

Article

## Partial Cooperative Equilibria: Existence and Characterization

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**Abstract:** We study the solution concepts of partial cooperative Cournot-Nash equilibria and partial cooperative Stackelberg equilibria. The partial cooperative Cournot-Nash equilibrium is axiomatically characterized by using notions of rationality, consistency and converse consistency with regard to reduced games. We also establish sufficient conditions for which partial cooperative Cournot-Nash equilibria and partial cooperative Stackelberg equilibria exist in supermodular games. Finally, we provide an application to strategic network formation where such solution concepts may be useful.

**Keywords:** partial cooperative equilibrium; supermodular games; network formation

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### 1. Introduction

The questions of coalition formation and cooperation are central to the theory of strategic behavior. When a subset of the agents forms a coalition, they often behave “cooperatively” in the sense that they choose and implement a joint course of action. Nevertheless, the players across those coalitions do not proceed jointly: their actions are chosen independently and non-cooperatively. Examples of such situations are the formation of a cartel of firms on a market, the signing of an environmental agreement

across countries, the cooperation in R&D between firms and the formation of a coalition of political parties. All these examples share a joint cooperate/compete feature that has a long and checkered history within the framework of non-cooperative games. In this literature, it is assumed that coalitions are given, and each coalition, rather than maximizing its individual payoff, maximizes a *group payoff function*, which can range from a simple sum of individual payoff functions (Mallozzi and Tijds [1–3]) to a vector valued function choosing points on the Pareto frontier of the individual payoffs of the members of the coalition (Ray and Vohra [4], Ray [5]). In order to allow for payoff transfers among the players, we choose the sum of payoffs as the cooperators's objective function rather than a point in the Pareto frontier.

The first way to analyze equilibria in these games is to assume that coalitions play in the spirit of the Nash equilibrium, namely an equilibrium is defined as a set of mutual best responses. The only difference is that individual players are now replaced by coalitions. Each coalition chooses a strategy vector to maximize the group payoff function. This has been the theme in articles such as Chakrabarti *et al.* [6], Chander and Tulkens [7], Deneckere and Davidson [8] and Ray and Vohra [4]. Other articles that have similar themes are Duggan [9], Salant *et al.* [10], Beaudry *et al.* [11], Carraro and Siniscalco [12,13] and Funaki and Yamato [14].

One can also look at a sequential structure, where coalitions do not proceed simultaneously but there is a sequence of moves and one coalition chooses its action followed by another coalition. Of course, the game becomes quite complicated and an open question would be: what is the order of moves? Due to this, one can analyze only very simple settings. The setting that has been analyzed here and also in Chakrabarti *et al.* [6] is that there is one non-singleton coalition and the rest of players are not part of any coalition. The coalition moves first, and the singleton players move subsequently. The issue in such games is how to define an equilibrium. Generally, while solving this game by backward induction, the coalition assumes that a Nash equilibrium will result in the second stage. If there is a unique Nash equilibrium for every choice made by the coalition, then the coalition can solve the game easily by backward induction. If on the other hand, there is more than one Nash equilibrium in the second stage of the game, the trend has been to focus on a certain *focal* Nash equilibrium. Two such focal equilibria are the one that is best from the point of view of the coalition, that is maximizes its group payoff function (the coalition is optimistic in its mental outlook) and the other is the worst Nash equilibrium, namely it minimizes the coalition's group payoff function (the coalition is pessimistic or risk averse in this mental outlook). These ideas have been explored in Mallozzi and Tijds [1–3] and Chakrabarti *et al.* [6]. Other articles that have similar themes are Barrett [15], D'Aspremont *et al.* [16] and Diamantoudi and Sartzetakis [17].

In this article, we study the two types of sequences of moves: simultaneous moves (Cournot-Nash), where the group of cooperators and the remaining set of independent players choose their strategies simultaneously, and sequential moves (Stackelberg), where the group of cooperators moves first and the independent players choose their strategies subsequently. In these settings, the associated equilibrium concepts are called partial cooperative Cournot-Nash equilibrium and partial cooperative Stackelberg equilibrium respectively.

The relevance of such new solution concepts is highlighted in two directions. Firstly, we point out the importance of partial cooperation in strategic environments by considering a model of strategic network formation. This model has the structure of a social dilemma in the sense that the private

interests are at odds with collective efficiency. The consequence is that the only Nash equilibrium corresponds to the Pareto-dominated empty network. On the other hand, whenever some players are allowed to cooperate, there is a partial cooperative Cournot-Nash equilibrium that results in the formation of a Bentham-efficient network, provided that the size of the set of cooperators is large enough compared to the cost of creating a link. Even though we focus on the partial cooperative Cournot-Nash equilibrium for the sake of exposition, a similar result can be obtained with the partial cooperative Stackelberg equilibrium.

Secondly, we characterize the concept of partial cooperative Cournot-Nash equilibrium using axioms that are similar in spirit to Peleg and Tijs [18]. Peleg and Tijs [18] have characterized the Nash equilibrium of a non-cooperative game using three axioms—one-person rationality, consistency and converse consistency. One player rationality is a basic notion of a player being an utility maximizer. For consistency, Peleg and Tijs [18] require that an equilibrium of a game is also the equilibrium of a reduced game, where the reduced game with respect to a certain coalition and a certain strategy tuple consists in fixing the strategies of players outside the coalition to be consistent with the strategy tuple. The remaining players are free to choose their strategies to maximize their reduced payoff function, which is in fact the original payoff function given the strategies of players outside the coalition. The interpretation is that if some players decide to choose their equilibrium strategies and leave, the members of the remaining coalition have no interest in reconsidering their strategies. Converse consistency says that if a strategy profile is an equilibrium in each reduced game, then it is also an equilibrium of the original game. The adaptation of consistency and converse consistency to our setting requires us to modify the reduction operator in order to account for the existence of cooperators. The basic idea is the following: if some cooperators choose their strategies and then leave the game, they transfer the sum of their payoffs to the remaining cooperators. In a sense, these transfers enable the remaining cooperators to evaluate the impact of their strategic choices on the whole group of cooperators. In other words, the group of cooperators never really breaks up when some of its members have left the game, which reflects the fact that it can be seen as having signed a binding agreement of cooperation. Payoff transfers can be found in other articles which mix cooperation and competition as for instance Jackson and Wilkie [19] and Kalai and Kalai [20]. In case the set of cooperators is empty, the reduced game considered in Peleg and Tijs [18] and ours coincide. However, we do not manage to find an axiomatic characterization of the partial cooperative Stackelberg equilibrium.

The application to strategic network formation and the characterization of the partial cooperative Cournot-Nash equilibrium conceal the fact that a partial cooperative equilibrium may fail to exist. In this article we also provide sufficient conditions for the existence of such equilibria by restricting ourselves to the class of supermodular games introduced by Topkis [21] and subsequently studied by Milgrom and Roberts [22], Vives [23], and Zhou [24], among others. The interesting feature of these games is that they allow existence of a Nash equilibrium while doing away with the assumptions of convexity of the strategy sets and quasi-concavity of the payoff functions as was the case in the analysis of Nash [25]. We prove the existence of partial cooperative Cournot-Nash and Stackelberg equilibria in supermodular games. In case these games are not supermodular, we show that a partial cooperative Stackelberg equilibrium exists if the strategy sets are compact, payoff functions are continuous and there exists a Nash equilibrium at the second stage of the game.

The rest of the article proceeds as follows. Section 2 introduces the notation and terminology. In section 3, we present a model of network formation and discuss the effect of introducing a set of cooperators on the set of stable and efficient networks. In Section 4, we provide the axiomatic characterization of the partial cooperative Cournot-Nash equilibrium. In Section 5, we explore existence of a partial cooperative equilibrium in supermodular games. Finally, Section 6 concludes.

## 2. Preliminaries

Weak set inclusion is denoted by  $\subseteq$ , proper set inclusion by  $\subset$ . A game in strategic form is a system  $\Gamma = (I, (X_i, f_i)_{i \in I})$ , where  $I$  is a nonempty and finite set of players,  $X_i$  is the non-empty set of strategies of player  $i$ , and  $f_i : X \rightarrow \mathbb{R}$  is the payoff function of player  $i$ , where  $X = \prod_{i \in I} X_i$ . Typical elements of  $X$  are strategy profiles denoted by  $x$  and  $y$ . For any nonempty subset  $S$  of  $I$ , we let  $X_S = \prod_{i \in S} X_i$ , with typical element  $x_S$ , and use  $X$  and  $X_i$  instead of  $X_N$  and  $X_{\{i\}}$  respectively. For any  $x \in X$  and any  $S$ , let  $x_{-S} = (x_i)_{i \in I \setminus S}$ . A strategy profile  $x \in X$  is a Nash equilibrium of  $\Gamma$  if  $f_i(x) \geq f_i(y_i, x_{-i})$  for any  $y_i \in X_i$  and any  $i \in I$ , where  $x_{-i}$  stands for  $x_{\{i\}}$ . We denote by  $G$  the set of all games in strategic form.

A group of cooperators for a game  $\Gamma \in G$  with player set  $I$  consists in a (possibly empty) subset  $C$  of  $I$ . A game in strategic form with partial cooperation is a pair  $(\Gamma, C)$  where  $\Gamma = (I, (X_i, f_i)_{i \in I}) \in G$  is a game in strategic form and  $C \subseteq I$  is a group of cooperators. Cooperators are assumed to choose their strategies by maximizing the joint payoff of the group members given by

$$F(x) = \sum_{i \in C} f_i(x) \tag{1}$$

for each  $x \in X$ . Non-cooperators play as singletons and maximize their individual payoff functions. There are two main assumptions regarding the sequence of moves in the above set-up that are popular in the literature: (a) simultaneous moves (Cournot-Nash), where all the players choose their strategies simultaneously and (b) sequential moves (Stackelberg), where the cooperators enjoy a first mover advantage and non-cooperators choose their strategies subsequently.

We begin with the first. A strategy profile  $x \in X$  is a partial cooperative Cournot-Nash equilibrium of  $(\Gamma, C)$  if

$$f_i(x) \geq f_i(y_i, x_{-i})$$

for any  $y_i \in X_i$  and any  $i \in I \setminus C$ , and

$$F(x) \geq F(y_C, x_{-C})$$

for any  $y_C \in X_C$ .

Coming to the second, there are two stages in the game. In the first stage, the cooperators choose a joint strategy  $y_C \in X_C$ . In the second stage, the players in  $I \setminus C$  play the reduced game  $\Gamma^{I \setminus C, y_C} = (I \setminus C, (X_i, g_i)_{i \in I \setminus C})$ , i.e., the  $|I \setminus C|$ -person game in strategic form where each  $i \in I \setminus C$  has strategy space  $X_i$  and payoff function being given by  $g_i(x_{I \setminus C}) = f_i(x_{I \setminus C}, y_C)$  for each  $x_{I \setminus C} \in X_{I \setminus C}$ . By  $D^{(\Gamma, C)} : X_C \rightarrow X_{I \setminus C}$ , we denote the correspondence mapping  $y_C \in X_C$  to the set  $D^{(\Gamma, C)}(y_C) \subseteq X_{I \setminus C}$  of Nash equilibrium profiles of game  $\Gamma^{I \setminus C, y_C}$ .

First, let us consider the class of games where the correspondence  $D^{(\Gamma, C)}$  is non-empty and single-valued. Then  $\Gamma^{I \setminus C, y_C}$  has a unique Nash equilibrium denoted by  $(\eta_1(y_C), \eta_2(y_C), \dots, \eta_k(y_C))$  for any  $y_C \in X_C$ . The cooperators maximize the joint payoff defined in (1) and solve the problem

$$\max_{y_C \in X_C} F(y_C, \eta_1(y_C), \eta_2(y_C), \dots, \eta_k(y_C)) \tag{2}$$

A vector  $x = (x_C, \eta_1(x_C), \eta_2(x_C), \dots, \eta_k(x_C)) \in X$  such that  $x_C$  solves (2) is called a *partial cooperative Stackelberg equilibrium* of  $(\Gamma, C)$ .

Next, let us consider the case where the correspondence  $D^{(\Gamma, C)}$  is a non-empty valued correspondence that is not necessary single-valued. Then given the choice  $y_C \in X_C$  of the cooperators, the game  $\Gamma^{I \setminus C, y_C}$  may have multiple Nash equilibria. If the group of cooperators have a pessimistic view about the non-cooperator choice, the cooperators will maximize the function

$$F^-(y_C) = \min_{x_{I \setminus C} \in D^{(\Gamma, C)}(y_C)} F(y_C, x_{I \setminus C})$$

On the other hand if the group members are optimistic, the cooperators will maximize the function

$$F^+(y_C) = \max_{x_{I \setminus C} \in D^{(\Gamma, C)}(y_C)} F(y_C, x_{I \setminus C})$$

The equilibria that result are called the *pessimistic* and *optimistic partial cooperative Stackelberg equilibria* of  $(\Gamma, C)$  respectively. Therefore, a vector  $x \in X$  such that

$$x_C \in \arg \max_{y_C \in X_C} F^-(y_C)$$

and  $x_{I \setminus C}(x_C) \in D^{(\Gamma, C)}(x_C)$  is called a *pessimistic partial cooperative equilibrium* of  $(\Gamma, C)$ . Similarly, a vector  $x \in X$  such that

$$x_C \in \arg \max_{y_C \in X_C} F^+(y_C)$$

and  $x_{I \setminus C}(x_C) \in D^{(\Gamma, C)}(x_C)$  is called an *optimistic partial cooperative Stackelberg equilibrium* of  $(\Gamma, C)$ . For simplicity, we will write  $x_{I \setminus C}$  instead of  $x_{I \setminus C}(x_C)$ . Also, for future reference, define correspondences  $\eta_{\min} : X_C \rightarrow X_{I \setminus C}$  and  $\eta_{\max} : X_C \rightarrow X_{I \setminus C}$  where for any arbitrary  $y_C \in X_C$ ,

$$\eta_{\min}(y_C) = \arg \min_{x_{I \setminus C} \in D^{(\Gamma, C)}(y_C)} F(y_C, x_{I \setminus C})$$

and

$$\eta_{\max}(y_C) = \arg \max_{x_{I \setminus C} \in D^{(\Gamma, C)}(y_C)} F(y_C, x_{I \setminus C})$$

These notions of equilibria, in the order they are presented have been depicted in Chakrabarti *et al.* [6], Mallozzi and Tijs [1], Chakrabarti *et al.* [6] and Mallozzi and Tijs [2] respectively. Mallozzi and Tijs [1] investigate a class of games in which a partial cooperative Stackelberg equilibrium always exists, namely the class of symmetric potential games with a strictly concave potential function. Mallozzi and Tijs [3] show that the joint payoff of the cooperators under an optimistic partial cooperative Stackelberg equilibrium is greater than or equal to that under a partial cooperative Cournot-Nash equilibrium. Chakrabarti *et al.* [6] have investigated the existence of partial cooperative Cournot-Nash

equilibrium and the pessimistic partial cooperative Stackelberg equilibrium in symmetric games under standard assumptions for existence of a Nash equilibrium.

Before we end this section, we would like to compare these notions to the notion of *coalitional equilibrium* of Ray [5].<sup>1</sup> A vector  $x \in X$  is called a coalitional equilibrium of the game  $(\Gamma, C)$ , if for all  $i \in I \setminus C$ ,

$$f_i(x) \geq f_i(y_i, x_{-i})$$

for all  $y_i \in X_i$ , and there does not exist any  $y_C \in X_C$  such that

$$f_i(x) < f_i(y_C, x_{-C})$$

for all  $i \in C$ .

The principal difference between the coalitional equilibrium and the partial cooperative Cournot-Nash equilibrium is that the former assumes that no transfers are possible among the group members. So, the group of cooperators chooses a payoff vector that lies in the Pareto frontier of the set of payoff vectors that are attainable by the group of cooperators given the strategies chosen by the non-cooperators.

### 3. Network Formation with Consent

In order to highlight the impact of partial cooperation on strategic interaction, we consider the non-cooperative model of costly network formation with consent proposed by Myerson [26]. Firstly, let  $I = \{1, 2, \dots, n\}$  be a finite player set. The bilateral communication possibilities between the players are represented by an *undirected graph* on  $I$ , denoted by  $(I, L)$ , where the set of nodes coincides with the set of players  $I$ , and the set of links  $L$  is a subset of the set of unordered pairs  $\{i, j\}$  of elements of  $I$ . For simplicity, we write  $ij$  to represent the link  $\{i, j\}$ . For each player  $i \in I$ , the set  $L_i = \{j \in I : ij \in L\}$  denotes the *neighborhood* of  $i$  in  $(I, L)$ . Player  $i$  is a *leaf* in  $(I, L)$  if  $|L_i| = 1$ . A sequence of distinct players  $(i_1, i_2, \dots, i_p)$  is a *path* in  $(I, L)$  if  $i_k i_{k+1} \in L$  for  $k = 1, \dots, p - 1$ . Let  $S(i)$  be the set of players  $j$  such that there exists a path between  $i$  and  $j$ .

Formally, the non-cooperative model of costly network formation with consent is described by a game  $\Gamma = (I, (X_i, f_i)_{i \in I})$  with the following features. Each player  $i \in I$  has an action set  $X_i = \{(x_{ij})_{j \neq i} : x_{ij} \in \{0, 1\}\}$ . Player  $i$  seeks contact with player  $j$  if  $x_{ij} = 1$ . Link  $ij$  forms if both players seek contact. Each profile  $x \in X$  induces a network  $(I, L(x))$  such that

$$L(x) = \{ij : x_{ij} = x_{ji} = 1\}$$

If link  $ij$  forms then both players support a cost  $c > 0$ . We assume that the value of network  $(I, L(x))$  for player  $i$  depends on the number  $|S(i)|$  of players to whom  $i$  is connected but not on the distance between two connected players. Player  $i$ 's payoff is equal to his value of the network minus the cost  $c$  for any created link as in the connection model without decay in Jackson and Wolinsky [27]:

$$f_i(x) = |S(i)| - c|L_i(x)| \tag{3}$$

A network  $(I, L(x))$  is an equilibrium network if  $x \in X$  is a partial cooperative equilibrium of  $(\Gamma, C)$  for some  $C \subseteq I$ . A network  $(I, L(x))$  is (Bentham) efficient if for each  $y \in X$

$$\sum_{i \in I} f_i(x) \geq \sum_{i \in I} f_i(y)$$

<sup>1</sup>See also Ray and Vohra [4].

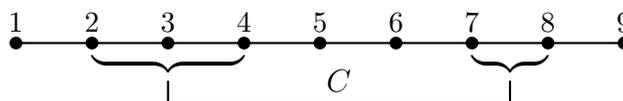
We assume  $c > 1$ , that is, the cost of creating a link is larger than the direct value of that link. We also impose the upper bound  $c \leq (n - 1)/2$ , which ensures the existence of a nonempty efficient network in which each player obtains a positive payoff. Given these assumptions, it is easy to see that every Nash equilibrium of game  $\Gamma$  induces the formation of the empty network.

The absence of nonempty networks arising from Nash equilibria can be avoided by several mechanisms. One of them consists in repeating the game over finitely many periods and assuming that the players have limited capacities to implement repeated game strategies (see Béal and Quérou [28]). In this paragraph, we show that the partial cooperation of a group yields an even stronger result: equilibrium networks can be not only nonempty but also efficient. In other words, partial cooperation can eliminate the tension between stability and efficiency of networks.

**Proposition 1** Assume  $c \in (1, (n - 1)/2]$ .

- (i) Every Nash equilibrium of  $\Gamma$  induces the empty network.
- (ii) For each group  $C \subseteq I$  such that  $|C| \geq 2c - 1$ , there exists a partial cooperative Cournot-Nash equilibrium of  $(\Gamma, C)$  that results in the formation of an efficient nonempty network.

The basic idea of the proof given in the appendix is as follows. For each group  $C$  such that  $|C| \geq c$ , we construct a line network in which the members of the group have strategic locations. More specifically, we can assume that the players on the line are ordered from 1 to  $n$ . If  $|C| = 2$ , the members of  $C$  are the neighbors, 2 and  $n - 1$ , of the two leaves. If  $|C| = 3$ , the group consists of players 2, 3 and  $n - 1$ . If  $|C| = 4$ , the group consists of players 2, 3,  $n - 2$  and  $n - 1$ , and so on until  $|C| \leq n - 2$ . An illustration of the locations of group members on a line network is given below for the case where  $|I| = 9$  and  $|C| = 5$ :



In the proof of Proposition 1, we show that an action profile that induces such a network is a partial cooperative Cournot-Nash equilibrium. Observe that a neighbor of a leaf, say player 2, would increase his own payoff if he cuts his link with player 1 since he would lose one connection and reduce his cost by  $c > 1$ . This is the reason why such a player has to belong to the group: taking into account the sum of the payoffs of the group members, it is not interesting for 2 to delete link 12 since  $|C|$  connections (with player 1) would be lost among group members while only player 2 reduces his cost by  $c$ . In other words, cutting link 12 affect the connections of each group member and this total loss is larger than the cost reduction of  $c$  provided that  $|C| \geq c$ .

#### 4. Axiomatization of the Partial Cooperative Cournot-Nash Equilibrium

The relevance of the partial cooperative equilibria can also be defended axiomatically. Peleg and Tijs [18] have characterized the Nash equilibrium of a non-cooperative game in terms of three properties: one-person rationality, consistency and converse consistency. In this section we show that a similar characterization is possible for the notion of partial cooperative Cournot-Nash equilibrium. To embark on such an endeavour, we need to define an appropriate concept of reduced game.

4.1. More Definitions

In this section, let  $\mathcal{U}$  denote the *universe of players*. For any game  $\Gamma = (I, (X_i, f_i)_{i \in I}) \in G$  and any subset  $K$  of  $\mathcal{U}$ ,  $(\Gamma, I \cap K)$  denotes the unique game with partial cooperation associated to  $\Gamma$  and  $K$ , where the group of cooperators consists in those players of  $I$  who belong to  $K$ , that is  $I \cap K$ . Recall that  $I \cap K$  can be empty. We denote by  $G^K$  the set of all games with partial cooperation induced by  $K$  and defined by:

$$G^K = \{(\Gamma, C) : \Gamma = (I, (X_i, f_i)_{i \in I}) \in G, C = I \cap K\}$$

Because game  $\Gamma \in G$  is associated to game  $(\Gamma, I \cap K) \in G^K$  in a unique way, we will henceforth refer to  $(\Gamma, I \cap K)$  as  $\Gamma$  in order to save on notations. A *solution concept* for  $G^K$  is a function  $\phi$  which assigns to every  $\Gamma \in G^K$  a subset  $\phi(\Gamma)$  of  $X$ . Let  $PCE^K(\Gamma)$  denote the solution concept on  $G^K$  which assigns to every game  $\Gamma \in G^K$  the set  $PCE^K(\Gamma)$  of strategy profiles  $x \in X$  such that  $x$  is a partial cooperative Cournot-Nash equilibrium of  $\Gamma$  with group  $I \cap K$ . We shall refer to PCE as the partial equilibrium correspondence.

We adapt the notion of a *reduced game* used in Peleg and Tijs [18] in order to account for the existence of a group of cooperators. Given a game  $\Gamma = (I, (X_i, f_i)_{i \in I}) \in G^K$ , any non-empty  $S \subset I$  and any  $x \in X$ , the reduced game of  $\Gamma$  with respect to  $S$  and  $x$ , denoted by  $\Gamma^{S,x}$ , is the system  $(S, (X_i, g_i)_{i \in S})$  where

$$g_i(y_S) = \begin{cases} f_i(y_S, x_{-S}) & \text{if } i \in S \setminus K \\ f_i(y_S, x_{-S}) + \frac{1}{|S \cap K|} \sum_{j \in (I \setminus S) \cap K} f_j(y_S, x_{-S}) & \text{if } i \in S \cap K \end{cases}$$

for each  $y_S \in X_S$ .<sup>2</sup> Note that the class of games with partial cooperation  $G^K$  is closed under the reduction operation, that is, if game  $\Gamma = (I, (X_i, f_i)_{i \in I})$  belongs to  $G^K$ , then  $S \subset I$ ,  $S \neq \emptyset$ , and  $x \in X$  imply  $\Gamma^{S,x} \in G^K$  with set of cooperators  $S \cap K$ .

The interpretation of the reduced game is as follows. In the reduced game  $\Gamma^{S,x}$ , each player in  $S$  obtains his payoff in the original game with partial cooperation  $\Gamma$  given that the members of  $I \setminus S$  play  $x_{-S}$  in accordance with  $x$ . In addition, if some but not all cooperators  $I \cap K$  belong to  $I \setminus S$ , they equally transfer the sum of the payoffs they would have received with  $(y_S, x_{-S})$  in  $\Gamma$  to the cooperators  $S \cap K$  who play the reduced game  $\Gamma^{S,x}$ . The group can be seen as a single entity interrelated by a binding agreement. In the reduced game, the departed members do not actively participate anymore in the choice of a strategy profile. Still they passively participate by transferring a part of their payoff to each of the remaining members. Thus the remaining members are able to measure the consequences of their choices on the whole group.

4.2. Axioms

We provide axioms of solution concepts on the class of games  $G^K$ . In order to take into account the particular structure of a game with partial cooperation, we consider variations of several axioms defined in Peleg and Tijs [18]. Throughout this section,  $\phi$  is an arbitrary solution concept on  $G^K$ .

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<sup>2</sup>Note that  $(I \setminus S) \cap K = (I \cap K) \setminus (S \cap K)$ .

**Group rationality:** For any game  $\Gamma = (I, (X_i, f_i)_{i \in I}) \in G^K$  such that  $|I| = 1$  or  $I \subseteq K$ , it holds that  $\phi(\Gamma) = \{\arg \max_{x \in X} \sum_{i \in I} f_i(x)\}$ .

**Group consistency:** For each  $\Gamma = (I, (X_i, f_i)_{i \in I}) \in G^K$  such that  $I \setminus K \neq \emptyset$ , each  $x \in \phi(\Gamma)$  and each  $S \subset I, S \neq \emptyset, x_S \in \phi(\Gamma^{S,x})$ .

For each  $\Gamma = (I, (X_i, f_i)_{i \in I}) \in G^K$  such that  $|I| \geq 2$  and  $I \setminus K \neq \emptyset$ , we define

$$\tilde{\phi}(\Gamma) = \{x \in X : x_S \in \phi(\Gamma^{S,x}), \forall S \notin \{\emptyset, I\}\}.$$

**Group converse consistency:** For each  $\Gamma = (I, (X_i, f_i)_{i \in I}) \in G^K$  such that  $|I| \geq 2, \tilde{\phi}(\Gamma) \subseteq \phi(\Gamma)$

Observe that if  $|K| \leq 1$ , then group rationality, group consistency and group converse consistency are equivalent to the axioms one-person rationality, consistency and converse consistency defined in Peleg and Tijs [18] respectively.

### 4.3. Characterization

The three axioms that we defined in the previous section uniquely characterize the partial cooperative Cournot-Nash equilibrium correspondence on the class of games  $G^K$ .

**Theorem 1** *The  $PCE^K$  correspondence is the unique solution on  $G^K$  that satisfies group rationality, group consistency and group converse consistency.*

[EXISTENCE] We show that  $PCE^K$  satisfies the three axioms on  $G^K$ .

*Group rationality:* Consider any game  $\Gamma = (I, (X_i, f_i)_{i \in I}) \in G^K$  such that  $|I| = 1$  or  $I \subseteq K$ . In case  $I \subseteq K$ , then all players in  $\Gamma$  belong to the group of cooperators  $I \cap K = I$  associated with  $\Gamma$ . Together with the definition of the payoff function of the group of cooperators in  $\Gamma$ , this means that the set of best responses of cooperators in  $\Gamma$  coincides with the set of maximizers of the sum of the payoff functions of players in  $I$ . Hence,  $x \in PCE^K(\Gamma)$  if and only if  $x \in \{\arg \max_{y \in X} F(y)\} = \{\arg \max_{y \in X} \sum_{i \in I} f_i(y)\}$ . In case  $|I| = 1$  but  $I \not\subseteq K$ , arguments are similar.

*Group consistency:* Consider any game  $\Gamma = (I, (X_i, f_i)_{i \in I}) \in G^K$  such that  $I \setminus K \neq \emptyset$ , any  $x \in PCE^K(\Gamma)$  and any  $S \subset I, S \neq \emptyset$ . First, pick a player  $i \in S \setminus K$  if such a player exists. By construction, player  $i$ 's payoff in the reduced game  $\Gamma^{S,x}$  is  $g_i(y_S) = f_i(y_S, x_{-S})$  for each  $y_S \in X_S$ . Because  $x \in PCE^K(\Gamma)$ , it holds that  $f_i(x) \geq f_i(y_i, x_{-i})$  for any  $y_i \in X_i$ . Hence,  $g_i(x_S) \geq g_i(y_i, x_{S \setminus \{i\}})$  for any  $y_i \in X_i$ . Second, in case  $S \cap K \neq \emptyset$ , consider the subset  $S \cap K$  of the group of cooperators  $I \cap K$  who play in  $\Gamma^{S,x}$ . We have

$$\begin{aligned} \sum_{i \in S \cap K} g_i(x_S) &= \sum_{i \in S \cap K} \left( f_i(x_S, x_{-S}) + \frac{1}{|S \cap K|} \sum_{j \in (I \cap K) \setminus (S \cap K)} f_j(x_S, x_{-S}) \right) \\ &= \sum_{i \in I \cap K} f_i(x) \\ &= F(x) \\ &\geq F(y_{I \cap K}, x_{-(I \cap K)}) \end{aligned}$$

for each  $y_{I \cap K} \in X_{I \cap K}$  since it is assumed that  $x \in \text{PCE}^K(\Gamma)$ . For the particular case when players in  $(I \setminus S) \cap K$  play  $x_{((I \setminus S) \cap K)}$ , this inequality is

$$\sum_{i \in S \cap K} g_i(x_S) \geq F(y_{S \cap K}, x_{-(S \cap K)})$$

for each  $y_{S \cap K} \in X_{S \cap K}$ , which is equivalent to

$$\sum_{i \in S \cap K} g_i(x_S) \geq \sum_{i \in S \cap K} g_i(y_{S \cap K}, x_{-(S \cap K)})$$

for each  $y_{S \cap K} \in X_{S \cap K}$ . We proved that  $x_S \in \text{PCE}^K(\Gamma^{S,x})$ , and because  $S$  was chosen arbitrarily, we conclude that  $\text{PCE}^K$  satisfies group consistency.

*Group converse consistency:* Consider a game  $\Gamma = (I, (X_i, f_i)_{i \in I}) \in G^K$  such that  $|I| \geq 2$  and  $I \setminus K \neq \emptyset$ , and consider  $x \in X$  such that  $x_S \in \text{PCE}^K(\Gamma^{S,x})$  for each  $S \subset I, S \neq \emptyset$ . To show:  $x \in \text{PCE}^K(\Gamma)$ .

First, pick a player  $i \in I \setminus K$ , and consider the reduced game  $\Gamma^{\{i\},x}$ . By assumption  $x_i \in \text{PCE}^K(\Gamma^{\{i\},x})$ , which implies that  $x_i \in \arg \max_{y_i \in X_i} g_i(y_i)$ . By definition of  $g_i$ , this is equivalent to  $x_i \in \arg \max_{y_i \in X_i} f_i(y_i, x_{-i})$ .

Second, in case  $I \cap K \neq \emptyset$ , consider the subset of cooperators  $I \cap K$  who play in  $\Gamma$  and the reduced game  $\Gamma^{I \cap K,x}$ . Note that  $I \setminus K \neq \emptyset$  implies  $I \cap K \subset I$ . Since  $x_S \in \text{PCE}^K(\Gamma^{S,x})$  for each  $S \subset I, S \neq \emptyset$ , we have  $x_{I \cap K} \in \text{PCE}^K(\Gamma^{I \cap K,x})$ , and so

$$x_{I \cap K} \in \arg \max_{y_{I \cap K} \in X_{I \cap K}} \sum_{i \in I \cap K} g_i(y_{I \cap K})$$

which is equivalent to

$$x_{I \cap K} \in \arg \max_{y_{I \cap K} \in X_{I \cap K}} F(y_{I \cap K}, x_{-(I \cap K)})$$

by construction of function  $g_i$  in  $\Gamma^{I \cap K,x}$ . This proves that  $x \in \text{PCE}^K(\Gamma)$  as desired.

[UNIQUENESS] Consider any solution  $\phi$  which satisfies the three axioms on  $G^K$ . To show:  $\phi = \text{PCE}^K$ . Let  $\Gamma = (I, (X_i, f_i)_{i \in I}) \in G^K$  be any game in  $G^K$ . We distinguish two cases. In a first case, suppose that  $I \setminus K = \emptyset$ . Then all players in  $I$  are members of the group. Because  $\phi$  satisfies group rationality, we have  $\phi(\Gamma) = \{\arg \max_{y \in X} F(y)\} = \text{PCE}^K(\Gamma)$ .

In a second case, suppose that  $I \setminus K \neq \emptyset$ . Firstly, we prove that  $\phi(\Gamma) \subseteq \text{PCE}^K(\Gamma)$  for each  $\Gamma \in G^K$ . So let  $\Gamma$  be any game in  $G^K$ . If  $\phi(\Gamma) = \emptyset$  then the inclusion is trivially satisfied. Otherwise, choose any  $x \in \phi(\Gamma)$ . Pick any  $i \in I \setminus K$  and consider the reduced game  $\Gamma^{\{i\},x}$ . By group consistency of  $\phi$ , we have  $x_i \in \phi(\Gamma^{\{i\},x})$ . In the reduced game  $\Gamma^{\{i\},x}$ , group rationality yields  $g_i(x_i) \geq g_i(y_i)$  for each  $y_i \in X_i$ , which is equivalent to  $f_i(x) \geq f_i(y_i, x_{-i})$  for each  $y_i \in X_i$ . Next, consider the group of cooperators  $I \cap K \subset I$  involved in game  $\Gamma$ . Once again, group consistency implies  $x_{I \cap K} \in \phi(\Gamma^{I \cap K,x})$ . In this reduced game, group rationality yields  $\sum_{i \in I \cap K} g_i(x_{I \cap K}) \geq \sum_{i \in I \cap K} g_i(y_{I \cap K})$  or equivalently  $F(x) \geq F(y_{I \cap K}, x_{-(I \cap K)})$  for each  $y_{I \cap K} \in X_{I \cap K}$ . We proved  $x \in \text{PCE}^K(\Gamma)$ .

Secondly, we prove that  $\phi(\Gamma) \supseteq \text{PCE}^K(\Gamma)$ . The proof is by induction on the number of players. So, consider any game  $\Gamma = (I, (X_i, f_i)_{i \in I}) \in G^K$  such that  $|I| = 1$ . Denote by  $i$  the unique element of  $I$ . By group rationality, it holds that  $\phi(\Gamma) = \{\arg \max_{x_i \in X_i} f_i(x_i)\} = \text{PCE}^K(\Gamma)$ , and it does not matter

whether  $i \in K$  or not. Now assume that  $\phi(\Gamma) \supseteq \text{PCE}^K(\Gamma)$  for any game  $\Gamma = (I, (X_i, f_i)_{i \in I}) \in G^K$  such that  $|I| \leq n$  and  $I \setminus K \neq \emptyset$ . Next, consider a game  $\Gamma = (I, (X_i, f_i)_{i \in I}) \in G^K$  such that  $|I| = n + 1$  and  $I \setminus K \neq \emptyset$ . Pick any  $x \in \text{PCE}^K(\Gamma)$ . By group consistency of  $\text{PCE}^K$ ,  $x_S \in \text{PCE}^K(\Gamma^{S,x})$  for each  $S \subset I$ ,  $S \neq \emptyset$ . Since  $|S| \leq n$ , the induction hypothesis implies that  $x_S \in \text{PCE}^K(\Gamma) \subseteq \phi(\Gamma^{S,x})$ . Since  $\phi$  satisfies group converse consistency,  $x \in \phi(\Gamma)$ . Hence,  $\phi(\Gamma) \supseteq \text{PCE}^K(\Gamma)$ . This completes the proof.

### 5. Existence in Supermodular Games

Before we formally define supermodular games, we start with the following result which proves existence of optimistic and pessimistic partial cooperative equilibria under standard assumptions of continuity of the payoff functions and compactness of the strategy sets, provided we can guarantee that the reduced game  $\Gamma^{I \setminus C, y_C}$  admits at least one Nash equilibrium for every  $y_C \in X_C$ . This Lemma will be used later on for proving the existence of an optimistic and a pessimistic partial cooperative Stackelberg equilibrium.

**Lemma 1** Consider a game  $(\Gamma, C)$ , where  $\Gamma = (I, (X_i, f_i)_{i \in I}) \in G$  and  $C \subseteq I$ . For each  $i \in I$ , suppose that  $X_i$  is a compact subset of a Euclidean space and that  $f_i$  is continuous. If the correspondence  $D^{(\Gamma, C)} : X_C \longrightarrow X_{I \setminus C}$  is non-empty valued, then both an optimistic and a pessimistic partial cooperative equilibrium of  $(\Gamma, C)$  exist.

The proof of Lemma 1 is quite involved, so we have relegated it to the Appendix.

#### 5.1. Supermodular Games

Consider a set  $W$  partially ordered by the binary relation  $\preceq$  and let  $V$  be a subset of  $W$ . If  $w$  is in  $W$  and  $v \preceq w$  ( $w \preceq v$ ) for each  $v$  in  $V$ , then we say  $w$  is an upper bound (lower bound) of  $V$  with respect to  $W$ . An upper bound (lower bound) of a set that belongs to the set itself is a maximum (minimum). If the set of upper bounds (lower bounds) of  $V$  with respect to  $W$  has a minimum (maximum), we say it is the least upper bound or supremum (greatest lower bound or infimum) of  $V$  with respect to  $W$  and is denoted by  $\sup_W(V)$  ( $\inf_W(V)$ ). For two elements,  $v$  and  $w$  of  $W$ , if their supremum (infimum) lies in  $W$ , it is their *join* (*meet*) and denoted by  $v \vee w$  ( $v \wedge w$ ).

A partially ordered set that contains the join and meet of each pair of its elements is a lattice. An example of a lattice is a Euclidean space  $\mathbb{R}^n$  where the relevant binary relation is simply the usual vector ordering relation  $\leq$ . Namely, for  $v = (v_1, v_2, \dots, v_n)$  and  $w = (w_1, w_2, \dots, w_n)$ ,  $v \leq w$  if  $v_i \leq w_i$  for all  $i \in I$ . The join and meet of a pair of elements  $v$  and  $w$  are simply the coordinate-wise maximum and minimum, namely,

$$v \wedge w = (\min\{v_1, w_1\}, \min\{v_2, w_2\}, \dots, \min\{v_n, w_n\})$$

and

$$v \vee w = (\max\{v_1, w_1\}, \max\{v_2, w_2\}, \dots, \max\{v_n, w_n\})$$

For a lattice  $W$ , let  $V \subseteq W$  be non-empty. If both  $\sup_W(V)$  and  $\inf_W(V)$  belong to  $V$  for all  $V \subseteq W$ , then we say that  $W$  is *complete lattice*. By putting  $V = W$ , it follows that a complete lattice has a greatest and least element. A lattice  $W$  for which either  $v \leq w$  or  $w \leq v$  for all  $v, w \in W$  is called a *chain*.

If  $V$  is a subset of a lattice  $W$  and  $V$  contains the join and meet (with respect to  $W$ ) of each pair of elements of  $V$ , then  $V$  is a sublattice of  $W$ .

Let  $V$  be some sublattice of  $\mathbb{R}^n$ . A function  $f : V \rightarrow \mathbb{R}$  is said to be supermodular on  $V$  (sometimes, we say supermodular in  $v \in V$ ) if for all  $v, w \in V$ , it is the case that

$$f(v) + f(w) \leq f(v \wedge w) + f(v \vee w)$$

Let  $W$  and  $\Theta$  be subsets of a Euclidean space partially ordered by  $\geq$ . A function  $f : W \times \Theta \rightarrow \mathbb{R}$  is said to satisfy *increasing differences in  $(w, \theta)$*  if for all  $w, w' \in W$  such that  $w \geq w'$  and all  $\theta, \theta' \in \Theta$  such that  $\theta \geq \theta'$ , it is the case that

$$f(w, \theta) - f(w', \theta) \geq f(w, \theta') - f(w', \theta')$$

Next, we present a lemma linking these notions.

**Lemma 2** Suppose  $f : W \times \Theta \rightarrow \mathbb{R}$  is supermodular in  $(w, \theta)$ . Then,

(i)  $f$  is supermodular in  $w$  for each fixed  $\theta$ , i.e., for any fixed  $\theta \in \Theta$ , and for any  $w$  and  $w'$  in  $W$ , we have

$$f(w, \theta) + f(w', \theta) \leq f(w \wedge w', \theta) + f(w \vee w', \theta)$$

(ii)  $f$  satisfies increasing differences in  $(w, \theta)$ .

For a proof, see Sundaram ([29], p. 257). In fact an equivalence can be derived by a slightly stronger notion of increasing differences. To see this, let  $V \subset \mathbb{R}^n$ . A function  $f : V \rightarrow \mathbb{R}$  satisfies increasing differences on  $V$  if for all distinct  $i$  and  $j$  in  $\{1, 2, \dots, n\}$ , all  $v_i, w_i$  such that  $w_i \geq v_i$  and all  $v_j, w_j$  such that  $w_j \geq v_j$ , it is the case that

$$f(v_{-\{i,j\}}, w_i, w_j) - f(v_{-\{i,j\}}, v_i, w_j) \geq f(v_{-\{i,j\}}, w_i, v_j) - f(v_{-\{i,j\}}, v_i, v_j)$$

Then, we can show the following.

**Lemma 3** Let  $V \subset \mathbb{R}^n$ . A function  $f : V \rightarrow \mathbb{R}$  is supermodular on  $V$  if and only if  $f$  has increasing differences on  $V$ .

For a proof, see Sundaram ([29], p. 264). Finally, the game  $\Gamma = (I, (X_i, f_i)_{i \in I})$  is supermodular if for each  $i \in I$ :

1.  $X_i$  is a sublattice of some Euclidean space;
2.  $f_i(\cdot, x_{-i})$  is supermodular on  $X_i$  for each  $x_{-i} \in X_{-i}$ ;
3.  $f_i$  has increasing differences in  $(x_i, x_{-i})$ .

Next, we present the following result.

**Theorem 2** (Zhou, 1994) Consider a supermodular game  $\Gamma = (I, (X_i, f_i)_{i \in I}) \in G$ . If  $X_i$  is compact and  $f_i(\cdot, x_{-i})$  is upper semi-continuous on  $X_i$  for each fixed  $x_{-i}$  and each  $i \in I$ , then the set of Nash equilibria constitutes a non-empty complete lattice, and hence has a maximum and a minimum.

### 5.2. The Partial Cooperative Stackelberg Equilibrium

First, we prove the existence of an optimistic and a pessimistic partial cooperative Stackelberg equilibrium for supermodular games. Using Lemma 1 and Theorem 2, the Theorem follows.

**Theorem 3** Consider a supermodular game  $\Gamma = (I, (X_i, f_i)_{i \in I})$ . For each  $i \in I$ , assume that  $X_i$  is compact and  $f_i$  is continuous. Then, for any  $C \subseteq I$ , both an optimistic and a pessimistic partial cooperative Stackelberg equilibrium of  $(\Gamma, C)$  exist.

Consider any  $C \subseteq I$ . Given Lemma 1, all that needs to be proven is that the correspondence  $D^{(\Gamma, C)} : X_C \rightarrow X_{I \setminus C}$  is non-empty valued. Consider an arbitrary  $y_C \in X_C$  and the reduced game  $\Gamma^{I \setminus C, y_C}$ . Clearly  $\Gamma^{I \setminus C, y_C}$  is a supermodular game and satisfies all the assumptions of Theorem 2. Hence,  $\Gamma^{I \setminus C, y_C}$  admits a Nash equilibrium for every  $y_C \in X_C$  making  $D^{(\Gamma, C)}$  non-empty valued.

We can weaken continuity of  $f_i$  to upper semi-continuity of  $f(\cdot, x_{-i})$  on  $X_i$  and get existence of an optimistic and a pessimistic partial cooperative Stackelberg equilibrium, but this requires additional assumptions regarding the nature of strategy sets and/or payoff functions. We summarize them in the form of the following proposition.

**Proposition 2** Consider a supermodular game  $\Gamma = (I, (X_i, f_i)_{i \in I}) \in G$ . For each  $i \in I$ , let  $X_i$  be compact and  $f_i(\cdot, x_{-i})$  be upper semi-continuous on  $X_i$  for each fixed  $x_{-i} \in X_{-i}$ . Then, for any  $C \subseteq I$ , both an optimistic and a pessimistic partial cooperative Stackelberg equilibrium of  $(\Gamma, C)$  exist if either of the following conditions hold:

- (a)  $X$  is finite.
- (b)  $X_C$  is finite and, for all  $y_C \in X_C$ ,  $F(\cdot, y_C)$  is monotone increasing (or monotone decreasing) in  $X_{I \setminus C}$ , i.e.,  $x_{I \setminus C} \geq z_{I \setminus C}$  implies  $F(x_{I \setminus C}, y_C) \geq F(z_{I \setminus C}, y_C)$  (or  $F(x_{I \setminus C}, y_C) \leq F(z_{I \setminus C}, y_C)$ ).

Firstly, for any given  $y_C \in X_C$ , consider the reduced game  $\Gamma^{I \setminus C, y_C}$ . It is a supermodular game and satisfies all the conditions of Theorem 2. Therefore, the set of Nash equilibria of  $\Gamma^{I \setminus C, y_C}$  is a non-empty complete lattice and admits a greatest and a least element.

(a) For every  $y_C \in X_C$ ,  $D^{(\Gamma, C)}(y_C)$  being a nonempty subset of a finite set  $X_{I \setminus C}$ , is finite. Since any finite set of real numbers has a maximum and minimum,  $\eta_{\min}(y_C)$  and  $\eta_{\max}(y_C)$  are non-empty. So,  $F^-$  and  $F^+$  are well-defined functions. Now, consider the sets of real numbers  $\{F^-(y_C) : y_C \in X_C\}$  and  $\{F^+(y_C) : y_C \in X_C\}$ . Given that  $X_C$  is finite, these are finite sets of real numbers as well and hence, admit a maximum.

(b) If  $F(\cdot, y_C)$  is monotone increasing in  $X_{I \setminus C}$ , then  $\eta_{\min}(y_C)$  contains the least element and  $\eta_{\max}(y_C)$  contains the greatest element of the complete lattice  $D^{(\Gamma, C)}(y_C)$ . So,  $F^-$  and  $F^+$  are well-defined functions. Once again, finiteness of  $X_C$  implies that  $\{F^-(y_C) : y_C \in X_C\}$  and  $\{F^+(y_C) : y_C \in X_C\}$  admit a maximum. The proof when  $F(\cdot, y_C)$  is monotone decreasing is similar and is omitted.

### 5.3. The Partial Cooperative Cournot-Nash Equilibrium

For a supermodular game to admit a partial cooperative Cournot-Nash equilibrium, we require the strategy sets to be compact, and the payoff functions to be upper semi-continuous on the entire domain

of all strategy tuples. To prove this, we begin by creating an artificial game and showing that the Nash equilibrium of this game corresponds to the partial cooperative Cournot-Nash equilibrium in the original game. To this end, consider the non-cooperative game  $\Gamma(C)$  corresponding to the game with partial cooperation  $(\Gamma, C)$  defined in Section 2 where

$$\Gamma(C) = (I \setminus C \cup \{C\}, (X_i, f_i)_{i \in I \setminus C}, X_C, F)$$

**Lemma 4** *Given  $C \subseteq I$ , a Nash equilibrium of  $\Gamma(C)$  is a partial cooperative Cournot-Nash equilibrium of  $(\Gamma, C)$ .*

Fix any  $C \subseteq I$ . Towards a contradiction, consider a Nash equilibrium  $x \in X$  of  $\Gamma(C)$  that is not a partial cooperative Cournot-Nash equilibrium of  $(\Gamma, C)$ . Then, either  $f_i(y_i, x_{-i}) > f_i(x)$  for some  $i \in I \setminus C$  and  $y_i \in X_i$  or  $F(y_C, x_{-C}) > F(x)$  for some  $y_C \in X_C$ . Both cases contradict that  $x$  is a Nash equilibrium of  $\Gamma(C)$ .

Lemma 4 directly leads us to our existence result by showing that  $\Gamma(C)$  under appropriate restrictions satisfies the conditions of Theorem 2 and hence admits a Nash equilibrium.

**Theorem 4** *Consider a supermodular game  $\Gamma = (I, (X_i, f_i)_{i \in I}) \in G$ . For each  $i \in I$ , suppose that  $X_i$  is compact and  $f_i$  is upper semi-continuous and supermodular on  $X$ . For any  $C \subseteq I$ , a partial cooperative Cournot-Nash equilibrium of  $(\Gamma, C)$  exists.*

Consider the non-cooperative game  $\Gamma(C)$  constructed from  $\Gamma = (I, (X_i, f_i)_{i \in I})$  and  $C \subseteq I$ . We will prove that  $\Gamma(C)$  satisfies all the conditions of Theorem 2. The cartesian product of sublattices is a sublattice, which makes  $X_C$  a sublattice of some Euclidean space. The sum of supermodular functions on a lattice is also supermodular. Since  $f_i$  is supermodular on  $X$  for each  $i \in I$ , it follows that  $F$  is supermodular on  $X$ . Hence, by Lemma 2,  $F(\cdot, x_{I \setminus C})$  is supermodular on  $X_C$  for all  $x_{I \setminus C} \in X_{I \setminus C}$ . Further, the fact that  $F$  is supermodular on  $X$  implies by Lemma 2 that  $F$  has increasing differences in  $(x_C, x_{I \setminus C})$ . The finite sum of upper semi-continuous functions is upper semi-continuous so that  $F$  is upper semi-continuous on  $X$  and  $F(\cdot, x_{I \setminus C})$  is upper semi-continuous on  $X_C$  for all  $x_{I \setminus C} \in X_{I \setminus C}$ . The game  $\Gamma(C)$  satisfies all the conditions of Theorem 2, and hence admits a Nash equilibrium. Therefore, using Lemma 4,  $(\Gamma, C)$  admits a partial cooperative Cournot-Nash equilibrium.

## 6. Conclusions

We would like to end by briefly considering some topics of further research. First, consider the partial cooperative Cournot-Nash equilibrium. Obviously, we are looking at the second stage of a two stage process, the first stage being a coalition formation game. A complete analysis would therefore make the formation of the group endogenous and look at *stable* coalitions. However, to do this kind of analysis, one needs to specify in significant details, for instance the nature of the payoff function. Such analysis has been carried out in Carraro and Siniscalco [12,13].

The other direction in which to take this kind of analysis is to consider other coalition structures. Here, we have analyzed the case of one non-singleton coalition because this is the dominant theme in several articles. But one can also look at multiple non-singleton coalitions. For instance, see Beaudry *et al.* [11]. Our results can be generalized into these settings without significant hurdles.

With regard to the partial cooperative Stackelberg equilibrium, one can look at settings where such an equilibrium would normally arise. For instance, see D'Aspremont *et al.* [16] and Barrett [15] for an analysis of stable coalitions in this setting. One can also look at axiomatization of these equilibria analogous to our axiomatization of the partial cooperative Cournot-Nash equilibria, an issue that we have tried to tackle without much success.

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## Appendix

### *Proof of Proposition 1*

The graph  $(S, L(S))$  is the *subgraph* of  $(I, L)$  induced by  $S$ . Two players  $i$  and  $j$  are *connected* in  $(I, L)$  if  $i = j$  or if there exists a path from  $i$  to  $j$ . A coalition  $T$  of  $I$  is a *component* of a graph  $(I, L)$  if the subgraph  $(T, L(T))$  is maximally connected, *i.e.*, if the subgraph  $(T, L(T))$  is connected and for each  $i \in I \setminus T$ , the subgraph  $(T \cup \{i\}, L(T \cup \{i\}))$  is not connected. Note that the collection of components

of  $(I, L)$  forms a partition of  $I$ . A *tree* is a graph  $(I, L)$  such that every pair of players is connected by a unique path. Note that a tree on  $I$  has exactly  $n - 1$  links. A *line* is a tree with exactly two leaves. A *forest* is a graph  $(I, L)$  such that each subgraph  $(T, L(T))$  induced by a component  $T$  is a tree.

[Proposition 1] For a proof of point (i), see Proposition 1 in Béal and Quérou [28]. For the proof of point (ii), we start by proving that under condition  $c \in (1, (n - 1)/2]$ , a network is efficient if and only if it is a tree network. Firstly, a tree network is more efficient than the empty network if  $(n - 1)(n - 2c) \geq 0$  and this inequality holds if  $c \leq (n - 1)/2$ . Secondly, any network that is not a forest cannot be efficient since some links can be deleted without altering the value of the network. Therefore, the set of efficient networks is included in the set of nonempty forests. Thirdly, we show that a tree is more efficient than any forest with at least two components. So consider any nonempty forest  $(I, L)$  with  $K$  components,  $K \in \{2, \dots, n - 1\}$ . We denote by  $n_k$  the number of players in component  $T_k$ ,  $k \in \{1, \dots, K\}$ . The total payoff of forest  $(I, L)$  is

$$\sum_{k=1}^K (n_k - 1)(n_k - 2c)$$

Because  $n > n_k$  for each  $k \in \{1, \dots, K\}$ , we obtain

$$(n - 2c) \sum_{k=1}^K (n_k - 1) > \sum_{k=1}^K (n_k - 1)(n_k - 2c)$$

Moreover,

$$n - 1 > \sum_{k=1}^K (n_k - 1) = n - K$$

so that

$$(n - 1)(n - 2c) > (n - 2c) \sum_{k=1}^K (n_k - 1) > \sum_{k=1}^K (n_k - 1)(n_k - 2c)$$

which proves efficiency of tree networks. Next, let  $|C|$  be any group of size at least  $2c - 1$ . For any  $k \in \mathbb{R}$ ,  $\lfloor k \rfloor$  denotes the largest integer smaller than or equal to  $k$ . Three cases have to be distinguished.

**Case (a).** If  $|C| = n$ , the result is immediate since the line network is an efficient network by assumption  $c \in (1, (n - 1)/2]$ .

**Case (b).** If  $|C| \leq n - 2$ , let  $C = \{2, 3, \dots, \lfloor |C|/2 \rfloor + 2, n - \lfloor |C|/2 \rfloor, \dots, n - 1\}$  if  $|C|$  is odd or  $C = \{2, \dots, |C|/2 + 1, n - |C|/2, \dots, n - 1\}$  if  $|C|$  is even. Consider the strategy profile  $x$  such that

$$x_{1,j} = \begin{cases} 1 & \text{if } j = 2 \\ 0 & \text{otherwise} \end{cases}$$

$$x_{n,j} = \begin{cases} 1 & \text{if } j = n - 1 \\ 0 & \text{otherwise} \end{cases}$$

and for each  $i \in I \setminus \{1, n\}$ ,

$$x_{i,j} = \begin{cases} 1 & \text{if } j \in \{i - 1, i + 1\} \\ 0 & \text{otherwise} \end{cases}$$

Remark that  $(I, L(x))$  is a line network described by the path  $(1, 2, \dots, n)$ . Let us prove that  $x$  is a partial cooperative Cournot-Nash equilibrium of  $(\Gamma, C)$ . A player  $i \in I \setminus C$  cannot add a link by himself

to  $(I, L(x))$  since  $x_{ji} = 0$  for each  $j \in I \setminus L_i$ . Similarly, it is not in the interest of the group to add more links since the extra links would not bring more value to the network.<sup>3</sup> Therefore, it is sufficient to test whether the group and players in  $I \setminus C$  have an incentive to cut some of their links. It is easy to check that the assumption  $c \leq (n - 1)/2$  implies  $f_i(x) \geq 0$  for each  $i \in I$  and thus  $F(x) \geq 0$ , which means that neither the group nor any player in  $I \setminus C$  would benefit from cutting all their links. Next, observe that  $f_1(x) = f_n(x) = n - 1 - c > 0$  for each  $c \leq (n - 1)/2$  while deleting their unique link entails a null payoff. It remains to check deviations for the group of cooperators and for players in  $I \setminus (C \cup \{1, n\})$ . Firstly, consider the case of a player  $i \in I \setminus (C \cup \{1, n\})$ . If player  $i$  cuts one of his two links, he will lose at least  $\lfloor |C|/2 \rfloor + 1$  connections while paying for one less link. Such a deviation is deterred if

$$\lfloor |C|/2 \rfloor + 1 \geq c$$

Since  $\lfloor |C|/2 \rfloor + 1 \geq |C|/2 + 1/2$ , this condition is met whenever  $|C| \geq 2c - 1$ . Secondly, consider the case of the group. The best deviation that the group can consider consists in cutting link  $12$  or link  $(n - 1)n$ . Cutting one such link implies a loss of one connection for each member of the group and a cost reduction of  $c$ . The deviation is deterred if  $|C| \geq c$ . Since both  $|C| \geq \lfloor |C|/2 \rfloor + 1$  for each  $|C| \geq 2$  and  $c > 1$ , this condition is verified whenever the condition in the statement of the proposition holds. Once again, the previous inequality prevents deviations in which the group deletes links  $12$  and  $(n - 1)n$ .

**Case (c).** If  $|C| = n - 1$ , the proof is similar except that one of the two leaves is added to the group.

This proves that  $x$  is a partial cooperative Cournot-Nash equilibrium assumption of  $(\Gamma, C)$  that induces the formation of an efficient nonempty network.

*Proof of Lemma 1*

In this section, we consider games  $\Gamma = (I, (X_i, f_i)_{i \in I}) \in G$  for which it is assumed that the correspondence  $D^{(\Gamma, C)} : X_C \rightarrow X_{I \setminus C}$  is non-empty valued for each  $C \subseteq I$  and each  $x_C \in X_C$ . First, we prove that the correspondence  $D^{(\Gamma, C)}$  is closed and compact-valued.

**Proposition 3** Consider a non-cooperative game  $\Gamma = (I, (X_i, f_i)_{i \in I}) \in G$  such that, for each  $i \in I$ ,  $X_i$  is a compact subset of a Euclidean space and  $f_i$  is continuous. For any  $C \subseteq I$ , if the correspondence  $D^{(\Gamma, C)} : X_C \rightarrow X_{I \setminus C}$  is upper hemi-continuous and compact-valued.

Consider any  $\Gamma \in G$  satisfying the conditions above and any  $C \subseteq I$ . First, we show that the correspondence  $D^{(\Gamma, C)} : X_C \rightarrow X_{I \setminus C}$  is closed. Consider two convergent sequences,  $(y_{C,p})_{p \in \mathbb{N}} \rightarrow y_C$  and  $(x_{I \setminus C,p})_{p \in \mathbb{N}} \rightarrow x_{I \setminus C}$  such that  $x_{I \setminus C,p} \in D^{(\Gamma, C)}(y_{C,p})$  for each  $p \in \mathbb{N}$ . We shall prove that  $x_{I \setminus C} \in D^{(\Gamma, C)}(y_C)$ . By the definition of a Nash equilibrium,

$$f_i(x_{I \setminus C,p}, y_{C,p}) - f_i(y_i, x_{I \setminus (C \cup \{i\}),p}, y_{C,p}) \geq 0$$

for each  $p \in \mathbb{N}$  and any arbitrary  $y_i \in X_i$ . Given  $f_i$  is continuous and the fact that  $(x_{I \setminus C,p})_{p \in \mathbb{N}} \rightarrow x_{I \setminus C}$  and  $(y_{C,p})_{p \in \mathbb{N}} \rightarrow y_C$ , it holds that for any  $y_i \in X_i$ ,

$$f_i(x_{I \setminus C}, y_C) - f_i(y_i, x_{I \setminus (C \cup \{i\})}, y_C) \geq 0$$

<sup>3</sup>For completeness, note that the group can simultaneously delete and add links between its members. However any such change in the network configuration cannot improve the value of the network for the group.

This proves that  $x_{I \setminus C} \in D^{(\Gamma, C)}(y_C)$ . Hence,  $D^{(\Gamma, C)}$  is a closed correspondence. Next, because a closed correspondence is closed-valued,  $D^{(\Gamma, C)}$  is closed-valued and hence  $D^{(\Gamma, C)}(y_C)$  is a closed set for all  $y_C \in X_C$ . Given that  $X_{I \setminus C}$  is compact,  $D^{(\Gamma, C)}(y_C)$  being a closed subset of a compact set is compact as well, making  $D^{(\Gamma, C)}$  compact-valued. Finally, since a closed and compact-valued correspondence is upper hemi-continuous, we conclude that  $D^{(\Gamma, C)}$  is upper hemi-continuous.

Next, we show that functions  $F^-$  and  $F^+$  are well-defined.

**Proposition 4** Consider a non-cooperative game  $\Gamma = (I, (X_i, f_i)_{i \in I}) \in G$  such that, for each  $i \in I$ ,  $X_i$  is a compact subset of a Euclidean space and  $f_i$  is continuous. For any  $C \subseteq I$ , functions  $F^-$  and  $F^+$  are well-defined.

Consider any  $\Gamma \in G$  satisfying the conditions above and any  $C \subseteq I$ . For any  $y_C \in X_C$ , consider the function  $F_{y_C} : X_{I \setminus C} \rightarrow \mathbb{R}$  defined as  $F_{y_C}(x_{I \setminus C}) = F(y_C, x_{I \setminus C})$  for each  $x_{I \setminus C} \in X_{I \setminus C}$ . From the continuity of  $f_i$ ,  $F_{y_C}$  is continuous. In addition, we would like to maximize  $F_{y_C}$  on the subset  $D^{(\Gamma, C)}(y_C) \subseteq X_{I \setminus C}$ , which is a compact set by the proof of Proposition 3. As a consequence, a maximum and a minimum of  $F_{y_C}$  on  $D^{(\Gamma, C)}(y_C)$  exists for each  $y_C \in X_C$ . Thus  $\eta_{\min}(y_C)$  and  $\eta_{\max}(y_C)$  are non-empty for every  $y_C \in X_C$ , which implies that functions  $F^-$  and  $F^+$  are well-defined.

The proof of Lemma 1 uses the following Berge's Theorem in order to show that the two correspondences  $\eta_{\max}$  and  $\eta_{\min}$  are upper hemi-continuous. Let  $A \subseteq \mathbb{R}^m$  and  $B \subseteq \mathbb{R}^n$ .

**Theorem 5 (Berge, 1963)** Let  $\phi : A \rightarrow B$  be an upper hemi-continuous and non-empty valued correspondence and  $\psi : A \times B \rightarrow \mathbb{R}$  an upper semi-continuous function. Then the function  $M : A \rightarrow \mathbb{R}$  defined as

$$M(a) = \max\{\psi(a, b) : b \in \phi(a)\}$$

is upper semi-continuous.

For a proof, see Theorem 2 pp. 116 in Berge [30] or Proposition 12.3 pp. 278 in Moore [31].

[Proof of Lemma 1] We only detail the proof of existence of an optimistic partial cooperative equilibrium since the proof of existence of a pessimistic partial cooperative equilibrium is similar. Consider any  $\Gamma \in G$  satisfying the conditions above and any  $C \subseteq I$ . From Proposition 3,  $D^{(\Gamma, C)}$  is upper hemi-continuous and compact-valued. Then, upper semi-continuity of  $F^+$  follows from an application of Theorem 5 to correspondence  $D^{(\Gamma, C)} : X_C \rightarrow X_{I \setminus C}$ , function  $F : X_C \times X_{I \setminus C} \rightarrow \mathbb{R}$  and  $F^+ : X_C \rightarrow \mathbb{R}$ . Since the upper semi-continuous function  $F^+$  is defined on the compact set  $X_C$ , it admits a maximum. Hence, an optimistic partial cooperative equilibrium exists.

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