) $2bN^2 + 2bN - 2bjN - 2bkN - 1 = 2bN^2 - 2bN - 1 + 2bN(2-j-k) < 0$; (v) $2bN^2 + 2bN - 4bjN - 1 = 2bN^2 - 2bN - 1 + 4bN(1-j) < 0$; (vi) $2bN^2 + 4bN - 4bkN - 1 = 2bN^2 - 2bN - 1 + 2bN(3-2k) < 0$ as $k \geq 2$; and (vii) $2bN^2 - 4bkN - 1 = 2bN^2 - 2bN - 1 + 2bN(1-2k) < 0$.

Finally, note that the factor,

$$\begin{aligned}A &= 12b^2N^4 - 36b^2kN^3 - 12jb^2N^3 + 24b^2N^3 + 32b^2k^2N^2 + 8jb^2N^2k + 4b^2N^2 - 36b^2N^2k - 12b^2jN^2 \\ &+8b^2j^2N^2 - 12bN^2 + 18bkN + 6bjN - 12bN + 3 \\ &= 12N^4b^2 - 12N^3b^2(3k+j-2) + 4N^2b^2[8k^2+2jk+1-9k-3j+2j^2] - 12N^2b + 6Nb(3k+j-2) + 3 \\ &= [12N^4b^2 - 12N^2b + 3] + (3k+j-2)[6Nb - 12N^3b^2] + 12N^2b^2(3k+j-2) \\ &+4N^2b^2[8k^2+2jk+1-9k-3j+2j^2] - 12N^2b^2(3k+j-2) \\ &= 3(2bN^2-1)^2 + 6bN(3k+j-2)\left[1+2bN-2bN^2\right] + 4N^2b^2[(k+j-3)^2 + 7k(k-2) + 2(k-1) + j^2],\end{aligned}$$

is > 0 as $1 + 2bN - 2bN^2 > 0$ (as $b < \frac{1}{2N(N-1)}$) and $k \geq 2$.

Hence, $\frac{\partial \mathcal{L}}{\partial p_{kj}} < 0$ for all $k \neq j$ and for all $k > 1$ when $(2bN^2 - 1) < 0$, i.e., when $b < \frac{1}{2N^2}$. □

Proof of Corollary 1. Note that in the initial problem, we have dropped the constraints that $\sum_{j=1}^{N-1}(p_{k+1j} - p_{kj})(y_j - y_N) > 0$ for all $k = 1, \ldots, N-1$, be satisfied and found candidate solutions of this modified constrained maximisation problem with a larger "feasible set". We add the constraints that $\sum_{j=1}^{N-1}(p_{k+1j} - p_{kj})(y_j - y_N) \geq 0$ for all $k = 1, \ldots, N-1$ to the above problem. However, notice that the above candidate solution, namely, the CS N-partition equilibrium, does satisfy these constraints and hence will be a candidate solution of the desired maximisation problem. □

Proof of Lemma 2.

$$\begin{aligned}&f(x_1, \ldots, x_{N-1}; p_{12}, p_{13}, \ldots, p_{1N}; p_{21}, p_{23}, \ldots, p_{2N}; \ldots\ldots; p_{N1}, \ldots, p_{NN-1}) \\ &= (1 - \textstyle\sum_{j\neq 1} p_{1j})\left[(y_1)^3 - (y_1 - x_1)^3\right] + \textstyle\sum_{j\neq 1} p_{1j}\left[(y_j)^3 - (y_j - x_1)^3\right] \\ &+(1 - \textstyle\sum_{j\neq 2} p_{2j})\left[(y_2 - x_1)^3 - (y_2 - x_2)^3\right] + \textstyle\sum_{j\neq 2} p_{2j}\left[(y_j - x_1)^3 - (y_j - x_2)^3\right] \\ &+(1 - \textstyle\sum_{j\neq 3} p_{3j})\left[(y_3 - x_2)^3 - (y_3 - x_3)^3\right] + \textstyle\sum_{j\neq 3} p_{3j}\left[(y_j - x_2)^3 - (y_j - x_3)^3\right] \\ &+ \ldots\ldots + (1 - \textstyle\sum_{j\neq k} p_{kj})\left[(y_k - x_{k-1})^3 - (y_k - x_k)^3\right] + \textstyle\sum_{j\neq k} p_{kj}\left[(y_j - x_{k-1})^3 - (y_j - x_k)^3\right] \\ &+ \ldots\ldots + (1 - \textstyle\sum_{j\neq N} p_{Nj})\left[(y_N - x_{N-1})^3 - (y_N - 1)^3\right] + \textstyle\sum_{j\neq N} p_{Nj}\left[(y_j - x_{N-1})^3 - (y_j - 1)^3\right].\end{aligned}$$

Note that $\frac{\partial y_i}{\partial p_{kj}} = 0$ for all $i \neq k, j$. Thus,

$$\begin{aligned}
&\tfrac{\partial f}{\partial p_{kj}} = p_{1k}\left[3(y_k)^2 - 3(y_k - x_1)^2\right]\tfrac{\partial y_k}{\partial p_{kj}} + p_{1j}\left[3(y_j)^2 - 3(y_j - x_1)^2\right]\tfrac{\partial y_j}{\partial p_{kj}}\\
&+p_{2k}\left[3(y_k - x_1)^2 - 3(y_k - x_2)^2\right]\tfrac{\partial y_k}{\partial p_{kj}} + p_{2j}\left[3(y_j - x_1)^2 - 3(y_j - x_2)^2\right]\tfrac{\partial y_j}{\partial p_{kj}}\\
&+\ldots\ldots + \left(-\left[(y_k - x_{k-1})^3 - (y_k - x_k)^3\right]\right) + (1 - \textstyle\sum_{j\neq k} p_{kj})\left[3(y_k - x_{k-1})^2 - 3(y_k - x_k)^2\right]\tfrac{\partial y_k}{\partial p_{kj}}\\
&+\left[(y_j - x_{k-1})^3 - (y_j - x_k)^3\right] + p_{kj}\left[3(y_j - x_{k-1})^2 - 3(y_j - x_k)^2\right]\tfrac{\partial y_j}{\partial p_{kj}}\\
&+\ldots\ldots + p_{Nk}\left[3(y_k - x_{N-1})^2 - 3(y_k - 1)^2\right]\tfrac{\partial y_k}{\partial p_{kj}} + p_{Nj}\left[3(y_j - x_{N-1})^2 - 3(y_j - 1)^2\right]\tfrac{\partial y_j}{\partial p_{kj}}.
\end{aligned}$$

Using the incentive compatibility condition for R (Equation (1) in the paper), we have

$$\frac{\partial f}{\partial p_{kj}} = -\left[(y_k - x_{k-1})^3 - (y_k - x_k)^3\right] + \left[(y_j - x_{k-1})^3 - (y_j - x_k)^3\right].$$

For $p_{kj} = 0$ for all $k \neq j$, $y_k = \frac{x_k + x_{k-1}}{2}$, $y_j = \frac{x_j + x_{j-1}}{2}$ and

$$\begin{aligned}
&\tfrac{\partial f}{\partial p_{kj}} = -\left[\left(\tfrac{x_k - x_{k-1}}{2}\right)^3 - \left(-\left(\tfrac{x_k - x_{k-1}}{2}\right)\right)^3\right] + \left[\left(\tfrac{x_j + x_{j-1} - 2x_{k-1}}{2}\right)^3 - \left(\tfrac{x_j + x_{j-1} - 2x_k}{2}\right)^3\right]\\
&= \tfrac{3}{4}(x_k - x_{k-1})\left(x_k - x_j + x_{k-1} - x_{j-1}\right)^2 > 0.
\end{aligned}$$

□

Proof of Lemma 3. We consider the first-order conditions for minimisation of f with respect to p_{kj}, $k \neq j$ for any $(x_1, \ldots, x_{N-1})$.

For all $k, j \in \{1, \ldots, N\}$ such that $k \neq j$, we have

$$\begin{aligned}
\tfrac{\partial f}{\partial p_{kj}} &= -\left[(y_k - x_{k-1})^3 - (y_k - x_k)^3\right] + \left[(y_j - x_{k-1})^3 - (y_j - x_k)^3\right]\\
&= 3(y_k - y_j)(x_k - x_{k-1})(x_k - y_k - y_j + x_{k-1}).
\end{aligned}$$

Thus, $\frac{\partial f}{\partial p_{kj}} = 0 \implies x_k - y_k - y_j + x_{k-1} = 0$, or, $y_k + y_j = x_k + x_{k-1}$. This means that if $\frac{\partial f}{\partial p_{kj}} = 0$, then $\frac{\partial f}{\partial p_{kl}} \neq 0$ for all $l \neq j$, because $y_l \neq y_j$. In addition, if $\frac{\partial f}{\partial p_{kj}} = 0$, then $\frac{\partial f}{\partial p_{jk}} \neq 0$. This proves that for a fixed $(x_1, \ldots, x_{N-1})$, there cannot be a minimum of f at a strictly interior point (i.e., $0 < p_{kj} < 1$ for all $k, j \in \{1, \ldots, N\}$ such that $k \neq j$).

We now check that for a fixed $(x_1, \ldots, x_{N-1})$, there is no other minimum of f at boundary points (i.e., where some of the p_{kj}s are equal to 0 or 1) which is strictly lower than the value of f at $p_{kj} = 0$ for all $k \neq j$.

Case 1. Suppose that $p_{il} = 0$ for all $(i, l) \neq (k, j)$ and $0 < p_{kj} < 1$. Then, $y_k = \frac{x_k + x_{k-1}}{2}$ and $y_j = \frac{1}{2}\left[\frac{\left(x_j^2 - x_{j-1}^2\right) + p_{kj}\left(x_k^2 - x_{k-1}^2\right)}{\left(x_j - x_{j-1}\right) + p_{kj}(x_k - x_{k-1})}\right]$, in which case $\frac{\partial f}{\partial p_{kj}} = 0 \implies y_k + y_j = x_k + x_{k-1} \implies x_j + x_{j-1} = x_k + x_{k-1}$, which is not possible.

So, if $p_{il} = 0$, for all $(i, l) \neq (k, j)$, then $\frac{\partial f}{\partial p_{kj}} \neq 0$. In fact, by continuity of f, $\frac{\partial f}{\partial p_{kj}}$ must be > 0 (since, from Lemma 2, we have $\frac{\partial f}{\partial p_{kl}} > 0$, for $p_{kl} = 0$, for all $k \neq l$). This shows that $0 < p_{kj} \leq 1$ and $p_{il} = 0$ for all $(i, l) \neq (k, j)$ cannot be a minimum of f.

Case 2. Suppose that $0 < p_{kj} < 1$ for some $k \neq j$, $k \in \{k_1, k_2, \ldots, k_l\}$, $l \geq 2$, where $\{k_1, k_2, \ldots, k_l\} \subseteq \{1, 2, \ldots, N\}$ and $p_{kj} = 0$, $k \neq j, k \notin \{k_1, k_2, \ldots, k_l\}$ satisfy the first-order conditions for the minimisation of f. Note from above that if $0 < p_{kj} < 1$ for some $k \neq j$, then $\frac{\partial f}{\partial p_{kj}} = 0$ and $\frac{\partial f}{\partial p_{kl}} \neq 0$ for all $l \neq j, k$, implying that $p_{kl} = 0$ for all $l \neq j, k$.

Since $\frac{\partial f}{\partial p_{kj}} = -\left[(y_k - x_{k-1})^3 - (y_k - x_k)^3\right] + \left[(y_j - x_{k-1})^3 - (y_j - x_k)^3\right] = 0$, if $0 < p_{kj} < 1$, it is easy to check that the value of f (evaluated at the above p_{kj}s) is the same as the value of f evaluated at $p_{kj} = 0$ for all $k \neq j$.

Case 3. Suppose now that $p_{kj} = 1$ for some $k \neq j$, $k \in \{k_1, k_2, \ldots, k_l\}$ where $\{k_1, k_2, \ldots, k_l\} \subset \{1, 2, \ldots N\}$ and $p_{kj} = 0$, $k \neq j$, $k \notin \{k_1, k_2, \ldots, k_l\}$. We now check if such a configuration is compatible with a global minimum of f. Without loss of generality, let $p_{k_1 j_1} = 1$ and $p_{j_1 l} = 0$ for all $l \neq j_1 \in \{1, \ldots, N\}$. Let us denote by Δ the following components in f, which are given by

$$\begin{aligned}\Delta = &(1-\textstyle\sum_{l\neq k_1} p_{k_1 l})\left[(y_{k_1}-x_{k_1-1})^3-(y_{k_1}-x_{k_1})^3\right]+\textstyle\sum_{l\neq k_1} p_{k_1 l}\left[(y_l-x_{k_1-1})^3-(y_l-x_{k_1})^3\right]\\ &+(1-\textstyle\sum_{l\neq j_1} p_{j_1 l})\left[(y_{j_1}-x_{j_1-1})^3-(y_{j_1}-x_{j_1})^3\right]+\textstyle\sum_{l\neq j_1} p_{j_1 l}\left[(y_l-x_{j_1-1})^3-(y_l-x_{j_1})^3\right].\end{aligned}$$

Note that in this particular case, Δ becomes

$$\left[(y_{j_1}-x_{k_1-1})^3-(y_{j_1}-x_{k_1})^3\right]+\left[(y_{j_1}-x_{j_1-1})^3-(y_{j_1}-x_{j_1})^3\right],$$

which will always be greater than the value of Δ evaluated at $p_{kj} = 0$, for all $k \neq j$, given by

$$\left[(y_{k_1}-x_{k_1-1})^3-(y_{k_1}-x_{k_1})^3\right]+\left[(y_{j_1}-x_{j_1-1})^3-(y_{j_1}-x_{j_1})^3\right].$$

Case 4. Suppose now that $p_{kj} = 1$ for some $k \neq j$ and $k \in \{k_1, k_2, \ldots, k_l\}$ and that $0 < p_{kj} < 1$ for some $k \neq j$, $k \in \{k_{l+1}, k_{l+2}, \ldots, k_M\}$ where $\{k_1, k_2, \ldots, k_l, k_{l+1}, k_{l+2}, \ldots, k_M\} \subset \{1, 2, \ldots N\}$ and $p_{kj} = 0$ for all $k \neq j$, $k \notin \{k_1, k_2, \ldots, k_l, k_{l+1}, k_{l+2}, \ldots, k_M\}$. Combining the arguments in Cases 2 and 3 above, it can be shown that this configuration cannot correspond to a global minimum of f.

Case 5. Finally, suppose for every k, $p_{kj} = 1$ for some $k \neq j$ and also, for every k, if $p_{kj} = 1$ then $p_{lj} = 0$ for all $l \neq k$. In this case, it is easy to check that the value of f is equal to the value of f when $p_{kj} = 0$ for all $k \neq j$. This is because such a configuration is equivalent to assigning a permutation of $\{y_1, \ldots, y_N\}$ to each element of the partition. □

Proof of Lemma 4. $f(x_1, \ldots, x_{N-1}; p_{12}, p_{13}, \ldots, p_{1N}; p_{21}, p_{23}, \ldots, p_{2N}; \ldots\ldots; p_{N1}, \ldots, p_{NN-1})$ evaluated at $p_{kj} = 0$, for all $k \neq j$, becomes

$$\begin{aligned}&\left[(y_1)^3-(y_1-x_1)^3\right]+\left[(y_2-x_1)^3-(y_2-x_2)^3\right]\\ &+\ldots\ldots+\left[(y_i-x_{i-1})^3-(y_i-x_i)^3\right]\\ &+\ldots\ldots+\left[(y_N-x_{N-1})^3-(y_N-1)^3\right]\\ &=\tfrac{x_1^3}{4}+\tfrac{(x_2-x_1)^3}{4}+\tfrac{(x_3-x_2)^3}{4}+\ldots\ldots+\tfrac{(1-x_{N-1})^3}{4}.\end{aligned}$$

Let $A = \left[a_{ij}\right]_{i,j=1\ldots N-1}$ denote the Hessian matrix for f, as a function of $\{x_k\}_{k=1}^{N-1}$, at $p_{kj} = 0$ for all $k \neq j$.

$$A = \frac{3}{2}\begin{bmatrix} x_2 & -(x_2-x_1) & 0 & 0 & . & . & . & 0\\ -(x_2-x_1) & (x_3-x_1) & -(x_3-x_2) & 0 & . & . & . & 0\\ 0 & -(x_3-x_2) & (x_4-x_2) & -(x_4-x_3) & 0 & . & . & .\\ 0 & 0 & . & . & . & . & . & 0\\ . & . & . & . & . & . & . & .\\ . & . & . & . & 0 & . & . & 0\\ . & . & . & . & 0 & 0 & . & .\\ 0 & . & . & . & . & 0 & . & (x_N-x_{N-2}) \end{bmatrix}_{(N-1)\times(N-1)}.$$

Let $A_1, A_2 \ldots, A_{N-1}$ denote the principal minors of A.

Clearly, $|A_1| = x_2 > 0$. In addition, $|A_2| = x_2(x_3-x_1)-(x_2-x_1)^2 > 0$ and $|A_3| = (x_4-x_2)|A_2| - x_2(x_3-x_2)^2 > 0$, because $|A_2| - (x_3-x_2)|A_1| > 0$.

One can check that for $k \geq 3$,

$|A_k| = a_{kk}|A_{k-1}| - a_{kk-1}a_{k-1k}|A_{k-2}| = (x_{k+1}-x_{k-1})|A_{k-1}| - (x_k-x_{k-1})^2|A_{k-2}|$.

Note that $|A_k| > 0$, if $|A_{k-1}| - (x_k-x_{k-1})|A_{k-2}| > 0$.

We now prove by induction that $|A_{k-1}| - (x_k-x_{k-1})|A_{k-2}| > 0$ for $k \geq 3$.

We know that $|A_2| - (x_3-x_2)|A_1| > 0$. Suppose that $|A_{k-1}| - (x_k-x_{k-1})|A_{k-2}| > 0$ for $k \geq 3$.

Then, $|A_k| - (x_{k+1}-x_k)|A_{k-1}| = (x_k-x_{k-1})|A_{k-1}| - (x_k-x_{k-1})^2|A_{k-2}|$

$= (x_k - x_{k-1})[|A_{k-1}| - (x_k - x_{k-1})|A_{k-2}|] > 0$ (by induction).

Therefore, $|A_k| > 0$ for all k. Hence, A is positive definite which implies that f, as a function of $\{x_k\}_{k=1}^{N-1}$, at $p_{kj} = 0$, for all $k \neq j$, is strictly convex. □

Proof of Theorem 1. We need to prove that for $b < \frac{1}{2N^2}$, the N-partition CS equilibrium is actually a global maximum among the set of N-simple mediated equilibria.

Lemma 3 shows that a global minimum of f exists at $p_{kj} = 0$ for all $k \neq j$. In the *final problem*, EU^R can be viewed as a function that is equal to $-\frac{1}{3}f$ with a further restriction on the domain of the variables given by the incentive compatibility constraint for S. Since the domain of f is less constrained than (and hence contains) the domain of the original constrained optimisation problem given by the *final problem*, the values $p_{kj} = 0$, for all $k \neq j$, should correspond to an optimum of the original optimisation problem as well.

In addition, by Lemma 4, f, as a function of $\{x_k\}_{k=1}^{N-1}$, is strictly convex at $p_{kj} = 0$, for all $k \neq j$, and hence has a unique global minimum in $\{x_k\}_{k=1}^{N-1}$ which will satisfy the necessary conditions for optimality.

From Corollary 1, we know that the N-partition CS equilibrium, given by $x_k = \frac{k}{N} + 2bk(k - N)$ for all $k \in \{1, \ldots, N\}$; $y_j = \frac{x_{j-1}+x_j}{2}$ for all $j \in \{1, \ldots, N\}$ and $p_{kj} = 0$ for all $k, j \in \{1, \ldots, N\}, k \neq j$, is a candidate solution to the *final problem* because it satisfies the necessary first order conditions.

Thus, for $b < \frac{1}{2N^2}$, the values of $\{x_k\}_{k=1}^{N-1}$ in the N-partition CS equilibrium given by $x_k = \frac{k}{N} + 2bk(k - N)$ for all $k \in \{1, \ldots, N\}$ is the unique global minimum of f in $\{x_k\}_{k=1}^{N-1}$, at $p_{kj} = 0$, for all $k \neq j$ and hence, for $b < \frac{1}{2N^2}$, the variables $x_k = \frac{k}{N} + 2bk(k - N)$ for all $k \in \{1, \ldots, N\}$ and $p_{kj} = 0$ for all $k, j \in \{1, \ldots, N\}, k \neq j$ must be a global minimum of the more constrained optimisation problem given by the *final problem*.

Hence, among the set of N-simple mediated equilibria, the N-partition CS equilibrium must attain the global maximum of EU^R for $b < \frac{1}{2N^2}$. □

Notes

1. Ref. [5] subsequently showed that this upper bound can also be implemented without any communication via a device or without any strategic mediator. Instead, it can be achieved by a strategy-correlated equilibrium of the game in which initially, both players privately receive a signal from a correlation device, and then the CS game is played.
2. Ref. [6] considers the role of a strategic mediator in the CS framework. He shows that for any bias b, there exists a strategic mediator who can help achieve the optimal payoffs obtained through a non-strategic mediator. Ref. [7] considered delegation to an intermediary in the CS framework. However, the role of his "intermediary" is different from that of "mediation" in our context. Ref. [8] studied the relative performance of noisy or stochastic mechanisms and deterministic mechanisms in a very similar principal-agent setting.
3. Ref. [9] also investigates optimal mediation in sender–receiver games and establishes a bound on the number of messages that the sender must convey to achieve the value of mediation.
4. We thank an anonymous referee for pointing this out.
5. For $\frac{1}{4} \leq b$, babbling is the only equilibrium.
6. Note that for any $b < \frac{1}{2N(N+1)}$, an N-partition CS equilibrium does exist. However, it is not the "best". Ref. [12] provide a formal selection argument for the "best" equilibrium.
7. This is the type of mechanism [2] also considered to construct an example in their paper. Ref. [16] used a "discrete" version of such a mechanism.
8. We are using a suitable version of the revelation principle [14] here to characterise the set of N-simple mediated equilibria involving direct messages only to cover all *simple* mediation schemes which can use any N inputs and any N outputs. As it turns out, considering only such direct mechanisms is not restrictive, as a revelation principle type result does hold in this context and can be proved using the methods constructed by [17].
9. We drop the subscript in x_1 for presentational simplicity.

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