games

Communication

A Note on Numerical Representations of Nested System of Strict Partial Orders

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Abstract: This note provides two numerical representations of a nested system of strict partial orders. The first representation is based on utility and threshold functions. We generalize the threshold representation of menu-dependent preferences by allowing the threshold to depend not only on the menu but also on the pair of alternatives under comparison. The threshold function can be interpreted as the distance between alternatives. The second representation is based on the aggregation of multi-dimensional preference. This representation describes a decision-making procedure where multiple criteria are gradually aggregated into an overall assessment.

Keywords: bounded rationality; satisficing behavior; menu-dependent preference; numerical representation

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1. Introduction

In classical economic theory, preference is represented by utility functions. Refs. [1,2] provide summaries on the finite alternative space and the infinite alternative space respectively. For a preference relation to admit a utility representation, it must be a weak order, i.e., complete and transitive. In many contexts, it is necessary to consider preferences that fails both completeness and transitivity. Refs. [3,4] provide utility representations for incomplete preferences. Ref. [5] provides utility representations for incomplete and nontransitive preferences. All of these representations assume that preference is menu independent. However, there is growing empirical evidence suggesting that the DM's preference is influenced by the menu, meaning that menu-independent preference cannot provide an accurate description. Ref. [6] provides a survey on the empirical evidence. One route to incorporate the empirical findings is Herbert Simon's idea of "satisficing" alternative to global maximization, which is summarized in [7]. Refs. [8,9] model the "satisficing" behavior with a nested system of menu-dependent preferences. Roughly speaking, a system of menu-dependent preferences is nested if the DM's perception of preference is more accurate in smaller menus. Nestedness captures the central idea of "satisficing": the DM is bounded by cognitive constraints and environmental complexity so that they can perceive their preference more accurately in less complex situations (small menus).

Theorem 7 of [8] takes choice as primitive and provides conditions for the choice correspondence to be represented by a nested system of strict partial orders. The objective of this note is to provide two numerical representations of a nested system of strict partial orders. In other words, our aim is to provide more concrete decision-making procedures that can generate a nested system of strict partial orders. Ref. [9] provides a numerical representation of the nested system of semiorders, which are stronger than strict partial orders. Strict partial orders require the transitivity of strict preference and decrease the transitivity of indifference. The latter is controversial, as a series of locally indiscernible alternatives may have discernible extremes. Refs. [10,11] provide further discussion on this issue. Ref. [12] describes semiorders as a relaxation of weak order by the intransitivity of indifferences originating from the inability to discern the difference between alternatives.

We further relax semiorders to strict partial orders for two reasons. First, the numerical representations offers both support and building blocks for the theory and application based on the nested system of strict partial orders. For example, theorem 7 of [8] provides a choice model with a nested system of strict partial orders. As an instrumental work, generality implies wider applicability. In the second place, we do not attempt to further generalize to acyclic preferences because there is experimental evidence supporting the transitivity of strict preference. In a recent study, [13] tested how WARP is violated by classifying its violation in two categories: pairwise inconsistency and the menu effect. They authors observed that the menu effect is by far the major source of the violation of WARP. That is, their study offers support for models postulating pairwise consistency, i.e., transitivity of strict preference.

The first representation is a generalization of [9]. Recall that menu-independent weak order can be encoded in a utility function in a sense that

$$x \succ y \Leftrightarrow u(x) - u(y) > 0$$

Adding a threshold function to the utility function can represent a nested system of menu-dependent preferences.

$$x \succ_S y \Leftrightarrow u(x) - u(y) > \theta(S)$$

The authors of [9] give representations of the case of weak order and semiorder with monotonic θ.

Our representation of the case of strict partial order generalizes the threshold function θ by allowing it to depend not only on the menu but also on the alternatives under comparison:

$$x \succ_S y \Leftrightarrow u(x) - u(y) > \theta(x, y, S)$$

We require θ to be monotone with respect to the menu. This assumption is in line with the idea of "satisficing", because the DM can better compare the alternatives when there are less of them due to the limit of total cognitive resources. We also require θ to be symmetric and satisfy triangular inequality for the pair of alternatives under comparison. Under such assumptions, θ can be interpreted as a distance function. The level of difficulty in comparing x and y may depend on the similarity in utility (in terms of u) and the differences in terms of which categories x and y belong to. For example, with similar utility levels, it is easier to compare chocolate bars with candies than chocolate bars with pens. Differences in categories can then be modeled by a distance function.

The second representation is based on the aggregation of multi-dimensional preference. Consider a DM who cares about multiple properties delivered by alternatives. Properties can be represented by real value functions $u_1, \ldots, u_n$. For example, when choosing a laptop, the DM may care about CPU performance, GPU performance, portability, and screen size. In other words, the DM possesses multiple criteria to assess the laptop. Facing large menus, aggregating all criteria into an overall assessment requires excessive cognitive resources. Therefore, the DM may only try to aggregate related properties, e.g., combining CPU and GPU performance into the overall performance. As the number of alternatives decreases, the DM may have enough resources for further aggregation. Consequently, smaller menus are associated with fewer dimensions. The DM then considers x to be considerably better than y in S if x is better in all dimensions. Our second representation result shows that this gradual aggregation procedure corresponds to a nested system of strict partial orders. To our knowledge, the gradual aggregation procedure is novel in the literature. In a different framework, refs. [14,15] provide a procedure where the choice is conducted in two stages and additive aggregation rule is applied to generate a complete preference in the second stage. However, the aggregation is not a gradual process.

The remainder of this note is organized as follows: Section 2 introduces the preliminaries, and Section 3 provides the representation results.

2. Preliminaries

Let X be a finite set of alternatives. A binary relation $\succ$ on $S \subseteq X$ is a subset of $S \times S$. When we define a binary relation without referring to S, we mean a binary relation on X. We write $(x,y) \in \succ$ and $x \succ y$ interchangeably throughout this note.

A binary relation $\succ$ is asymmetric if $x \succ y \Leftrightarrow y \not\succ x$. A binary relation $\succ$ is transitive if $x \succ y \succ z \Rightarrow x \succ z$. It is negatively transitive if $x \not\succ y \not\succ z \Rightarrow x \not\succ z$. A binary relation $\succ$ is connected if for all $x \neq y$, either $x \succ y$ or $y \succ x$ holds. A binary relation is semi-transitive if $x \succ y \succ z$, then either $x \succ w$ or $w \succ z$ for any w. It satisfies the interval order property if $x \succ z$ and $y \succ w$ imply either $x \succ w$ or $y \succ z$.

An asymmetric and transitive binary relation is called a strict partial order. A negatively transitive strict partial order is called a weak order. A connected weak order is called a linear order. An asymmetric and semi-transitive binary relation satisfying the interval order property is called a semiorder.

A menu-dependent preference $\succ_S$ is a collection of preferences where $\succ_S$ is a binary relation on S. Given $\succ_S$, we define its base relation as follows:

$$\succ_b = \{(x,y) : x \succ_{xy} y\}$$

We deem a menu-dependent preference to be nested if $\succ_S \cap S' \times S' \subseteq \succ_{S'}$ for all $S' \subseteq S$. We are interested in the numerical representations of menu-dependent strict partial orders when their base relation is acyclic. Acyclicity of $\succ_b$ is equivalent to the existence of a linear-order extension $\succ$ of $\succ_b$. Because of nestedness,

$$\bigcup_{S \subseteq X} \succ_S \subseteq \succ$$

Therefore, it is equivalent to discuss numerical representations of the pair $<\succ_S, \succ>$ where $\succ_S$ is a nested menu-dependent strict partial orders and $\succ$ is a linear order such that $\succ_S \subseteq \succ$ for all $S \subseteq X$. We call such pair a nested system of strict partial orders.

3. Results

3.1. Threshold Representation

Ref. [9] introduces the following representation for the nested system of semiorders $<\succ_S, \succ>$:

$$x \succ_S y \Leftrightarrow u(x) - u(y) > \theta(S)$$

where $\theta(S) \leq \theta(S')$ for all $S' \subseteq S$.

We show that the nested system of strict partial order can be represented by a utility function and a more general threshold function that depends on both the menu and the alternatives under comparison. Let $\Phi = \{(x,y,S) \in X \times X \times Pow(X) : x,y \in S\}$. We deem $<\succ_S, \succ>$ to admit a general threshold representation if there exist a utility function $u : X \to \mathcal{R}$ and a threshold function $\theta : \Phi \to \mathcal{R}^+$ such that

$$S \subseteq S' \Rightarrow \theta(x,y,S) \leq \theta(x,y,S') \tag{1}$$

$$\theta(x,y,S) = \theta(y,x,S) \tag{2}$$

$$\theta(x,y,S) + \theta(y,z,S) \geq \theta(x,z,S) \tag{3}$$

and

$$For\ x,y \in S,\ x \succ_S y \Leftrightarrow u(x) - u(y) > \theta(x,y,S) \tag{4}$$

$$x \succ y \Leftrightarrow u(x) - u(y) > 0 \tag{5}$$

Equation (1) is a monotone requirement in the same spirit of [9]. Equations (2) and (3) are symmetry and triangular inequality properties. These properties are compelling if the threshold depends on the distance between two alternatives, i.e., $\theta(x,y,S) = \theta(\mu * d(x,y), S)$, where d is the distance function and μ is some positive real number.

Proposition 1. *Given a preference system* $<\succ_S, \succ>$*, the followings are equivalent.*

1. $<\succ_S, \succ>$ *is a nested system of strict partial order.*
2. $<\succ_S, \succ>$ *admits a general threshold representation.*

Proof. $2 \Rightarrow 1$: Suppose $<\succ_S, \succ>$ has a general threshold representation with u and θ. We need to establish nestedness, asymmetry, and transitivity.

Suppose $x, y \in S' \subseteq S$ and $x \succ_S y$. By (4), $u(x) - u(y) > \theta(x, y, S)$. By (1), $u(x) - u(y) > \theta(x, y, S')$. This establishes nestedness.

Suppose $x \succ_S y$. By (4), $u(x) - u(y) > \theta(x, y, S)$. By (2), $\theta(x, y, S) = \theta(y, x, S)$. Therefore, $u(x) - u(y) > \theta(y, x, S)$. By (4), $y \not\succ_S x$. This establishes asymmetry.

Suppose $x \succ_S y \succ_S z$. By (4), $u(x) - u(y) > \theta(x, y, S)$ and $u(y) - u(z) > \theta(y, z, S)$. Combining these inequality gives $u(x) - u(z) > \theta(x, y, S) + \theta(y, z, S)$. By (3), $\theta(x, y, S) + \theta(y, z, S) \geq \theta(x, z, S)$. Therefore, $u(x) - u(z) > \theta(x, z, S)$. By (4), $x \succ_S z$. This establishes transitivity.

$1 \Rightarrow 2$: Suppose $<\succ_S, \succ>$ is a nested system of strict partial order. We construct the representation.

Suppose $|X| = m$ and enumerate $X = \{x_1, \ldots, x_m\}$ such that $x_i \succ x_j$ for all $i > j$. This is possible because $\succ$ is a linear order. Construct a utility function u to represent $\succ$ by $u(x_i) = i$. That is, the utility difference equals to 1 for adjacent alternatives. Let

$$\delta = \frac{1}{4m}$$

For all $i > j$, let

$$\delta_{ij}^+ = (m - (i - j)) * \delta$$

and

$$\delta_{ij}^- = (i - j) * \delta$$

Note that $\delta_{ij}^+, \delta_{ij}^- < 1/4$ for all i, j. That is, $\delta_{ij}^+, \delta_{ij}^-$ is always smaller than the minimal utility gap of distinct alternatives (which is 1).

For $x_i, x_j \in S$ and $i > j$, construct the threshold function as follows:

$$\theta(x_i, x_j, S) = \theta(x_j, x_i, S) = \begin{cases} (i - j) - \delta_{ij}^-, & x_i \succ_S x_j \\ (i - j) + \delta_{ij}^+, & \text{otherwise.} \end{cases}$$

Note that θ is always non-negative by construction.

For $x_i, x_j \in S$, suppose $x_i \succ_S x_j$. Because $\succ_S \subseteq \succ$, we must have $i > j$. By construction,

$$u(x_i) - u(x_j) = i - j > (i - j) - \delta_{ij}^- = \theta(x_i, x_j, S)$$

Reversely, suppose $u(x_i) - u(x_j) > \theta(x_i, x_j, S)$. Note that we must have $i > j$; otherwise, we will have a negative number bigger than a non-negative number. By construction, we must have $\theta(x_i, x_j, S) = (i - j) - \delta_{ij}^-$ and in turn $x_i \succ_S x_j$.

We have shown that u and θ can represent $<\succ_S, \succ>$. By construction, θ is symmetric; i.e., (2) is satisfied.

Suppose $x_i, x_j \in S \subseteq S'$ and assume $i > j$ without loss of generality. If $\theta(x_i, x_j, S') = (i - j) + \delta_{ij}^+$, then $\theta(x_i, x_j, S') \geq \theta(x_i, x_j, S)$ by construction. If $\theta(x_i, x_j, S') = (i - j) - \delta_{ij}^-$, then $x_i \succ_{S'} x_j$. By nestedness, $x_i \succ_S x_j$. By construction, $\theta(x_i, x_j, S) = (i - j) - \delta_{ij}^- = \theta(x_i . x_j, S')$. This establishes (1).

Now, triangular inequality must be established, i.e., (3). This is equivalent to showing that given $x_i, x_j, x_k \in S$ such that $i > j > k$, the following inequalities hold:

$$\theta(x_i, x_j, S) + \theta(x_j, x_k, S) \geq \theta(x_i, x_k, S) \tag{6}$$

and

$$\theta(x_i, x_k, S) \geq |\theta(x_i, x_j, S) - \theta(x_j, x_k, S)| \tag{7}$$

This is because (7) implies that

$$\theta(x_i, x_k, S) + \theta(x_j, x_k, S) \geq \theta(x_i, x_j, S)$$

and

$$\theta(x_i, x_k, S) + \theta(x_i, x_j, S) \geq \theta(x_j, x_k, S)$$

We verify (6) first.

Case 1: $x_i \not\succ_S x_k$. By transitivity, either $x_i \not\succ_S x_j$ or $x_j \not\succ_S x_k$. If $x_i \not\succ_S x_j$, then

$$\begin{aligned}\theta(x_i, x_j, S) + \theta(x_j, x_k, S) - \theta(x_i, x_k, S) &\geq \delta^+_{ij} - \delta^-_{jk} - \delta^+_{ik} \\ &= ((m - (i-j)) - (j-k) - (m - (i-k))) * \delta \\ &= 0\end{aligned}$$

If $x_j \not\succ_S x_k$, then

$$\begin{aligned}\theta(x_i, x_j, S) + \theta(x_j, x_k, S) - \theta(x_i, x_k, S) &\geq -\delta^-_{ij} + \delta^+_{jk} - \delta^+_{ik} \\ &= (-(i-j) + (m - (j-k)) - (m - (i-k))) * \delta \\ &= 0\end{aligned}$$

Case 2: $x_i \succ_S x_k$.

$$\begin{aligned}\theta(x_i, x_j, S) + \theta(x_j, x_k, S) - \theta(x_i, x_k, S) &\geq -\delta^-_{ij} - \delta^-_{jk} + \delta^-_{ik} \\ &= (-(i-j) - (j-k) + (i-k)) * \delta \\ &= 0\end{aligned}$$

Next, we verify (7). Recall that $\delta^+_{ij}, \delta^-_{ij} < 1/4 < 1/2 * |u(x_k) - u(x_s)|$ for all i, j, k, s.
Case 1: $\theta(x_i, x_j, S) \geq \theta(x_j, x_k, S)$.

$$\begin{aligned}|\theta(x_i, x_j, S) - \theta(x_j, x_k, S)| &= \theta(x_i, x_j, S) - \theta(x_j, x_k, S) \\ &\leq (i-j) - (j-k) + \delta^+_{ij} + \delta^-_{jk} \\ &< (i-j) + \delta^+_{ij} + \delta^-_{jk} \\ &< (i-j) + 1/2 \\ &\leq (i-k) - 1/2 \\ &< (i-k) - \delta^-_{ik} \\ &\leq \theta(x_i, x_k, S)\end{aligned}$$

Case 2: $\theta(x_i, x_j, S) < \theta(x_j, x_k, S)$.

$$\begin{aligned}|\theta(x_i, x_j, S) - \theta(x_j, x_k, S)| &= \theta(x_j, x_k, S) - \theta(x_i, x_j, S) \\ &\leq (j-k) - (i-j) + \delta^+_{jkj} + \delta^-_{jk} \\ &< (i-j) + \delta^+_{ij} + \delta^-_{ij} \\ &< (j-k) + 1/2 \\ &\leq (i-k) - 1/2 \\ &< (i-k) - \delta^-_{ik} \\ &\leq \theta(x_i, x_k, S)\end{aligned}$$

This concludes the proof. □

This result can be used to give numerical representations of choice procedures involved in nested systems of strict partial orders. For example, combining Proposition 1 and Theorem 7 of [8] leads to the following corollary.

We use $Pow(X)$ to denote all nonempty subsets of X. A choice correspondence c on X is a function $c : Pow(X) \to Pow(X)$ such that $c(S) \subseteq S$ for all $S \subseteq X$. We define the base relation $\succ^b$ associated with c by

$$\succ_b^c = \{(x, y) \in X \times X : c(\{x, y\}) = \{x\}\}$$

A choice correspondence satisfies Weak Expansion if

$$c(S_1) \cap c(S_2) \subseteq c(S_1 \cup S_2)$$

A choice correspondence satisfies Aizerman Expansion if for all $S' \subset S$

$$c(S) \subseteq S' \Rightarrow c(S') \subseteq c(S)$$

Corollary 1. *Given a choice correspondence c, the followings are equivalent.*

1. *c satisfies Weak Expansion AND Aizerman Expansion, and its associated base relation is acyclic.*
2. *There exist a utility function $u : X \to \mathcal{R}$ and a threshold function $\theta : \Phi \to \mathcal{R}^+$ satisfying (1), (2), and (3) such that*

$$c(S) = \{y \in S : \nexists x \in S, \ s.t. \ u(x) - u(y) > \theta(x, y, S)\}$$

Let us turn to the identification of θ. It is observable that we can reveal that $\theta(x, y, S) > 0$ by

$$x \succ_{xy} y$$

and

$$x \not\succ_S y$$

For the choice procedure of Corollary 1, we can reveal $\theta(x, y, S) > 0$ by

$$c(\{x, y\}) = \{x\}$$

and

$$y \in c(S)$$

Therefore, we can have $\theta(x, y, S) > 0$ and $\theta(w, z, S) = 0$. Only if θ depends on S can we reveal that $\theta(S) > 0$ whenever we can reveal $\theta(x, y, S) > 0$ for some $x, y \in S$.

We impose no requirement on the functional form of $\theta(x, y, S)$. A natural question is whether it is possible to decompose θ additively, that is,

$$\theta(x, y, S) = \theta(x, y) + \theta(S)$$

We illustrate by the following example that the decomposition is not always possible.

Example 1. *Let $X = \{x, y, z, w, u\}$ and consider the following preferences*

$$\succ_{xyz} = \{(x, y), (x, z)\}$$

$$\succ_{xyzw} = \{(x, z)\}$$

$$\succ_{xyzu} = \{(x, y)\}$$

These preferences can be revealed from the following choice correspondence satisfying the conditions of Corollary 1.

$$c(\{x, y\}) = c(\{x, z\}) = c(\{x, y, z\}) = \{x\}$$

$$c(\{x, y, z, w\}) = \{x, y\}$$
$$c(\{x, y, z, u\}) = \{x, z\}$$

and for all other S,
$$c(S) = S$$

If the threshold function is additively decomposable, then we must have

$$u(x) - u(y) - \theta(x, y) \leq \theta(\{x, y, z, w\}) < u(x) - u(z) - \theta(x, z)$$

and
$$u(x) - u(z) - \theta(x, z) \leq \theta(\{x, y, z, u\}) < u(x) - u(y) - \theta(x, y)$$

which is a contradiction.

3.2. Multi-Dimensional Utility and Aggregation

The second representation is based on aggregation of multi-dimensional preference. Let U denote all real-value injective functions on the domain of X. Given a set A, a partition PA_A of A is a collection of disjoint subsets of A such that $\bigcup_{S \in PA_A} S = A$. We deem a preference system $<\succ_S, \succ>$ to admit a multi-utility aggregation representation if there exist (i) a set of real-value injections $\{u_1, \ldots, u_n\} = \hat{U} \subset U$ on X and an aggregation function $F : Pow(\hat{U}) \to U$ satisfying

$$F(\{u\}) = u \tag{8}$$

and

$$F(\hat{u})(x) > F(\hat{u})(y) \;\; \& \;\; F(\hat{u}')(x) > F(\hat{u}')(y) \Rightarrow F(\hat{u} \cup \hat{u}')(x) > F(\hat{u} \cup \hat{u}')(y) \tag{9}$$

(ii) For each $S \subseteq X$, there exists a partition $PA_S = \{\hat{u}_S^1, \ldots, \hat{u}_S^m\}$ of $\hat{U}$ such that if $S \subseteq S'$, then
$$\hat{u}_{S'} \in PA_{S'} \Rightarrow \exists \hat{u}_S \in PA_S \; s.t. \; \hat{u}_{S'} \subseteq \hat{u}_S \tag{10}$$

such that for all S with $PA_S = \{\hat{u}_S^1, \ldots, \hat{u}_S^m\}$

$$For \; x, y \in S, \; x \succ_S y \Leftrightarrow [\forall 1 \leq i \leq m, F(\hat{u}_S^i)(x) > F(\hat{u}_S^i)(y)] \tag{11}$$

and
$$x \succ y \Leftrightarrow F(\hat{U})(x) > F(\hat{U})(y) \tag{12}$$

The set $\{u_1, \ldots, u_n\}$ is the collection of utility functions each representing one criterion (e.g., portability, battery life, and performance). The aggregation function $F(\hat{u})$ gives a new criterion that represents the overall assessment of alternatives based on the collection $\hat{u}$ of criteria. $F(\hat{u})(x)$ is the "utility" of alternative x according to criterion $F(\hat{u})$. For example, when $\hat{u}$ contains the criteria of CPU performance, GPU performance, and SSD performance, F gives an overall performance index. The aggregation function is required to satisfy the basic property of (9), which is satisfied by all reasonable aggregation rules (e.g., additive and lexicographical maximin). PA_S partitions all criteria such that criteria in the same partition can be aggregated by F into a single measure when the menu is S. Equation (10) requires that the partition is more coarse in smaller menus. This property guarantees that the number of remaining criteria becomes smaller when the menu shrinks. Finally, $\succ_S$ is generated by the remaining criteria via the unanimity rule. The unanimity rule is used because it is the least demanding in terms of cognitive cost, as Herbert Simon points out himself ([7], p. 295):

> Frequently, a course of action satisfying a number of constraints, even a sizeable number, is far easier to discover than a course of action maximizing some function.

Proposition 2. *Given a preference system* $<\succ_S, \succ>$*, the followings are equivalent.*

1. $<\succ_S, \succ>$ *is a nested system of strict partial order.*
2. $<\succ_S, \succ>$ *admits a multi-utility aggregation representation.*

Proof. $2 \Rightarrow 1$: Suppose $<\succ_S, \succ>$ admits a multi-utility aggregation representation. Note that (11) implies that $\succ_S$ is the intersection of linear orders, which in turn implies that $\succ_S$ must be a strict partial order. Now suppose $x, y \in S \subseteq S'$ and $x \succ_{S'} y$. By (11), for all $\hat{u}^i_{S'} \in PA_{S'}$, $F(\hat{u}^i_{S'})(x) > F(\hat{u}^i_{S'})(y)$. Because $PA_{S'}$ and PA_S are partitions of $\hat{U}$, (10) implies that for all $\hat{u}_S \in PA_S$ can be partitioned into $\hat{u}^1_{S'}, \ldots, \hat{u}^m_{S'} \in PA_{S'}$ for some positive integer m. By (8) and (9), we must have for all $\hat{u}^i_S \in PA_S$, $F(\hat{u}^i_S)(x) > F(\hat{u}^i_S)(y)$. By (11), $x \succ_S y$.

$1 \Rightarrow 2$: Suppose $<\succ_S, \succ>$ is a nested system of strict partial order. We construct the representation. The case when $\succ = \succ_X$ is trivial, because by nestedness, it is reduced to menu-independent preference. Such a case can be represented by a utility representation u of $\succ$, an additive aggregation rule, and a trivial partition containing $\{u\}$ for all S. We thus focus on cases where $\succ_X \subset \succ$.

Let $|X| = m$. Let $\mathcal{E} = \{\succ_1, \ldots, \succ_n\}$ be the set of all linear order extensions of $\succ_X$. In particular, $\succ \in \mathcal{E}$.

For each $\succ_j \in \{\succ_1, \ldots, \succ_n\} \setminus \{\succ\}$, enumerate $X = \{x^j_1, \ldots, x^j_m\}$ in ascending order with respect to $\succ^j$, i.e., $x_s \succ_j x_k$ if and only if $s > k$. Construct u^j to represent $\succ^j$ by letting $u^j(x^j_k) = k$. Let the set of constructed utility functions be $\{u_1, \ldots, u_{n-1}\}$.

For $\succ$, enumerate $X = \{x_1, \ldots, x_m\}$ in ascending order with respect to $\succ$ and construct $\bar{u}$ by letting $u(x_k) = m * n * k$. Let $\hat{U} = \{u_1, \ldots, u_{n-1}, \bar{u}\}$.

Let us turn to partition PA_S of $\hat{U}$. Given S, construct PA_S as follows. Let

$$P_S = \{\{u\} \subset \hat{U} \setminus \{\bar{u}\} : \nexists x, y \in S' \supseteq S, \text{ s.t. } x \succ_{S'} y \;\; \& \;\; u(y) > u(x)\}$$

$$PA_S = P_S \cup \{u \in \hat{U} : \{u\} \notin P_S\} \tag{13}$$

That is, for all u, except $\bar{u}$, not contradicting $\succ_{S'}$ for any superset S' of S, we leave it in a singleton partition. We then put all other u, including $\bar{u}$, in one partition. Let us refer to the partition containing $\bar{u}$ as $\hat{u}^*$. Note that (10) is satisfied by construction.

Finally, let $F_S(u_1, \ldots, u_t)(x) = \sum_{0 \le j \le t} u_j(x)$, i.e., the additive aggregation rule. The additive function clearly satisfies (8) and (9).

Equations (11) and (12) must now be established.

Equation (12) is verifiable by noting that combining all u additively is equivalent to $\bar{u}$. To see this, recall that the minimal utility difference with respect to $\bar{u}$ is $m * n$, whereas the largest utility difference with respect to any other $u \in \hat{U}$ is $m - 1$. Therefore, if $\bar{u}(x) > \bar{u}(y)$, then

$$\sum_{u \in \hat{U}} u(x) - \sum_{u \in \hat{U}} u(y) \ge m * n - (m-1) * (n-1) > 0 \tag{14}$$

The reverse is also true because $\bar{u}$ is constructed to represent $\succ$.

Finally, we verify (11). Suppose $x \succ_S y$ and thus $x \succ y$. Via the construction of PA_S, all u such that $u(y) > u(x)$ is in the same partition with $\bar{u}$. By (14),

$$F(\hat{u}^*)(x) - F(\hat{u}^*)(y) = \sum_{u \in \hat{u}^*} u(x) - u(y) > 0$$

Because singleton partitions agree with $\succ_S$ by construction, (11) must hold.

Note that because $\succ_X$ is a strict partial order, for any acyclic $\succ'$ such that $(x, y) \in \succ_X \Rightarrow (x, y), (y, x) \notin \succ'$, there exists $\succ^* \in \mathcal{E}$ such that $\succ' \subseteq \succ^*$.

Now suppose $\forall \hat{u} \in PA_S$, $\sum_{u \in \hat{u}} u(x) > \sum_{u \in \hat{u}} u(y)$. This implies $x \succ y$. Assume the contrary that $x \not\succ_S y$. This implies that $x \not\succ_X y$ by nestedness. Let

$$\mathcal{E}_{yx} = \{\succ \in \mathcal{E} : y \succ x\}$$

By construction, for all $\succ_i \in \mathcal{E}_{yx}$, there exist $z^i, w^i \in S^i \supseteq S$ such that $w^i \succ_i z^i$ and $z^i \succ_{S^i} w^i$. Let us enumerate such $\mathcal{E}_{yx}$ as $\{\succ_1, \ldots, \succ_t\}$. We also enumerate the corresponding pairs (w^i, z^i) and sets S^i as $\{(w^1, z^1), \ldots, (w^t, z^t)\}$ and $\{S^1, \ldots, S^t\}$, respectively. Let

$$\succ' = tc(\succ_X \cup \{(z^1, w^1), \ldots, (z^t, w^t)\})$$

where $tc(\succ)$ is the transitive closure of $\succ$. By definition and because $\succ_S \subseteq \succ$ for all S, $\succ'$ is a strict partial order extension of $\succ_X$. We show that $(x, y), (y, x) \notin \succ'$. First, $(y, x) \notin \succ'$ because $(x, y) \in \succ \supseteq \succ'$. Second, if $(x, y) \in tc(\succ_X \cup \{(z^1, w^1), \ldots, (z^t, w^t)\}$, then by transitivity and nestedness, $(x, y) \in \succ_{S^1 \cap \cdots \cap S^t}$. However, this is impossible, because $S \subseteq S^1 \cap \cdots \cap S^t$ and $x \not\succ_S y$. This establishes that $(x, y), (y, x) \notin \succ'$.

Because $(x, y), (y, x) \notin \succ'$ and $\succ'$ is a strict partial order extension of $\succ_X$, there exists $\succ^* \in \mathcal{E}$ such that $\succ' \cup \{(y, x)\} \in \succ^*$. By definition, $\succ^* \in \mathcal{E}_{yx}$. However, for any $(w^i, z^i) \in \{(w^1, z^1), \ldots, (w^t, z^t)\}$, we cannot have $w^i \succ^* z^i$, because $\succ' \subseteq \succ^*$, which is a contradiction. $\square$

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