

Article

High-Resolution Imaging of Multi-Beam Uniform Linear Array Sonar Based on Two-Stage Sparse Deconvolution Method

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Highlights

What are the main findings?

- Beamforming methods in underwater sonar imaging often produce a wide main-lobe, which degrades the accuracy and quality of the acoustic images. This broadening reduces spatial resolution, introduces high sidelobes that lead to energy leakage, and creates a point-spread effect that blurs the imaged targets. Therefore, it is crucial to enhance imaging precision while effectively removing the influence of point spread.
- Incorporating prior knowledge of the spatial sparsity of targets while mitigating the point-spread effect enables a substantial improvement in imaging accuracy and quality.

What are the implications of the main findings?

- Owing to the spatial sparsity in the distribution of target echoes, a reconstruction method for directional information, formulated under dual sparse constraints, is proposed to effectively suppress interference from non-target directions.
- An auxiliary-parameter-based iterative soft-thresholding method is proposed for solving high-resolution sparse reconstruction models, allowing for efficient, stable signal recovery and high-precision imaging.

Abstract

Classical beamforming (CBF) beamforming constrains the accuracy and quality of underwater acoustic imaging by producing wide main-lobes that reduce resolution, high sidelobes that cause leakage, and point-spread functions that blur targets. Existing approaches typically address only one of these issues at a time, limiting their ability to resolve multiple, interrelated problems simultaneously. In this study, we introduce a double-compression deconvolution high-resolution beamforming method designed to enhance multi-beam sonar imaging using an underwater uniform linear array. The proposed approach formulates imaging as a sparse deconvolution problem and suppresses off-target interference through two sparse constraints, thereby improving the sonar's resolving capability. During sparse reconstruction, an auxiliary-parameter iterative shrinkage-threshold algorithm is employed to recover azimuthal sparse signals with higher accuracy. Simulations and controlled pool experiments demonstrate that, relative to classical beamforming, the proposed method significantly improves resolution, suppresses off-target interference, expands the imaging intensity dynamic range, and yields clearer target representations. This study provides an effective strategy to mitigate intrinsic limitations in high-resolution underwater sonar imaging.

Keywords: uniform linear array; multi-beam sonar; sparse deconvolution; high resolution



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1. Introduction

Underwater detection relies on technologies such as acoustics, optics, electricity, and magnetism to collect information about the underwater environment, objects, and resources. It plays a crucial role in marine research (e.g., submarine observation networks, climate change monitoring), resource development (oil and gas exploration, mineral assessment), environmental monitoring (ecological protection and restoration), engineering operations (pipeline deployment, marine construction, and maintenance), and national defense (underwater surveillance, anti-submarine warfare). Among these technologies, acoustics remains the most effective modality, offering precise imaging of underwater targets and accurate environmental characterization [1–3]. Multi-beam sonar systems use digital beamforming (BF) technology to independently control beam transmission and reception, enabling wide coverage and high-precision acquisition of underwater information, and thus playing a central role in underwater sensing [4,5]. The horizontal linear array, owing to its one-dimensional structure, supports a large physical aperture. This configuration provides high horizontal resolution and strong array gain in target detection, allowing long-range sensing of low-frequency sound waves. Moreover, linear arrays are relatively simple to process, manufacture, and integrate, resulting in manageable system costs. Consequently, most contemporary research on multi-beam sonar favors linear array architectures. The linear array multi-beam sonar system introduces challenges in array signal processing, which form the core focus of this study. In terms of range resolution, a single-frequency pulse yields a resolution determined by pulse width, whereas wideband waveforms can achieve improved range resolution through pulse compression. For azimuth resolution, CBF sums the signals received by each array element after applying time delays determined by the element positions and steering angle. When the steering angle matches the direction of arrival (DOA) of a target, constructive interference strengthens the target signal, whereas non-target noise is attenuated through incoherent summation. Although CBF provides good robustness and desirable array gain (AG), its performance is fundamentally constrained by the main-lobe width and sidelobe level. Strong interference may leak energy through high sidelobes, complicating the detection of two adjacent targets of similar intensity and often yielding suboptimal imaging results [6].

From a waveform perspective, the theory of CBF typically relies on the plane wave assumption to leverage the computational efficiency of fast Fourier transform in linear arrays. However, this assumption causes defocusing in near-field applications. While near-field spherical wave assumptions allow BF in the wavenumber domain, they forgo the computational advantages of fast Fourier transform. To mitigate this drawback, the Fresnel approximation phase compensation method, when integrated with fast Fourier transform, improves near-field CBF focusing [7]. From the perspective of BF imaging, spatial angular resolution is intrinsically linked to array aperture and operating frequency. Although resolution can be improved by increasing frequency or aperture size, these changes can adversely impact the sonar detection range and overall system complexity. Consequently, recent research has focused on developing super-resolution BF techniques that operate within fixed frequency and aperture constraints.

To further address the limitations of CBF and enhance its effectiveness, a minimum-variance distortionless response (MVDR) approach focused on azimuth filter design is proposed. MVDR constructs a filter that preserves the target-direction signals while suppressing interference from other directions. Specifically, the white noise gain-constrained MVDR and the distributed source-constrained MVDR approaches described in [8,9] improve CBF by leveraging the covariance matrix of the received signals. Similarly, multiple signal classification (MUSIC) is a technique for DOA estimation that uses the covariance matrix. It emphasizes the orthogonality within the signal's internal structure and accurately

estimates the target azimuth through eigenvalue decomposition. An enhanced version, root MUSIC, provides improved tolerance to signal outliers compared with the original MUSIC method. Each method offers unique advantages: the MVDR method excels in robustness and stability under low signal-to-noise ratio (SNR) conditions, whereas the MUSIC method performs effectively in high-SNR scenarios when the number of signal sources is known. Nonetheless, both methods rely heavily on the covariance matrix of the observed signals and perform poorly under low-snapshot conditions with insufficient samples, particularly when addressing coherent signal sources [10,11]. Roy R. et al. [12] introduced the subspace direction of arrival (DOA) estimation technique known as estimation of signal parameters via rotational invariance techniques (ESPRIT). This approach leverages the array's inherent rotational invariance to directly compute the DOA of signals, circumventing the spectral peak search required by the MUSIC method. Additionally, variations of this technique, such as least squares ESPRIT (LS-ESPRIT) and total least squares ESPRIT (TLS-ESPRIT), have been developed based on different strategies for solving rotational invariance equations. Yardibi, T. et al. [13–17] proposed a semi-parametric iterative adaptive power spectrum estimation technique called the iterative adaptive approach (IAA). This approach operates without strict assumptions about the number of signal sources and demonstrates notable stability and robustness.

Recently, Yang T. C. [18,19] redefined the CBF method under the far-field assumption by converting the spatial filtering task into a source-reconstruction problem using a known point-spread function (PSF). The source is estimated through the Richardson–Lucy deconvolution. Bahr C. et al. [20] further examined the feasibility of this deconvolution framework. Xenaki A. et al. [21,22] extended deconvolution BF to three-dimensional planar array imaging, while Tiana-Roig E. et al. [23] adapted it for circular array imaging. Sun D.J. et al. [24] applied a non-negative least squares solution to deconvolution BF for vector arrays. Mei J. et al. [25] investigated near-field target imaging using deconvolution-based BF. These studies confirm the superior performance of deconvolution technology in high-resolution imaging. Compressed Sensing (CS) has been extensively applied in areas such as image restoration, denoising, and ultrasound imaging due to its powerful sparse reconstruction capabilities [26–28]. The incorporation of CS has provided BF and DOA estimation problems with enhanced super-resolution capabilities relative to traditional power spectrum estimation methods [29]. Studies have further shown that CS improves DOA estimation accuracy under low-snapshot conditions [30], although this improvement is accompanied by increased computational complexity.

Related Work

The existing literature on BF for underwater uniform linear arrays has primarily focused on two critical factors: controlling the main-lobe width and sidelobe height, and mitigating the imaging blur induced by the PSF. Addressing these challenges concurrently during the imaging process remains an unresolved issue that requires further investigation. This study addresses the dual challenges of resolution loss and blurring by using sparse deconvolution to enhance CBF performance. By accounting for both factors, the proposed approach aims to substantially increase the intensity dynamic range and achieve high-resolution imaging of underwater targets.

The remainder of this paper is organized as follows.

The linear array CBF model is presented in Section 2.1. Section 2.2 introduces the deconvolution beamforming (dCv-BF) model, and Section 2.3 derives the solution for the proposed Compressed Sensing deconvolution beamforming method (Cs-dCv-CBF method). Section 3 presents the angular resolution simulation test, point source simulation test,

and water tank experiment for the proposed Cs-dCv-CBF. Sections 4 and 5 summarize the research findings.

2. Methods

2.1. Conventional Beamforming (CBF)

(1) Uniform Horizontal Linear Array (UHLA) Received Signal Model

The UHLA sonar system consists of multiple receiving array elements arranged uniformly along a straight line. During operation, the transmitting array emits acoustic pulses into the surveillance area using pre-formed beams. These pulses are scattered by underwater targets and subsequently received by the array elements. Due to slight geometric differences in the distance between each element and the target, echo signals reach the array elements with progressively increasing or decreasing time delays. The beamforming algorithm applies delay weighting and signal synthesis to these echoes, forming a high-gain aggregated receiving beam in a specific direction. This process enables directional filtering of the received signals. By computing the intensity and delay of echoes from each direction, the system infers the range and reflection strength of targets at various azimuths. Ultimately, by combining echo data from all directions within the sensing area, an underwater acoustic image is constructed, as illustrated in Figure 1a. Under far-field conditions, narrow-band echo signals arrive at the array as plane waves, as illustrated in Figure 1b.

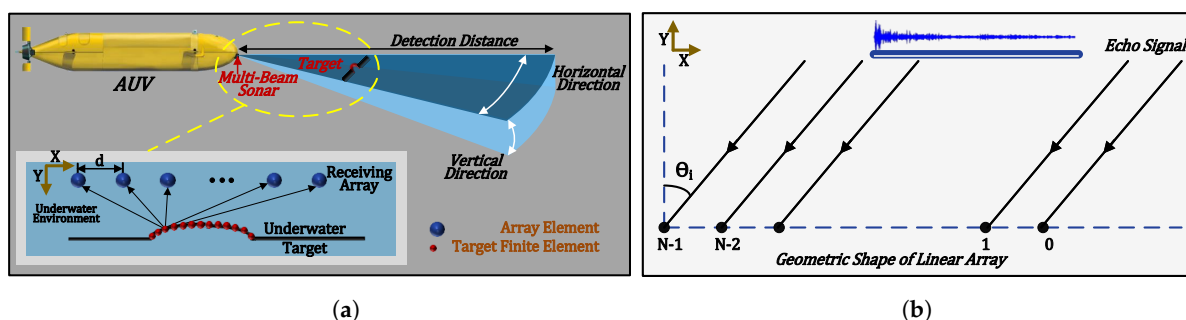


Figure 1. UHLA sonar working principle and diagram of far-field echo reception. (a) UHLA Sonar Target Detection Process. (b) UHLA Far-Field Received Signal Model.

The UHLA receiving array places its N elements uniformly at equal intervals in a d configuration. Each element simultaneously detects the acoustic scattering echoes from the target. To compensate for the phase shifts of echoes from the same scatterer at different elements, arising from their relative positions, the geometric relationships illustrated in Figure 1b are used.

In this context, θ_i refers to acquiring and measuring echo signals at an angle of θ_i relative to the normal direction of HLA, and A_i denotes the strength of the target echo signal. The phase-shift vector associated with the N array elements in the i -th target direction is denoted as $\mathbf{p}_i = [p_1^i, p_2^i, \dots, p_n^i, \dots, p_N^i]^T$, $\mathbf{p}_i \in \mathbb{C}^{N \times 1}$, with p_n^i defined as $p_n^i = e^{-jk(n-1)d \sin \theta_i}$, $n = 1, 2, \dots, N-1, N$, representing the phase shift of the i -th target direction at the n -th array element, and $k = 2\pi/\lambda$ indicating the wave number. The superscript T represents the transpose of the vector, while j denotes the imaginary unit, and $\mathbf{n} \in \mathbb{C}^{N \times 1}$ represents the noise vector for N array elements. The reception model $\mathbf{r}_i \in \mathbb{C}^{N \times 1}$ for the signal from the i -th target direction is given by

$$\begin{aligned} \mathbf{r}_i &= A_i \mathbf{p}_i + \mathbf{n} = A_i \cdot [p_1^i, p_2^i, \dots, p_n^i, \dots, p_N^i]^T + \mathbf{n} \\ &= A_i \cdot [0, -jkd \sin \theta_i, \dots, -jk(n-1)d \sin \theta_i, \dots, -jk(N-1)d \sin \theta_i]^T + \mathbf{n}. \end{aligned} \quad (1)$$

(2) CBF and Beam Power Mode

CBF, also known as delay-and-sum beamforming, is based on array signal processing principles. It employs the steering vector to coherently combine target echo signals received at each array element, thereby steering the array's reception toward a specific direction. This enhances the desired signal while suppressing interference from non-target directions. The steering vector compensates for phase differences resulting from echo incidence at angle θ , maximizing the signal response at this angle. The steering vector is defined as $\mathbf{s} = [s_1, s_2, \dots, s_n, \dots, s_{N-1}, s_N]^T, n = 1, 2, \dots, N$, with $s_n = (1/N) \cdot e^{-jk(n-1)d \sin \theta}$ denoting the phase delay of the n -th array element. The BF output in the θ direction is given by

$$\begin{aligned}
 Y(\sin(\theta)) &= \mathbf{s}^H \mathbf{r}_i \\
 &= \left[[s_1, s_2, \dots, s_n, \dots, s_{N-1}, s_N]^T \right]^H \\
 &\quad \times A_i \cdot [0, -jkd \sin \theta_i, \dots, -jk(n-1)d \sin \theta_i, \dots, -jk(N-1)d \sin \theta_i]^T + \mathbf{n} \\
 &= \left[[0, -jkd \sin \theta, \dots, -jk(n-1)d \sin \theta, \dots, -jk(N-1)d \sin \theta]^T \right]^H \\
 &\quad \times A_i \cdot [0, -jkd \sin \theta_i, \dots, -jk(n-1)d \sin \theta_i, \dots, -jk(N-1)d \sin \theta_i]^T + \mathbf{n} \\
 &\approx \frac{A_i}{N} \cdot \sum_{n=0}^{N-1} e^{i(2\pi)n(d/\lambda)(\sin \theta - \sin \theta_i)} \\
 &= A_i \frac{\sin(\pi N(d/\lambda)(\sin \theta - \sin \theta_i))}{N \sin(\pi(d/\lambda)(\sin \theta - \sin \theta_i))} \\
 &= A_i \frac{\sin c(\pi N(d/\lambda)X_i)}{\sin c(\pi(d/\lambda)X_i)}.
 \end{aligned} \tag{2}$$

For notational convenience, assume that $X_i = \sin \theta - \sin \theta_i$, with $\sin c = \sin(x)/x$ and H representing the conjugate transposes. From the beamformer output in the θ direction given in Equation (2), the corresponding output power $B(\sin(\theta))$ in the θ direction is obtained as follows:

$$\begin{aligned}
 B(\sin(\theta)) &= |Y(\sin(\theta))|^2 = |A_i|^2 \left| \frac{\sin c(\pi N(d/\lambda)X_i)}{\sin c(\pi(d/\lambda)X_i)} \right|^2, \\
 X_i &= \sin \theta - \sin \theta_i.
 \end{aligned} \tag{3}$$

Figure 2 illustrates BF output capability diagrams for arrays with different element counts and a targeting direction of $\theta_i = 30^\circ$, with a steering vector scanned from 0 to 90° . When $\sin(\theta) = \sin(\theta_i)$, the BF energy peak achieves the maximum output energy of $|A_i|^2$. Thus, increasing the number of array elements and identifying the energy peak effectively estimates direction, enhances echoes in the desired direction, and suppresses those from other directions. As N increases, the main beam becomes narrower, improving directional filtering and enhancing target resolution.

The area illuminated by the UHLA sonar transmitting beam is continuous. The signals collected by the receiving array are a superposition of target echoes from multiple directions, with $\mathbf{r} = \sum_{k=1}^K \mathbf{r}_k = \sum_{k=1}^K A_k \mathbf{p}_k, k = 1, 2, \dots, K$, and K indicating the number of echo directions. The terms “output of BF in the θ direction” and “output power of BF in the direction” refer to the corresponding measurements:

$$\begin{aligned}
 Y(\sin(\theta)) &= \mathbf{s}^H \mathbf{r} = \mathbf{s}^H \sum_{k=1}^K \mathbf{r}_k = \sum_{k=1}^K \mathbf{s}^H A_k \mathbf{p}_k \\
 &= \sum_{k=1}^K A_k \frac{\sin c(\pi N(d/\lambda) X_k)}{\sin c(\pi(d/\lambda) X_k)}, \\
 k &= 1, 2, \dots, K, X_k = \sin \theta - \sin \theta_k,
 \end{aligned} \tag{4}$$

$$\begin{aligned}
 B(\sin(\theta)) &= \langle |\mathbf{s}^H \mathbf{r}| \rangle = \mathbf{s}^H \langle \mathbf{r} \mathbf{r}^H \rangle \mathbf{s} \\
 &= \mathbf{s}^H \left\langle \left(\sum_{k=1}^K A_{k-i} \mathbf{p}_{k-i} \right) \left(\sum_{k=j=1}^K A_{k-j} \mathbf{p}_{k-j} \right)^H \right\rangle \mathbf{s}, \\
 &= \mathbf{s}^H \left[\sum_{k=i=1, k=j=1}^K \mathbf{p}_{k-i} \langle A_{k-i} A_{k-j}^* \rangle \mathbf{p}_{k-j}^H \right] \mathbf{s},
 \end{aligned} \tag{5}$$

where $\langle \cdot \rangle$ denotes the sample mean. Given that the different source signals are incoherent, $\langle A_{k-i} A_{k-j}^* \rangle = \langle |A_{k-i}|^2 \rangle \delta_{k-i, k-j}$ allows Equation (5) to be extrapolated to Equation (6). Thus, estimating the multi-target source energy spectrum reduces to estimating the sum of several single-target energy spectra:

$$\begin{aligned}
 B(\sin(\theta)) &= \mathbf{s}^H \left[\sum_{k=i=1, k=j=1}^K \mathbf{p}_{k-i} \langle A_{k-i} A_{k-j}^* \rangle \mathbf{p}_{k-j}^H \right] \mathbf{s} \\
 &= \sum_{k=i=1}^K \langle |A_{k-i}|^2 \rangle |\mathbf{s}^H \mathbf{p}_{k-i}|^2 \\
 &= \sum_{k=i=1}^K \langle |A_{k-i}|^2 \rangle \left| \frac{\sin c(\pi N(d/\lambda) X_{k-i})}{\sin c(\pi(d/\lambda) X_{k-i})} \right|^2, \\
 X_{k-i} &= \sin \theta - \sin \theta_{k-i}
 \end{aligned} \tag{6}$$

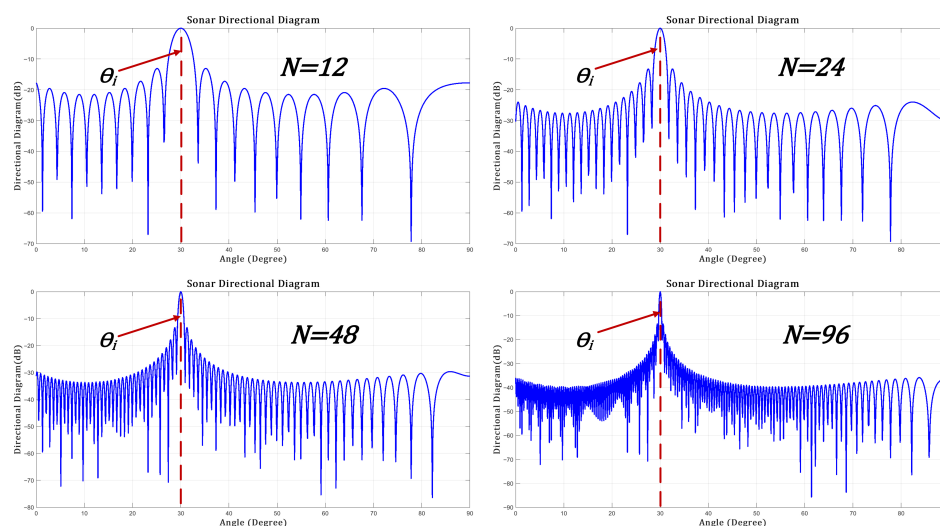


Figure 2. BF energy output at varying angles with different numbers of array elements.

2.2. Deconvolved BF (dCv-BF)

(1) Establishment of the CBF convolutional model

In applying BF to the target, assuming complete non-coherence is overly idealized, introducing non-coherent conditions leads to difficulties when addressing coherence-related

issues. Therefore, starting with Equation (4), we develop a more comprehensive continuous BF output model that does not impose the non-coherence assumption:

$$\begin{aligned}
 Y(\sin(\theta)) &= \sum_{k=1}^K A_k \frac{\sin c(\pi N(d/\lambda)X_k)}{\sin c(\pi(d/\lambda)X_k)} \\
 &= \int_{-1}^1 Y_p(\sin \theta - \sin \varphi) A(\sin \varphi) d(\sin \varphi) \\
 Y_p(X) &= \frac{\sin c(\pi N(d/\lambda)X)}{\sin c(\pi(d/\lambda)X)} \\
 A(\sin \varphi) &= \sum_{k=1}^K A_k \delta(\sin \varphi - \sin \theta_k) \\
 X &= \sin \theta - \sin \varphi
 \end{aligned} \tag{7}$$

Based on Equation (6), and using the convolutional representation of the BF output expressed in Equation (7), we derive the convolution-based formulation for the BF energy output as follows:

$$\begin{aligned}
 B(\sin(\theta)) &= \sum_{k=1}^K \langle |A_k|^2 \rangle \left| \frac{\sin c(\pi N(d/\lambda)X_k)}{\sin c(\pi(d/\lambda)X_k)} \right|^2 \\
 &= \int_{-1}^1 B_p(\sin \theta - \sin \varphi) S(\sin \varphi) d(\sin \varphi) \\
 B_p(X) &= \left| \frac{\sin c(\pi N(d/\lambda)X)}{\sin c(\pi(d/\lambda)X)} \right|^2 \\
 S(\sin \varphi) &= \sum_{k=1}^K |A_k|^2 \delta(\sin \varphi - \sin \theta_k) \\
 X &= \sin \theta - \sin \varphi
 \end{aligned} \tag{8}$$

(2) Total Variation L1 Deconvolved BF model

From the convolution description of $\int_{-1}^1 B_p(\sin \theta - \sin \varphi) S(\sin \varphi) d(\sin \varphi)$ regarding BF energy output, it is evident that B represents the observation obtained through CBF, while S denotes the true azimuth energy of the target. The two are linked through a convolution operation. The azimuth ambiguity function B_p impacts the true azimuth energy, resulting in inaccurate azimuth estimation when using B in CBF, as illustrated in Figure 3.

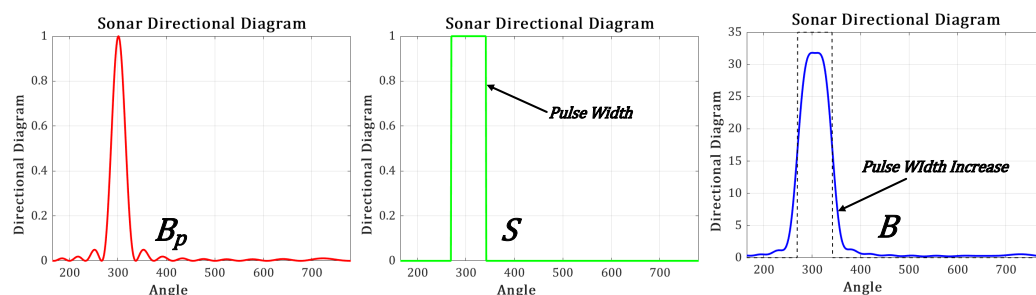


Figure 3. Schematic diagram of BF energy output broadening.

In sonar array processing, the azimuth ambiguity function serves as the point-spread function (PSF), defining the system's intrinsic azimuth-resolution limit and reflecting constraints imposed by the array aperture and signal processing chain. The PSF characterizes how the energy of an ideal point target diffuses into neighboring azimuth units. Thus, ac-

curately estimating a target's azimuth can be reformulated as recovering the target azimuth energy spectrum using the forward model expressed in Equation (8). This requires deconvolving the known PSF from the observed azimuth energy. Conventional methods often fall short due to the constraints imposed by real target scenarios. To improve robustness in estimating target azimuth energy, this paper introduces the Total Variation L1 BF model, a regularized deconvolution framework. As shown in Equation (9), the $\|S \otimes B_p - B\|$ term quantifies the discrepancy between the estimated azimuth energy spectrum after convolution and the observed spectrum, thereby preserving the physical consistency of the process. In addition, the $\|\nabla S\|$ and $\|S\|$ terms regulate the model's overall noise resistance, and λ represents the weighting coefficient of the constraint term:

$$E = \|S \otimes B_p - B\| + \lambda \|\nabla S\| + \lambda \|S\| \quad (9)$$

2.3. Compressed Sensing dCv-CBF (Cs-dCv-CBF)

(1) Transformation of the coupled and non-differentiable Cs-dCv-CBF model

Minimizing E (Equation (9)) allows the method to satisfy both the ∇S and S sparsity requirements while still meeting reconstruction constraints. Although minimizing Equation (9) appears straightforward, two major challenges in Compressed Sensing arise. First, the objective function contains several L1 norms, which are non-differentiable at zero, making gradient-descent-based optimization methods ineffective. Second, the data fidelity term $\|S \otimes B_p - B\|$ and the regularization terms $\|\nabla S\|$ and $\|S\|$ are strongly coupled through the optimization variable, preventing an analytical solution. To address these issues, Equation (9) is transformed into Equation (10) by introducing auxiliary variables:

$$\begin{aligned} E(S, \omega, o) &= \frac{1}{2\beta} \|S \otimes B_p - B\|_2^2 + \frac{1}{2\theta} \|S - \omega\|_2^2 + \frac{1}{2\eta} \|\nabla S - o\|_2^2 + \lambda_1 \|\omega\|_1 + \lambda_2 \|o\|_1 \\ E(S, \omega, o) &= \frac{1}{2\beta} \|\mathbf{H}S - B\|_2^2 + \frac{1}{2\theta} \|S - \omega\|_2^2 + \frac{1}{2\eta} \|\mathbf{A}S - o\|_2^2 + \lambda_1 \|\omega\|_1 + \lambda_2 \|o\|_1 \end{aligned} \quad (10)$$

In this study, we apply half-quadratic splitting to introduce auxiliary variables, thereby transforming a complex, coupled, and non-differentiable optimization problem into a series of simpler, decoupled, and smooth sub-problems. By incorporating the auxiliary variables ω and o , the original single-variable optimization problem is reformulated as a multi-variable problem with relaxation constraints. This reformulated problem can then be efficiently solved for the variables ω and o using alternating minimization. Additionally, the algorithm's convergence improves as the relaxation parameters β , θ , and η are gradually reduced. λ_1 and λ_2 represent the weighting coefficients for the constraint components.

(2) Solution of the dCv-CBF transformation model

After the reformulation, Equation (10) enables parameter estimation through the alternating minimization (AM) method. The solver sequentially addresses the optimization problems for S , ω , and o , initially setting the values of ω and o to 0.01. Next, the terms associated with S in Equation (10) are isolated, while unrelated terms are removed. With ω and o held constant, we optimize and solve for S , as shown in Equation (11). Since Equation (11) is differentiable, its analytical solution for minimization is given in Equation (12):

$$\begin{aligned} E(S; \omega, o) &= \|S \otimes B_p - B\|_2^2 + \frac{\beta}{\theta} \|S - \omega\|_2^2 + \frac{\beta}{\eta} \|\nabla S - o\|_2^2 \\ E(S; \omega, o) &= \|\mathbf{H}S - B\|_2^2 + \frac{\beta}{\theta} \|S - \omega\|_2^2 + \frac{\beta}{\eta} \|\mathbf{A}S - o\|_2^2 \end{aligned} \quad (11)$$

Following the same procedure, we solve for ω and o individually under a fixed S . Since ω and o serve as auxiliary constraint variables and are mutually indepen-

dent, three sub-optimization objective functions arise. These functions, denoted as $E(S; \omega, o)$, $E(\omega; S)$ and $E(o; S)$, are derived from Equation (10) and are presented in Equations (11)–(14):

$$S = [\mathbf{H}^T \mathbf{H} + (\beta/\theta)I + (\beta/\eta)\mathbf{A}^T \mathbf{A}]^{-1} [\mathbf{H}^T B + (\beta/\theta)\omega + (\beta/\eta)\mathbf{A}^T o]. \quad (12)$$

$$E(\omega; S) = \frac{1}{2} \|S - \omega\|_2^2 + \theta \lambda_1 \|\omega\|_1 \quad (13)$$

$$E(o; S) = \frac{1}{2} \|\mathbf{A}S - o\|_2^2 + \eta \lambda_2 \|o\|_1. \quad (14)$$

Each objective function in Equations (13) and (14) represents a univariate optimization problem that includes an L1 norm constraint term. These auxiliary variables can be optimized using the Shrinkage method, as shown in Equations (15) and (16), where λ_1 and λ_2 are initialized to 1. Analysis of Equation (11) shows that the solver sequentially updates S , ω , and o while holding the remaining variables fixed, and that the final solution for S is obtained through minimization. The relaxation parameters β , θ , and η are small positive values used during the iterative process to enforce consistency between each auxiliary variable and its corresponding term. To accelerate convergence and dynamically adjust the constraints, these relaxation parameters are gradually reduced, thus enhancing the sparsity of the reconstruction result. The auxiliary variables ω and o are, respectively, expressed as

$$\omega = \text{sign}(S) \max(\|S\| - \theta \lambda_1, 0), \quad (15)$$

$$o = \text{sign}(\mathbf{A}S) \max(\|\mathbf{A}S\| - \eta \lambda_2, 0). \quad (16)$$

The overall execution flow of the algorithm is summarized in Algorithm 1.

Algorithm 1: Solution of the dCv-CBF transformation model.

INPUT:

B : Beam Power,

S : Source Power Distribution

$\mathbf{H}$: Point Scattering Function Matrix

$\mathbf{A}$: Gradient Function Matrix

$S \leftarrow B$ $\beta \leftarrow \beta_0$

repeat

Solve for S using Equation (12)

$\theta \leftarrow \theta_0$

repeat

Solve for ω using Equation (15)

$\eta \leftarrow \eta_0$

repeat

Solve for o using Equation (16)

$\eta \leftarrow \eta/2$

until $\eta < \eta_{\min}$

$\theta \leftarrow \theta/2$

until $\theta < \theta_{\min}$

$\beta \leftarrow \beta/2$

until $\beta < \beta_{\min}$

OUTPUT: Source Power Distribution S

3. Results

To evaluate the performance of the proposed method, this section examines the multiple sparse deconvolution imaging algorithm using two approaches: simulation data validation and field test validation. Simulation tests enable detailed analysis and one-on-one comparisons of imaging performance indicators, such as angular resolution of the uniform linear array, PSF deconvolution-based information restoration, and point target imaging ability. Field test validation involves water-pool experiments designed to assess the imaging performance for different targets. All analyses in this section were conducted using Matlab R2022b 9.13 on a Microsoft Windows 10 workstation equipped with an NVIDIA GTX TITAN-XP graphics processor and 64 GB of memory, ensuring a thorough evaluation of the proposed scheme.

3.1. Simulation Data Test

(1) Analysis of UHLA's Angle Resolution Performance

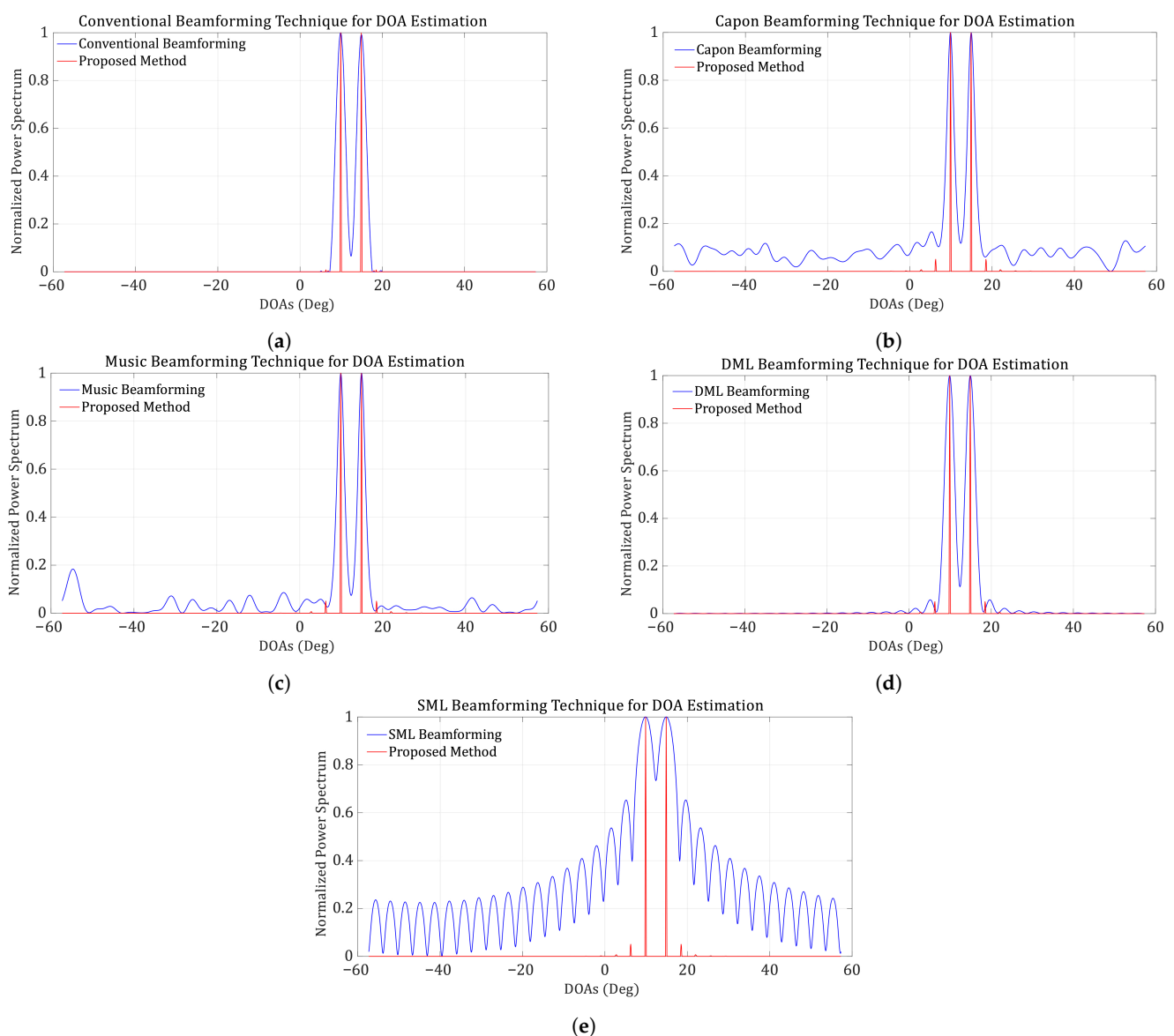
This section evaluates the performance of the proposed imaging algorithm by simulating the angular resolution capability of the UHLA for targets located at angles of 10° and 15° . For a more comprehensive performance comparison, we include the CBF, Capon, MUSIC, Deterministic Maximum Likelihood (DML), and Stochastic Maximum Likelihood (SML) methods. Each of these five methods offers distinct advantages in terms of signal-to-noise ratio performance, achievable resolution, and robustness. The simulation parameters are as follows: the number of array elements is $N = 32$, the spacing between them is $d = \lambda/2$, and the signal frequency is 0.125 kHz. The corresponding simulation results are presented in Figure 4.

Figure 4 shows that the CBF, Capon, MUSIC, DML, and SML methods all produce two noticeable main-lobe peaks at 10° and 15° , indicating that each method can resolve the two targets. However, compared with the method proposed in this paper, the CBF, Capon, and MUSIC methods exhibit broader main-lobe angular ranges. Such broadening implies weaker separation ability, making it more difficult to distinguish closely spaced targets and increasing the likelihood of localization errors. Additionally, there is a specific amplitude range in the sidelobe directions. When strong interference occurs within these sidelobe directions, it can leak through and cause the sidelobe energy of strong targets to overwhelm nearby weak targets. This reduces both the dynamic range of the detection system and the contrast of the imaging results. Furthermore, the DML and SML methods rely on model assumptions, treating the signal as a deterministic process and a Gaussian process, respectively, and their performance can fluctuate significantly when these assumptions are not met. When quantitatively comparing angular resolution performance across different methods, the 3 dB main-lobe width is used as a key evaluation metric. Table 1 summarizes the 3 dB main-lobe widths for Capon, MUSIC, DML, and SML methods, which are 2.91° , 2.86° , 3.64° , and 13.17° , respectively. These results indicate that Capon and MUSIC provide superior angular resolution and can mitigate resolution degradation in low-signal-to-noise underwater environments. By contrast, DML and SML rely heavily on their underlying model assumptions, and their performance degrades significantly under model mismatch. The method proposed in this study explicitly exploits angular sparsity during deconvolution-based reconstruction and introduces auxiliary variables to improve the sparse reconstruction solution, thereby further enhancing angular resolution. The experimental results demonstrate that the proposed method achieves a 3 dB main-lobe width of 0.26° , a substantial improvement over the conventional approaches.

Table 1. Quantitative analysis of angular resolution performance.

Method	3 dB Main-Lobe Width (°)
Capon	2.91
MUSIC	2.86
DML	3.64
SML	13.17
Proposed	0.26

Using the CBF model, the proposed method incorporates a reasonable double sparsity prior, specifically a spatial sparsity constraint, into the inversion problem. This approach transforms an ill-posed or poorly posed inverse problem into a well-posed optimization problem, substantially enhancing main-lobe resolution while significantly reducing side-lobes. As a result, it achieves target resolution performance far superior to that of the comparison methods.

**Figure 4.** Performance analysis of dual-target angular resolution. (a) CBF. (b) Capon. (c) Music. (d) DML. (e) SML.

(2) Analysis of the Performance of Sparse Deconvolution in Angle Recovery

This paper presents a method for addressing the convolutional beamforming problem using the CBF model. Recognizing the inherent sparsity of the signal space, we introduce a deconvolution beamforming approach that incorporates dual sparse constraints and applies optimization techniques rooted in sparse reconstruction theory. To evaluate the effectiveness of the proposed method in reconstructing spatial information, we compare three Tikhonov regularization variants with squared 2-norm smoothing constraints, 1-norm LASSO regression, iterative shrinkage-thresholding algorithm (ISTA), and fast ISTA (FISTA) with total variation (TV) constraints, including its accelerated variant. Additionally, we assess the 1-norm and 2-norm constraint methods of our proposed model for deconvolution-based recovery of simulated signals across two dimensions [31–35]. The PSF represents a 32-element, 500 kHz acoustic array with a half-wavelength element spacing, evaluated across 1333 angular grid points from -90° to 90° . We assume that the two targets are separated by 15 angular grid units, located at 16° and 18.02° . Figure 5 illustrates the simulated impact signal, the PSF, and the observed signal.

- The inherent angular resolution limit of the CBF method

Figure 5 demonstrates, through numerical simulation, the intrinsic angular resolution limitations of the CBF method. In the ideal scenario, two discrete targets (Target 1 at 16.0° and Target 2 at 18.02°) appear as distinct peaks in the original target-impulse subplot. However, after propagation through a point-spread-function system with a given main-lobe width, the two peaks merge into a single broad main-lobe in the observed signal, preventing angular discrimination. This phenomenon shows that CBF-based observation produces substantial main-lobe broadening in the spatial spectrum and consequently a significant loss of resolution. Thus, deconvolution and sparse-reconstruction techniques are necessary to compensate for system-induced blur and to achieve accurate separation and parameter estimation of closely spaced targets.

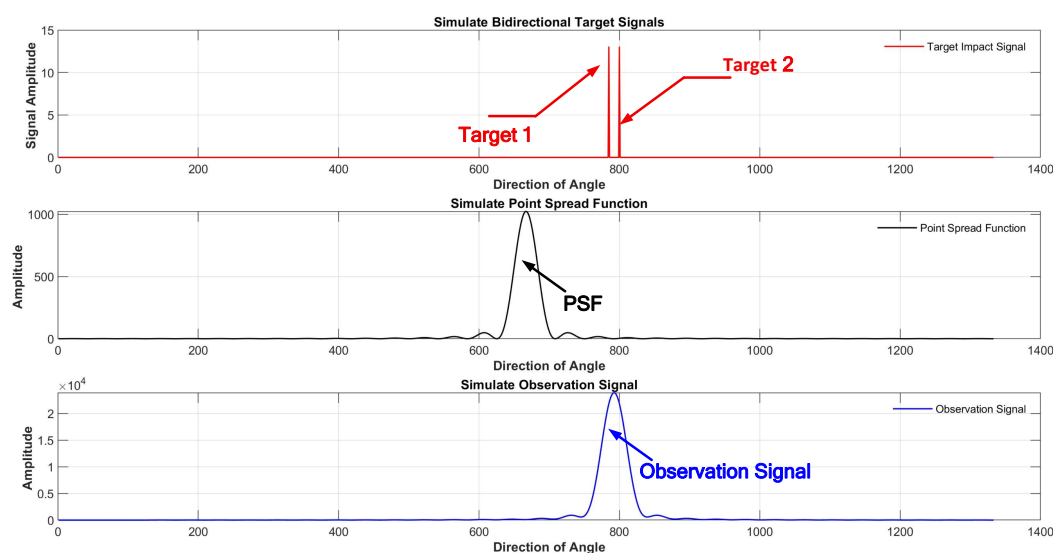


Figure 5. Diagram of relationships between two impact target signals, PSF, and observation signal.

- Square of 2-norm $\|\cdot\|_2^2$ smoothing Tikhonov regularization deconvolution

In Equation (9), the original signal S is represented in three forms: the signal itself, its first-order gradient, and its second-order gradient. Additionally, the constraint term may adopt different norm formulations, each offering distinct advantages in promoting sparsity in the variables targeted for regularization. This section compares three methods—the identity Tikhonov constraint under the 2-norm constraint, the gradient Tikhonov

constraint, and the Laplacian Tikhonov constraint—against the method proposed in this paper. The experimental results are shown in Figure 6.

Using the identity Tikhonov constraint shown in Figure 6 for ambiguity resolution yields a recovered signal in which the two spectral peaks are poorly separated. Under interference, the targets become difficult to distinguish, indicating limited resolution. This weakness arises from the 2-norm constraint, which emphasizes global sparsity and suppresses signal amplitudes, thereby reducing the ability to resolve closely spaced targets. Applying gradient Tikhonov or Laplacian Tikhonov constraints improves recovery as the sparse regularization acts on the signal's gradient or Laplacian, increasing sparsity in the recovered information and revealing two azimuth spectral peaks. Nevertheless, under the 1-norm constraint used in this paper, the recovered signal is sparser, the azimuth peaks are clearer, and the peak widths are narrower. These characteristics indicate higher target resolution and improved discrimination of targets with small angular separations.

- 1-norm $\|\cdot\|_1$ LASSO regression shrinkage-threshold deconvolution

Based on the analysis of the square of the 2-norm above, this paper compares its performance with three representative methods that impose 1-norm constraints: ISTA, FISTA, and the TV-1norm method.

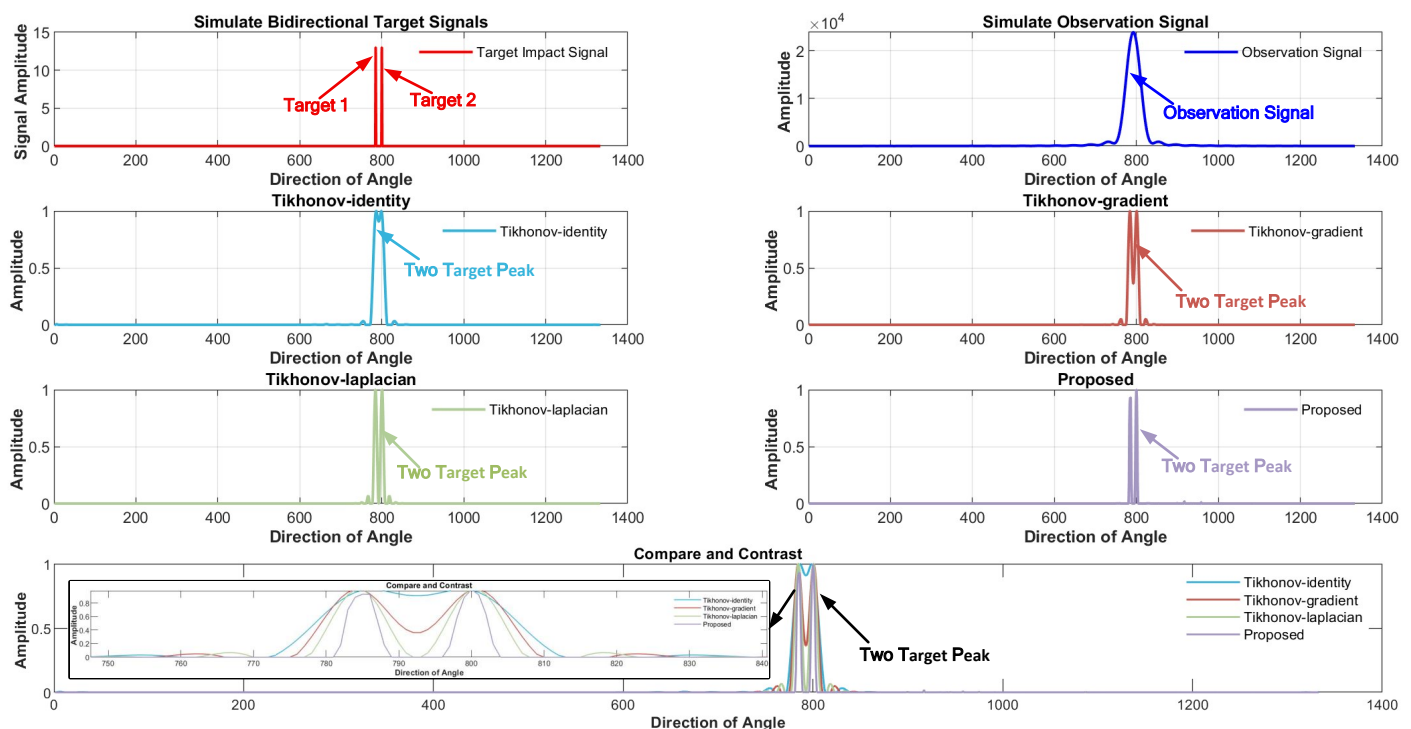


Figure 6. 2-norm identity, gradient, and Laplacian Tikhonov constraints: angle recovery results.

Figure 7 shows that ISTA, FISTA, and TV-Norm1 fail to recover the angle or azimuth information of the dual targets with the accuracy achieved by the method proposed here. Although these methods enforce sparsity through a 1-norm constraint, their performance depends strongly on the choice of control parameters, and improper parameter selection leads to substantial degradation in reconstruction fidelity and target resolution. By contrast, the proposed method preserves target detail through dynamic adjustment of control parameters and by combining dual sparsity constraints with auxiliary-parameter constraints. Consequently, it achieves superior target resolution.

- Comparison of 1-norm $\|\cdot\|_1$ and 2-norm $\|\cdot\|_2$ constraints in this paper

The method proposed in this paper also raises the question of how different pairs of constraint conditions affect information reconstruction. This section evaluates the performance of the two constraint forms introduced here, the 1-norm and the square root of the 2-norm, as shown in Figure 8.

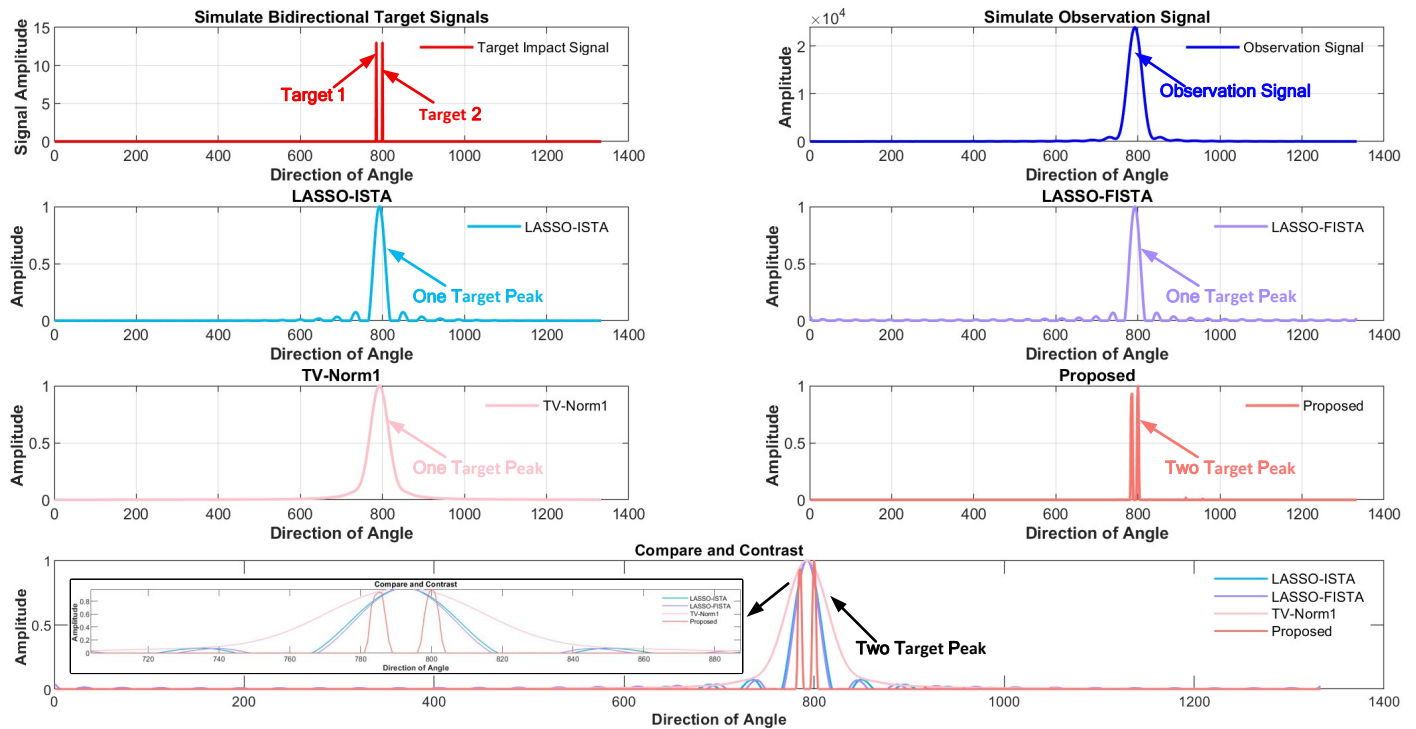


Figure 7. 1-norm ISTA, FISTA, and TV: angle recovery results.

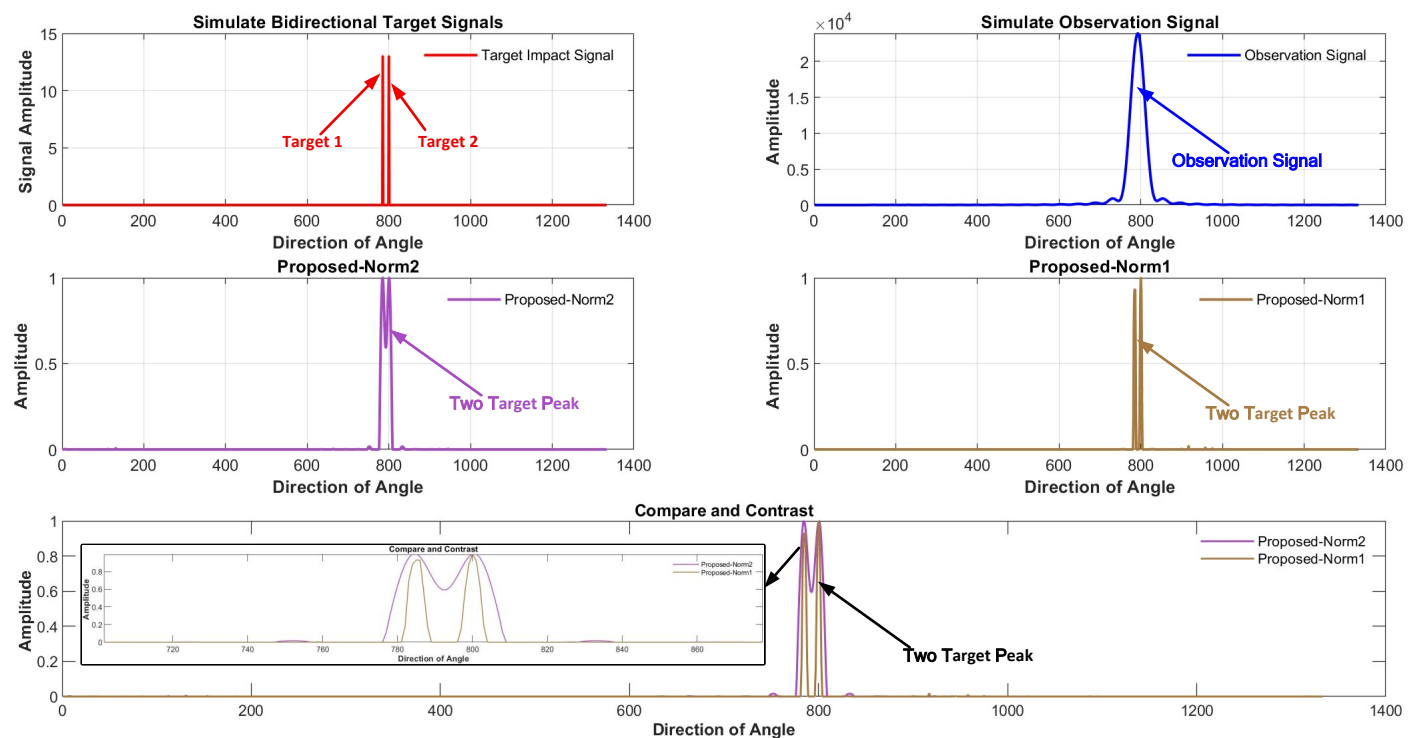


Figure 8. Angle recovery results under different norm constraints in this paper.

Figure 8 shows that both the 1-norm and 2-norm constraints accurately recover the dual-target angle information. The 2-norm approach emphasizes smoothing and therefore limits sparsity, producing a wider main-lobe in the reconstructed spatial spectrum and reduced resolution for closely spaced targets. The 1-norm constraint used in this paper emphasizes target sparsity, and the introduction of auxiliary-variable constraints resolves the non-differentiability of the 1-norm optimization. The experimental results indicate that this method yields a narrower main-lobe, improved azimuth reconstruction, and superior angle resolution.

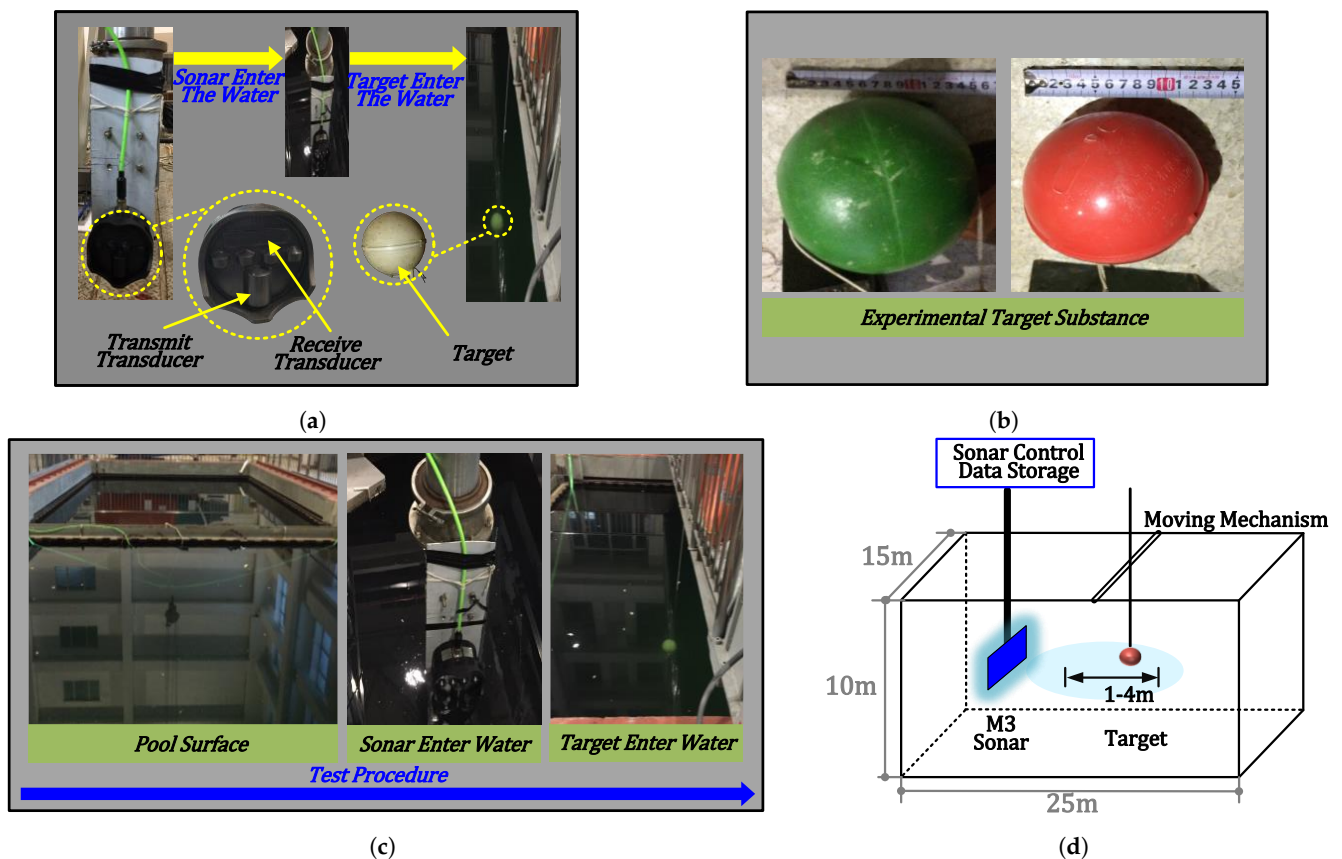
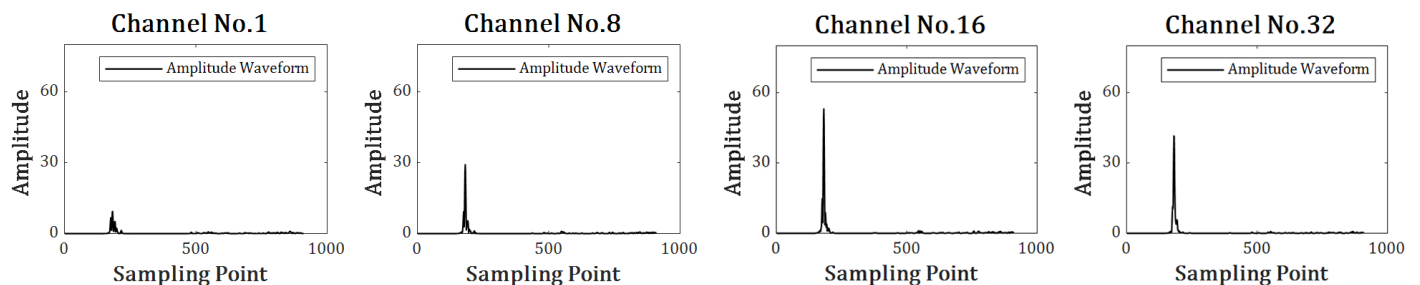
In summary, among the deconvolution-based sparse reconstruction approaches considered, methods based on the 1-norm constraint (ISTA, FISTA, and TV-Norm1) impose a 1-norm penalty directly on the reconstructed signal and therefore depend heavily on the choice of regularization parameters. As a result, they may yield unstable or inaccurate solutions in nonsmooth sparse reconstruction problems. In contrast, squared 2-norm-based methods (identity Tikhonov, gradient Tikhonov, and Laplacian Tikhonov) emphasize smoothness and tend to perform poorly when target features are closely spaced. The method proposed in this paper combines the advantages of both the 1-norm and 2-norm approaches, enabling reliable sparse reconstruction while consistently improving recovery accuracy.

3.2. Water-Pool Data Test

To evaluate imaging performance on real targets and validate the advantages of the proposed method, we conducted water-pool experiments comparing the relevant approaches. Conventional CBF imaging and Richardson–Lucy (R-L) deconvolution imaging were selected as baseline methods for comparison. The sonar system used was a Kongsberg M3 multi-beam sonar operating at 500 kHz with a transmitted pulse width of 20 μ s. The number of array elements was 32. The range spanned 0.2–150 m, with a range resolution of 1 cm, and supported two signal types: CW and chirp. The horizontal field of view was 120°. The detailed sonar operating parameters are listed in Table 2. Tests were performed in a pool measuring 25 m (length) $\times$ 15 m (width) $\times$ 10 m (depth). The equipment layout and test procedure are shown in Figure 9. As illustrated in Figure 9a, the multi-beam sonar system consists of two parts: a transmitting array and a receiving array. During transmission, the sonar emits acoustic pulses from a vertically arranged transmitting array, forming a beam of a specified width. Echoes are received by a horizontally arranged 32-element receiving array, enabling high-resolution azimuth discrimination. The experimental target is a plastic sphere with a diameter of approximately 10 cm. Its simple geometry and uniform acoustic properties make it suitable for echo-characteristic analysis as a point target or standard scatterer. The target arrangement is depicted in Figure 9b. Figure 9c presents the data-acquisition workflow: after the sonar is placed in the water and powered on, the target sphere is hoisted to the designated position, and the sonar records the target echo signals continuously, completing equipment preparation, target deployment, and data recording. To evaluate the imaging sonar's performance at different detection distances, we conducted a controlled verification experiment in a static-water environment. The sonar transducer was fixed at the center of the pool along the width axis. The target was mounted on a precision linear displacement platform and positioned at the same depth as the sonar's acoustic center to eliminate depth-related effects. Using the platform, the target was moved in 0.5 m increments from 1 m to 4 m. At each position the system synchronously recorded raw echo data in multi-channel mode. A schematic diagram of the sonar equipment and target placement is shown in Figure 9d. Figure 10 presents the multi-channel data obtained from the pool experiment, showing the target echo waveforms recorded by the sonar array in channels 1, 8, 16, and 32.

Table 2. Operating parameters of the Kongsberg M3 multi-beam sonar.

Operating Parameter	Value
Range	0.2–150 m
Frequency	500 kHz
Range Resolution	1 cm
Pulse Types	CW, Chirp
Pulse Width	20 μ s
Horizontal Field of View	120°/140°

**Figure 9.** Equipment and procedures of the pool experiment. (a) Experimental scene. (b) Experimental target. (c) Experimental workflow. (d) Schematic diagram of sonar equipment and target placement.**Figure 10.** Multi-channel signal acquisition waveform result graph.

(1) Comparison and analysis of single-target imaging results

Figure 11 shows the imaging results for a single spherical target. Figure 11a presents the output of conventional delay-and-sum beamforming. The sphere appears relatively blurred, and spatial resolution is limited. High sidelobes in the spatial spectrum cause

leakage into non-target directions, producing spurious artifacts and reducing the reliability of target recognition. Figure 11b shows the result of Richardson–Lucy deconvolution. Although iterative deconvolution reduces sidelobe energy and partially mitigates leakage, spatial resolution remains limited, and the target’s edge definition and localization accuracy are still inadequate. In contrast, the proposed method in Figure 11c demonstrates clear advantages. By exploiting the azimuthal sparsity of the echo signal and incorporating sparse constraints into the optimization model, the method produces a more accurate inversion of the true target scene. Consequently, it achieves superior sidelobe suppression, fewer artifacts, and enhanced resolution, yielding the clearest target image and significantly improving reconstruction quality.

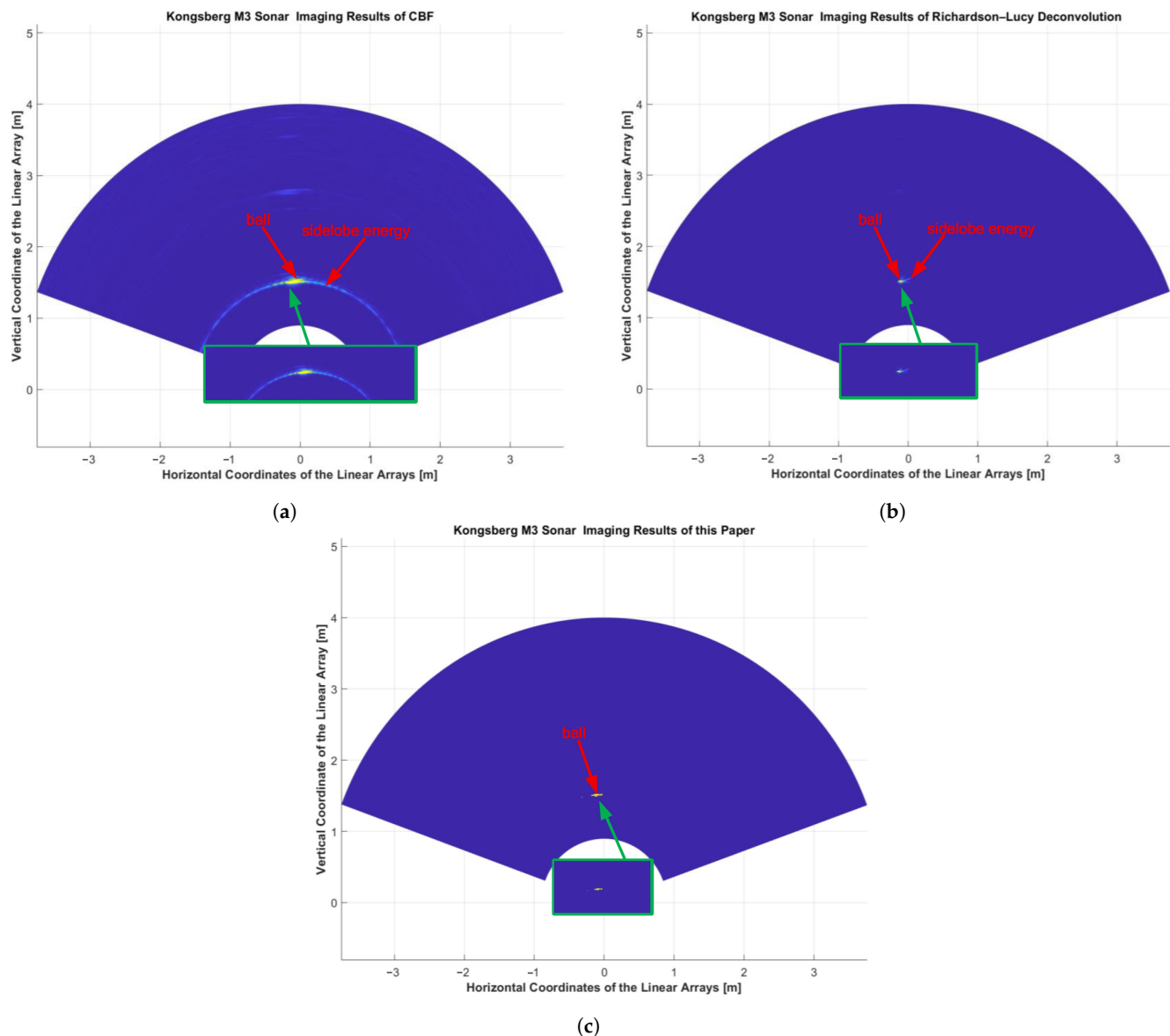


Figure 11. Single-sphere imaging results. (a) CBF. (b) R-L deconvolution. (c) Proposed.

(2) Comparison and analysis of dual-target imaging results

Figure 12 presents the imaging results for the dual-sphere target. Figure 12a shows the CBF result, while Figure 12b shows the R-L deconvolution result. Both methods produce blurred images of the two spheres. The CBF output shows the poorest resolution and fails to separate the two spheres reliably. The R-L deconvolution offers some improvement, but a noticeably blurred region remains between the spheres due to mutual interference

of their echoes. As the angle spacing between the spheres decreases, both methods lose their ability to distinguish the targets, eventually merging them into a single elongated response. By contrast, Figure 12c shows the result of the proposed method. Under a sparse-constraint framework, it effectively suppresses non-target echo energy in the central region and markedly improves the resolvability of the two spheres. Incorporating an azimuthal sparsity prior better aligns the imaging model with the physical scene, resulting in significantly improved spatial resolution for the dual-sphere target.

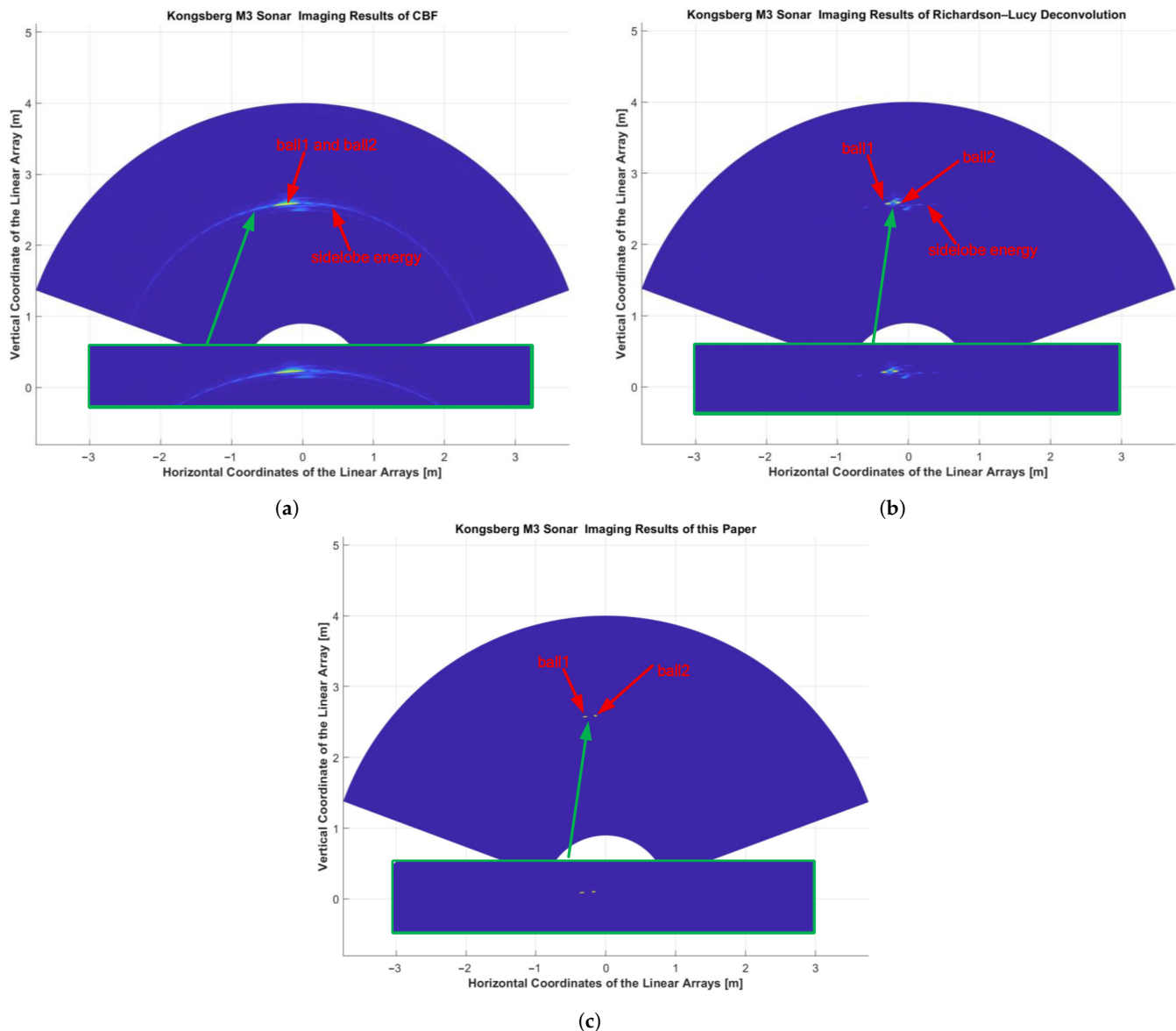


Figure 12. Dual-sphere imaging results. (a) CBF. (b) R-L deconvolution. (c) Proposed.

(3) Comparison and analysis of wood target imaging results

To evaluate the proposed method's adaptability to different target materials, we imaged a wooden target; the results are shown in Figure 13. Consistent with the single-sphere and double-sphere experiments, our method demonstrates clear advantages for wooden targets. The conventional beamforming result contains numerous false targets caused by sidelobe leakage, which obscures the true structure. The R-L deconvolution reduces some artifacts, but its model does not incorporate a spatial sparsity prior and therefore cannot fully suppress noise and interference. By introducing an azimuthal sparsity constraint, our method enhances focus on the target region during inversion and substantially re-

duces responses and artifacts in non-target areas. Consequently, it achieves superior target contour definition and background-noise suppression, demonstrating improved material adaptability and reliable target resolution.

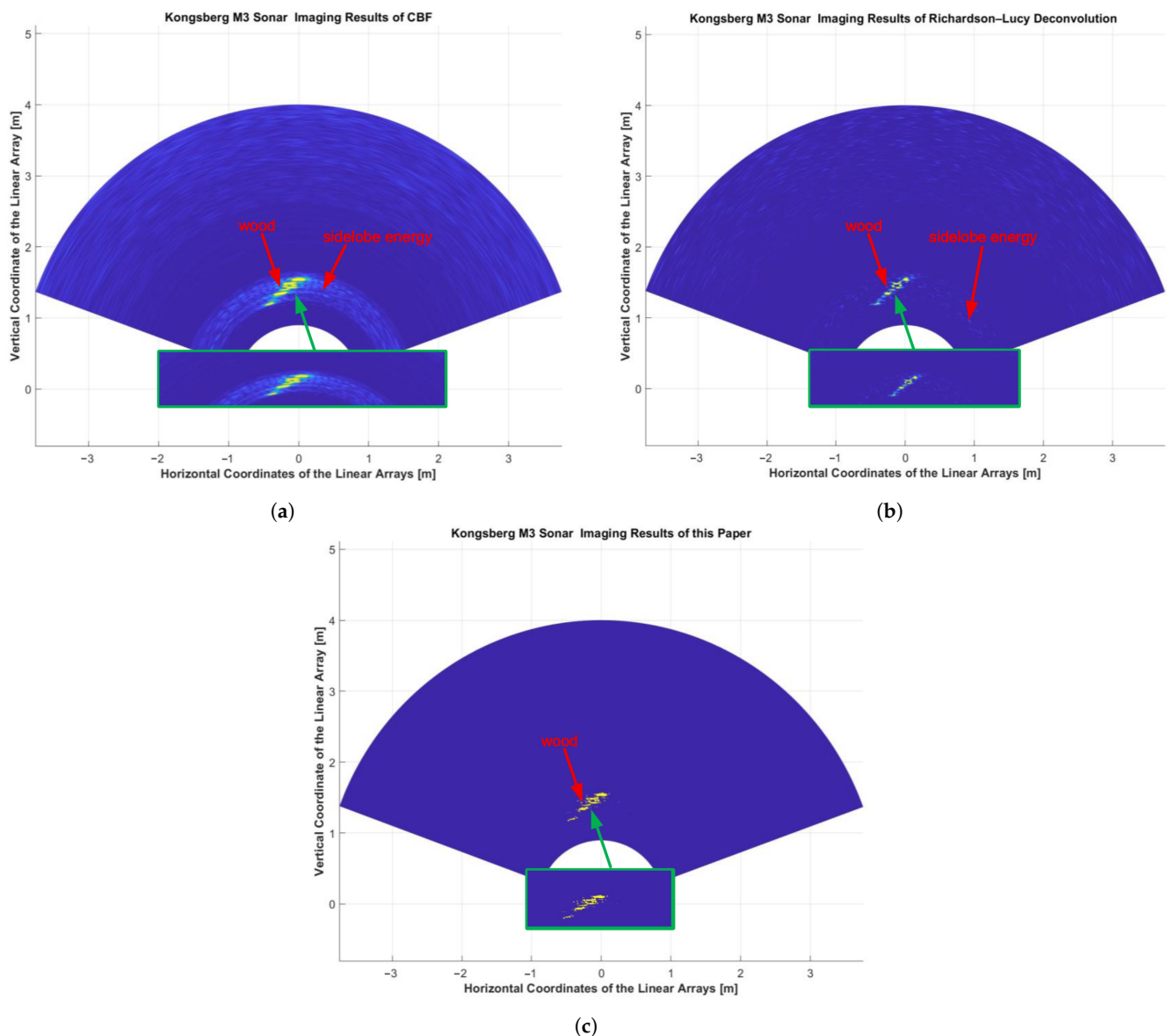


Figure 13. Wood target imaging results. (a) CBF. (b) R-L deconvolution. (c) Proposed.

4. Discussion

This study introduces a two-stage sparse deconvolution beamforming method for high-resolution imaging with multi-beam uniform linear array sonar. The method mitigates typical limitations of conventional beamformers, including main-lobe broadening and sidelobe leakage. By casting imaging as a sparse deconvolution problem with dual constraints on the signal and its gradient, the approach enhances azimuthal resolution and suppresses off-target interference; these improvements are confirmed by simulations and pool experiments. The results demonstrate substantially improved angular resolution, yielding a main-lobe width of 0.26° , and outperforming methods such as Capon and MUSIC. Computational complexity, however, complicates real-time deployment, so future work should prioritize algorithmic optimization to achieve a practical trade-off between efficiency and accuracy.

5. Conclusions

This paper addresses the challenge of high-resolution imaging of underwater targets by proposing a deconvolution-based high-resolution beamforming method with dual sparse constraints, using a uniform linear array multi-beam sonar for data acquisition. The main contributions of this study are threefold. First, within the far-field beamforming framework, a deconvolution beamforming technique is introduced to mitigate imaging blur caused by the array's point-spread function, thereby enhancing image quality. Second, recognizing the sparse nature of target echoes in spatial distribution, an azimuth information reconstruction method with dual sparse constraints is proposed to effectively suppress interference from non-target directions. Third, to address the challenge of solving the sparse reconstruction model, an iterative soft-threshold optimization algorithm with auxiliary parameters is developed to achieve efficient and stable signal reconstruction. To validate the effectiveness of the proposed method, systematic evaluations were conducted through simulation experiments and pool-based imaging tests. The results indicate that, compared with conventional beamforming and standard deconvolution methods, the proposed approach significantly enhances target resolution and image clarity, enabling higher-quality underwater target imaging. However, despite the notable improvements in performance, the method still faces limitations in computational complexity, requiring substantial computational resources and posing challenges for extremely fast real-time implementation. Future research should therefore focus on optimizing the algorithmic structure to reduce computational cost while maintaining imaging accuracy, which will be essential for advancing the practical application of this method.

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