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Egalitarian-Equivalence and Strategy-Proofness in the Object Allocation Problem with Non-Quasi-Linear Preferences

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Abstract: We consider the problem of allocating heterogeneous objects to agents with money, where the number of agents exceeds that of objects. Each agent can receive at most one object, and some objects may remain unallocated. A bundle is a pair consisting of an object and a payment. An agent's preference over bundles may not be quasi-linear, which exhibits income effects or reflects borrowing costs. We investigate the class of rules satisfying one of the central properties of fairness in the literature, *egalitarian-equivalence*, together with the other desirable properties. We propose (i) a novel class of rules that we call the *independent second-prices rules with variable constraints* and (ii) a novel condition on constraints that we call *respecting the valuation coincidence.* Then, we establish that the independent second-prices rule with variable constraints that respects the valuation coincidence is the only rule satisfying *egalitarian-equivalence*, *strategy-proofness*, *individual rationality*, and *no subsidy for losers*. Our characterization result implies that in the case of three or more agents, there are few opportunities for agents to receive objects under a rule satisfying *egalitarian-equivalence* and the other desirable properties, which highlights the strong tension between *egalitarian-equivalence* and *efficiency*. In contrast, in the case of two agents and a single object, *egalitarian-equivalence* is compatible with *efficiency*.

Keywords: egalitarian-equivalence; strategy-proofness; efficiency; non-quasi-linear preferences; second-price rule; independent second-prices rule; independent second-prices rule with variable constraints; multi-object auctions

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1. Introduction

1.1. Motivation

Governments in many countries make use of auctions in order to allocate scarce resources such as spectrum licenses, vehicle ownership licenses, public houses, etc. One of the most important goals of government auctions is efficiency. However, a government is often more concerned with the other goals such as fairness of an allocation, promotion of competition, raising the revenue, etc. In this paper, we focus on the fairness of an allocation. In particular, we investigate the implications of one of the most important properties of fairness in the literature, *egalitarian-equivalence* [1], in the auction model with unit-demand agents and non-quasi-linear preferences.

1.2. Main Result

We consider the object allocation problem with money, where the number of agents exceeds that of objects. An agent can obtain at most one object. We allow the possibility that some objects remain unallocated. A *bundle* of an agent is a pair consisting of the object that he receives and the amount of payments made by him. Each agent has a preference over bundles which are not necessarily quasi-linear. Non-quasi-linear preferences reflects the important factors in practical auctions such as income effects and borrowing costs.

An *allocation* specifies a bundle to each agent. An *allocation rule*, or a *rule* for short, is a mapping from the set of preference profiles (the *domain*) to the set of allocations. A rule is *egalitarian-equivalent* if for each preference profile, there is a reference bundle to which each agent finds his outcome bundle of the rule indifferent. A rule is *strategy-proof* if no agent ever benefits by misrepresenting his preferences. A rule is *individually rational* if no agent ever gets worse off than receiving no object and making no monetary transfer. A rule satisfies *no subsidy for losers* if an agent who receives no object always makes non-negative payments. We regard these four properties as desiderata and investigate the class of rules satisfying the four properties.

A *second-price rule for an object* is a rule such that an agent with the highest valuation of the object receives the object and pays the second highest valuation, and the other agents receive and pay nothing. Note that all the other objects are allocated to no agent under the rule. An *independent second-prices rule* is a rule whose outcome allocation is determined by running a second-price rule for each object independently. More precisely, it determines an allocation for each preference profile as follows. First, a second-price rule for each object is conducted. Second, each winner in second-price rules chooses a best bundle among the set of bundles that he won in the first step. The outcome allocation of the rule is as follows. Each winner in the first step receives the bundle that he chose in the second step, and each loser in the first step receives and pays nothing.

We incorporate constraints into an independent second-prices rule. A *constraint* on each agent restricts the set of available objects to him *in the second step of the above procedure*. Thus, if an agent faces a constraint in an independent second-prices rule, then he participates in a second-price rule for each agent as in the case without a constraint, but in the second step, he chooses the best bundle among the set of bundles that he won in the first step *and are available to him under the constraint*. A *variable constraint* on an agent is a mapping from other agents' preferences to a constraint. Thus, it allows a constraint to vary depending on other agents' preferences. A rule is an *independent second-prices rule with variable constraints* if it is an independent second-prices rule where each agent faces a variable constraint.

If agents face no constraints, an independent second-prices rule violates *egalitarian-equivalence* (Example 3). We introduce a condition on constraints that we call *respecting the valuation coincidence*. It ensures that an independent second-prices rule with variable constraints satisfies *egalitarian-equivalence*. The basic idea is as follows: if an independent second-prices rule violates *egalitarian-equivalence* for a preference profile, then the constraints forbid agents to win objects in an independent second-prices rule with variable constraints. We show that respecting the valuation coincidence is a necessary and sufficient condition for an independent second-prices rule with variable constraints to satisfy *egalitarian-equivalence* (Proposition 3).

In our result, we require a domain to be sufficiently rich. Our richness condition of a domain is borrowed from Kazumura et al. [2], which is satisfied by many domains of interest such as the quasi-linear domain, the positive income effects domain, the borrowing costs domain, etc.

The main result of this paper is a characterization of the class of rules satisfying *egalitarian-equivalence* together with the other desirable properties. We establish that a rule on a rich domain satisfies *egalitarian-equivalence*, *strategy-proofness*, *individual rationality*, and *no subsidy for losers* if and only if it is an independent second-prices rule with variable constraints that respects the valuation coincidence (Theorem 1).

In the case of three or more agents, the respecting the valuation coincidence condition severely restricts the set of available objects to each agent so that agents have almost no chances to win objects (Example 6). Thus, in such a case, our characterization theorem (Theorem 1) highlights the strong tension between *egalitarian-equivalence* and *efficiency* under the other three desirable properties.

In contrast, in the case of two agents and a single object, the condition always holds and places no restrictions. Thus, in such a case, *egalitarian-equivalence* is compatible with *efficiency* under the other desirable properties (Proposition 4).

1.3. Related Literature

This paper belongs to the two strands of research: object allocation problems with money and fair allocation theory. The papers on object allocation problems with money mainly investigate *efficiency* [3–6]. In the model studied in this paper (i.e., the model with unit-demand agents and non-quasi-linear preferences), the generalized Vickrey rule [4,5] and the minimum price Warlasian rule [6,7] occupy the central positions. Remarkably, they are the only rules satisfying *efficiency*, *strategy-proofness*, *individual rationality*, and *no subsidy for losers* in the different settings.[1] The minimum price Warlasian rule is the only rule satisfying the four properties in the heterogeneous objects model . The class of independent second-prices rules with variable constraints is novel because the two central rules in this model do not belong to it.

In the literature on fair allocation theory, there are at least two important properties of fairness: *egalitarian-equivalence* and *envy-freeness*.[2] Many authors have investigated the implications of *envy-freeness* in object allocation problems with money [9–12].

Compared with the papers on *envy-freeness*, there are a few papers that investigate *egalitarian-equivalence* in object allocation problems with money. The following is the complete list of papers that investigate *egalitarian-equivalence* and *strategy-proofness* to our knowledge. Ohseto [9] considers the identical objects model with unit-demand agents and quasi-linear preferences, and they characterize the class of rules satisfying *egalitarian-equivalence*, *efficiency*, and *strategy-proofness*. Yengin [12,13] considers the heterogeneous objects model with multi-demand agents and quasi-linear preferences, and they identify the class of rules satisfying the same properties as Ohseto [14]. Chun et al. [15] consider the queueing model (a special case of the model with unit-demand agents) with quasi-linear preferences, and they characterize the class of rules satisfying the same three properties.

There are three differences between this paper and the above papers. First, we impose additional properties that are crucial for the practical auction design, i.e., *individual rationality* and *no subsidy for losers*, while they do not. Note that they obtain positive characterization results without such properties, while our main result with the properties for auction design (Theorem 1) can be seen as negative. Thus, in conjunction with their positive results, our main result suggests that it is the properties for auction design that yield a negative conclusion. Second, we do not impose any property of efficiency in our characterization result (Theorem 1), while the above papers assume *efficiency* in their results. In general, characterizing a class of rules without any property of efficiency is difficult because such a class of rules is usually so large that a tractable characterization is almost impossible. We overcome such a difficulty by exploiting the strong implications of *egalitarian-equivalence* compared with the other properties of fairness. Finally, we consider general preferences that are not necessarily quasi-linear, while their results and proofs crucially depend on the assumption of quasi-linear preferences.

Apart from object allocation problems with money, to the best of our knowledge, the unique paper that investigates a class of rules satisfying *egalitarian-equivalence* and *strategy-proofness* is Hayashi [16]. He establishes that in the pure exchange economy, the equal division rule is the only rule satisfying *egalitarian-equivalence*, *strategy-proofness*, and *non-bossiness*. Our characterization result is different from his in that we do not impose *non-bossiness* which prevents agents from joint manipulation of preferences if it is combined with *strategy-proofness*. In contrast, we impose properties that are important for the practical auction design: *individual rationality* and *no subsidy for losers*.

In object allocation problems with money, some authors investigate other properties of fairness such as *anonymity in welfare* [17] and *equal treatment of equals* [2]. Again, the difference between this paper and such papers is that they impose a minimal property of efficiency such as *no wastage* in their characterization results, while we do not.

1.4. Organization

The remainder of the paper is organized as follows. Section 2 sets up the model. Section 3 introduces the independent second-prices rule with variable constraints. Section 4 provides the main result. Section 5 discusses the relationships between *egalitarian-equivalence* and the other important properties. Section 6 concludes. All the proofs are relegated to Appendices A–C.

2. Model

There are n agents and m objects, where $n > m$. Let $N = \{1, \ldots, n\}$ denote the set of agents. Let M denote the set of objects. Typical agents are denoted by i, j, etc., and typical objects are denoted by a, b, etc. Each agent receives at most one object. An agent who receives no "real" object consumes the **null object** denoted by 0. Let $L = M \cup \{0\}$.

The amount of money paid by agent $i \in N$ is $t_i \in \mathbb{R}$. The **consumption set** of agent $i \in N$ is $L \times \mathbb{R}$, and his (**consumption**) **bundle** is $z_i = (x_i, t_i) \in L \times \mathbb{R}$. Let $\mathbf{0} = (0, 0) \in L \times \mathbb{R}$ denote the status quo bundle.

Each agent has a complete and transitive preference R_i over $L \times \mathbb{R}$. Let P_i and I_i be the strict and indifference parts of R_i, respectively. Throughout the paper, we consider the class of preferences satisfying the following four properties.

Money monotonicity. For each $x_i \in L$ and each pair $t_i, t'_i \in \mathbb{R}$ with $t'_i < t_i$, we have $(x_i, t'_i)\ P_i\ (x_i, t_i)$.
Desirability of objects. For each $x_i \in M$ and each $t_i \in \mathbb{R}$, we have $(x_i, t_i)\ P_i\ (0, t_i)$.
Possibility of compensation. For each $z_i \in L \times \mathbb{R}$ and each $x_i \in L$, there is a pair $t_i, t'_i \in \mathbb{R}$ such that $(x_i, t_i)\ R_i\ z_i$ and $z_i\ R_i\ (x_i, t'_i)$.
Continuity. For each $z_i \in L \times \mathbb{R}$, its upper contour set, $\{z'_i \in L \times \mathbb{R} : z'_i\ R_i\ z_i\}$, and its lower contour set, $\{z'_i \in L \times \mathbb{R} : z_i\ R_i\ z'_i\}$, are both closed.

A preference is **classical** if it satisfies the above four properties. Let $\mathcal{R}^C$ denote the class of classical preferences. A typical subset of $\mathcal{R}^C$ is denoted by $\mathcal{R}$.

Definition 1. *A preference $R_i \in \mathcal{R}$ is* **quasi-linear** *if for each pair $(x_i, t_i), (x'_i, t'_i) \in L \times \mathbb{R}$ and each $\delta \in \mathbb{R}$, $(x_i, t_i)\ I_i\ (x'_i, t'_i)$ implies $(x_i, t_i + \delta)\ I_i\ (x'_i, t'_i + \delta)$.*

Let $\mathcal{R}^Q$ denote the class of quasi-linear preferences. Given a quasi-linear preference $R_i \in \mathcal{R}^Q$, there is a **valuation function** $v_i : L \to \mathbb{R}_+$ such that (i) $v_i(0) = 0$, (ii) for each $x_i \in M$, $v_i(x_i) > 0$, and (iii) for each pair $(x_i, t_i), (x'_i, t'_i) \in L \times \mathbb{R}$, $(x_i, t_i)\ R_i\ (x'_i, t'_i)$ if and only if $v_i(x_i) - t_i \geq v_i(x'_i) - t'_i$.

Given $R_i \in \mathcal{R}$, $z_i \in L \times \mathbb{R}$, and $x_i \in L$, the possibility of compensation and continuity together imply the existence of a payment $V_i(x_i, z_i) \in \mathbb{R}$ such that $(x_i, V_i(x_i, z_i))\ I_i\ z_i$. By money monotonicity, such a payment is unique. We call the payment $V_i(x_i, z_i)$ the **valuation of x_i at z_i**. If $R_i \in \mathcal{R}^Q$, then for each $(x_i, t_i) \in L \times \mathbb{R}$ and each $x'_i \in L$, $V_i(x'_i, (x_i, t_i)) - t_i = v_i(x'_i) - v_i(x_i)$.

Given a preference $R_i \in \mathcal{R}$ and a set of bundles $A \subseteq L \times \mathbb{R}$, let

$$B(R_i, A) = \{z_i \in A : z_i\ R_i\ z'_i \text{ for each } z'_i \in A\}$$

denote the set of *best* bundles in A according to R_i.

An **object allocation** is an n-tuple $x = (x_i)_{i \in N} \in L^n$ such that for each distinct pair $i, j \in N$, $x_i \neq 0$ implies $x_i \neq x_j$. Let X denote the set of object allocations. Note that we allow the possibility that some objects remain unallocated.

An **allocation** is an n-tuple $z = (z_i)_{i \in N} = (x_i, t_i)_{i \in N} \in (L \times \mathbb{R})^n$ such that $(x_i)_{i \in N} \in X$. Let Z denote the set of allocations. We may write $z = (x, t) \in Z$, where $x \in X$ and $t \in \mathbb{R}^n$ are the object allocation and the profile of payments associated with z, respectively.

A **preference profile** is an n-tuple $R = (R_i)_{i \in N} \in \mathcal{R}^n$. Given a distinct pair $i, j \in N$, let $R_{-i} = (R_k)_{k \in N \setminus \{i\}}$ and $R_{-i,j} = (R_k)_{k \in N \setminus \{i,j\}}$.

We call $\mathcal{R}^n$ a **domain**. An (**allocation**) **rule** on $\mathcal{R}^n$ is a function $f : \mathcal{R}^n \to Z$. With an abuse of notation, we may write $f = (x, t)$, where $x : \mathcal{R}^n \to X$ and $t : \mathcal{R}^n \to \mathbb{R}^n$ are the object allocation and the payment rules associated with the rule f, respectively.

Now, we introduce the properties of rules.

Given a rule f on $\mathcal{R}^n$, $R \in \mathcal{R}^n$, and $z_0 \in L \times \mathbb{R}$, if $f_i(R)\ I_i\ z_0$ for each $i \in N$, then we call the bundle z_0 a **reference bundle for** R (**under** f). The first property is a central property of fairness in the literature which was introduced by Pazner and Shmeidler [1]. It requires that for each preference profile, there is a reference bundle to which each agent finds his outcome bundle of the rule indifferent.

Egalitarian-equivalence. For each $R \in \mathcal{R}^n$, there is a reference bundle $z_0 \in L \times \mathbb{R}$ for R such that $f_i(R)\ I_i\ z_0$ for each $i \in N$.

Another central property of fairness was introduced by Foley [18]. The second property requires that no agent prefer another agent's bundle to his own.

Envy-freeness. For each $R \in \mathcal{R}^n$ and each pair $i, j \in N$, $f_i(R)\ R_i\ f_j(R)$.

The third property requires that no agent ever benefit by misrepresenting his preferences.

Strategy-proofness. For each $R \in \mathcal{R}^n$, each $i \in N$, and each $R'_i \in \mathcal{R}$, $f_i(R)\ R_i\ f_i(R'_i, R_{-i})$.

The fourth property requires that each agent find his bundle at least as desirable as the status quo bundle $\mathbf{0}$.

Individual rationality. For each $R \in \mathcal{R}^n$ and each $i \in N$, $f_i(R)\ R_i\ \mathbf{0}$.

The fifth property requires that an agent who receives the null object make non-negative payments.

No subsidy for losers. For each $R \in \mathcal{R}^n$ and each $i \in N$, if $x_i(R) = 0$, then $t_i(R) \geq 0$.

The last four properties are concerned with the efficiency of an allocation. The sixth property is a standard property of the (Pareto) efficiency in our model.

Efficiency. For each $R \in \mathcal{R}^n$, there is no $z = (x, t) \in Z$ such that (i) $z_i\ R_i\ f_i(R)$ for each $i \in N$, (ii) $z_j\ P_j\ f_j(R)$ for some $j \in N$, and (iii) $\sum_{i \in N} t_i \geq \sum_{i \in N} t_i(R)$.

Given a rule $f = (x, t)$ on $\mathcal{R}^n$ and $R \in \mathcal{R}^n$, let $L^f(R) = \{a \in L : \exists i \in N \text{ s.t. } x_i(R) = a\}$ denote the set of objects that are already allocated to agents at R under f. By $n > m$, for each rule f on $\mathcal{R}^n$ and each $R \in \mathcal{R}^n$, $0 \in L^f(R)$.

The seventh property requires that for each preference profile, there be no other allocation at which each agent receives an object already allocated to some agent at the preference profile under the rule, and which Pareto dominates the allocation chosen by the rule.

Constrained efficiency. For each $R \in \mathcal{R}^n$, there is no $z = (x, t) \in Z$ such that (i) $x_i \in L^f(R)$ for each $i \in N$, (ii) $z_i\ R_i\ f_i(R)$ for each $i \in N$, (iii) $z_j\ P_j\ f_j(R)$ for some $j \in N$, and (iv) $\sum_{i \in N} t_i \geq \sum_{i \in N} t_i(R)$.

The next characterization of *constrained efficiency* is useful.

Remark 1. *A rule $f = (x, t)$ on $\mathcal{R}^n$ is constrained efficient if and only if for each $R \in \mathcal{R}^n$,*

$$\sum_{i \in N} t_i(R) = \max\Big\{ \sum_{i \in N} V_i(x_i, f_i(R)) : x = (x_i)_{i \in N} \in X,\ x_i \in L^f(R) \text{ for each } i \in N \Big\}.$$

The eighth property requires that for each preference profile, each object be allocated to some agent.

No wastage. For each $R \in \mathcal{R}^n$ and each $a \in M$, there is $i \in N$ such that $x_i(R) = a$.

The last property requires that for each preference profile, at least one object be allocated to some agent.

Minimal no wastage. For each $R \in \mathcal{R}^n$, there are $a \in M$ and $i \in N$ such that $x_i(R) = a$.

The next remark reveals the relationships between the above properties concerned with the efficiency of an allocation.

Remark 2. *(i) If a rule f on $\mathcal{R}^n$ satisfies efficiency, then it satisfies constrained efficiency.*
(ii) If a rule f on $\mathcal{R}^n$ satisfies efficiency, then it satisfies no wastage.
(iii) If a rule f on $\mathcal{R}^n$ satisfies no wastage, then it satisfies minimal no wastage.
(iv) A rule f on $\mathcal{R}^n$ satisfies efficiency if and only if it satisfies constrained efficiency and no wastage.
(v) Suppose $m = 1$. A rule f on $\mathcal{R}^n$ satisfies minimal no wastage if and only if it satisfies no wastage.

3. The Independent Second-Prices Rule with Variable Constraints

In this section, we introduce a novel class of rules that we call the independent second-prices rules with variable constraints.

First, we introduce a second-price rule for each object.

Definition 2. *Given $a \in M$, a rule f on $\mathcal{R}^n$ is a* **second-price rule for object** *a if for each $R \in \mathcal{R}^n$ and each $i \in N$,*

$$f_i(R) = \begin{cases} (a, \max_{j \in N \setminus \{i\}} V_j(a, \mathbf{0})) & \text{if } V_i(a, \mathbf{0}) > \max_{j \in N \setminus \{i\}} V_j(a, \mathbf{0}), \\ \mathbf{0} & \text{if } V_i(a, \mathbf{0}) < \max_{j \in N \setminus \{i\}} V_j(a, \mathbf{0}). \end{cases}$$

Note that the above definition may be slightly different from the standard definition of a second-price rule in that in case of ties, we allow the possibility that an object is not allocated to anyone. The next remark states that in the case of a single object, if a second-price rule for the object always gives the object to some agent, then it coincides with the **generalized Vickrey rule** [4,5].[3]

Remark 3. *Assume $m = 1$. Let $\mathcal{R} \subseteq \mathcal{R}^C$. A rule f on $\mathcal{R}^n$ is a second-price rule for an object satisfying no wastage if and only if it is a generalized Vickrey rule.*

Next, we introduce the no-trade rule.

Definition 3. *A rule f on $\mathcal{R}^n$ is the* **no-trade rule** *if for each $R \in \mathcal{R}^n$ and each $i \in N$, $f_i(R) = \mathbf{0}$.*

Given $a \in M$, let f^a denote a second-price rule for object a. Furthermore, let f^0 denote the no-trade rule. We regard the no-trade rule f^0 as the second-price rule for the null object, and we call a profile of rules $(f^a)_{a \in L}$ a **profile of second-price rules**.

Now, we introduce the independent second-prices rule.

Definition 4. *A rule f on $\mathcal{R}^n$ is an* **independent second-prices rule** *if there is a profile of second-price rules $(f^a)_{a \in L}$ such that for each $R \in \mathcal{R}^n$ and each $i \in N$,*

$$f_i(R) \in B\Big(R_i, \{f_i^{x_i}(R) : x_i \in L\}\Big).$$

Given a preference profile, the outcome allocation of an independent second-prices rule is determined by a second-price rule for each object as follows.

Step 1: A second-price rule for each object is conducted.
Step 2: Each winner of some object(s) in the first step chooses a best bundle among the bundles that he won in the first step.
Step 3: The outcome allocation of the rule is as follows. Each winner in the first step receives the bundle chosen by him in the second step. Each loser in the first step receives no object and pays nothing.

If an agent wins several objects in second-price rules, then the objects that are not chosen by him in the second step are wasted. The next example illustrates this point.

Example 1. *Let $N = \{1,2,3,4\}$ and $M = \{a,b,c\}$. Let $\mathcal{R} = \mathcal{R}^C$. Let f be an independent second-prices rule on $\mathcal{R}^n$ associated with a profile of second-price rules $(f^d)_{d\in L}$. Let $R_1 \in \mathcal{R}^Q$ be such that $v_1(a) = 10$, $v_1(b) = 7$, and $v_1(c) = 5$. Let $R_2 \in \mathcal{R}^Q$ be such that $v_2(a) = 6$, $v_2(b) = 4$ and $v_2(c) = 10$. Let $R_3 \in \mathcal{R}^Q$ be such that $v_3(x_3) = 1$ for each $x_3 \in M$. Let $R_4 = R_3$.*

We first determine the outcome allocations of the second-price rules for R. By $v_1(a) = 10 > 6 = \max_{i\in N\setminus\{1\}} v_i(a)$, $f_1^a(R) = (a,6)$ and $f_i^a(R) = \mathbf{0}$ for each $i \in N\setminus\{1\}$. By $v_1(b) = 7 > 4 = \max_{i\in N\setminus\{1\}} v_i(b)$, $f_1^b(R) = (b,4)$ and $f_i^b(R) = \mathbf{0}$ for each $i \in N\setminus\{1\}$. Further, by $v_2(c) = 10 > 5 = \max_{i\in N\setminus\{2\}} v_i(c)$, $f_2^c(R) = (c,5)$, and $f_i^c(R) = \mathbf{0}$ for each $i \in N\setminus\{2\}$.

Next, we determine the outcome allocation of f for R. For each $i \in \{3,4\}$, by $\{f_i^{x_i}(R) : x_i \in L\} = \{\mathbf{0}\}$, $f_i(R) = \mathbf{0}$. By $v_2(c) - t_2^c(R) = 5 > 0$, $f_2^c(R)\ P_2\ \mathbf{0}$. Thus, by $\{f_2^{x_2}(R) : x_2 \in L\} = \{\mathbf{0}, f_2^c(R)\}$, $f_2(R) = f_2^c(R) = (c,5)$. By $v_1(a) - t_1^a(R) = 4 > 3 = v_1(b) - t_1^b(R)$, $f_1^a(R)\ P_1\ f_1^b(R)$. Further, by $v_1(b) - t_1^b(R) = 3 > 0$, $f_1^b(R)\ P_1\ \mathbf{0}$. Thus, by $\{f_1^{x_1}(R) : x_1 \in L\} = \{\mathbf{0}, f_1^a(R), f_1^b(R)\}$, $f_1(R) = f_1^a(R) = (a,6)$. Note that agent 1 wins both objects a and b in the second-price rules for the objects, but he finally receives object a, and object b is wasted.

Next, we restrict the set of available objects to each agent in an independent second-prices rule. Given $i \in N$, a **constraint on** (the set of available objects to) **agent** i is a subset of L such that $0 \in L_i$. A typical constraint on agent i is denoted by L_i. Given $i \in N$, let $\mathcal{L}_i$ denote the set of constraints on agent i.

Definition 5. *A rule f on $\mathcal{R}^n$ is an* **independent second-prices rule with constraints** *if there is a profile of constraints $(L_i)_{i\in N} \in \times_{i\in N}\mathcal{L}_i$ and a profile of second-price rules $(f^a)_{a\in L}$ such that for each $R \in \mathcal{R}^n$ and each $i \in N$,*

$$f_i(R) \in B\Big(R_i, \{f_i^{x_i}(R) : x_i \in L_i\}\Big).$$

Note that if $L_i = L$ for each $i \in N$, then an independent second-prices rule with constraints associated with $(L_i)_{i\in N}$ coincides with an independent second-prices rule.

A constraint on each agent restricts the set of available objects to him not in the first step of the above procedure but in the second step. In other words, if agents face constraints, then all the agents participate in a second-price rule for each object as in the case without constraints, but the constraints on the winners restrict the set of available objects to them in the second step. Thus, in the second step, a winner in the first step chooses his most preferred bundle among the bundles that he won in the first step *and whose objects are available to him under the constraint*.

The next example illustrates an independent second-prices rule with constraints.

Example 2. *Let $N = \{1,2,3,4\}$ and $M = \{a,b,c\}$. Let $(L_i)_{i\in N} \in \times_{i\in N} \mathcal{L}_i$ be such that $L_1 = \{0,b\}$, $L_2 = \{0,a,b\}$, and $L_3 = L_4 = L$. Let $\mathcal{R} = \mathcal{R}^C$. Let f be an independent second-prices rule on $\mathcal{R}^n$ with constraints associated with a profile of constraints $(L_i)_{i\in N}$ and a profile of second-price rules $(f^d)_{d\in L}$. Consider the same preference profile R as in Example 1.*

In Example 1, we showed that $f_1^a(R) = (a,6)$, $f_1^b(R) = (b,4)$, and $f_2^c(R) = (c,5)$. For each $i \in \{3,4\}$, by $\{f_i^x(R) : x \in L_i\} = \{\mathbf{0}\}$, $f_i(R) = \mathbf{0}$. Note that $x_3^c(R) = c$, but by $c \notin L_2$, $\{f_2^{x_2}(R) : x_2 \in L_2\} = \{\mathbf{0}\}$. Thus, $f_2(R) = \mathbf{0}$. Recall that in Example 1, we showed that $f_1^a(R)\ P_1\ f_1^b(R)\ P_1\ \mathbf{0}$. By $\{f_1^{x_1}(R) : x_1 \in L_1\} = \{\mathbf{0}, f_1^b(R)\}$, $f_1(R) = f_1^b(R) = (b,4)$. Thus, agent 1 does not receive the best bundle $f_1^a(R)$ among the bundles that he won in the second-price rules because it is not available to him (i.e., $a \notin L_1$). Notice that $(L_i)_{i\in N}$ does not affect the outcome allocation of the second-price rule for each object $d \in M$: $f^d(R)$. Instead, L_i restricts the set of available objects to agent i when he chooses the best bundle that he won in the second-price rules.

Now, we extend an independent second-prices rule with constraints so that the constraint on each agent could depend on other agents' preferences. Given $i \in N$, a **variable constraint on agent** i is a function $L_i : \mathcal{R}^{n-1} \to \mathcal{L}_i$.

Definition 6. *A rule f on $\mathcal{R}^n$ is an* **independent second-prices rule with variable constraints** *if there is a profile of variable constraints $(L_i(\cdot))_{i \in N}$ and a profile of second-price rules $(f^a)_{a \in L}$ such that for each $R \in \mathcal{R}^n$ and each $i \in N$,*

$$f_i(R) \in B\Big(R_i, \{f_i^{x_i}(R) : x_i \in L_i(R_{-i})\}\Big).$$

The class of independent second-prices rules with variable constraints is novel in that central rules in our model such as the *minimum price Warlasian rule* [7] and the *generalized Vickrey rule* [4,5] do not belong to it.

4. Main Result

In this section, we provide the main result of this paper.

4.1. Rich Domain

First, we introduce a domain richness condition that will be used in the main result.

A **price vector** is $p = (p^a)_{a \in L} \in \mathbb{R}_+^{|L|}$ such that $p^0 = 0$. Given a pair of price vectors $p, \hat{p} \in \mathbb{R}_+^{|L|}$, we may write $p > \hat{p}$ if $p^a > \hat{p}^a$ for each $a \in M$.

Given a preference $R_i \in \mathcal{R}$, a price vector $p \in \mathbb{R}_+^{|L|}$, and a set $L_i \subseteq L$, the **demand set of** R_i **at** p **on** L_i is defined as

$$D(R_i, p, L_i) = \{x_i \in L_i : (x_i, p^{x_i})\ R_i\ (x_i', p^{x_i'}) \text{ for each } x_i' \in L_i\}.$$

If $L_i = L$, then let $D(R_i, p) = D(R_i, p, L_i)$.

Our main result requires a domain to be rich. The following richness condition of a domain is introduced by Kazumura et al. [2].

Definition 7. *A class of preferences $\mathcal{R}$ is* **rich** *if for each $a \in M$, for each pair of price vectors $p, \hat{p} \in \mathbb{R}_+^{|L|}$ such that (i) $p^a > 0$ and $p^b = 0$ for each $b \in L \backslash \{a\}$, and (ii) $\hat{p} > p$, there is a preference $R_i \in \mathcal{R}$ such that*

$$D(R_i, p) = \{a\} \quad \text{and} \quad D(R_i, \hat{p}) = \{0\}.$$

A domain $\mathcal{R}^n$ is **rich** *if $\mathcal{R}$ is rich.*

Clearly, $\mathcal{R}^Q$ and $\mathcal{R}^C$ are both rich.

An example of a rich domain is the positive income effects domain. We do not take into account an agent's income explicitly in our model, but the zero payment can be regarded as the initial income of the agent. Then, a payment level can be interpreted as the negative of an agent's income level relative to his initial income. Thus, the increase of an income corresponds to the decrease of a payment. In other words, a preference exhibits a positive income effect if the increase of an income (or equivalently, the decrease of a payment) makes a preferred object more preferable. Formally, R_i exhibits the **positive income effect** if for each pair $(x_i, t_i), (x_i', t_i') \in L \times \mathbb{R}$ and each $\delta \in \mathbb{R}_{++}$, $(x_i, t_i)\ R_i\ (x_i', t_i')$ and $t_i > t_i'$ imply $(x_i, t_i - \delta)\ P_i\ (x_i', t_i' - \delta)$. Let $\mathcal{R}^+$ denote the class of preferences that exhibits positive income effects. Then, it is straightforward to verify that $\mathcal{R}^+$ is rich.

Another example of a rich domain is the quasi-linear domain with borrowing costs. Suppose that an agent has a quasi-linear preference but faces a soft budget constraint $I_i > 0$. Then, he has to borrow money at an interest rate $r > 0$ if a payment exceeds his budget I_i. Then, there is a valuation function $v_i : L \to \mathbb{R}_+$ such that for each pair

$(x_i, t_i), (x_i', t_i') \in L \times \mathbb{R}$, $(x_i, t_i)\ R_i\ (x_i', t_i')$ if and only if $v_i(x_i) - c_i(t_i, I_i) \geq v_i(x_i') - c_i(t_i, I_i)$, where $c_i : \mathbb{R} \times \mathbb{R}_{++} \to \mathbb{R}$ is a function such that for each $t_i \in \mathbb{R}$ and each $I_i \in \mathbb{R}_{++}$,

$$c_i(t_i, I_i) = \begin{cases} t_i & \text{if } t_i \leq I_i, \\ I_i + (t_i - I_i)(1+r) & \text{if } t_i > I_i. \end{cases}$$

Let $\mathcal{R}^B$ denote the class of quasi-linear preferences with borrowing costs. Then, it is rich.

Kazumura et al. (2020) include other examples of rich domains of interest. For further examples and the detailed discussion of a rich domain, see Kazumura et al. [2].

4.2. Constraints

The goal of this paper is to characterize the class of rules satisfying *egalitarian-equivalence*, *strategy-proofness*, *individual rationality*, and *no subsidy for losers*. The purpose of this subsection is to provide building blocks for a characterization.

We begin with the following proposition. It states that the class of rules satisfying *egalitarian-equivalence*, *strategy-proofness*, *individual rationality*, and *no subsidy for losers* is a subset of the class of independent second-prices rules with variable constraints.

Proposition 1. *Let $\mathcal{R}$ be rich. If a rule f on $\mathcal{R}^n$ satisfies egalitarian-equivalence, strategy-proofness, individual rationality, and no subsidy, then it is an independent second-prices rule with variable constraints.*

Thus, the problem of characterizing the class of rules satisfying the four properties reduces to the problem of whether the converse of Proposition 1 is true, i.e., whether an independent second-prices rule with variable constraints satisfies the four properties. The next proposition states that it satisfies all the properties except for *egalitarian-equivalence*.

Proposition 2. *Let $\mathcal{R} \subseteq \mathcal{R}^C$. An independent second-prices rule with variable constraints satisfies strategy-proofness, individual rationality, and no subsidy for losers.*

The next example shows that an independent second-price rule violates *egalitarian-equivalence*. Thus, some independent second-prices rules with variable constraints violate it, and so the converse of Proposition 1 is not necessarily true.

Example 3 (Independent second-prices rule violates egalitarian-equivalence)**.** *Let $N = \{1, 2, 3\}$ and $M = \{a, b\}$. Let $\mathcal{R} = \mathcal{R}^C$. Let f be an independent second-prices rule on $\mathcal{R}^3$ associated with a profile of second-price rules $(f^c)_{c \in L}$. Let $R_1 \in \mathcal{R}^Q$ be such that $v_1(a) = 10$ and $v_1(b) = 9$. Let $R_2 \in \mathcal{R}^Q$ be such that $v_2(a) = v_2(b) = 5$. Let $R_3 \in \mathcal{R}^Q$ be such that $v_3(a) = v_3(b) = 1$. Then, agent 1 wins both objects a and b in the second-price rules: $f_1^a(R) = (a, 5)$ and $f_1^b(R) = (b, 5)$. It is straightforward to check that $f_1^a(R)\ P_1\ f_1^b(R)\ P_1\ \mathbf{0}$. Thus, $f_1(R) = f_1^a(R) = (a, 5)$. Further, $f_2(R) = f_3(R) = \mathbf{0}$.*

We show that there is no reference bundle $z_0 \in L \times \mathbb{R}$ for R such that $f_i(R)\ I_i\ z_0$ for each $i \in N$. By contradiction, suppose there is a reference bundle $z_0 = (x_0, t_0) \in L \times \mathbb{R}$ for R such that $f_i(R)\ I_i\ z_0$ for each $i \in N$. If $x_0 = 0$, then by $x_3(R) = 0$, $z_0 = f_3(R) = \mathbf{0}$. However, by $v_1(a) = 10 > 5 = t_1(R)$, $f_1(R)\ P_1\ \mathbf{0} = z_0$, a contradiction. If $x_0 = a$, then by $x_1(R) = a$, $z_0 = f_1(R) = (a, 5)$. By $v_3(a) = 1 < 5 = t_0$, $f_3(R) = \mathbf{0}\ P_3\ (a, 5) = z_0$, a contradiction. Finally, if $x_0 = b$, then by $v_2(b) - t_0 = 0$, $t_0 = 5$. By $v_3(b) = 1 < 5 = t_0$, $f_3(R) = \mathbf{0}\ P_3\ (b, 5) = z_0$, a contradiction. Thus, f violates egalitarian-equivalence.

Given Propositions 1 and 2 and Example 3, the problem of characterizing the class of rules satisfying *egalitarian-equivalence* and the other properties reduces to the problem of identifying a necessary and sufficient condition on constraints for an independent second-prices rule with variable constraints to satisfy *egalitarian-equivalence*. Such a condition will be as follows: for a preference profile at which an independent second-prices rule violates *egalitarian-equivalence*, the constraints forbid agents to win objects. Thus, the problem is to identify preference profiles at which an independent second-prices rule violates *egalitarian-*

equivalence. Instead of solving this problem directly, we identify preference profiles at which an independent second-prices rule *satisfies egalitarian-equivalence*.

We introduce the two examples of preference profiles at which an independent second-prices rule satisfies *egalitarian-equivalence*. The first example is concerned with the case of a single winner.

Example 4 (Single winner)**.** *Let $N = \{1, 2, 3\}$ and $M = \{a, b\}$. Let f be an independent second-prices rule on $\mathcal{R}^3$ associated with a profile of second-price rules $(f^c)_{c \in L}$. Let $R \in \mathcal{R}^3$ be such that $V_1(a, \mathbf{0}) > \max_{i \in N \setminus \{1\}} V_i(a, \mathbf{0})$, $V_1(b, \mathbf{0}) > \max_{i \in N \setminus \{1\}} V_i(b, \mathbf{0})$, and $(a, \max_{i \in N \setminus \{1\}} V_i(a, \mathbf{0}))$ P_1 $(b, \max_{i \in N \setminus \{1\}} V_i(b, \mathbf{0}))$. Then, agent 1 wins both objects a and b in the second-price rules: $f_1^a(R) = (a, \max_{i \in N \setminus \{1\}} V_i(a, \mathbf{0}))$ and $f_1^b(R) = (b, \max_{i \in N \setminus \{1\}} V_i(b, \mathbf{0}))$. By $f_1^a(R)$ P_1 $f_1^b(R)$, $f_1(R) = f_1^a(R)$. Since agents 2 and 3 win no object in the second-price rules, for each $i \in N \setminus \{1\}$, $f_i(R) = \mathbf{0}$.*

Now, suppose that for each pair $i, j \in N \setminus \{1\}$, $V_i(a, \mathbf{0}) = V_j(a, \mathbf{0})$. We show that f satisfies egalitarian-equivalence for R. Let $z_0 = f_1(R)$. Then, z_0 I_1 $f_1(R)$. For each $i \in N \setminus \{1\}$, by $V_i(a, \mathbf{0}) = t_1(R)$, $f_i(R) = \mathbf{0}$ I_i $(a, V_i(a, \mathbf{0})) = (a, t_1(R)) = z_0$. Thus, the example shows that in the case of a single winner, if the losers' valuations of the winner's object at $\mathbf{0}$ coincide with each other, then an independent second-prices rule satisfies egalitarian-equivalence.

Next, we provide an example of a preference profile at which an independent second-prices rule satisfies *egalitarian-equivalence* in the case of several winners.

Example 5 (Several winners)**.** *Let $N = \{1, 2, 3, 4\}$ and $M = \{a, b, c\}$. Let f be an independent second-prices rule on $\mathcal{R}^4$ associated with a profile of second-price rules $(f^d)_{d \in L}$. Let $R \in \mathcal{R}^4$ be such that $V_1(a, \mathbf{0}) > \max_{i \in N \setminus \{1\}} V_i(a, \mathbf{0})$, $V_2(b, \mathbf{0}) > \max_{i \in N \setminus \{2\}} V_i(b, \mathbf{0})$, $V_1(c, \mathbf{0}) > \max_{i \in N \setminus \{c\}} V_i(c, \mathbf{0})$, and $(a, \max_{i \in N \setminus \{1\}} V_i(a, \mathbf{0}))$ P_1 $(c, \max_{i \in N \setminus \{1\}} V_i(c, \mathbf{0}))$. Then, agents 1 wins both objects a and c, and agent 2 wins object b in the second-price rules: $f_1^a(R) = (a, \max_{i \in N \setminus \{1\}} V_i(a, \mathbf{0}))$, $f_2^b(R) = (b, \max_{i \in N \setminus \{2\}} V_i(b, \mathbf{0}))$, and $f_1^c(R) = (c, \max_{i \in N \setminus \{1\}} V_i(c, \mathbf{0}))$. By $f_1^a(R)$ P_1 $f_1^c(R)$, $f_1(R) = f_1^a(R)$. Furthermore, $f_2(R) = f_2^b(R)$, and for each $i \in N \setminus \{1, 2\}$, $f_i(R) = \mathbf{0}$.*

Now, suppose that there is an object $d \in M$ such that for each pair $i, j \in N$, it holds that $V_i(d, f_i(R)) = V_j(d, f_j(R))$.[4] *We show that f satisfies egalitarian-equivalence for R. By the assumption, we can choose $v \in \mathbb{R}$ such that $v = V_i(d, f_i(R))$ for each $i \in N$. Let $z_0 = (d, v)$. Then, for each $i \in N$, $f_i(R)$ I_i $(d, V_i(d, f_i(R))) = (d, v) = z_0$. Thus, the example shows that in the case of several winners, if all the agents' valuations of some (real) object at their outcome bundles of the rule coincide with each other, then an independent second-prices rule satisfies egalitarian-equivalence.*

The above two examples show that when the agents' valuations coincide with each other, an independent second-prices rule with variable constraints satisfies *egalitarian-equivalence*. Then, we introduce the condition on constraints such that agents win objects only when the valuations coincide with each other as in the above two examples, i.e., if the agents' valuations do not coincide with each other, then agents cannot win objects.

In order to introduce the condition, we need to prepare a notation. Given $R \in \mathcal{R}^n$ and $(L_i)_{i \in N} \in \times_{i \in N} \mathcal{L}_i$, let

$$W(R, (L_i)_{i \in N}) = \left\{ i \in N : \exists x_i \in L_i \setminus \{0\} \text{ s.t. } V_i(x_i, \mathbf{0}) > \max_{j \in N \setminus \{i\}} V_j(x_i, \mathbf{0}) \right\}$$

denote the set of *strict* winners who win an object for sure in an independent second-prices rule with variable constraints associated with $(L_i)_{i \in N}$ for R.

Now, we are ready to introduce the condition on the constraints. In other words, it states that in an independent second-prices rule with variable constraints, (i) there is a single strict winner only if the losers' valuations of the winner's object at $\mathbf{0}$ coincide with each other as in Example 4, and (ii) there are several strict winners only if all the agent's valuations of some (real) object at their outcome bundles of the rule coincide with each other as in Example 5.

Definition 8. *An independent second-prices rule with variable constraints associated with* $(L_i(\cdot))_{i \in N}$ **respects the valuation coincidence** *if for each* $R \in \mathcal{R}^n$*, the following conditions hold.*

(i) *If* $|W(R, (L_i(R_{-i}))_{i \in N})| = 1$*, then for each* $i \in W$ *and each pair* $j, k \in N \backslash \{i\}$*,* $V_j(x_i(R), \mathbf{0}) = V_k(x_i(R), \mathbf{0})$*.*

(ii) *If* $|W(R, (L_i(R_{-i}))_{i \in N})| \geq 2$*, then there is* $a \in M$ *such that for each pair* $i, j \in N$*,* $V_i(a, f_i(R)) = V_j(a, f_j(R))$*.*

The next proposition states that respecting the valuation coincidence is a necessary and sufficient condition for an independent second-prices rule with variable constraints to satisfy *egalitarian-equivalence*.

Proposition 3. *Let* $\mathcal{R}$ *be rich. An independent second-prices rule with variable constraints satisfies egalitarian-equivalence if and only if it respects the valuation coincidence.*

When $n = 2$ and $m = 1$, the valuations always coincide with each other in the sense of the above definition, and so any independent second-prices rule with variable constraints respects the valuation coincidence. However, when $n \geq 3$, the valuations almost never coincide with each other, and so in an independent second-prices rule with variable constraints, agents have few opportunities to win objects. In the example below, we confirm this fact by providing a few examples of independent second-prices rules with variable constraints that respect the valuation coincidence.

Example 6. *Let* f *be an independent second-prices rule with variable constraints on* $\mathcal{R}^n$*.*

(i) *Let* $(L_i(\cdot))_{i \in N}$ *be such that for each* $i \in N$ *and each* $R_{-i} \in \mathcal{R}^{n-1}$*,* $L_i(R_{-i}) = \{0\}$*. In other words, each agent never has a chance to win an object under the profile of variable constraints* $(L_i(\cdot))_{i \in N}$*. Then,* f *associated with* $(L_i(\cdot))_{i \in N}$ *coincides with the no-trade rule.*

(ii) *Let* $i \in N$*. Let* $(L_j(\cdot))_{j \in N}$ *be such that for each* $R_{-i} \in \mathcal{R}^{n-1}$*,* $L_i(R_{-i}) = \{x_i \in L : V_j(x_i, \mathbf{0}) = V_k(x_i, \mathbf{0})$ *for each pair* $j, k \in N \backslash \{i\}\}$*, and for each* $j \in N \backslash \{i\}$ *and each* $R_{-j} \in \mathcal{R}^{n-1}$*,* $L_j(R_{-j}) = \{0\}$*. In other words, agent* i *can receive an object only if all the other agents' valuations of the object at* $\mathbf{0}$ *coincide with each other, and no other agent has an opportunity to win objects.*

(iii) *Let* $a \in M$*. Let* $(L_i(\cdot))_{i \in N}$ *be such that for each* $i \in N$ *and each* $R_{-i} \in \mathcal{R}^{n-1}$*,* $L_i(R_{-i}) = \{0, a\}$ *if* $V_j(a, \mathbf{0}) = V_k(a, \mathbf{0})$ *for each pair* $j, k \in N \backslash \{i\}$*, and* $L_i(R_{-i}) = \{0\}$ *otherwise. In words, each agent has an opportunity to win the object* a *only if all the other agents' valuations of* a *at* $\mathbf{0}$ *coincide with each other, but it has no access to all the other real objects.*

(iv) *Let* $(L_i(\cdot))_{i \in N}$ *be such that for each* $i \in N$ *and each* $R_{-i} \in \mathcal{R}^{n-1}$*,* $L_i(R_{-i}) = L$ *if for each pair* $j, k \in N \backslash \{i\}$ *and each* $a \in M$*,* $V_j(a, \mathbf{0}) = V_k(a, \mathbf{0})$*, and* $L_i(R_{-i}) = \{0\}$ *otherwise. In words, each agent has access to all the objects when all the other agents' valuations of each object coincide with each other, but it has no access to a real object otherwise.*

In all the above cases, an independent second-prices rule with variable constraints f *associated with* $(L_i(\cdot))_{i \in N}$ *respects the valuation coincidence.*

4.3. Main Result

Now, we are ready to provide the main result of this paper.

The main result of this paper is a characterization of a class of rules on a rich domain satisfying *egalitarian-equivalence*, *strategy-proofness*, *individual rationality*, and *no subsidy for losers*.

Theorem 1. *Let* $\mathcal{R}$ *be rich. A rule* f *on* $\mathcal{R}^n$ *satisfies egalitarian-equivalence, strategy-proofness, individual rationality, and no subsidy for losers if and only if it is an independent second-prices rule with variable constraints that respects the valuation coincidence.*

Note that Theorem 1 follows from Propositions 1–3. Instead of proving the three propositions, we prove Theorem 1 directly in Appendix B and omit the proofs of the three propositions.[5]

Since $(\mathcal{R}^C)^n$, $(\mathcal{R}^Q)^n$, $(\mathcal{R}^+)^n$, and $(\mathcal{R}^B)^n$ are all rich, Theorem 1 is valid on these domains. Thus, Theorem 1 is valid not only on the quasi-linear domain but also on non-quasi-linear domains.

All the properties in Theorem 1 are indispensable for the characterization result. In the examples below, let $\mathcal{R}$ be an arbitrary rich class of preferences.

Example 7 (Dropping egalitarian-equivalence)**.** *Let f be a minimum price Warlasian rule on $\mathcal{R}^n$.*[6] *Then, f satisfies all the properties in Theorem 1 other than egalitarian-equivalence.*

Example 8 (Dropping strategy-proofness)**.** *Let $f = (x, t)$ be a generalized pay-as-bid rule on $\mathcal{R}^n$ such that for each preference profile $R \in \mathcal{R}^n$, (i) the objects are allocated so as to maximize the sum of valuations at* **0***, and (ii) each agent pays his valuation of $x_i(R)$ at* **0***. Then, f satisfies all the properties in Theorem 1 other than strategy-proofness.*

Example 9 (Dropping individual rationality)**.** *Let f be the no-trade rule with a fixed entry fee $e > 0$ on $\mathcal{R}^n$. Then, f satisfies all the properties in Theorem 1 other than individual rationality.*

Example 10 (Dropping no subsidy for losers)**.** *Let f be the no-trade rule with a fixed subsidy $s < 0$ on $\mathcal{R}^n$. Then, f satisfies all the properties in Theorem 1 other than no subsidy for losers.*

Recall that when $n = 2$ and $m = 1$, any independent second-prices rule with variable constraints respects the valuation coincidence. Thus, we obtain the following characterization.

Corollary 1. *Assume $n = 2$ and $m = 1$. Let $\mathcal{R}$ be rich. A rule on $\mathcal{R}^n$ satisfies egalitarian-equivalence, strategy-proofness, individual rationality, and no subsidy for losers if and only if it is an independent second-prices rule with variable constraints.*

5. Discussion

In this section, we discuss the relationships between *egalitarian-equivalence* and the other important properties under *strategy-proofness*, *individual rationality*, and *no subsidy for losers*.

5.1. Efficiency

Efficiency is arguably one of the most important properties in economic design. Thus, it is important to clarify the relationships between *egalitarian-equivalence* and properties of efficiency under the other three desirable properties.

Recall that Remark 3 states that in the case of a single object, a second-price rule for the object that satisfies *no wastage* coincides with the generalized Vickrey rule. Saitoh and Serizawa [4] and Sakai [5] establish that in the case of a single object, the generalized Vickrey rule satisfies *efficiency*. Thus, by Corollary 1 and Remark 2 (ii), we obtain the following.

Proposition 4. *Assume $n = 2$ and $m = 1$. Let $\mathcal{R}$ be rich.*

(i) A rule on $\mathcal{R}^n$ satisfies egalitarian-equivalence, efficiency, strategy-proofness, individual rationality, and no subsidy for losers if and only if it is a generalized Vickrey rule.

(ii) A rule on $\mathcal{R}^n$ satisfies egalitarian-equivalence, no wastage, strategy-proofness, individual rationality, and no subsidy for losers if and only if it is a generalized Vikcrey rule.

By Remark 2 (v), *no wastage* in Proposition 4 (ii) can be replaced by *minimal no wastage*.

Proposition 4 implies that in the case of two agents and a single object, *egalitarian-equivalence* is compatible with *efficiency* under the other three desirable properties. This positive observation comes from the fact that in such a case, the respecting the valuation coincidence condition is always true.

In contrast with the case of two agents and a single object, in the case of three or more agents, the respecting valuation coincidence condition severely restricts the set of available objects to each agent, which may yield significant inefficiency. Indeed, the next proposition states that *egalitarian-equivalence* is incompatible with a minimal property of efficiency under the other three desirable properties.

Proposition 5. *Assume $n \geq 3$. Let $\mathcal{R}$ be rich. Then, no rule on $\mathcal{R}^n$ satisfies egalitarian-equivalence, minimal no wastage, strategy-proofness, individual rationality, and no subsidy for losers.*

The following corollary is immediate from Proposition 5 and Remark 2 (ii) and (iii).

Corollary 2. *Assume $n \geq 3$. Let $\mathcal{R}$ be rich.*

(i) *No rule on $\mathcal{R}^n$ satisfies egalitarian-equivalence, efficiency, strategy-proofness, individual rationality, and no subsidy for losers.*
(ii) *No rule on $\mathcal{R}^n$ satisfies egalitarian-equivalence, no wastage, strategy-proofness, individual rationality, and no subsidy for losers.*

Recall that Remark 2 (iv) states that *efficiency* can be decomposed into *constrained efficiency* and *no wastage*. The next proposition states that an independent second-prices rule with variable constraints satisfies *constrained efficiency*.

Proposition 6. *Let $\mathcal{R} \subseteq \mathcal{R}^C$. An independent second-prices rule with variable constraints on $\mathcal{R}^n$ satisfies constrained efficiency.*

When $n \geq 3$, *egalitarian-equivalence* is incompatible with one of the two components of *efficiency*, i.e., *no wastage*, under the other three properties (Proposition 5). In contrast, *egalitarian-equivalence* is compatible with the other component of *efficiency*, i.e., *constrained efficiency*, under the other three properties (Theorem 1 and Proposition 6). Thus, the inefficiency that arises from *egalitarian-equivalence* can be fully attributed to the few opportunities for agents to receive objects.

5.2. Envy-Freeness

In this paper, we have so far investigated the class of rules satisfying *egalitarian-equivalence* together with the other desirable properties. Another important property of fairness in the literature is *envy-freeness*. Thus, it is worthwhile to discuss the relationship between *egalitarian-equivalence* and *envy-freeness*.

The next proposition states that an independent second-prices rule with variable constraints satisfies *envy-freeness*.

Proposition 7. *Let $\mathcal{R} \subseteq \mathcal{R}^C$. An independent second-prices rule with variable constraints on $\mathcal{R}^n$ satisfies envy-freeness.*

In many models, *egalitarian-equivalence* neither implies nor is implied by *envy-freeness*. In particular, they are often incompatible under mild assumptions [19]. However, Theorem 1 and Proposition 7 together imply that *egalitarian-equivalence* implies *envy-freeness* under the other three desirable properties in our model.

Corollary 3. *Let $\mathcal{R}$ be rich. Let f be a rule on $\mathcal{R}^n$ satisfying strategy-proofness, individual rationality, and no subsidy for losers. If f satisfies egalitarian-equivalence, then it satisfies envy-freeness.*

The converse is obviously not true; i.e., *envy-freeness* does not necessarily imply *egalitarian-equivalence* under the other three properties. For example, the minimum price Warlasian rule satisfies *envy-freeness* and the other three properties, but it violates *egalitarian-equivalence* (Example 7). Since the rule is not an independent second-prices rule with variable constraints, this implies that the class of rules satisfying *envy-freeness* and the other three properties is not equivalent to the class of independent second-prices rules with variable constraints (indeed, the latter is a propert subset of the former).

6. Conclusions

In this paper, we have investigated the implications of *egalitarian-equivalence* along with the other desirable properties in the auction model with unit-demand agents and non-quasi-linear preferences. We characterize the class of rules satisfying *egalitarian-equivalence*, *strategy-proofness*, *individual rationality*, and *no subsidy for losers* (Theorem 1). Our characterization result reveals that in the case of three or more agents, the cost of *egalitarian-equivalence* in the auction model is enormous as it gives agents few opportunities to receive objects (Proposition 5). In contrast, in the case of two agents and a single object, *egalitarian-equivalence* is compatible with *efficiency* (Proposition 4).

Our characterization result is of independent interest in that it suggests if we impose a strong property of fairness such as *egalitarian-equivalence*, then the independent second-prices rule stands out instead of the minimum price Warlasian rule and the generalized Vickrey rule, both of which are the central rules in the literature.[7]

An interesting and fruitful direction of future research will be to drop the assumption of the model or to weaken properties of rule. For example, to drop the assumption on the numbers of agents and objects would change not only the proofs but also the results. In addition, allowing randomization in our model or weakening the concept of *egalitarian-equivalence* may enable us to escape from the negative result in the case of three or more agents in this paper (Proposition 5).[8] Although such directions of research are beyond the scope of this paper, we believe that our results and proof technique serve as a benchmark for future research.

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Appendix A. Preliminaries

In this section, we provide the lemmas that will be used to prove the results.

The next lemma gives a useful characterization of the independent second-prices rule with variable constraints. The proof of the following lemma is trivial. Thus, we omit it.

Lemma A1. *A rule f on $\mathcal{R}^n$ is an independent second-prices rule with variable constraints if and only if for each $i \in N$, there is $L_i : \mathcal{R}^{n-1} \to \mathcal{L}_i$ such that for each $R \in \mathcal{R}^n$, $x_i(R) \in D(R_i, p_i, L_i(R_{-i}))$ and $t_i(R) = p^{x_i}$, where $p_i \in \mathbb{R}_+^{|L|}$ is a price vector such that for each $x_i \in M$, $p_i^{x_i} = \max_{j \in N \setminus \{i\}} V_j(x_i, \mathbf{0})$.*

Now, we introduce the lemmas that provide the implications of the properties of rules.

The next lemma states that an agent who receives the null object makes no monetary transfer. Since its proof is trivial, we omit it.

Lemma A2 (Zero payment for losers)**.** *Let f be a rule on $\mathcal{R}^n$ satisfying individual rationality and no subsidy for losers. Let $R \in \mathcal{R}^n$ and $i \in N$. If $x_i(R) = 0$, then $t_i(R) = 0$.*

The following lemma states that if a reference bundle includes the null object, then it coincides with the status quo bundle $\mathbf{0}$.

Lemma A3. *Let f be a rule on $\mathcal{R}^n$ satisfying individual rationality and no subsidy for losers. Let $R \in \mathcal{R}^n$ and $z_0 = (x_0, t_0) \in L \times \mathbb{R}$ be a reference bundle for R such that $f_i(R)\ I_i\ z_0$ for each $i \in N$. If $x_0 = 0$, then $z_0 = \mathbf{0}$.*

Proof. Suppose $x_0 = 0$. By $n > m$, there is $i \in N$ such that $x_i(R) = 0$. By Lemma A2, $f_i(R) = \mathbf{0}$. By $\mathbf{0} = f_i(R)\ I_i\ z_0 = (0, t_0)$, $t_0 = 0$. Thus, $z_0 = \mathbf{0}$. □

The following lemma states that the payment at a reference bundle is non-negative.

Lemma A4. *Let f be a rule on $\mathcal{R}^n$ satisfying individual rationality and no subsidy for losers. Let $R \in \mathcal{R}^n$ and $z_0 = (x_0, t_0) \in L \times \mathbb{R}$ be a reference bundle for R such that $f_i(R)\ I_i\ z_0$ for each $i \in N$. Then, $t_0 \geq 0$.*

Proof. By contradiction, suppose $t_0 < 0$. By $n > m$, there is $i \in N$ such that $x_i(R) = 0$. By Lemma A2, $f_i(R) = \mathbf{0}$. We show that $x_0 \neq 0$. Suppose $x_0 = 0$. By $t_0 < 0$, $z_0 = (0, t_0)\ P_i\ \mathbf{0} = f_i(R)$, which contradicts that $f_i(R)\ I_i\ z_0$.

Thus, $x_0 \neq 0$. By $\mathbf{0} = f_i(R)\ I_i\ z_0$, $V_i(x_0, \mathbf{0}) = t_0 < 0$. Then,

$$\mathbf{0}\ I_i\ (x_0, V_i(x_0, \mathbf{0}))\ P_i\ (x_0, 0),$$

which contradicts desirability of objects. □

The next lemma states that if an agent receives a real object and has a preference whose valuations at $\mathbf{0}$ are sufficiently small compared with other agents, then his payment is equal to his valuation of the object that he receives at $\mathbf{0}$.

Lemma A5. *Let f be a rule on $\mathcal{R}^n$ satisfying egalitarian-equivalence, individual rationality and no subsidy for losers. Let $R \in \mathcal{R}^n$ and $i \in N$ be such that $x_i(R) \neq 0$, and for each $x_i \in M$, $V_i(x_i, \mathbf{0}) < \min_{j \in N \setminus \{i\}} \min_{x_j \in M} V_j(x_j, \mathbf{0})$. Then, $t_i(R) = V_i(x_i(R), \mathbf{0})$.*

Proof. By $n > m$ and $x_i(R) \neq 0$, there is $j \in N \setminus \{i\}$ such that $x_j(R) = 0$. By Lemma A2, $f_j(R) = \mathbf{0}$. By *egalitarian-equivalence*, there is a reference bundle $z_0 = (x_0, t_0) \in L \times \mathbb{R}$ for R such that $f_k(R)\ I_k\ z_k$ for each $k \in N$.

Now, we claim that $x_0 = 0$. Suppose by contradiction that $x_0 \neq 0$. By $\mathbf{0} = f_j(R)\ I_j\ z_0$, $t_0 = V_j(x_0, \mathbf{0})$. Then,

$$t_0 = V_j(x_0, \mathbf{0}) > V_i(x_0, \mathbf{0}), \tag{A1}$$

where the inequality follows from the assumption on R_i. Then,

$$f_i(R)\ R_i\ \mathbf{0}\ I_i\ (x_0, V_i(x_0, \mathbf{0}))\ P_i\ (x_0, t_0) = z_0,$$

where the first relation follows from *individual rationality*, and the third one follows from (A1). This contradicts $f_i(R)\ I_i\ z_0$.

Thus, $x_0 = 0$. By Lemma A3, $z_0 = \mathbf{0}$. By $f_i(R)\ I_i\ z_0 = \mathbf{0}$, $t_i(R) = V_i(x_i(R), \mathbf{0})$. □

Given a rule f on $\mathcal{R}^n$, $i \in N$ and $R_{-i} \in \mathcal{R}^{n-1}$, agent i's **option set under f for R_{-i}** is defined by

$$o_i^f(R_{-i}) = \{z_i \in L \times \mathbb{R} : \exists R_i \in \mathcal{R} \text{ s.t. } f_i(R_i, R_{-i}) = z_i\}.$$

Furthermore, given a rule f on $\mathcal{R}^n$, $i \in N$ and $R_{-i} \in \mathcal{R}^{n-1}$, let $L_i^f(R_{-i}) = \{x_i \in L : \exists R_i \in \mathcal{R} \text{ s.t. } x_i(R_i, R_{-i}) = x_i\}$ and $M_i^f(R_{-i}) = L_i^f(R_{-i}) \setminus \{0\}$.

Let f be a rule on $\mathcal{R}^n$ satisfying *strategy-proofness*. Given $i \in N$, $R_{-i} \in \mathcal{R}^{n-1}$, and $x_i \in L_i^f(R_{-i})$, let $t_i^f(R_{-i}; x_i) \in \mathbb{R}$ be a payment such that $(x_i, t_i^f(R_{-i}; x_i)) \in o_i^f(R_{-i})$. By *strategy-proofness*, such a payment must be unique. Then, for each $i \in N$ and each $R_{-i} \in \mathcal{R}^{n-1}$, it holds that

$$o_i^f(R_{-i}) = \{(x_i, t_i^f(R_{-i}; x_i)) : x_i \in L_i^f(R_{-i})\}.$$

Furthermore, given $i \in N$, $R_{-i} \in \mathcal{R}^{n-1}$, and $x_i \in L_i^f(R_{-i})$, let $z_i^f(R_{-i}; x_i) = (x_i, t_i^f(R_{-i}; x_i))$.

The following lemma states that each agent receives the best bundle among his option set. Since its proof is trivial, we omit it.

Lemma A6. *Let f be a rule on $\mathcal{R}^n$ satisfying strategy-proofness. For each $R \in \mathcal{R}^n$, each $i \in N$, and each $x_i \in L_i^f(R_{-i})$, $f_i(R)\ R_i\ z_i^f(R_{-i}; x_i)$.*

The following lemma states that the payment of an agent who receives a real object is positive.

Lemma A7 (Positive payments for real objects)**.** *Let f be a rule on $\mathcal{R}^n$ satisfying strategy-proofness, individual rationality, and no subsidy for losers. Let $i \in N$ and $R_{-i} \in \mathcal{R}^{n-1}$ be such that $0 \in L_i^f(R_{-i})$. For each $x_i \in M_i^f(R_{-i})$, $t_i^f(R_{-i}; x_i) > 0$.*

Proof. Let $x_i \in M_i^f(R_{-i})$. Suppose by contradiction that $t_i^f(R_{-i}; x_i) \leq 0$. By $0 \in L_i^f(R_{-i})$, there is $R_i \in \mathcal{R}$ such that $x_i(R) = 0$. By Lemma A2, $f_i(R) = \mathbf{0}$. Then,

$$z_i^f(R_{-i}; x_i)\ R_i\ (x_i, 0)\ P_i\ \mathbf{0} = f_i(R),$$

where the first relation follows from $t_i^f(R_{-i}; x_i) \leq 0$, and the second one follows from the desirability of objects. However, this contradicts Lemma A6. □

The next corollary states that the payment of each agent is non-negative.

Corollary A1 (No subsidy)**.** *Let f be a rule on $\mathcal{R}^n$ satisfying strategy-proofness, individual rationality, and no subsidy for losers. Let $i \in N$ and $R_{-i} \in \mathcal{R}^{n-1}$ be such that $0 \in L_i^f(R_{-i})$. For each $R_i \in \mathcal{R}$, $t_i(R) \geq 0$.*

Proof. Let $R_i \in \mathcal{R}$. If $x_i(R) = 0$, then by Lemma A2, $t_i(R) = 0$. If $x_i(R) \neq 0$, then by Lemma A7, $t_i(R) = t_i^f(R_{-i}; x_i(R)) > 0$. □

Given a bundle $z_i = (x_i, t_i) \in M \times \mathbb{R}_{++}$, a preference R_i is z_i**-favoring** if for each $x_i' \in L \backslash \{x_i\}$, $V_i(x_i', z_i) < 0$. Given $z_i \in M \times \mathbb{R}_{++}$, let $\mathcal{R}^{NV}(z_i)$ denote the class of preferences that are z_i-favoring.

The next lemma states that for each $R \in \mathcal{R}^n$, if agent $i \in N$ replaces his preference by a $f_i(R)$-favoring one, then the outcome bundle of the agent does not change by such a preference replacement.

Lemma A8. *Let f be a rule on $\mathcal{R}^n$ satisfying strategy-proofness, individual rationality, and no subsidy for losers. Let $i \in N$ and $R_{-i} \in \mathcal{R}^{n-1}$ be such that $0 \in L_i^f(R_{-i})$. Let $R_i \in \mathcal{R}$ be such that $f_i(R) \in M \times \mathbb{R}_{++}$. Let $R_i' \in \mathcal{R} \cap \mathcal{R}^{NV}(f_i(R))$. Then, $f_i(R_i', R_{-i}) = f_i(R)$.*

Proof. By $R_i' \in \mathcal{R}^{NV}(f_i(R))$, for each $x_i \in L \backslash \{x_i(R)\}$,

$$V_i'(x_i, f_i(R)) < 0. \tag{A1}$$

Now, we show that $x_i(R_i', R_{-i}) = x_i(R)$. By contradiction, suppose $x_i(R_i', R_{-i}) \neq x_i(R)$. Then. by *strategy-proofnsss*, $f_i(R_i', R_{-i})\ R_i'\ f_i(R)$. This implies

$$t_i(R_i', R_{-i}) \leq V_i'(x_i(R_i', R_{-i}), f_i(R)) < 0,$$

where the last inequality follows from (A1). However, this contradicts Corollary A1.

Thus, $x_i(R_i', R_{-i}) = x_i(R)$. This implies that $f_i(R_i', R_{-i}) = z_i^f(R_{-i}; x_i(R_i', R_{-i})) = z_i^f(R_{-i}; x_i(R)) = f_i(R)$. □

The next lemma states that for each $R \in \mathcal{R}^n$, if agent $i \in N$ replaces his preference by a $f_i(R)$-favoring one R_i', then a reference bundle for (R_i', R_{-i}) coincides with $f_i(R_i', R_{-i})$.

Lemma A9. *Let f be a rule on $\mathcal{R}^n$ satisfying strategy-proofness, individual rationality, and no subsidy for losers. Let $i \in N$ and $R_{-i} \in \mathcal{R}^{n-1}$ be such that $0 \in L_i^f(R_{-i})$. Let $R_i \in \mathcal{R}$ be such* *that $f_i(R) \in M \times \mathbb{R}_{++}$. Let $R'_i \in \mathcal{R} \cap \mathcal{R}^{NV}(f_i(R))$. Let $z_0 = (x_0, t_0) \in L \times \mathbb{R}$ be a reference bundle for (R'_i, R_{-i}) such that $f_i(R'_i, R_{-i})\ I'_i\ z_0$ and $f_j(R'_i, R_{-i})\ I_j\ z_0$ for each $j \in N\backslash\{i\}$. Then, $f_i(R'_i, R_{-i}) = z_0$.*

Proof. We show $x_0 = x_i(R'_i, R_{-i})$. Suppose by contradiction that $x_0 \neq x_i(R'_i, R_{-i})$. By $R'_i \in \mathcal{R} \cap \mathcal{R}^{NV}(f_i(R))$, Lemma A8 implies $f_i(R'_i, R_{-i}) = f_i(R)$. Thus, by $x_0 \neq x_i(R'_i, R_{-i})$, $x_0 \neq x_i(R)$. Thus, by $R'_i \in \mathcal{R}^{NV}(f_i(R))$, $V'_i(x_0, f_i(R)) < 0$. Thus, by $f_i(R'_i, R_{-i}) = f_i(R)$,

$$V'_i(x_0, f_i(R'_i, R_{-i})) = V'_i(x_0, f_i(R)) < 0.$$

By $f_i(R'_i, R_{-i})\ I'_i\ z_0$, $t_0 = V'_i(x_0, f_i(R'_i, R_{-i}))$. Thus, $t_0 < 0$, which contradicts Lemma A4.

Thus, $x_0 = x_i(R'_i, R_{-i})$. By $f_i(R'_i, R_{-i})\ I'_i\ z_0$, $t_i(R'_i, R_{-i}) = t_0$. Thus, we obtain $f_i(R'_i, R_{-i}) = z_0$. □

Finally, the next lemma states that for each $z_i \in M \times \mathbb{R}_{++}$, a rich class of preferences includes at least one z_i-favoring preference.

Lemma A10. *Let $\mathcal{R}$ be rich. Let $z_i = (x_i, t_i) \in M \times \mathbb{R}_{++}$. Then, $\mathcal{R} \cap \mathcal{R}^{NV}(z_i) \neq \varnothing$.*

Proof. Let $p \in \mathbb{R}_+^{|L|}$ be a price vector such that $p^{x_i} = t_i$, and for each $x'_i \in L\backslash\{x_i\}$, $p^{x'_i} = 0$. By richness, there is a preference $R_i \in \mathcal{R}$ such that $D(R_i, p) = \{x_i\}$. Let $x'_i \in L\backslash\{x_i\}$. Then, by $D(R_i, p) = \{x_i\}$, $z_i = (x_i, p^{x_i})\ P_i\ (x'_i, p^{x'_i}) = (x'_i, 0)$. This implies $V_i(x'_i, z_i) < 0$. Thus, $R_i \in \mathcal{R}^{NV}(z_i)$. By $R_i \in \mathcal{R}$, we have $R_i \in \mathcal{R} \cap \mathcal{R}^{NV}(z_i)$. □

Appendix B. Proof of Theorem 1

In this section, we provide the proof of Theorem 1. Let $\mathcal{R}$ be rich.

Appendix B.1. Proof of the "If" Part

In this subsection, we show the "if" part of Theorem 1. Let f be an independent second-prices rule with variable constraints on $\mathcal{R}^n$ respecting the equality for a single winner and the equality for several winners. Let $(L_i(\cdot))_{i\in N}$ be a profile of variable constraints associated with f.

It is straightforward to show that f satisfies *individual rationality* and *no subsidy for losers*. Thus, we here show *egalitarian-equivalence* and *strategy-proofness*.

EGALITARIAN-EQUIVALENCE. Let $R \in \mathcal{R}^n$. Let $W = W(R, (L_i(R_{-i})_{i\in N}))$. For each $i \in N$, let $p_i \in \mathbb{R}_+^{|L|}$ be a price vector such that for each $x_i \in M$, $p_i^{x_i} = \max_{j\in N\backslash\{i\}} V_j(x_i, \mathbf{0})$.

First, we show the following claim.

Claim A1. *Let $i \in N\backslash W$. Then, $f_i(R)\ I_i\ \mathbf{0}$.*

Proof. By $i \notin W$, for each $x_i \in L_i(R_{-i})$, $V_i(x_i, \mathbf{0}) \leq p_i^{x_i}$. Thus, by $x_i(R) \in L_i(R_{-i})$, $V_i(x_i(R), \mathbf{0}) \leq p_i^{x_i(R)} = t_i(R)$, where the equality follows from Lemma A1. This implies $\mathbf{0}\ R_i\ f_i(R)$. This, together with *individual rationality*, implies $f_i(R)\ I_i\ \mathbf{0}$. □

We consider the following three cases.

CASE 1. $|W| = 0$.

By Claim A1, for each $i \in N$, $f_i(R)\ I_i\ \mathbf{0}$. Thus, $\mathbf{0}$ can be taken as a reference bundle for R.

CASE 2. $|W| = 1$.

By $|W| = 1$, there is $i \in N$ such that $W = \{i\}$. Since f respects the valuation coincidence, for each $j \in N\backslash\{i\}$,

$$V_j(x_i(R), \mathbf{0}) = p_i^{x_i(R)} = t_i(R),$$

which implies $\mathbf{0}\ I_j\ f_i(R)$. By Claim A1, for each $j \in N \backslash \{i\}$, $f_j(R)\ I_j\ \mathbf{0}$. Thus, for each $j \in N \backslash \{i\}$, $f_j(R)\ I_j\ f_i(R)$. Thus, $f_i(R)$ can be chosen as a reference bundle for R.

CASE 3. $|W| \geq 2$.

Since f respects the valuation coincidence, there is $a \in M$ such that for each pair $i, j \in N$, $v = V_i(a, f_i(R)) = V_j(a, f_j(R))$. Then, for each $i \in N$, $f_i(R)\ I_i\ (a, V_i(a, f_i(R))) = (a, v)$. Thus, (a, v) can be taken as a reference bundle for R.

STRATEGY-PROOFNESS. Let $R \in \mathcal{R}^n$, $i \in N$, and $R_i' \in \mathcal{R}$. Then, by Lemma A1, it holds that $x_i(R) \in D(R_i, p_i, L_i(R_{-i}))$, $t_i(R) = p_i^{x_i(R)}$, and $t_i(R_i', R_{-i}) = p_i^{x_i(R_i', R_{-i})}$, where $p_i \in \mathbb{R}_+^{|L|}$ is a price vector such that for each $x_i \in M$, $p_i^{x_i} = \max_{j \in N \backslash \{i\}} V_j(x_i, \mathbf{0})$. By $x_i(R) \in D(R_i, p_i, L_i(R_{-i}))$ and $x_i(R_i', R_{-i}) \in L_i(R_{-i})$,

$$f_i(R) = (x_i(R), p_i^{x_i(R)})\ R_i\ (x_i(R_i', R_{-i}), p_i^{x_i(R_i', R_{-i})}) = f_i(R_i', R_{-i}),$$

as desired.

Appendix B.2. Proof of the "Only If" Part

In this subsection, we show the "only if" part of Theorem 1. Let f be a rule on $\mathcal{R}^n$ satisfying *egalitarian-equivalence*, *strategy-proofness*, *individual rationality*, and *no subsidy for losers*. The proof is in five steps.

STEP 1. We show that for each i and each $R_{-i} \in \mathcal{R}^{n-1}$, $L_i^f(R_{-i}) \in \mathcal{L}_i$, i.e., $0 \in L_i^f(R_{-i})$.

Let $i \in N$ and $R_{-i} \in \mathcal{R}^{n-1}$. By contradiction, suppose $0 \notin L_i^f(R_{-i})$. Let $p \in \mathbb{R}_+^{|L|}$ be a price vector such that $p^{x_i} = \min_{j \in N \backslash \{i\}} \min_{x_j \in M} V_j(x_j, \mathbf{0})$ for each $x_i \in M$. By richness, there is a preference $R_i \in \mathcal{R}$ such that $D(R_i, p) = \{0\}$. Then, for each $x_i \in M$, $\mathbf{0}\ P_i\ (x_i, p^{x_i})$, which implies

$$V_i(x_i, \mathbf{0}) < p^{x_i} = \min_{j \in N \backslash \{i\}} \min_{x_j \in M} V_j(x_j, \mathbf{0}). \tag{A1}$$

Let $\hat{p} \in \mathbb{R}_+^{|L|}$ be a price vector such that $\hat{p}^{x_i} = V_i(x_i, \mathbf{0})$ for each $x_i \in M$. Again, by richness, there is a preference $R_i' \in \mathcal{R}$ such that $D(R_i', \hat{p}) = \{0\}$. Then, for each $x_i \in M$, $\mathbf{0}\ P_i'\ (x_i, \hat{p}^{x_i})$, which implies

$$V_i'(x_i, \mathbf{0}) < \hat{p}^{x_i} = V_i(x_i, \mathbf{0}). \tag{A2}$$

For each $x_i \in M$, by (A1) and (A2),

$$V_i'(x_i, \mathbf{0}) < \min_{j \in N \backslash \{i\}} \min_{x_j \in M} V_j(x_j, \mathbf{0}). \tag{A3}$$

By $0 \notin L_i^f(R_{-i})$, $x_i(R), x_i(R_i', R_{-i}) \neq 0$. Thus, by (A1) and (A3), Lemma A5 implies that $t_i(R) = V_i(x_i(R), \mathbf{0})$ and $t_i(R_i', R_{-i}) = V_i'(x_i(R_i', R_{-i}), \mathbf{0})$. Then,

$$f_i(R_i', R_{-i})\ P_i\ (x_i(R_i', R_{-i}), V_i(x_i(R_i', R_{-i}), \mathbf{0})\ I_i\ \mathbf{0}\ I_i\ (x_i(R), V_i(x_i(R), \mathbf{0})) = f_i(R),$$

where the first relation follows from (A2). However, this contradicts *strategy-proofness*.

STEP 2. We show that for each $i \in N$, each $R_{-i} \in \mathcal{R}^{n-1}$, and each $x_i \in M$, if $x_i \in L_i^f(R_{-i})$, then $V_j(x, \mathbf{0}) = V_k(x_i, \mathbf{0})$ for each pair $j, k \in N \backslash \{i\}$.

Let $i \in N$, $R_{-i} \in \mathcal{R}^{n-1}$, and $x_i \in M_i^f(R_{-i})$. Suppose by contradiction that for some distinct pair $j, k \in N \backslash \{i\}$, it holds that $V_j(x_i, \mathbf{0}) \neq V_k(x_i, \mathbf{0})$.

Let $p \in \mathbb{R}_+^{|L|}$ be a price vector such that $p^{x_i} = t_i^f(R_{-i}; x_i)$ and $p^{x_i'} = 0$ for each $x_i' \in L \backslash \{x_i\}$. Furthermore, let $\hat{p} \in \mathbb{R}_+^{|L|}$ be a price vector such that $\hat{p}^{x_i} = t_i^f(R_{-i}; x_i) + \varepsilon$, $\hat{p}^{x_i'} = t_i^f(R_{-i}; x_i')$ for each $x_i' \in M_i^f(R_{-i}) \backslash \{x_i\}$, and $\hat{p}^{x_i'} = \varepsilon$ for each $x_i' \in M \backslash M_i^f(R_{-i})$, where $\varepsilon > 0$ is a positive number. By Step 1 and $x_i \in M_i^f(R_{-i})$, Lemma A7 implies $p^{x_i} > 0$. Furthermore, for each $x_i' \in M_i^f(R_{-i}) \backslash \{x_i\}$, by Step 1, Lemma A7 implies $\hat{p}^{x_i'} > 0 = p^{x_i'}$. Thus, $\hat{p} > p$. By richness, there is $R_i \in \mathcal{R}$ such that $D(R_i, p) = \{x_i\}$ and $D(R_i, \hat{p}) = \{0\}$.

By $D(R_i, p) = \{x_i\}$,

$$z_i^f(R_{-i}; x_i) = (x_i, p^{x_i})\ P_i\ \mathbf{0} = z_i^f(R_{-i}; 0),$$

where the last equality follows from Step 1 and Lemma A2. Furthermore, for each $x_i' \in M_i^f(R_{-i}) \backslash \{x_i\}$, by $D(R_i, \hat{p}) = \{0\}$,

$$z_i^f(R_{-i}; 0) = \mathbf{0}\ P_i\ (x_i', \hat{p}^{x_i'}) = z_i^f(R_{-i}; x_i').$$

Thus, $z_i^f(R_{-i}; x_i)\ P_i\ z_i^f(R_{-i}; x_i')$ for each $x_i' \in L_i^f(R_{-i}) \backslash \{x_i\}$. Thus, by Lemma A6, $x_i(R) = x_i$. Note that by $x_i \in M$ and $t_i(R) = t_i^f(R_{-i}; x_i(R)) = p^{x_i} > 0$, $f_i(R) \in M \times \mathbb{R}_{++}$.

Now, we will show that for each $j \in N \backslash \{i\}$, $x_j(R) \neq 0$. Note that by $x_i(R) \neq 0$ and $n > m$, this will give a contradiction.

By contradiction, suppose $x_j(R) = 0$ for some $j \in N \backslash \{i\}$. By Lemma A2, $f_j(R) = \mathbf{0}$. Let $\overline{N} = \{k \in N \backslash \{i, j\} : V_k(x_i, \mathbf{0}) \neq V_j(x_i, \mathbf{0})\}$. By our assumption that $V_k(x_i, \mathbf{0}) \neq V_l(x_i, \mathbf{0})$ for some pair $k, l \in N \backslash \{i\}$, $\overline{N} \neq \varnothing$. The proof is in five substeps.

STEP 2-1. By *egalitarian-equivalence*, there is a reference bundle $z_0 = (x_0, t_0) \in L \times \mathbb{R}$ for R such that $f_k(R)\ I_k\ z_k$ for each $k \in N$. We show $z_0 = f_i(R)$. Let $x_i' \in L \backslash \{x_i\}$. By $D(R_i, p) = \{x_i\}$, $f_i(R) = (x_i, p^{x_i})\ P_i\ (x_i', p^{x_i'}) = (x_i', 0)$. This implies

$$V_i(x_i', f_i(R)) < 0.$$

Thus, $R_i \in \mathcal{R}^{NV}(f_i(R))$. By Step 1, Lemma A9 gives $z_0 = f_i(R)$.

STEP 2-2. We show that for each $k \in \overline{N}$, $V_k(x_i, \mathbf{0}) > V_j(x_i, \mathbf{0})$. By contradiction, suppose there is $k \in \overline{N}$ such that $V_k(x_i, \mathbf{0}) \leq V_j(x_i, \mathbf{0})$. By $k \in \overline{N}$, $V_k(x_i, \mathbf{0}) < V_j(x_i, \mathbf{0})$. By Step 2-1, $z_0 = f_i(R)$. Thus, by $x_i(R) = x_i$, $x_0 = x_i$. By $\mathbf{0} = f_j(R)\ I_j\ z_0 = (x_i, t_0)$, $t_0 = V_j(x_i, \mathbf{0})$. Thus, by $V_k(x_i, \mathbf{0}) < V_j(x_i, \mathbf{0})$, $V_k(x_i, \mathbf{0}) < t_0$. This implies $\mathbf{0}\ P_k\ (x_i, t_0)$. Thus, by $f_k(R)\ I_k\ z_0 = (x_i, t_0)$, $\mathbf{0}\ P_k\ f_k(R)$, which contradicts *individual rationality*.

STEP 2-3. We show that for each $R_i' \in \mathcal{R} \cap \mathcal{R}^{NV}(f_i(R))$ and each $k \in \overline{N}$, $x_k(R_i', R_{-i}) \neq 0$. Let $R_i' \in \mathcal{R} \cap \mathcal{R}^{NV}(f_i(R))$ and $k \in \overline{N}$. Suppose by contradiction that $x_k(R_i', R_{-i}) = 0$. By Lemma A2, $f_k(R_i', R_{-i}) = \mathbf{0}$. By Step 1, Lemma A8 implies $f_i(R_i', R_{-i}) = f_i(R)$. By Step 2-1, $f_i(R) = z_0$. Thus, $\mathbf{0} = f_j(R)\ I_j\ z_0 = f_i(R) = f_i(R_i', R_{-i})$. Thus, by $x_i(R_i', R_{-i}) = x_i(R) = x_i$,

$$t_i(R_i', R_{-i}) = V_j(x_i, \mathbf{0}). \tag{A4}$$

By *egalitarian-equivalence*, there is a reference bundle $z_0' = (x_0', t_0') \in L \times \mathbb{R}$ for (R_i', R_{-i}) such that $f_i(R_i', R_{-i})\ I_i'\ z_0'$ and $f_l(R_i', R_{-i})\ I_l\ z_0'$ for each $l \in N \backslash \{i\}$. By Step 1, Lemma A9 gives $z_0' = f_i(R_i', R_{-i})$. Thus, by (A4) and Step 2-2,

$$t_0' = t_i(R_i', R_{-i}) = V_j(x_i, \mathbf{0}) < V_k(x_i, \mathbf{0}),$$

which implies $(x_i, t_0')\ P_k\ \mathbf{0} = f_k(R_i', R_{-i})$. By $z_0' = f_i(R_i', R_{-i}) = f_i(R)$ and $x_i(R) = x_i$, $z_0' = (x_i, t_0')$. Thus, $z_0'\ P_k\ f_k(R_i', R_{-i})$. However, this contradicts $f_k(R_i', R_{-i})\ I_k\ z_0'$.

STEP 2-4. We show that for each $R_i' \in \mathcal{R} \cap \mathcal{R}^{NV}(f_i(R))$ and each $k \in \overline{N}$, we have $V_k(x_k(R_i', R_{-i}), f_i(R)) > 0$. Let $R_i' \in \mathcal{R} \cap \mathcal{R}^{NV}(f_i(R))$ and $k \in \overline{N}$. By Step 2-3, we have $x_k(R_i', R_{-i}) \neq 0$.

By *egalitarian-equivalence*, there is a reference bundle $z_0' \in L \times \mathbb{R}$ for (R_i', R_{-i}) such that $f_i(R_i', R_{-i})\ I_i'\ z_0$ and for each $l \in N \backslash \{i\}$, $f_l(R_i', R_{-i})\ I_l\ z_0'$. By Step 1, Lemma A9 implies $z_0' = f_i(R_i', R_{-i})$. By Step 1, Lemma A8 gives $f_i(R_i', R_{-i}) = f_i(R)$. Thus,

$$f_k(R_i', R_{-i})\ I_k\ z_0' = f_i(R_i', R_{-i}) = f_i(R),$$

which implies $t_k(R'_i, R_{-i}) = V_k(x_k(R'_i, R_{-i}), f_i(R))$. By Step 1 and $x_k(R'_i, R_{-i}) \neq 0$, Lemma A7 implies that

$$V_k(x_k(R'_i, R_{-i}), f_i(R)) = t_k(R'_i, R_{-i}) = t_k^f(R'_i, R_{-i,k}; x_k(R'_i, R_{-i})) > 0,$$

as desired.

STEP 2-5. Let $k \in \overline{N}$. Let $p_i \in \mathbb{R}_+^{|L|}$ be a price vector such that $p_i^{x_i} = t_i(R)$ and $p_i^{x'_i} = 0$ for each $x'_i \in L \setminus \{x_i\}$. By Step 1 and $x_i = x_i(R) \in M_i^f(R_{-i})$, Lemma A7 implies $p^{x_i} > 0$.
By Step 2-3,

$$M_k = \bigcup_{R'_i \in \mathcal{R} \cap \mathcal{R}^{NV}(f_i(R))} \{x_k(R'_i, R_{-i})\} \subseteq M.$$

By $f_i(R) \in M \times \mathbb{R}_{++}$ and Lemma A10, $\mathcal{R} \cap \mathcal{R}^{NV}(f_i(R)) \neq \varnothing$. Thus, M_k is well-defined, and $M_k \neq \varnothing$. Note that $x_i \notin M_k$.[9]

Let $\hat{p}_i \in \mathbb{R}_+^{|L|}$ be a price vector such that $\hat{p}_i^{x_i} = t_i(R) + \varepsilon'$, $\hat{p}_i^{x'_i} = V_k(x'_i, f_i(R))$ for each $x'_i \in M_k$, and $\hat{p}^{x'_i} = \varepsilon'$ for each $x'_i \in M \setminus (M_k \cup \{x_i\})$, where $\varepsilon' > 0$ is a positive number. Let $x_k \in M_k$. Then, there is $R'_i \in \mathcal{R} \cap \mathcal{R}^{NV}(f_i(R))$ such that $x_k = x_k(R'_i, R_{-i}) \in M$. By Step 2-4,

$$\hat{p}_i^{x_k} = V_k(x_k, f_i(R)) = V_k(x_k(R'_i, R_{-i}), f_i(R)) > 0 = p_i^{x_k}.$$

Thus, it holds that $\hat{p}_i > p_i$.

By richness, there is a preference $R'_i \in \mathcal{R}$ such that $D(R'_i, p_i) = \{x_i\}$ and $D(R'_i, \hat{p}_i) = \{0\}$. Then, by $D(R'_i, p_i) = \{x_i\}$, $R'_i \in \mathcal{R}^{NV}(f_i(R))$. Thus, by Step 1, Lemma A8 gives $f_i(R'_i, R_{-i}) = f_i(R)$.

By *egalitarian-equivalence*, there is a reference bundle $z'_0 = (x'_0, t'_0) \in L \times \mathbb{R}$ for (R'_i, R_{-i}) such that $f_i(R'_i, R_{-i})\ I'_i\ z'_0$ and $f_l(R'_i, R_{-i})\ I_l\ z'_0$ for each $l \in N \setminus \{i\}$. By Step 1, Lemma A9 implies $z'_0 = f_i(R'_i, R_{-i})$. Thus, by $f_i(R'_i, R_{-i}) = f_i(R)$, $z'_0 = f_i(R)$. Thus, $f_k(R'_i, R_{-i})\ I_k\ z'_0 = f_i(R)$, which implies

$$t_k(R'_i, R_{-i}) = V_k(x_k(R'_i, R_{-i}), f_i(R)). \tag{A5}$$

By Step 2-4, $x_k(R'_i, R_{-i}) \neq 0$. Thus, by Step 1, Lemma A7 gives

$$t_k(R'_i, R_{-i}) = t_k^f(R'_i, R_{-i,k}; x_k(R'_i, R_{-i})) > 0.$$

Thus, $f_k(R'_i, R_{-i}) \in M \times \mathbb{R}_{++}$. Thus, by Lemma A10, there is $R'_k \in \mathcal{R} \cap \mathcal{R}^{NV}(f_k(R'_i, R_{-i}))$. By Step 1, Lemma A8 implies $f_k(R'_i, R'_k, R_{-i,k}) = f_k(R'_i, R_{-i})$.

By *egalitarian-equivalence*, there is a reference bundle $z''_0 = (x''_0, t''_0) \in L \times \mathbb{R}$ for $(R'_i, R'_k, R_{-i,k})$ such that for each $l \in \{i, k\}$, $f_l(R'_i, R'_k, R_{-i,k})\ I'_l\ z''_0$, and for each $l \in N \setminus \{i, k\}$, $f_l(R'_i, R'_k, R_{-i})\ I_l\ z''_0$. By Step 1, Lemma A9 gives $z''_0 = f_k(R'_i, R'_k, R_{-i,k})$. By $f_k(R'_i, R'_k, R_{-ik}) = f_k(R'_i, R_{-i})$, $z''_0 = f_k(R'_i, R_{-i})$.

Let $x_k = x_k(R'_i, R_{-i})$. By $R'_i \in \mathcal{R} \cap \mathcal{R}^{NV}(f_i(R))$, $x_k \in M_k$. Thus, by $D(R'_i, \hat{p}_i) = \{0\}$,

$$\mathbf{0}\ P'_i\ (x_k, \hat{p}_i^{x_k}) = (x_k, V_k(x_k, f_i(R))),$$

which implies

$$V'_i(x_k, \mathbf{0}) < V_k(x_k, f_i(R)) = t_k(R'_i, R_{-i}),$$

where the equality follows from (A5). Thus,

$$\mathbf{0}\ P'_i\ (x_k, t_k(R'_i, R_{-i})) = f_k(R'_i, R_{-i}) = z''_0.$$

This, together with *individual rationality*, implies that

$$f_i(R'_i, R'_k, R_{-i,k})\ P'_i\ z''_0.$$

However, this contradicts $f_i(R'_i, R'_k, R_{-i,k})\ I'_i\ z''_0$.

Now, we complete the proof of Step 2. We have shown that for each $j \in N \setminus \{i\}$, $x_j(R) \neq 0$. Recall that $x_i(R) \neq 0$. Thus, for each $j \in N$, $x_j(R) \neq 0$. However, this is impossible because $n > m$.

STEP 3. We show that for each $i \in N$, each $R_{-i} \in \mathcal{R}^{n-1}$, and each $x_i \in L_i^f(R_{-i})$, $t_i^f(R_{-i}; x_i) = \max_{j \in N \setminus \{i\}} V_j(x_i, \mathbf{0})$. Let $i \in N$, $R_{-i} \in \mathcal{R}^{n-1}$, and $x_i \in L_i^f(R_{-i})$. If $x_i = 0$, then by Lemma A2, $t_i^f(R_{-i}; x_i) = 0$. Thus, assume $x_i \neq 0$.

By Step 1 and $x_i \in M_i^f(R_{-i})$, Lemma A7 gives $t_i^f(R_{-i}; x_i) > 0$. Let $p \in \mathbb{R}_+^{|L|}$ be a price vector such that $p^{x_i} = t_i^f(R_{-i}; x_i)$, and $p^{x_i'} = 0$ for each $x_i' \in L \setminus \{x_i\}$. By richness, there is $R_i \in \mathcal{R}$ such that $D(R_i, p) = \{x_i\}$. Then, for each $x_i' \in L_i^f(R_{-i}) \setminus \{x_i\}$,

$$z_i^f(R_{-i}; x_i) = (x_i, p^{x_i}) \ P_i \ (x_i', p^{x_i'}) = (x_i', 0) \ R_i \ (x_i', t_i^f(R_{-i}; x_i')) = z_i^f(R_{-i}; x_i'),$$

where the first relation follows from $D(R_i, p) = \{x_i\}$, and the second one from Step 1 and Corollary A1. Thus, by Lemma A6, $f_i(R) = z_i^f(R_{-i}; x_i)$. By $x_i \neq 0$ and $t_i^f(R_{-i}; x_i) > 0$, $f_i(R) \in M \times \mathbb{R}_{++}$.

By *egalitarian-equivalence*, there is a reference bundle $z_0 \in L \times \mathbb{R}$ for R such that $f_j(R) \ I_j \ z_0$ for each $j \in N$. For each $x_i' \in L \setminus \{x_i\}$, by $D(R_i, p) = \{x_i\}$,

$$f_i(R) = (x_i, p^{x_i}) \ P_i \ (x_i', p^{x_i'}) = (x_i', 0),$$

which implies

$$V_i(x_i', f_i(R)) < 0.$$

Thus, $R_i \in \mathcal{R}^{NV}(f_i(R))$. Thus, by $f_i(R) \in M \times \mathbb{R}_{++}$ and Step 1, Lemma A9 implies $z_0 = f_i(R)$.

By $n > m$, there is $j \in N \setminus \{i\}$ such that $x_j(R) = 0$. By Lemma A2, $f_j(R) = \mathbf{0}$. By $\mathbf{0} = f_j(R) \ I_j \ z_0 = f_i(R)$, $t_i(R) = V_j(x_i(R), \mathbf{0})$. Thus,

$$t_i(R_{-i}; x_i) = t_i(R) = V_j(x_i(R), \mathbf{0}) = V_j(x_i, \mathbf{0}) = \max_{k \in N \setminus \{i\}} V_k(x_i, \mathbf{0}),$$

where the last equality follows from Step 2.

STEP 4. We show that f is an independent second-prices rule with variable constraints. Let $R \in \mathcal{R}^n$ and $i \in N$. Let $p_i \in \mathbb{R}_+^{|L|}$ be a price vector such that for each $x_i \in M$, $p_i^{x_i} = \max_{j \in N \setminus \{j\}} V_j(x_i, \mathbf{0})$. By Step 3, for each $x_i \in L_i^f(R_{-i})$, $p_i^{x_i} = t_i^f(R_{-i}; x_i)$. By Lemma A6, $x_i(R) \in D(R_i, p_i, L_i^f(R_{-i}))$. Furthermore, $t_i(R) = t_i^f(R_{-i}; x_i(R)) = p_i^{x_i(R)}$. Thus, by Lemma A1 and Step 1, f is an independent second-prices rule with variable constraints associated with $(L_i^f(\cdot))_{i \in N}$.

STEP 5. Now, we complete the proof of Theorem 1. By Step 4, f is an independent second-prices rule with variable constraints associated with $(L_i^f(\cdot))_{i \in N}$. Thus, we finally show that it respects the valuation coincidence.

Let $R \in \mathcal{R}^n$. We show that the two conditions of respecting the valuation coincidence holds. Let $W = W(R, (L_i^f(R_{-i}))_{i \in N})$.

Suppose $|W| = 1$. Then, there is $i \in W$. In addition, there is $x_i \in L_i^f(R_{-i})$ such that $V_i(x_i, \mathbf{0}) > \max_{j \in N \setminus \{i\}} V_j(x_i, \mathbf{0})$. This implies $(x_i, \max_{j \in N \setminus \{i\}} V_j(x_i, \mathbf{0})) \ P_i \ \mathbf{0}$. By Step 4 and Lemma A1,

$$f_i(R) \ R_i \ (x_i, \max_{j \in N \setminus \{i\}} V_j(x_i, \mathbf{0})) \ P_i \ \mathbf{0}.$$

Thus, by Lemma A2, $x_i(R) \neq 0$. By $x_i(R) \in L_i^f(R_{-i})$, $x_i(R) \in M_i^f(R_{-i})$. By Step 2, for each pair $j, k \in N \setminus \{i\}$,

$$V_j(x_i(R), \mathbf{0}) = V_k(x_i(R), \mathbf{0}),$$

as desired.

Next, suppose $|W| \geq 2$. By *egalitarian-equivalence*, there is a reference bundle $z_0 = (x_0, t_0) \in L \times \mathbb{R}$ for R such that $f_i(R)\ I_i\ z_0$ for each $i \in N$. We show that $x_0 \in M$. By $|W| \geq 2$, there is $i \in W$. By the same argument as above, $f_i(R)\ P_i\ \mathbf{0}$. Thus, by $f_i(R)\ I_i\ z_0$, $z_0\ P_i\ \mathbf{0}$. In addition, $z_0 \neq \mathbf{0}$. By Lemma A3, this implies $x_0 \neq 0$. For each $j \in N$, $f_j(R)\ I_j\ z_0 = (x_0, t_0)$ implies $t_0 = V_j(x_0, f_j(R))$. Thus, for each pair $j, k \in N$,

$$V_j(x_0, f_j(R)) = t_0 = V_k(x_0, f_k(R)),$$

as desired.

Appendix C. Proof of Propositions

In this section, we provide the proofs of Propositions 5–7.

Appendix C.1. Proof of Proposition 5

Suppose by contradiction that there is a rule f on $\mathcal{R}^n$ satisfying *egalitarian-equivalence*, *minimal no wastage*, *strategy-proofness*, *individual rationality*, and *no subsidy for losers*. Note that by Step 1 of the proof of the "only if" part of Theorem 1, for each $i \in N$ and each $R_{-i} \in \mathcal{R}^{n-1}$, $0 \in L_i^f(R_{-i})$. Note also that by Step 2 of the proof of the "only of" part of Theorem 1, for each $i \in N$, each $R_{-i} \in \mathcal{R}^{n-1}$, and each $x_i \in M$, if $x_i \in L_i^f(R_{-i})$, then for each pair $j, k \in N \backslash \{i\}$, $V_j(x_i, \mathbf{0}) = V_k(x_i, \mathbf{0})$.

Fix a preference $R_1 \in \mathcal{R}$ of agent 1. Let $p_2 \in \mathbb{R}_+^{|L|}$ be a price vector such that $p_2^{x_2} = V_1(x_2, \mathbf{0})$ for each $x_2 \in M$. By richness, we can pick a preference $R_2 \in \mathcal{R}$ of agent 2 such that $D(R_2, p_2) = \{0\}$. For each $x_2 \in M$, $D(R_2, p_2) = \{0\}$ implies $\mathbf{0}\ P_2\ (x_2, p_2^{x_2})$. Thus, for each $x_2 \in M$,

$$V_2(x_2, \mathbf{0}) < p_2^{x_2} = V_1(x_2, \mathbf{0}).$$

Next, we construct a preference of agent 3. Let $p_3 \in \mathbb{R}_+^{|L|}$ be a price vector such that $p_3^{x_3} = V_2(x_3, \mathbf{0})$ for each $x_3 \in M$. By richness, there is a preference $R_3 \in \mathcal{R}$ of agent 3 such that $D(R_3, p_3) = \{0\}$. Thus, for each $x_3 \in M$, $\mathbf{0}\ P_3\ (x_3, p_3^{x_3})$, which implies

$$V_3(x_3, \mathbf{0}) < p_3^{x_3} = V_2(x_3, \mathbf{0}).$$

By repeating the same arguments inductively, we can construct a preference profile $R \in \mathcal{R}^n$ such that for each $i \in N \backslash \{1\}$ and each $x_i \in M$,

$$V_i(x_i, \mathbf{0}) < p_i^{x_i} = V_{i-1}(x_i, \mathbf{0}).$$

Thus, for each $a \in M$ and each distinct pair $i, j \in N$,

$$V_i(a, \mathbf{0}) \neq V_j(a, \mathbf{0}). \quad \text{(A1)}$$

By Step 2 of the proof of the "only if" of Theorem 1, (A1) implies that for each $i \in N$ and each $x_i \in M$, $x_i \notin L_i^f(R_{-i})$. Thus, for each $i \in N$, by $0 \in L_i^f(R_{-i})$, $L_i^f(R_{-i}) = \{0\}$. Thus, for each $i \in N$, $x_i(R) = 0$. However, this contradicts *minimal no wastage*.

Appendix C.2. Proof of Proposition 6

Let f be an independent second-prices rule with variable constraints on $\mathcal{R}^n$. Let $R \in \mathcal{R}^n$. The next claim is immediate from the definition of the rule. Thus, we omit the proof

Claim A2. *Let $i \in N$. (i) $t_i(R) \geq 0$. (ii) If $x_i(R) = 0$, then $t_i(R) = 0$.*

Then, we show the following claim.

Claim A3. *For each pair* $i, j \in N$, $V_i(x_j(R), f_i(R)) \leq t_j(R)$

Proof. Let $i, j \in N$ be a pair. If $i = j$, then the conclusion is trivial. Thus, suppose $i \neq j$.

CASE 1. $x_j(R) = 0$.

By Claim A2 (ii), $t_j(R) = 0$. By Theorem 1, f satisfies *individual rationality*. Thus, $f_i(R)\ R_i\ \mathbf{0}$, which implies $V_i(0, f_i(R)) \leq 0$.

CASE 2. $x_j(R) \neq 0$.

Then, $f_j(R) = (x_j(R), \max_{k \in N \setminus \{j\}} V_k(x_j(R), \mathbf{0}))$. Thus,

$$t_j(R) = \max_{k \in N \setminus \{j\}} V_k(x_j(R), \mathbf{0}) \geq V_i(x_j(R), \mathbf{0}). \tag{A1}$$

By *individual rationality*,

$$(x_j(R), V_i(x_j(R), f_i(R)))\ I_i\ f_i(R)\ R_i\ \mathbf{0}\ I_i\ (x_j(R), V_i(x_j(R), \mathbf{0})),$$

which implies

$$V_i(x_j(R), \mathbf{0}) \geq V_i(x_j(R), f_i(R)). \tag{A2}$$

(A1) and (A2) together imply $t_j(R) \geq V_i(x_j(R), f_i(R))$. □

Let $x \in X$ be such that $x_i \in L^f(R)$ for each $i \in N$. Then, for each $i \in N$, there is $\iota(i) \in N$ such that $x_i = x_{\iota(i)}(R)$. Let $N_1 = \{i \in N : x_i(R) \neq 0\}$ and $N_2 = \{\iota(i) \in N : x_{\iota(i)}(R) \neq 0\} = \{i \in N : \exists j \in N \text{ s.t. } i = \iota(j) \text{ and } x_i(R) \neq 0\}$.

The proof of the following claim is trivial, and so we omit it.

Claim A4. $N_2 \subseteq N_1$.

Then,

$$\sum_{i \in N} V_i(x_i, f_i(R)) \leq \sum_{i \in N} t_{\iota(i)}(R) = \sum_{i \in N_2} t_i(R) \leq \sum_{i \in N_1} t_i(R) = \sum_{i \in N} t_i(R),$$

where the first inequality follows from Claim A3, the first equality follows from Claim A2 (ii), the second inequality follows from Claims A2 (i) and A4, and the second equality follows from Claim A2 (ii). Thus, by Remark 1, f is *constrained efficient*.

Appendix C.3. Proof of Proposition 7

Let f be an independent second-prices rule with variable constraints on $\mathcal{R}^n$. Let $R \in \mathcal{R}^n$ and $i, j \in N$ be a pair. Then,

$$f_i(R)\ I_i\ (x_j(R), V_i(x_j(R), f_i(R)))\ R_i\ (x_j(R), t_j(R)) = f_j(R),$$

where the second relation follows from Claim A3 in the proof of Proposition 6.

Notes

1 The generalized Vickrey rule is the only rule satisfying the four properties in the identical objects model [4,5].

2 For the comprehensive survey on fair allocation theory, see [8]

3 A rule $f = (x, t)$ on $\mathcal{R}^n$ is a *generalized Vickrey rule* if it holds that for each $R \in \mathcal{R}^n$, $x(R) \in \arg\max_{x \in X} \sum_{i \in N} V_i(x_i, \mathbf{0})$, and for each $i \in N$, $t_i(R) = \max_{x \in X} \sum_{j \in N \setminus \{i\}} V_j(x_j, \mathbf{0}) - \sum_{j \in N \setminus \{i\}} V_j(x_j(R), \mathbf{0})$.

4 By $f_1(R)\ P_1\ f_2(R)$ and $f_2(R)\ P_2\ f_1(R)$, in this case it must hold that $d = c$.

5 Note that all Propositions 1–3 follow from Theorem 1. The purpose of the three propositions in Section 4.2 was to clarify the motivation of the respecting the valuation coincidence condition.

6 For the formal definition of the minimum price Warlasian see, for example, Morimoto and Serizawa [6]

[7] In the companion paper [20], motivated by the observation that real-life bidders usually have neither the full access to the outcomes of auctions nor full confidence that the published data are correct, we propose a new property of fairness that we call *obvious envy-freeness*. It extends *envy-freeness* to the agents who has only partial access to or partial confidence in the other agents' outcome bundles. In Shinozaki [20], we establish that the independent second-prices rule with variable *reserve prices* is the only rule satisfying *obvious envy-freeness*, *strategy-proofness*, *individual rationality*, and *no subsidy for losers*.

[8] We are grateful to an anonymous referee for suggesting such interesting directions of future research.

[9] Indeed, suppose by contradiction that $x_i \in M_k$. Then, there is $R'_i \in \mathcal{R} \cap \mathcal{R}^{NV}(f_i(R))$ such that $x_k(R'_i, R_{-i}) = x_i$. By Step 1, Lemma A8 implies $x_i(R'_i, R_{-i}) = x_i(R) = x_i$. However, this contradicts $x_k(R'_i, R_{-i}) = x_i$ since $k \neq i$ and $x_i \neq 0$.

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