

A Note on Binary Strategy-Proof Social Choice Functions

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Abstract: Let Φ_n be the set of the binary strategy-proof social choice functions referred to a group of n voters who are allowed to declare indifference between the alternatives. We provide a recursive way to obtain the set Φ_{n+1} from the set Φ_n . Computing the cardinalities $|\Phi_n|$ presents difficulties as the computation of the Dedekind numbers. The latter give the analogous number of social choice functions when only strict preferences are admitted. A comparison is given for the known values. Based on our results, we present a graphical description of the binary strategy-proof social choice functions in the case of three voters.

Keywords: binary social choice functions; strategy-proofness; indifference; monotonicity; Dedekind numbers

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1. Introduction

Let V , A be sets representing, respectively, a collection of decision makers (whom we refer to as individuals, voters, agents and so on) and a set of possible alternatives. Exactly one of the alternatives must be implemented as the choice of the collectivity V . Since alternatives do not have identical value for the agents, the problem arises of deducing from the aggregation of the agents' preferences the social choice to implement.

Let $\mathcal{W}$ be the set of all preferences on the set A . The underlying mathematical model of collective choice is a function $\phi : \mathcal{W}^V \rightarrow A$, which is named social choice function.

So, asking every agent $i \in V$ to report his preference $P_i \in \mathcal{W}$ for the alternative to choose, a profile $P = (P_i)_{i \in V}$ of preferences forms, and the function ϕ selects the alternative $\phi((P_i)_{i \in V})$ to realize.

The main issue in this framework of Social Choice Theory is to determine which social choice functions guarantee that no agent has an incentive to deviate from declaring his true preference about the elements of A . Indeed, if the society V adopts an “arbitrary” aggregation rule, it is possible that some agent is induced to false reporting his preference with the purpose of implementing some alternative he considers better, but not an alternative that would be socially desirable. In other words, a social choice function not carefully fixed can be manipulated. Social desirability here has a basic, elementary, interpretation: the political decision of the society is about the rule ϕ to adopt; once ϕ is fixed, it is desirable that the implemented alternative mirrors the profile of the true preferences of the individuals.

The non-manipulability property we would like a social choice function to enjoy is named *strategy-proofness* and corresponds to the fact that we look for social choice functions ϕ such that every profile P is an equilibrium in the Nash sense. Namely:

for every voter $i \in V$, for every profile $P \in \mathcal{W}^V$, and every preference $W \in \mathcal{W}$, letting Q be the profile identical to P except for the voter i , where $Q_i = W$, one has that the alternative $\phi(P)$ is at least as good as $\phi(Q)$ according to P_i .

For many relevant applications, the case of a binary set $A = \{a, b\}$ of alternatives plays an important role. Think of a community that has to approve or reject a new proposal, or think of ballots between two candidates. This paper deals with such a binary framework. Then, it is worthwhile to observe that if we restrict our attention to social choice functions defined on the smaller domain $\mathcal{S}^V$, where $\mathcal{S}$ consists of strict preferences, then the strategy-proof social choice functions can be identified with the *monotone boolean functions*. Consequently, it is known that there are $\text{Ded}(|V|)$ -many strategy-proof social choice functions that can be adopted for determining a binary collective choice in case the voters are not allowed to report an indifference between alternatives (see [1,2]). The symbol $\text{Ded}(n)$ stands for the Dedekind number, i.e., the number that counts the monotone boolean functions of n variables.

The situation is significantly different when we consider strategy-proof social choice functions on $\mathcal{S}^V$ having range of cardinality three or more. In this case, the celebrated Gibbard–Satterthwaite theorem [3,4] gives that there are only $|V|$ -many strategy-proof social choice functions without indifference.

Our article is about a more complex, though still binary, scenario. We assume that each voter not only has the possibility to say that he strictly prefers one of the two alternatives to the other, but he can also declare to be indifferent between them. This possibility results in a larger number of strategy-proof social choice functions. Translating the problem in mathematical terms to allow for indifference between alternatives corresponds to a step from considering the monotone functions from $\{-1, 1\}^n$ to $\{-1, 1\}$, to the monotone functions from $\{-1, 0, 1\}^n$ to $\{-1, 1\}$.

When the voters can declare indifference among the alternatives, a further notion of strategy-proofness naturally arises: strong group strategy-proofness. This notion is widely studied in [5], where the authors determine the functional form of social choice functions that are strong group strategy-proof and have range of cardinality two, although the set A may contain more than two alternatives. The same question was already addressed, for the case where A has exactly two elements, in the paper [6]. In both cases, when strong group strategy-proofness is imposed to social choice functions, only very few of them remain, and anonymity emerges if there are three or more agents (see [5,6] for details).

The sensitivity of the number of strategy-proof binary social choice functions to the introduction of further axioms is quite strong. The previous is not the only possible example. The addition of a few further, still simple, axioms can reduce this number drastically. In [7], the Positive Welfare Association (PWA) property is proposed for introducing solidarity in the model under consideration. This axiom, when joined with strategy-proofness, dramatically reduces the number of functions: only the two status quo rules and the two constant functions remain.

We believe that counting the binary strategy-proof social choice functions when indifference is admitted is of interest for the following two reasons. One reason is that we contribute to substantiate the sentence “indifference may help in obtaining positive results” (see [8], p. 148). Naturally, we are thinking about the existence of more strategy-proofs social choice functions as a positive, and we see how the transition from strict to weak preferences moves up the number of the binary strategy-proof social choice functions.

Another reason deals with the combinatorics involved in the model. Whereas in the case of strict preferences, it is known that one obtains the famous Dedekind numbers sequence, there are no sequences in The On-Line Encyclopedia of Integer Sequences ([9], accessed on September and November 2021) that can be referred to for characterizing the number of the binary strategy-proof social choice functions when indifference is admitted.

The content of the paper is the following. After the description in Section 2 of the model, in Section 3, we present a procedure to obtain the set Φ_{n+1} of all strategy-proof social choice functions in the case of $|V| = n + 1$ voters from the analogous set Φ_n of

the functions in case the number of voters is n . The procedure is illustrated for the cases $n = 1, 2$. In Section 4, we use the characterization of strategy-proofness based on the notion of “covering”, a relation between elements of partially ordered sets, to determine Φ_3 , following a different approach that allows an intuitive geometrical visualization of social choice functions. Finally, in Section 5, we comment on the number $|\Phi_n|$ of the binary strategy-proof social choice functions. In particular, we shall argue that its computation, analogously to that of the Dedekind numbers, is not an elementary task. We provide the values $|\Phi_3|$, and $|\Phi_4|$, whereas, due to the computational complexity involved, we do not compute the expression in decimal form of the numbers $|\Phi_n|$ for $n > 4$.

2. The Model, Notations, Definitions

The set A is the set of *alternatives* and the set V is the set of *voters*, or, we say, it is the *society*. Both sets are assumed to be finite.

Different agents may have different opinions about the alternatives, but the society is called to select exactly one alternative to be realized. The problem is then the aggregation of different points of view in a unique outcome. The taste of the voters is modeled by means of complete, transitive binary relations on the set A . We call them *weak preferences* or just *preferences* for the sake of brevity. The set of all preferences on A is denoted by $\mathcal{W}(A)$ or simply by $\mathcal{W}$. Preferences that are also antisymmetric are named *strict preferences*. They form the subset $\mathcal{S}$ of $\mathcal{W}$.

Suppose $W \in \mathcal{W}$, and a, b are two alternatives. The expression “ a is preferred to b ” and the expression “ a is at least as good as b ” are used both as equivalent to $(a, b) \in W$. The expression “ a is strictly preferred to b ” and the expression “ a is better than b ” are used both as equivalent to $[(a, b) \in W, \text{ and } (b, a) \notin W]$. The case that both (a, b) and (b, a) belong to W is referred to as a and b are *indifferent*.

When each voter expresses his own preference, the *state* of the society is known. We also say that a *profile* is formed. State and profile are used synonymously. Formally, a profile is an element P of the set $\mathcal{W}^V$. We sometimes use the notations:

- $P = (P_i)_{i \in V}$, where P_i is the weak preference of the voter i in the state P ;
- $P = (P_i; P_{-i})$; where P_{-i} is the restriction of P to the complement of the singleton $\{i\}$.

In the theory of collective choice, the society is called to aggregate preferences, in other words to select the unique alternative to be realized on the basis of each profile. Hence, a rule to aggregate preferences has to be chosen and applied. The rule is, in fact, a function associating an alternative to each profile. We introduce the definition formally and fix the main property it has to obey.

Definition 1. A **social choice function** is a function $\phi : \mathcal{W}^V \rightarrow A$ from the set $\mathcal{W}^V$ of all states of the society to the set A of alternatives.

Once a social choice function is fixed, the alternative selected on the basis of a certain profile may change if a voter changes his preference, since this corresponds to a change of the profile. This could suggest to some voter to lie (i.e., to declare a false preference) if he can obtain, on the basis of the new profile, the selection of a different alternative that he considers better. Clearly, it is socially desirable to avoid such a possibility. Therefore, big efforts are deserved to the identification of mechanisms that prevent this occurrence. Social choice functions that cannot be manipulated in the described sense (which means that do not give any voter incentive to lie in any profile) are called *strategy-proof*. The formal definition follows.

Definition 2. A social choice function ϕ is said to be **strategy-proof** (SP, for brevity) if for each voter i , for each profile P and for each profile $Q = (Q_i; P_{-i})$ identical to P in all the components different from i , the alternative $\phi(P)$ is at least as good as the alternative $\phi(Q)$ according to the preference P_i .

The previous definition is manifestly inspired to the Nash equilibrium notion. It implies that even if a voter is called to declare his preference knowing all the others' preferences, he has no incentive to declare any preference different from his own (the true one). The change would not result in the realization of an alternative he considers better.

Stricto sensu, Definition 2 does not cover the case that the domain of social choice function is a subset, call it $\mathcal{P}$, of $\mathcal{W}^V$. However, if we consider social choice functions defined on arbitrary subsets of $\mathcal{W}^V$, the strategy-proofness is defined exactly as in Definition 2 integrated with the explicit requirement that the profiles P and Q belong to $\mathcal{P}$.

We concentrate our attention to the so-called *binary* case, which means that there are exactly two alternatives

$$A = \{a, b\}.$$

The case of two alternatives, interesting per se from the point of view of applications, offers a set of strategy-proof functions rich enough to make its investigation interesting.

Obviously, in the binary case, there are three preferences ($|\mathcal{W}| = 3$):

- Alternative a is better than alternative b ;
- Alternative b is better than alternative a ;
- Alternatives a and b are indifferent to each other,

that we denote, respectively, by a , b and $\sim$. Hence, from now on, it is

$$\mathcal{W} = \{a, b, \sim\}.$$

The strategy-proofness of a social choice function can be described in terms of “monotonicity” of the function itself. Although the relation between strategy-proofness and (various) notions of monotonicity is often discussed in the literature, we explain it in detail both for the sake of completeness and to have the exact terms which we refer to.

Let us first order the sets A and $\mathcal{W} = \{a, b, \sim\}$ by setting $a < b$ between the alternatives and $a < \sim < b$ among the preferences. Then, we order the set $\mathcal{W}^V$ of profiles componentwise: for profiles P and Q , we set

$$P \leq Q \stackrel{\text{def}}{\iff} P_i \leq Q_i \quad \forall i \in V.$$

The relation $\leq$ on the set of the profiles is a partial order (i.e., reflexive, transitive, and antisymmetric) which is not complete. In fact, two profiles P and Q are not comparable ($P \not\leq Q$ and $Q \not\leq P$) when for two different agents i and j it is $P_i < Q_i$ and $Q_j < P_j$.

Definition 3. A social choice function ϕ is said to be **monotone** if

$$P < Q \Rightarrow \phi(P) \leq \phi(Q).$$

Our aim is to present a result connecting monotonicity to strategy proofness. With the help of two concepts from the theory of partially ordered sets (posets, for short), we clarify that:

- To check the strategy-proofness (equivalently monotonicity) of a function, it is sufficient to refer to pairs of profiles “close” to each other, in the sense that there are no other profiles between them (this is the notion of *covering*);
- The equivalence between monotonicity and strategy-proofness does not need the universal domain assumption. Indeed, the equivalence holds true over domains without “gaps” (this is the notion of *convexity*).

Definition 4. Let s and t be two elements of a partially ordered set $(S, \leq)$. We say that s **covers** t , and we write $t \triangleleft s$, if $t < s$ and there is no element u in S such that $t < u < s$.

Definition 5. We say that a subset S' of a partially ordered set S is **convex** if

$$[p, q \in S', r \in S, p \leq r \leq q] \Rightarrow r \in S'.$$

Of course, in $\mathcal{W}$, the preference $\mathbf{b}$ covers $\sim$ and the preference $\sim$ covers $\mathbf{a}$.

It is not difficult to describe explicitly the meaning of covering among profiles in a convex subset of $\mathcal{W}^V$: if $Q \leq P$, the two profiles only differ for the preference of a single agent, say i , who is indifferent between the two alternatives in one of the two profiles. More precisely, either $[P_i = \mathbf{b} \text{ and } Q_i = \sim]$, or $[P_i = \sim \text{ and } Q_i = \mathbf{a}]$. Formally

Proposition 1. Let $\mathcal{P}$ be a convex subset of $\mathcal{W}^V$. Suppose $P, Q \in \mathcal{P}$. Then, P covers Q if and only if there exists $i \in V$ such that $Q = (Q_i; P_{-i})$, and $Q_i \leq P_i$.

Observe that since the set of profiles is finite, when $Q < P$, there is always between them a chain of profiles $R^1, R^2, \dots, R^h$ each covering the previous:

$$Q = R^1 \leq R^2 \leq \dots \leq R^h = P.$$

This reveals the central role of indifference: moving from one strict preference to another, we *pass through* indifference. The same is true moving from a profile to a bigger one.

We are now ready to relate the strategy-proofness of a function with its monotonicity:

Theorem 1. Let $\phi : \mathcal{P} \rightarrow \{a, b\}$ be a function whose domain is a convex subset of $\mathcal{W}^V$. Then, the following are equivalent:

- (i) ϕ is SP;
- (ii) if $Q \leq P$ then $\phi(Q) \leq \phi(P)$;
- (iii) ϕ is monotone.

Proof. (i) $\Rightarrow$ (ii): Assume that $\phi(P) = a$ on a profile P that covers profile Q , and, by contradiction, that $\phi(Q) = b$. Looking at Proposition 1, there is a unique voter i such that P_i differs from Q_i .

If $P_i = \mathbf{b}$, agent i may manipulate ϕ (violate SP) declaring Q_i in the state P . If $Q_i = \mathbf{a}$, he may do it declaring P_i in the state Q . A contradiction in both cases.

(ii) $\Rightarrow$ (iii): If $Q < P$, choose profiles $R^1, R^2, \dots, R^h$ such that

$$Q = R^1 \leq R^2 \leq \dots \leq R^h = P.$$

For each profile R^k it is $Q \leq R^k \leq P$; hence, all profiles R^k are elements of the convex set $\mathcal{P}$.

From the relation $a = \phi(P) = \phi(R^h)$, it follows that $\phi(R^k) = a$ for all k , in particular $\phi(Q) = \phi(R^1) = a$, as wanted.

(iii) $\Rightarrow$ (i) Assume by contradiction that ϕ is not SP. Then, there exists a voter i and two profiles P and Q in $\mathcal{P}$ only differing in the component i , such that according to the preference P_i the alternative $\phi(Q)$ is better than the alternative $\phi(P)$. Without loss of generality, assume $\phi(Q) = b$, $\phi(P) = a$ and b better than a according to P_i , which means that $P_i = \mathbf{b}$. Given the hypothesis, the two relations $Q < P$ and $\phi(P) = a$ imply that $\phi(Q) = a$, which is a contradiction. The result is the same assuming $\phi(Q) = a$, $\phi(P) = b$ and $P_i = \mathbf{a}$. $\square$

The previous characterization is crucial in all that follows: both the recursive procedure described in the next section and the graph representation of Section 4 are based on it.

Remark 1. In the paper [10], the authors obtain a characterization of strategy-proofness of social choice functions through a notion of monotonicity. It is based on a relation among

profiles denoted by $\geq^b$. It is sufficient to observe that $P \geq^b Q \Leftrightarrow P \geq Q$ to derive the correspondence between ([10], Theorem 2.5) and the relations (i) $\Leftrightarrow$ (iii) in our Theorem 1.

The characterization of SP obtained in [11] is based on a binary relation $\sqsupset$ called dominance among profiles:

$$P \sqsupset Q \Leftrightarrow (P \sqsupset_a Q \text{ or } P \sqsupset_b Q).$$

We rewrite the previous relations using our notations:

$$P \sqsupset_a Q \Leftrightarrow [(P_i, Q_i) = (\sim, a) \quad \forall i : P_i \neq Q_i];$$

$$P \sqsupset_b Q \Leftrightarrow [(P_i, Q_i) = (\sim, b) \quad \forall i : P_i \neq Q_i].$$

In ([11], Theorem 2.12), the authors prove that SP is equivalent to the following implication, called *compatibility with dominance*. It is a sort of monotonicity:

$$(*) \quad P \sqsupset_{\phi(P)} Q \Rightarrow \phi(P) = \phi(Q).$$

Let us make a comparison between their result and our Theorem 1. From the relation

$$P \text{ covers } Q \Rightarrow (P \sqsupset_a Q \text{ or } Q \sqsupset_b P)$$

it follows that (*) implies (ii) of Theorem 1. On the other side, from the relations

$$\begin{aligned} P \sqsupset_a Q &\Rightarrow P \geq Q \\ Q \sqsupset_b P &\Rightarrow P \geq Q \end{aligned}$$

it follows that (iii) implies (*). Hence, the two mentioned results can be derived from each other. It is worth observing that compatibility with dominance reveals to be equivalent to the simultaneous validity of the two properties (introduced by Barberà et al. in [5]) of a social choice function of being *essentially ab-based* and *essentially ab-monotonic*.

3. A Recursive Construction of Φ_n

Recall that Φ_n denotes the set of all SP binary social choice functions with reference to a set V of n voters. In this section, we describe a recursive procedure to obtain Φ_{n+1} from Φ_n . Without loss of generality, we name the individuals as $1, 2, \dots, n$ in a society of cardinality n and consider the further agent $n+1$ to form the larger society.

Since the restrictions of SP social choice functions remain SP, there is a natural way to obtain from ϕ in Φ_{n+1} three elements of the set Φ_n . It is sufficient to consider the restrictions of ϕ to the sets of profiles having the last component fixed. Precisely, we set:

$$\phi_a(P_1, P_2, \dots, P_n) = \phi(P_1, P_2, \dots, P_n, a),$$

$$\phi_{\sim}(P_1, P_2, \dots, P_n) = \phi(P_1, P_2, \dots, P_n, \sim),$$

$$\phi_b(P_1, P_2, \dots, P_n) = \phi(P_1, P_2, \dots, P_n, b).$$

Hence, the map, injective by definitions,

$$T : \phi \in \Phi_{n+1} \mapsto (\phi_a, \phi_{\sim}, \phi_b) \in \Phi_n \times \Phi_n \times \Phi_n$$

embeds Φ_{n+1} into $[\Phi_n]^3$.

On the converse, an element of $[\Phi_n]^3$ belongs to the range of T if and only if its three components are connected as shown in the next theorem.

Theorem 2. Let $(g, h, k) \in [\Phi_n]^3$. The function ϕ defined on $\mathcal{W}^{n+1}$ as

$$\phi = \phi_{[g,h,k]}(P_1, P_2, \dots, P_n, P_{n+1}) = \begin{cases} g(P_1, P_2, \dots, P_n) & \text{if } P_{n+1} = a \\ h(P_1, P_2, \dots, P_n) & \text{if } P_{n+1} = \sim \\ k(P_1, P_2, \dots, P_n) & \text{if } P_{n+1} = b \end{cases}$$

belongs to Φ_{n+1} if and only if the following two implications hold true:

$$h(P_1, P_2, \dots, P_n) = a \Rightarrow g(P_1, P_2, \dots, P_n) = a, \quad (1)$$

$$h(P_1, P_2, \dots, P_n) = b \Rightarrow k(P_1, P_2, \dots, P_n) = b. \quad (2)$$

Moreover, $T(\phi) = (g, h, k)$.

Proof. Let ϕ be defined as described above and assume that both implications (1) and (2) hold true. To verify the strategy-proofness of ϕ , we use characterization (ii) of Theorem 1. Assume that for profiles $P = (P_1, P_2, \dots, P_{n+1})$ and $Q = (Q_1, Q_2, \dots, Q_{n+1})$, it is $\phi(P) = a$ and P covers Q .

If $P_{n+1} = Q_{n+1}$, then for an element ℓ in the set $\{g, h, k\}$, it is $\phi(P) = \ell(P_1, P_2, \dots, P_n)$ and $\phi(Q) = \ell(Q_1, Q_2, \dots, Q_n)$. Moreover, the profile $(P_1, P_2, \dots, P_n)$ covers $(Q_1, Q_2, \dots, Q_n)$. Being ℓ SP, Theorem 1 ensures that from $\ell(P) = a$, it follows that $\ell(Q) = a$, as wanted. If $P_{n+1} \neq Q_{n+1}$, using Proposition 1, we have that $(P_1, P_2, \dots, P_n) = (Q_1, Q_2, \dots, Q_n)$ and either

$$(P_{n+1}, Q_{n+1}) = (b, \sim), \text{ or}$$

$$(P_{n+1}, Q_{n+1}) = (\sim, a).$$

In the first case, it is $\phi(P) = k(P_1, P_2, \dots, P_n)$ and $\phi(Q) = h(P_1, P_2, \dots, P_n)$. Then, the relation $\phi(Q) = a$ follows from (2). In the second case, it is $\phi(P) = h(P_1, P_2, \dots, P_n)$ and $\phi(Q) = g(P_1, P_2, \dots, P_n)$, and the same relation follows from (1).

Conversely, let ϕ be SP. To prove that implication (1) holds, assume that $h(P_1, P_2, \dots, P_n) = a$. This means that $\phi(P_1, P_2, \dots, P_n, \sim) = a$. Since profile $(P_1, P_2, \dots, P_n, \sim)$ covers $(P_1, P_2, \dots, P_n, a)$, and since ϕ is SP, it is also $\phi(P_1, P_2, \dots, P_n, a) = a$, which means $g(P_1, P_2, \dots, P_n) = a$, as wanted. An analogous reasoning proves implication (2). $\square$

The two displayed implications allow to obtain all the elements of Φ_{n+1} from the elements of Φ_n . Let us see how.

Let $h \in \Phi_n$. Define the sets $\Phi_n(h, a)$ and $\Phi_n(h, b)$ as follows

$$\Phi_n(h, a) = \{g \in \Phi_n : (1) \text{ holds true}\},$$

$$\Phi_n(h, b) = \{k \in \Phi_n : (2) \text{ holds true}\}.$$

Then, for every $(g, k) \in \Phi_n(h, a) \times \Phi_n(h, b)$, after Theorem 2, the function $\phi^{[h]}(g, k) := \phi_{[g,h,k]}$ belongs to Φ_{n+1} . Hence, finally, we can write

$$\Phi_{n+1} = \bigcup_{h \in \Phi_n} \phi^{[h]}(\Phi_n(h, a) \times \Phi_n(h, b)). \quad (3)$$

From the operational point of view, formula (3) can be used as the justification of the following algorithm.

0 Preliminarily, the elements of Φ_n (i.e., all the SP binary social choice functions defined on $\{a, \sim, b\}^n$) and all the profiles are enumerated. This allows to display the elements of Φ_n in a matrix $M(n)$ whose entry $m_{u,v}$ stands for the value of the function (numbered) u in the profile (numbered) v .

Next, for every row h of $M(n)$, we proceed by analyzing each component.

1 For every element $m_{h,j}$ in the selected row, we highlight the elements $m_{\cdot,j}$ of the column j that differ from $m_{h,j}$. This identifies the set of rows $\{u : m_{u,j} \neq m_{h,j}\}$.

- 2 The highlighting is realized in a way that distinguishes the case $m_{u,j} = a$ from the case $m_{u,j} = b$. To fix the ideas, we can realize this by coloring red $m_{u,j}$ if it is a , and blue if it is b .
- 3 The set $\Phi_n(h, a)$ (respectively, $\Phi_n(h, b)$) consists of all functions occupying rows not containing any blue (respectively, red) component.

Example. We show how the above steps work in the cases $n = 1, 2$, i.e., with reference to formula (3) in the cases $n = 1, 2$.

Case $n = 1$.

It is straightforward to recognize that $|\Phi_1| = 4$. In fact, apart from the two constant functions (that are obviously SP), a (non-constant) SP function necessarily assumes value a on the profile a and value b on the profile b . The value on the remaining profile $\sim$ can be any of the two. We list below these four functions obtaining the (4×3) matrix $M(1)$ with entries a and b . The idea is to number the functions arbitrarily from 1 to 4 to order the profiles arbitrarily (we do it here from the smallest to the biggest) and set $m_{u,v}$ as the value of function (numbered) u on the profile (numbered) v .

Applying the described procedure to function 1 (row 1): The first two elements in row 1 are a ; hence, we color blue each element b that appears in column 1 or in column 2. The third element of row 1 is b ; hence, we color red each element a in the third column. This is shown next.

profiles $\rightarrow$	a	$\sim$	b
functions $\downarrow$			
1	a	a	b
2	a	b	b
3	a	a	a
4	b	b	b

Then, the set $\Phi_1(1, a)$ consists of all the rows without blue components, and the set $\Phi_1(1, b)$ consists of all the rows without red components:

$$\Phi_1(1, a) = \{1, 3\},$$

$$\Phi_1(1, b) = \{1, 2, 4\}.$$

The $(|\Phi_1(1, a)| \cdot |\Phi_1(1, b)| = 6)$ functions ϕ of Φ_2 having as $\phi \sim$ the function in the first row of $M(1)$ are (for brevity, we list profiles avoiding commas between preferences):

profiles $\rightarrow$	aa	$a\sim$	ab	$\sim a$	$\sim\sim$	$\sim b$	ba	$b\sim$	bb
functions $\downarrow$									
$\phi_{[1,1,1]}$	a	a	a	a	a	a	b	b	b
$\phi_{[1,1,2]}$	a	a	a	a	a	b	b	b	b
$\phi_{[1,1,4]}$	a	a	b	a	a	b	b	b	b
$\phi_{[3,1,1]}$	a	a	a	a	a	a	a	b	b
$\phi_{[3,1,2]}$	a	a	a	a	a	b	a	b	b
$\phi_{[3,1,4]}$	a	a	b	a	a	b	a	b	b

Repeating the procedure for the remaining rows, we obtain:

$$\begin{aligned} \Phi_1(2, a) &= \{1, 2, 3\}, & \Phi_1(2, b) &= \{2, 4\}, \\ \Phi_1(3, a) &= \{3\}, & \Phi_1(3, b) &= \{1, 2, 3, 4\}, \\ \Phi_1(4, a) &= \{1, 2, 3, 4\}, & \Phi_1(4, b) &= \{4\}. \end{aligned}$$

We end up with the case where $n = 1$ by listing all the (20) elements of Φ_2 , organized in the matrix $M(2)$ below.

profiles →	aa	a~	ab	~a	~~	~b	ba	b~	bb
functions ↓									
1	a	a	a	a	a	a	a	a	a
2	a	a	a	a	a	a	a	a	b
3	a	a	a	a	a	a	a	b	b
4	a	a	a	a	a	b	a	a	b
5	a	a	a	a	a	b	a	b	b
6	a	a	a	a	a	a	b	b	b
7	a	a	b	a	a	b	a	a	b
8	a	a	a	a	a	b	b	b	b
9	a	a	b	a	a	b	a	b	b
10	a	a	b	a	a	b	b	b	b
11	a	a	a	a	b	b	a	b	b
12	a	a	b	a	b	b	a	b	b
13	a	a	a	a	b	b	b	b	b
14	a	b	b	a	b	b	a	b	b
15	a	a	a	b	b	b	b	b	b
16	a	a	b	a	b	b	b	b	b
17	a	b	b	a	b	b	b	b	b
18	a	a	b	b	b	b	b	b	b
19	a	b	b	b	b	b	b	b	b
20	b	b	b	b	b	b	b	b	b

Case $n = 2$.

Starting from the latter matrix, we can proceed to obtain Φ_3 . As in the previous case, we shall present only the analysis for a single row. Fix, as an example, the row number 7. Applying the described procedure, we obtain:

profiles →	aa	a~	ab	~a	~~	~b	ba	b~	bb
functions ↓									
1	a	a	a	a	a	a	a	a	a
2	a	a	a	a	a	a	a	a	b
3	a	a	a	a	a	a	a	b	b
4	a	a	a	a	a	b	a	a	b
5	a	a	a	a	a	b	a	b	b
6	a	a	a	a	a	a	b	b	b
7	a	a	b	a	a	b	a	a	b
8	a	a	a	a	a	b	b	b	b
9	a	a	b	a	a	b	a	b	b
10	a	a	b	a	a	b	b	b	b
11	a	a	a	a	b	b	a	b	b
12	a	a	b	a	b	b	a	b	b
13	a	a	a	a	b	b	b	b	b
14	a	b	b	a	b	b	a	b	b
15	a	a	a	b	b	b	b	b	b
16	a	a	b	a	b	b	b	b	b
17	a	b	b	a	b	b	b	b	b
18	a	a	b	b	b	b	b	b	b
19	a	b	b	b	b	b	b	b	b
20	b	b	b	b	b	b	b	b	b

from which we have:

$$\Phi_2(7, a) = \{1, 2, 4, 7\}, \text{ and } \Phi_2(7, b) = \{7, 9, 10, 12, 14, 16, 17, 18, 19, 20\}.$$

We have then determined all the functions ϕ of Φ_3 having as restriction $\phi_{\sim}$ the element of Φ_2 corresponding to row 7. There are exactly $(|\Phi_2(7, a)| \cdot |\Phi_2(7, b)| =) 40$ such functions. Repeating the procedure for all the rows of the matrix, we obtain the whole Φ_3 .

4. Graphic Representation of 3-Voters SP Social Choice Functions

An expressive way, as it is visualized, to describe binary SP social choice functions by means of diagrams, can be adopted in case of three voters. We devote the present section to this. We present a simple way to translate into a graph the characterization (ii) of strategy-proofness contained in Theorem 1. This is possible since at the same time, we point out the symmetry of the (componentwise) partially ordered set $\mathcal{W}^V$, and the pivotal role played by the state, denoted as $\sim_V$, of unanimous indifference, i.e., the state where all voters are indifferent between a and b . A similar, a_V, b_V , notation is used for the other two unanimity states of the society. We also adopt the terminology defined next.

Definition 6. A subset C of a poset $(S, \leq)$ is said to be **super order-closed** if it contains the majorants of its elements. Formally: C is super order-closed if

$$x \in S, c \in C, c \leq x \Rightarrow x \in C.$$

Analogously, we say that C is **sub order-closed** if

$$x \in S, c \in C, x \leq c \Rightarrow x \in C.$$

Lemma 1. Let $\phi : \mathcal{P} \rightarrow \{a, b\}$ be a binary function. Take two sets $\mathcal{D}_a \subseteq \phi^{-1}(a)$ and $\mathcal{D}_b \subseteq \phi^{-1}(b)$ such that $\mathcal{D}_a$ is sub order-closed and $\mathcal{D}_b$ is super order-closed in the poset $\mathcal{P}$. Then, ϕ is SP if (and only if) its restriction to $\mathcal{P} \setminus (\mathcal{D}_a \cup \mathcal{D}_b)$ is SP.

Proof. Assume the restriction of ϕ to the set $\mathcal{P} \setminus (\mathcal{D}_a \cup \mathcal{D}_b)$ is SP. If for two profiles $P = (P_i; P_{-i})$ and $Q = (Q_i; P_{-i})$ belonging to $\mathcal{P}$ one has that $\phi(Q)$ is better than $\phi(P)$ according to P_i , then either

$$\begin{aligned} P_i = a, & \quad (\text{hence } \phi(P) = b \quad \phi(Q) = a), \text{ or} \\ P_i = b, & \quad (\text{hence } \phi(P) = a \quad \phi(Q) = b). \end{aligned}$$

In both cases, we obtain a contradiction. Let us consider the first case, the second case being analogous.

Note that

$$\phi(P) = b \Rightarrow P \notin \mathcal{D}_a,$$

$$\phi(Q) = a \Rightarrow Q \notin \mathcal{D}_b.$$

Moreover, since $P < Q$, the two order-closedness entail $P \notin \mathcal{D}_b$ and $Q \notin \mathcal{D}_a$. This means that both profiles P and Q are in $\mathcal{P} \setminus (\mathcal{D}_a \cup \mathcal{D}_b)$, violating the strategy-proofness of the restriction of ϕ . $\square$

With the aim of describing the SP functions $\phi : \{a, \sim, b\}^V \rightarrow \{a, b\}$, we assume:

$$\phi(a_V) = a = \phi(\sim_V), \text{ and } \phi(b_V) = b. \quad (4)$$

Remark 2. There is no loss of generality in the above assumption. Indeed, first, observe that given the strategy-proofness, each of the two relations $\phi(a_V) = b, \phi(b_V) = a$ implies that the social choice function ϕ is constant. With the purpose of describing the non-constant ϕ , we can then assume $\phi(a_V) = a$, and $\phi(b_V) = b$. Moreover, considering the symmetric role played by the two alternatives, limiting to the SP social choice functions assuming value a on the profile of unanimous indifference suffices. Obviously, exchanging a with b everywhere in what follows will give the SP social choice functions assuming value b on the profile of unanimous indifference.

A further simplification is possible in the representation of the non-constant ϕ 's satisfying (4).

Observe that the relation $\phi(\sim_V) = a$ implies that $\phi(Q) = a$ for each profile $Q \leq \sim_V$, which means for each profile in $\{a, b, \sim\}^V$ not containing any b . In other words, under assumption (4), the function ϕ takes value a on the set $\{a, \sim\}^V$. We can therefore restrict our attention to the remaining profiles: those in $\{a, b, \sim\}^V \setminus \{b_V\}$ having at least one component equal to b .

Let us set

$$\mathcal{P}_b := \{P \in \{a, b, \sim\}^V \setminus \{b_V\} : \exists i \in V \text{ such that } P_i = b\},$$

and notice the following straightforward consequence of Lemma 1.

Proposition 2. *For a binary social choice function ϕ , the following conditions are equivalent:*

- (i) ϕ is SP and (4) hold true.
- (ii) ϕ takes value a on $\{a, \sim\}^V$, $\phi(b_V) = b$, and its restriction to $\mathcal{P}_b$ is SP.

So, after Proposition 2, the study of $\Phi_{|V|}$ reduces to that of the strategy-proof functions on the smaller domain $\mathcal{P}_b$. It is a poset under the componentwise ordering adopted in $\mathcal{W}^V$ and we can represent it as a directed graph. For $|V| > 3$, the diagram of the graph is rather complex, so we exemplify the possibility in the case of three agents.

Example. The diagram of $\mathcal{P}_b$ for $|V| = 3$.

For a given profile P , let $c[P]$ denote the set of all profiles in $\mathcal{P}_b$ covered by P .

The way to obtain the elements of $c[P]$ easily follows from Proposition 1: we have to select one component (which means preference) of the given profile different from a and replace it by the preference it covers ($\sim$ if it is b , a if it is $\sim$). The resulting profile belongs to $c[P]$ only if it is an element of $\mathcal{P}_b$, which means only if at least one of its components is b . We indicate below the sets $c[P]$ for all the elements P in $\mathcal{P}_b$:

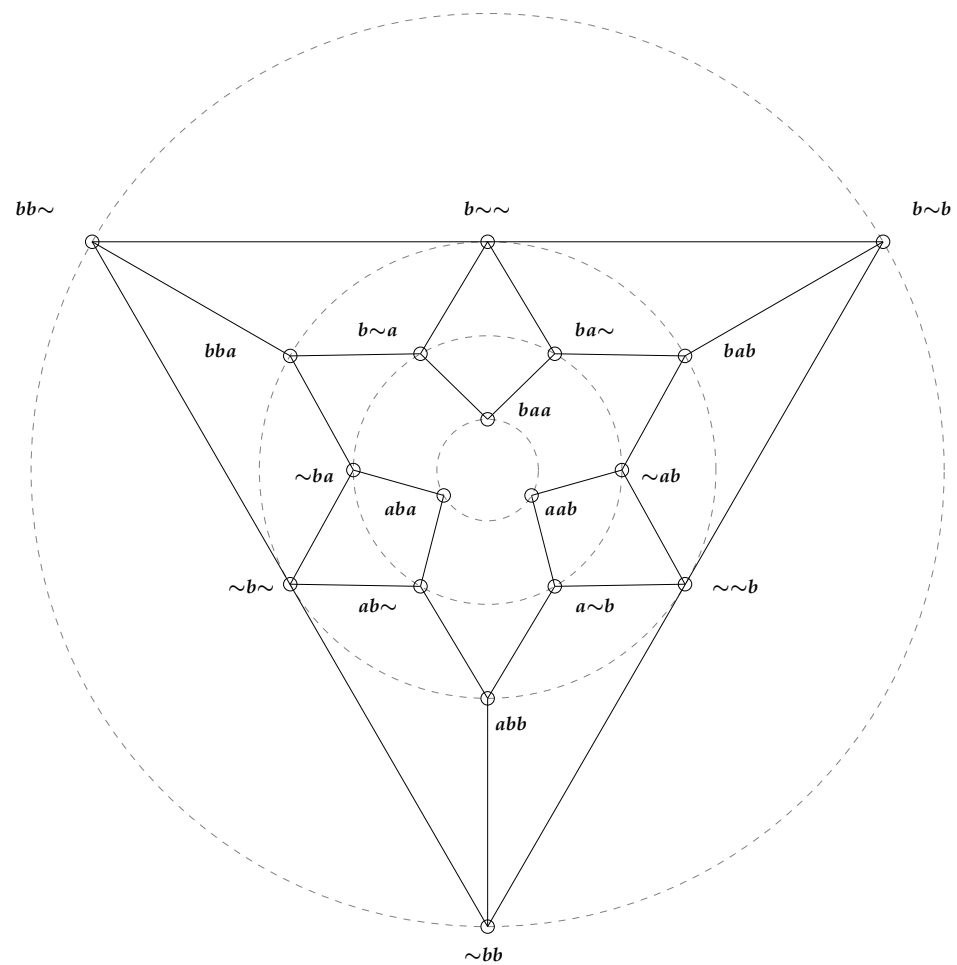
$$\begin{aligned} c[(b, \sim, b)] &= \{(\sim, \sim, b), (b, a, b), (b, \sim, \sim)\}, & c[(\sim, a, b)] &= \{(a, a, b)\}, \\ c[(\sim, b, b)] &= \{(a, b, b), (\sim, \sim, b), (\sim, b, \sim)\}, & c[(a, \sim, b)] &= \{(a, a, b)\}, \\ c[(b, b, \sim)] &= \{(\sim, b, \sim), (b, \sim, \sim), (b, b, a)\}, & c[(a, b, \sim)] &= \{(a, b, a)\}, \\ c[(b, a, b)] &= \{(\sim, a, b), (b, a, \sim)\}, & c[(\sim, b, a)] &= \{(a, b, a)\}, \\ c[(a, b, b)] &= \{(a, \sim, b), (a, b, \sim)\}, & c[(b, \sim, a)] &= \{(b, a, a)\}, \\ c[(b, b, a)] &= \{(\sim, b, a), b, \sim, a\}, & c[(b, a, \sim)] &= \{b, a, a\}, \\ c[(\sim, b, \sim)] &= \{(a, b, \sim), (\sim, b, a)\}, & c[(a, a, b)] &= \emptyset, \\ c[(b, \sim, \sim)] &= \{(b, a, \sim), (b, \sim, a)\}, & c[(a, b, a)] &= \emptyset, \\ c[(\sim, \sim, b)] &= \{(a, \sim, b), (\sim, a, b)\}, & c[(b, a, a)] &= \emptyset. \end{aligned}$$

The edges of $\mathcal{P}_b$ as a graph are the pairs of profiles joined when one covers the other. To provide an intuitive view of $\mathcal{P}_b$, we conveniently draw four concentric circles numbered from 1 to 4 according to increasing rays (1 is the smallest, 4 is the biggest), and set the vertices (i.e., the eighteen profiles of $\mathcal{P}_b$) on the circles according to the following rule:

- Profiles in $\mathcal{P}_b$ that are not covered by any other profile in $\mathcal{P}_b$ are on circle 4;
- Profiles in $\mathcal{P}_b$ covered by some profile on circle 4 are on circle 3;
- Profiles in $\mathcal{P}_b$ covered by some profile on circle 3 are on circle 2;
- Profiles in $\mathcal{P}_b$ covered by some profile on circle 2 are on circle 1.

Consequently, the edges in the following diagrams are directed from outside to inside coherently with the covering relation.

Again, for brevity, we avoid commas among preferences when displaying profiles in the figures.

The graph $\mathcal{P}_b$ **Visual examples of SP social choice functions for $|V| = 3$.**

A social choice function associates one of the two alternatives to each of the eighteen vertices of the graph $\mathcal{P}_b$. We again use colors: we color red the profiles having image a and we color blue the profiles having image b . Of course, not all the (2^{18}) functions that we obtain coloring the vertices are SP. Since $\mathcal{P}_b$ is a convex subset of $\mathcal{W}^V$, according to (ii) of Theorem 1, we have to fulfill the condition

$$[P, Q \in \mathcal{P}_b \quad \phi(P) = a, Q \in c[P]] \Rightarrow \phi(Q) = a.$$

This happens if (and only if) the two endpoints of an edge can be differently colored only when red is on the smaller circle. In other words, to recognize whether or not a coloring of the vertices of the graph $\mathcal{P}_b$ is an SP function, it is sufficient (and necessary) to check that all the segments with different colors on the endpoints have the two colors in the “right” order. A few examples follow.

Example 1. Only one SP function having all red vertices on the largest circle (respectively, all blue vertices on the smallest) exists: it is the one with also all the remaining vertices colored red (respectively, blue).

Example 2. If two of the three vertices on the largest circle are red and the third one is blue, necessarily all but one of the remaining vertices will be red. The value on what is left can be assigned either red or blue in any case preserving strategy-proofness. So, this example really deals with six different SP functions. Consider, to fix ideas, the case

$\phi(\sim, b, b) = \phi(b, \sim, b) = a$ and $\phi(b, b, \sim) = b$, in the next Figure 1. The value a has to be assigned to all profiles except (b, b, a) for which the value of ϕ cannot be fixed. We can color it red ($\phi(b, b, a) = a$) or blue ($\phi(b, b, a) = b$).

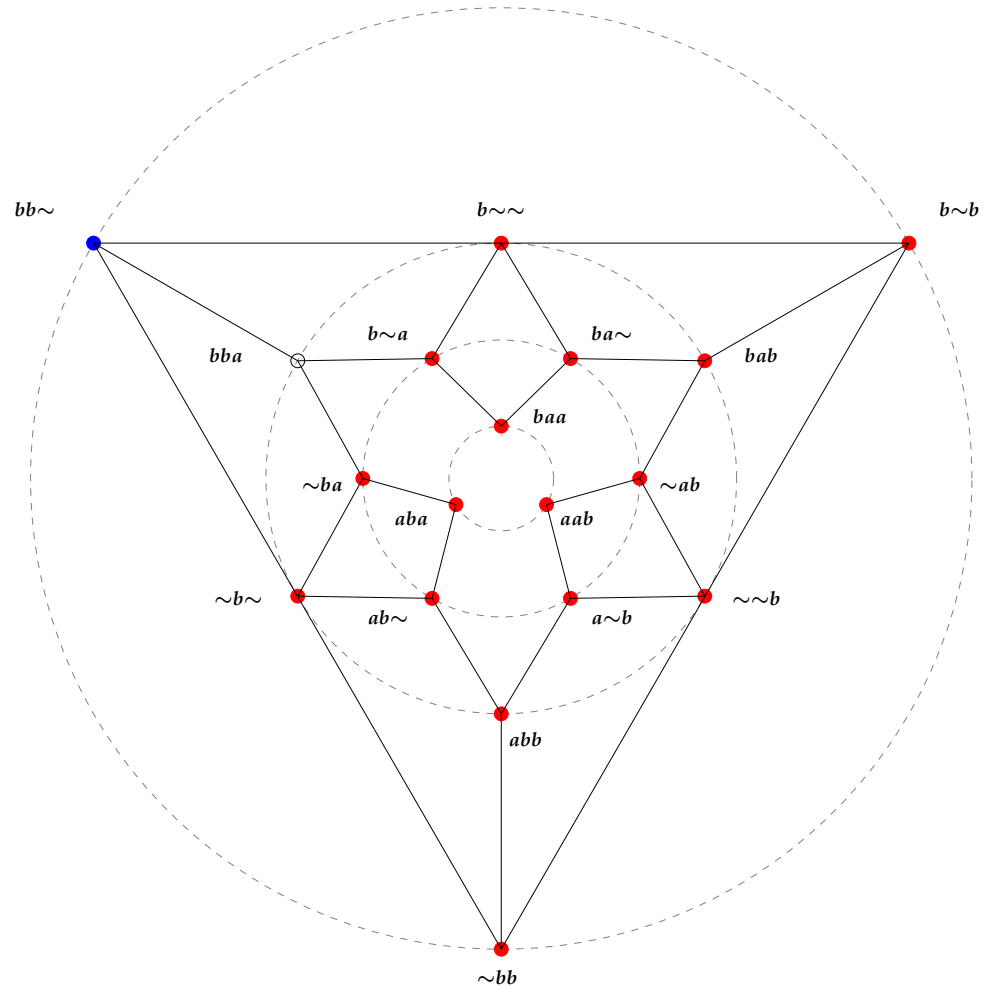


Figure 1. The case $\phi(\sim bb) = \phi(b \sim b) = a$ and $\phi(bb \sim) = b$.

Example 3. If there is exactly one red vertex on circle 4 (in the next Figure 2 we assume, to fix ideas, that $\phi(b, \sim, b) = \phi(\sim, b, b) = b$ and $\phi(b, b, \sim) = a$), then six of the remaining profiles remain uncolored, i.e., the value of ϕ is not assigned. However, those blank vertices cannot be colored arbitrarily in order to represent an SP function.

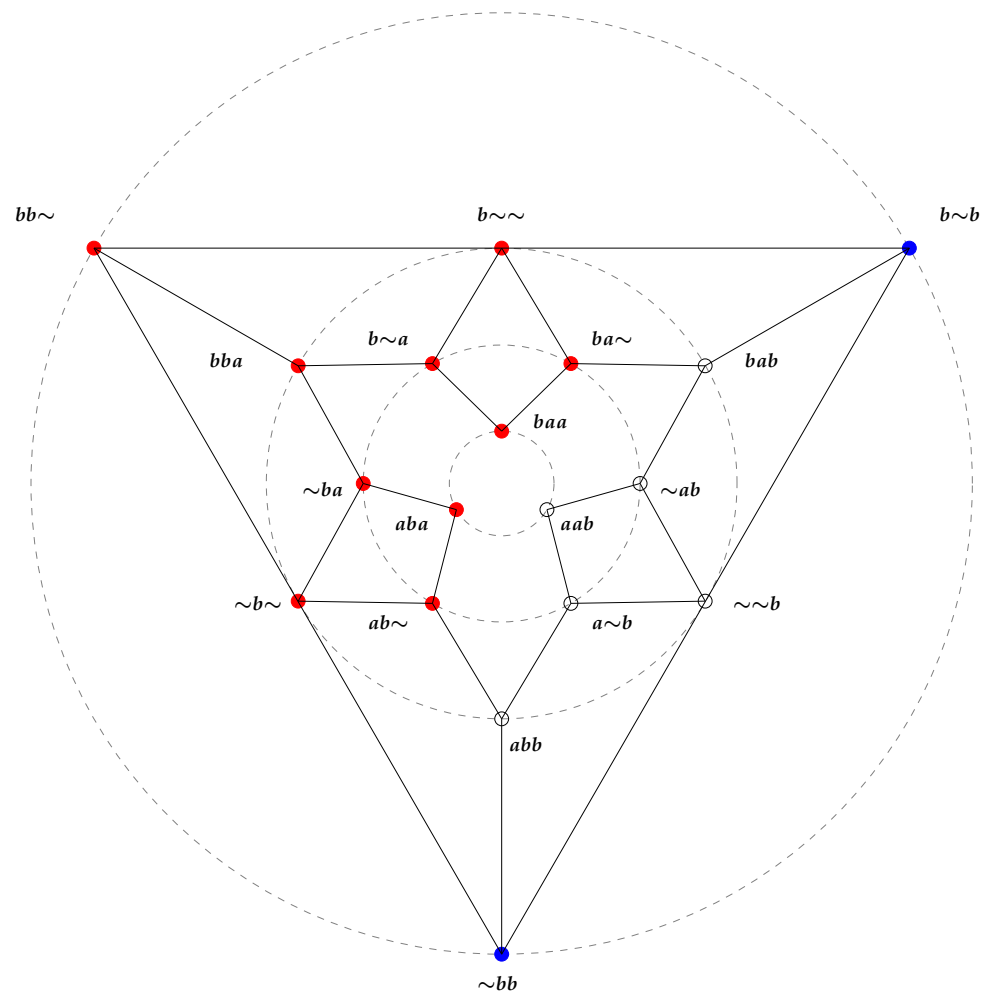


Figure 2. The case $\phi(b\sim b) = \phi(\sim bb) = b$ and $\phi(bb\sim) = a$.

Lemma 1 allows one to obtain all SP functions consistent with the values already assigned in Figure 2 (in its turn consistent with Assumption 4) by isolating the subgraph $\mathcal{P}'_b$ of Figure 3 and considering all SP functions having $\mathcal{P}'_b$ as the domain.

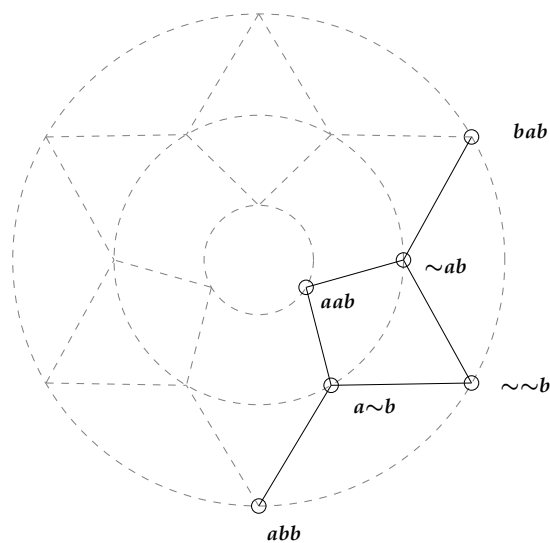


Figure 3. The graph $\mathcal{P}'_b$.

Applying Theorem 1, it can be verified that there are 42 such functions. Some of the possible functions are plotted below (Figures 4 and 5), where for the blank vertices (in these cases!), any blue/red color can be assigned.

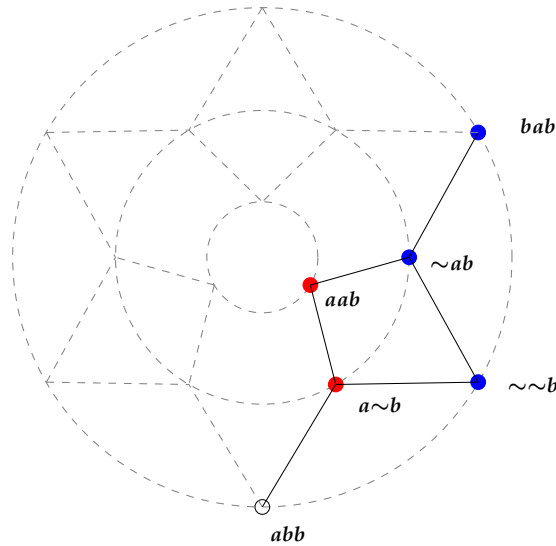


Figure 4. The subcase $\phi(a \sim b) = a$ and $\phi(\sim ab) = b$.

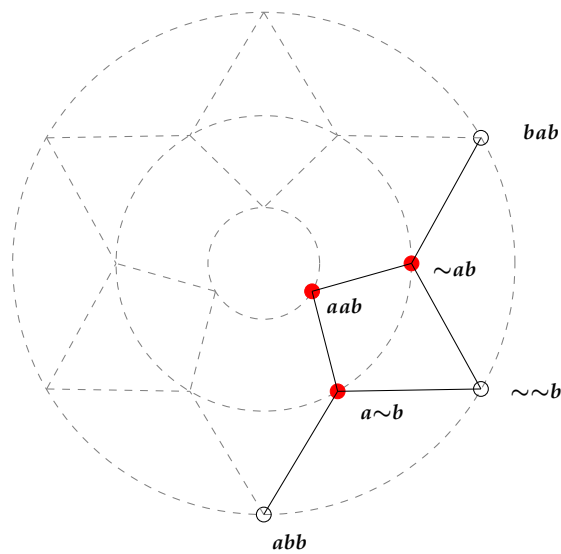


Figure 5. The subcase $\phi(a \sim b) = \phi(\sim ab) = a$.

Example 4. The case of three blue vertices on the largest circle is possible. Again, Lemma 1 can be suitably applied, and one can limit to analyze the SP assignments on the subgraph $\mathcal{P}_b''$ in Figure 6 (i.e., we can avoid plotting the largest circle).

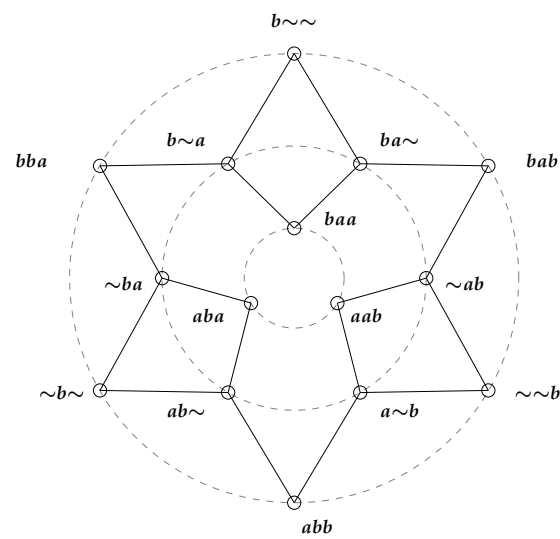


Figure 6. The graph $\mathcal{P}_b''$.

We conclude with a final example.

Example 5. Let us fix the ideas with the case that the value of ϕ is b on exactly two vertices of the smallest circle (only $\phi(a, b, a) = a$). The values of ϕ remain not fixed on three uncolored profiles (see Figure 7). Differently from the last two cases (of figures 4 and 5), this time, clearly, not all the ($2^3 = 8$) possibilities are allowed. Only five (obvious) assignments give rise to an SP function (Theorem 1).

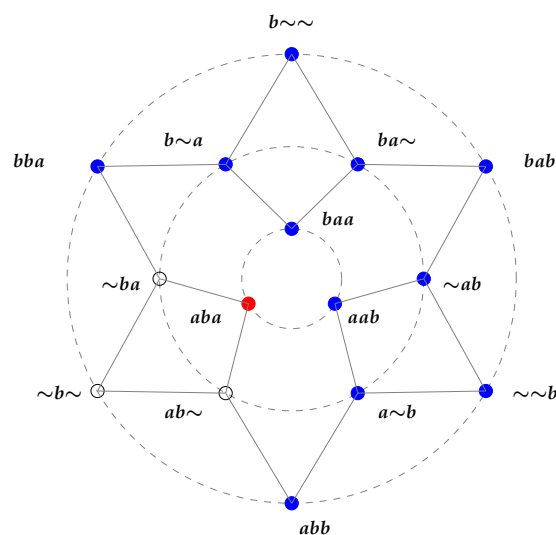


Figure 7. The subcase $\phi(aba) = a$ and $\phi(aab) = \phi(baa) = b$.

5. On the Number of Binary SP Social Choice Functions with Indifference

In our setting, that permits indifference, the number of binary SP social choice functions has been denoted by $|\Phi_n|$. This number can be described in different ways. We present three of them. Note that the first two evoke one of the known characterizations of the Dedekind numbers ([9], Sequence A000372, accessed on September and November 2021).

Recall that a subset of a partially ordered set is said to be an antichain if it consists of pairwise incomparable distinct elements. Let V be a set with n elements. The number $\text{Ded}(n)$ can be defined as the cardinality of the set of all antichains of the partially ordered set

$\{-1, 1\}^V$ (componentwise order). Since the elements of $\{-1, 1\}^V$ can be easily interpreted as the indicator functions of the subsets of V , we can equivalently refer to $\text{Ded}(n)$ as the number of the antichains of the power set of V ordered by set inclusion.

Our first way of describing $|\Phi_n|$ is the following.

Proposition 3. $|\Phi_n|$ is the cardinality of the set of all antichains of the partially ordered set $\{-1, 0, 1\}^V$ or, equivalently, of $\mathcal{W}^V$.

Proof. Let us first observe that the set $\mathcal{W}^V$ is obviously order isomorphic to $\{-1, 0, 1\}^V$. In addition, note that any $\{a, b\}$ -valued function defined on this set can be described by the inverse image of the element b . Theorem 1 says that such a function is SP exactly when $\phi^{-1}(b)$ is super order-closed. This implies that the number $|\Phi_n|$ is the number of the super order-closed parts of $\mathcal{W}^V$. At this point, it is sufficient to recall that for finite posets, the super order-closed subsets can be identified with the antichains of the poset. $\square$

A second characterization of the number $|\Phi_n|$ is related to the poset of *veto pairs* introduced in [10]. This is the following set

$$\mathcal{V} := \{(S, W) : S \subseteq W \subseteq V\}$$

(whose elements are the veto pairs) endowed with the partial order corresponding to the componentwise set inclusion. We have that:

Proposition 4. $|\Phi_n|$ is the cardinality of the set of all antichains of the partially ordered set $\mathcal{V}$.

Proof. It is sufficient to apply ([10], Theorem 3.3). $\square$

Proposition 4 also gives meaning to $|\Phi_0|$, as it happens for $\text{Ded}(0)$. Of course, both numbers have value 2.

Remark 3. The super order-closed subsets of the power set of V (ordered by inclusion) are referred to as *simple games* (respectively, *committees*) in the Game (respectively, Social Choice) Theory literature. In other words, ([1], Theorem 2) characterizes the binary SP social choice functions, in case of strict preferences, as simple games. Since in finite posets, antichains and super order-closed subsets are identifiable, the above-mentioned ([10], Theorem 3.3) (as it can be verified) extends this simple games characterization to the case of weak preferences, via a parallel extension of the notion of simple games to that of super order-closed subsets of the poset of veto pairs.

One more equivalent characterization of $|\Phi_n|$ can derive from [12] where the authors aim to emphasize the role of the groups of agents having no power to determine the values of a given SP social choice function. This is the motivation to introduce some families of functions, named “powerless revealing”, that are proved to be all and only the SP social choice functions ([12], Theorem 4). Moreover, they are also proved to be in a one-to-one correspondence with the class of “ineffective” families on the society. For the reader’s convenience, we give the definition:

A map $\mathcal{F} = (\mathcal{F}_M)_{M \subseteq V}$ that associates to each subset M of V a part of its power set $\mathcal{P}(M)$ is said to be an *ineffective family on V* if

- (i) $X \in \mathcal{F}_M, Y \subseteq X \Rightarrow Y \in \mathcal{F}_M$, and
- (ii) $X \in \mathcal{F}_M, N \subseteq M \Rightarrow X \cap N \in \mathcal{F}_N$.

Proposition 5. $|\Phi_n|$ is the number of ineffective families on a set V of n elements.

Concerning the values of $|\Phi_n|$, since the sets $\phi^{[h]}(\Phi_n(h, a) \times \Phi_n(h, b))$ are pairwise disjoint, a consequence of the formula (3) in Section 3 is that we can express the cardinality of Φ_{n+1} as follows:

$$|\Phi_{n+1}| = \sum_{h \in \Phi_n} |\Phi_n(h, a)| \cdot |\Phi_n(h, b)|. \quad (5)$$

In particular, we have

$$|\Phi_3| = 980,$$

and

$$|\Phi_4| = 17,792,748.$$

Remark 4. We have to observe that the procedure furnishes the cardinality of Φ_{n+1} on the basis not only of the cardinality of Φ_n but on the whole set of functions. This suggests that the cardinality of Φ_n may not suffice to determine even the number of functions belonging to Φ_{n+1} .

As for the sequence of Dedekind numbers, also the sequence $(|\Phi_n|)_{n \in \mathbb{N}}$ grows quite rapidly and presents similar computational difficulties. We recall that there is an explicit formula for $Ded(n)$, due to Kisielewicz [13]. However, its complexity makes it difficult to be used for any practical purposes. An expression in decimal form for $n > 8$ has not been computed. Analogously, finding the numbers $|\Phi_n|$ is not an elementary task.

The next table compares, on the second row, the number of SP binary social choice functions when n (up to eight) voters are not allowed to declare indifference, and on the third row, the number of SP binary social choice functions when declaring indifference is allowed.

$ V $	1	2	3	4	5	6	7	8
$Ded(V)$	3	6	20	168	7581	7828354	2414682040998	56130437228687557907788
$ \Phi_{ V } $	4	20	980	17792748	—	—	—	—

The values $|\Phi_3|$ and $|\Phi_4|$ have been obtained by means of the Python implementation of the procedure presented in Section 3. In an informal communication, Giovanni Cutolo (Università Federico II di Napoli) informed us that they can be also obtained by elaborating on the results presented in [14] (in particular Corollary 3.2 and Theorem 3.3). We are grateful to him for having pointed out this to us. Because of the computational complexity, we have not computed the decimal form expression of the numbers $|\Phi_n|$ for $n > 4$.

Two concluding comments are now in order. First, the Gibbard–Satterthwaite theorem is often seen by social choice theorists as a “negative” result in the sense that the small number of SP social choice functions with range of cardinality at least three makes non-dictatorship incompatible with strategy-proofness. So, one possible hint from the Barberà [8] general suggestion that introducing indifference may help in obtaining “positive” results is that proposing to voters the option of indifference increases the number of social choice functions with desirable properties. We notice that the possible comparisons between the numbers $Ded(n)$ and $|\Phi_n|$ give an idea of up to what magnitude one can confirm this. With as few as three or four voters, the binary SP social choice functions with indifference are, respectively, 49 or 10^5 times those without indifference. There exists a more precise result in this direction if one limits to considering SP functions which are also invariant under permutations of voters: in [10], it is shown that stepping from strict to weak preferences, the number of anonymous, strategy-proof, binary social choice functions depend exponentially on the number of voters, rather than linearly.

A second comment concerns the distinction between our paper and Campbell et al. [15]. While the latter is focused on discrediting the belief that the SP functions are few, we have focused on binary social choice functions with indifference, for which we provided formula (5) to compute $|\Phi_n|$. We in fact computed $|\Phi_3|$ and $|\Phi_4|$ here.

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