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Comparative Analysis of Prior and Posterior Integrity Monitoring Techniques for Enhanced Global Navigation Satellite System Positioning Continuity and Accuracy

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Abstract: GNSS integrity is an essential component for ensuring the reliability of safety-critical applications using Global Navigation Satellite Systems (GNSSs). These applications, such as use in aviation and autonomous vehicles, demand high precision and dependability. There are two major GNSS integrity monitoring techniques, namely prior and posterior integrity monitoring. The principles of the two approaches, however, differ significantly, each influencing the GNSS positioning system's continuity and accuracy performance in unique ways. In this study, we conduct a thorough evaluation and comparison of these two approaches to integrity monitoring, focusing on their effects on continuity and accuracy performance. We assess the probability of false alarms and continuity risks associated with posterior integrity monitoring by defining specific geometric spheres, both inside and outside the contours of the parity set, where the integrity risk requirement is satisfied. By using these defined spheres, we determine the lower and upper bounds for the probability of false alarms and continuity risks in posterior integrity monitoring. These spheres provide a novel and effective framework for comparing the continuity performance between the Chi-squared residual-based prior and posterior integrity monitoring. Our analysis highlights that, under fault-free scenarios, posterior integrity monitoring offers superior accuracy compared with the Chi-squared residual-based prior integrity monitoring approach. This finding underscores the critical importance of selecting an appropriate integrity monitoring strategy to enhance GNSS positioning system performance, particularly in environments where safety and precision are paramount. The insights gained from this study contribute to the advancement of GNSS technologies, supporting their implementation in an increasingly wide range of safety-critical applications.

Keywords: global navigation satellite system (GNSS); continuity risk; integrity monitoring



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1. Introduction

Global navigation satellite systems (GNSSs) play a critical role in providing positioning, navigation, and timing (PNT) services for a wide range of applications, including transportation, communication, and financial transactions [1,2]. However, GNSSs are vulnerable to various interferences and disruptions [3], which can impact the accuracy and reliability of PNT and pose a threat to applications such as autonomy and aviation [4–6]. High GNSS integrity, therefore, must be ensured for applications that rely on trusted positioning information [7,8]. The receiver autonomous integrity monitoring (RAIM) method

has been widely adopted to compute the integrity risk and provide alerts if application requirements cannot be met. The RAIM method can be implemented based on prior integrity monitoring [9–11] or posterior integrity monitoring [12–14].

The probabilities of the prior integrity risk and the posterior integrity risk, however, are different, which would then lead to differences in performance in terms of positioning continuity and accuracy. In the prior integrity monitoring, the integrity risk computation is independent of real-time GNSS measurements and is only based on prior GNSS information, such as measurement error models and fault occurrence probabilities [15,16]. In comparison, the integrity risk computation in the posterior integrity monitoring takes real-time GNSS measurements into account. The prior integrity monitoring includes fault detection (FD), which refers to the detection of any anomalies or errors in the received GNSS signals according to the continuity requirement of applications [17,18], but no fault detection is involved in the posterior integrity monitoring. Further, the continuity will be interrupted when the fault detection statistic exceeds the threshold or the integrity risk is greater than the integrity risk requirement in the prior integrity monitoring [19]. Meanwhile, in the posterior integrity monitoring, the availability of the position solution is determined only by checking if the integrity risk is less than the integrity risk requirement.

Since prior integrity monitoring and posterior integrity monitoring are different from each other in principle, they have distinct accuracy and continuity performance when they are applied to a GNSS positioning system [20,21]. Each approach has its advantages and also performance trade-offs. In prior integrity monitoring, the probability of false alarm under the fault-free case is required to be preset for fault detection [22]. The continuity risk, therefore, can be conveniently evaluated in the prior integrity monitoring. However, it is challenging to precisely compute the probability of false alarm and continuity risk in posterior integrity monitoring [12]. The relationship between the accuracy and continuity of prior integrity monitoring and those of posterior integrity monitoring should be investigated, but limited work has been conducted so far in the GNSS community.

In this contribution, we introduce an innovative method for assessing the probability of false alarm in posterior integrity monitoring by defining two distinct spheres within the parity space. This approach allows us to establish precise lower and upper bounds for both the probability of false alarm and continuity risk in posterior integrity monitoring. By leveraging these defined spheres, we provide a robust framework for evaluating and comparing the continuity risks associated with different integrity monitoring strategies. Specifically, we compare the continuity risks of prior integrity monitoring with those of Chi-squared residual-based prior integrity monitoring by examining the relative sizes of the defined spheres and the parity sphere determined by the fault detection test in the Chi-squared residual-based approach. This comparative analysis offers valuable insights into the performance differences between these monitoring techniques. Additionally, we evaluate the accuracy of both prior and posterior integrity monitoring under fault-free conditions, ensuring that the comparisons are performed at equivalent probabilities of false alarm. Our findings demonstrate that posterior integrity monitoring tends to achieve better accuracy in fault-free scenarios, highlighting its potential advantages for applications demanding high precision. This work contributes to the field by providing a systematic and quantifiable method for analyzing and comparing the effectiveness of different integrity monitoring strategies in GNSS systems, ultimately enhancing their reliability and performance in safety-critical contexts.

2. Methodology

In the local east–north–up (ENU) coordinate system, the normalized GNSS pseudorange measurement equation, involving n different satellites and m states, can be expressed in a linearized form as follows [23]:

$$\rho = \mathbf{G}\mathbf{x} + \mathbf{f} + \mathbf{v} \quad (1)$$

where ρ is the $n \times 1$ pseudorange measurement vector, $\mathbf{G}$ is the $n \times m$ observation matrix, $\mathbf{x}$ denotes the $m \times 1$ state vector to be estimated, $\mathbf{v}$ is the $n \times 1$ Gaussian white measurement noise vector following $N(0, 1)$, and $\mathbf{f}$ is a $n_f \times 1$ fault vector where n_f represents the number of faults. Given the measurement vector ρ , the weighted least-squares (WLS) estimate of $\mathbf{x}$ is obtained as [24]:

$$\hat{\mathbf{x}} = (\mathbf{G}^T \mathbf{G})^{-1} \mathbf{G}^T \rho \quad (2)$$

Then, the corresponding estimation errors are equal to:

$$\Delta \mathbf{x} = \hat{\mathbf{x}} - \mathbf{x} = (\mathbf{G}^T \mathbf{G})^{-1} \mathbf{G}^T (\mathbf{f} + \mathbf{v}) \quad (3)$$

The parity vector $\mathbf{p}$ of the WLS positioning is obtained by [25]:

$$\mathbf{p} = \mathbf{Q}(\mathbf{f} + \mathbf{v}) \quad (4)$$

where $\mathbf{Q}$ represents the parity matrix. In this work, the parity vector $\mathbf{p}$ is assumed to be at least two-dimensional. The covariances of $\Delta \mathbf{x}$ and $\mathbf{p}$ are equivalent to $(\mathbf{G}^T \mathbf{G})^{-1}$ and $\mathbf{I}_{(n-m) \times (n-m)}$, respectively.

2.1. Prior Integrity Monitoring

In the prior integrity monitoring, the prior integrity risk is expressed as follows under multiple hypotheses [26]:

$$P_{\text{HML},q} = \sum_{j=0}^h P_{\text{HML},q,j} \quad (5)$$

where the subscript $q \in \{E, N, U\}$ indicates the component associated with the East, North, or Up direction, and h indicates the number of hypotheses that should be monitored in the integrity monitoring. The hypotheses are denoted as $h_j (0 \leq j \leq h)$, with h_0 standing for the fault-free hypothesis. Fault detection in prior integrity monitoring can be conducted in either the measurement domain, primarily utilizing positioning residuals or parity vector, or in the position domain, which commonly employs solution separations between the satellite full-set and subsets. In this study, we employ Chi-squared residual-based prior integrity monitoring for the comparative analysis of prior and posterior integrity monitoring, as both integrity monitoring methods are computed in the measurement domain. In the Chi-squared residual-based prior integrity monitoring, the prior integrity risk under hypothesis h_j is represented as follows [27]:

$$P_{\text{HML},q,j} = P\{|\Delta \mathbf{x}_q| > AL_q, \mathbf{p}^T \mathbf{p} < T_{th} | h_j\} P_{h_j} = P\{|\Delta \mathbf{x}_q| > AL_q | h_j\} P\{\mathbf{p}^T \mathbf{p} < T_{th} | h_j\} P_{h_j} \quad (6)$$

where P_{h_j} stands for the prior occurrence probability of hypothesis h_j , which is computed according to the prior satellite fault probability P_{sat} , and AL_q is the alert limit. $\mathbf{p}^T \mathbf{p}$ follows a Chi-squared distribution with $n - m$ degrees of freedom, and the non-centrality parameter is equal to:

$$\lambda = \mathbf{f}^T \left(\mathbf{I} - \mathbf{G} (\mathbf{G}^T \mathbf{G})^{-1} \mathbf{G}^T \right) \mathbf{f} \quad (7)$$

When $\mathbf{f} = 0$, $\mathbf{p}^T \mathbf{p}$ is assumed to follow a central Chi-squared distribution [28]. The fault detection test is carried out as follows:

$$\mathbf{p}^T \mathbf{p} < T_{th} \triangleq \chi^{-2}(P_{FA}, n - m, 0) \quad (8)$$

where P_{FA} is the preset probability of false alarm, and function $\chi^{-2}(t, \gamma, \lambda)$ means the t quantile of the Chi-squared distribution with degrees of freedom γ and non-centrality parameter λ .

2.2. Posterior Integrity Monitoring

Given ρ , the posterior integrity risk is defined as follows [12]:

$$P_{HMI,q,\rho} = P\{|\Delta \mathbf{x}_q| > AL_q | \rho\} = \sum_{j=0}^h P_{HMI,q,j,\rho} \quad (9)$$

where $P_{HMI,q,j,\rho}$ refers to the posterior integrity risk under hypothesis h_j and can be expressed as follows:

$$P_{HMI,q,j,\rho} = P\{|\Delta \mathbf{x}_q| > AL_q | \rho, h_j\} P\{h_j | \rho\} \quad (10)$$

Considering that the parity vector $\mathbf{p}$ encapsulates the fault and noise information of ρ , $P_{HMI,q,j,\rho}$ can be rewritten into:

$$P_{HMI,q,j,\rho} = P_{HMI,q,j,\mathbf{p}} \triangleq P\{|\Delta \mathbf{x}_q| > AL_q | \mathbf{p}, h_j\} P\{h_j | \mathbf{p}\} \quad (11)$$

And hence, we have:

$$P_{HMI,q,\rho} = P_{HMI,q,\mathbf{p}} = \sum_{j=0}^h P_{HMI,q,j,\mathbf{p}} \quad (12)$$

The $P_{HMI,q,j}$ indicates a probability for a set of parity $\mathbf{p}^T \mathbf{p} < T_{th}$. In comparison, $P_{HMI,q,\mathbf{p}}$ is a probability computed under a single parity vector $\mathbf{p}$.

3. Impacts on Positioning Continuity and Accuracy

As shown in Equations (5), (6) and (9), the definitions of prior integrity risk and posterior integrity risk are different, which indicates that the prior integrity monitoring and the posterior integrity monitoring will have different impacts on the continuity and accuracy of GNSS positioning.

3.1. Impact on Continuity

In the prior integrity monitoring, the position solution is deemed unavailable if either the fault detection statistic does not pass the fault detection test $\mathbf{p}^T \mathbf{p} \geq T_{th}$ or the prior integrity risk is not less than the integrity risk requirement $P_{HMI,q} \geq P_{HMI,q,IR}$, where $P_{HMI,q,IR}$ indicates the integrity risk requirement. In contrast, the position solution in the posterior integrity monitoring is declared to unavailable if $P_{HMI,q,\mathbf{p}} \geq P_{HMI,q,IR}$. Let us use Θ to denote the $\mathbf{p}$ set where the position solution is considered available. Then, Θ can be expressed as follows:

$$\Theta = \begin{cases} \{\mathbf{p} | \mathbf{p}^T \mathbf{p} < T_{th} \cap P_{HMI,q} < P_{HMI,q,IR}\}, \text{prior integrity monitoring} \\ \{\mathbf{p} | P_{HMI,q,\mathbf{p}} < P_{HMI,q,IR}\}, \text{posterior integrity monitoring} \end{cases} \quad (13)$$

$P_{HMI,q}$ is related to P_{FA} , AL_q , P_{sat} , $\mathbf{G}$, and GNSS measurement error models. In order to guarantee $P_{HMI,q} < P_{HMI,q,IR}$ under given AL_q , P_{sat} , $\mathbf{G}$, and GNSS measurement error

models, P_{FA} should be greater than the lower bound $P_{FA,lower}$. When $P_{HML,q} < P_{HML,q,IR}$ is met, Θ in the prior integrity monitoring then can be rewritten into the following.

$$\Theta = \left\{ \mathbf{p} \mid \mathbf{p}^T \mathbf{p} < T_{th} \right\} \quad (14)$$

Continuity may be interrupted by either fault-free events or those that are faulted. The continuity risk (for detection only) in both prior integrity monitoring (when $P_{HML,q} < P_{HML,q,IR}$ is satisfied) and posterior integrity monitoring can be represented as follows [23]:

$$P_{CONT,q} = \sum_{j=0}^h P\{\mathbf{p} \notin \Theta \mid h_j\} P_{h_j} \quad (15)$$

Evaluating $P\{\mathbf{p} \notin \Theta \mid h_j\} (1 \leq j \leq h)$ is difficult, since it depends on the distribution of $\mathbf{f}$. Considering (15), we have:

$$P_{CONT,q} = P_{FA,q} P_{h_0} + \sum_{j=1}^h P\{\mathbf{p} \notin \Theta \mid h_j\} P_{h_j} \leq P_{FA,q} P_{h_0} + \sum_{j=1}^h P_{h_j} \quad (16)$$

where $P_{FA,q}$ is the probability of false alarm of direction q under the fault-free condition. For Chi-squared residual-based prior integrity monitoring, it is equal to P_{FA} . The P_{FA} in the prior integrity monitoring is preset, while $P_{FA,q}$ in the posterior integrity monitoring needs to be evaluated. In order to meet the continuity risk requirement, P_{FA} in prior integrity monitoring should be less than the upper bound $P_{FA,upper}$. Hence, a feasible P_{FA} should satisfy $P_{FA,lower} < P_{FA} < P_{FA,upper}$. Equation (16) provides an upper bound for $P_{CONT,q}$. The lower bound for $P_{CONT,q}$ can be obtained as follows:

$$P_{CONT,q} = P_{FA,q} P_{h_0} + \sum_{j=1}^h P\{\mathbf{p} \notin \Theta \mid h_j\} P_{h_j} \geq P_{FA,q} P_{h_0} + \sum_{j=1}^h P_{FA,q} P_{h_j} \quad (17)$$

Set Θ defines a contour of $\mathbf{p}$, and the false alarm contribution to continuity risk is the total probability when $\mathbf{p}$ falls outside this contour under fault-free conditions. The contour of Θ in the prior integrity monitoring is a multi-dimensional sphere $\mathbf{p}^T \mathbf{p} = T_{th}$. However, the contour of Θ in the posterior integrity monitoring has a complex shape, and it is challenging to precisely compute the contour and the exact probability of false alarm $P_{FA,q}$. The $P_{FA,q}$ in the posterior integrity monitoring can be conservatively evaluated by finding a large sphere $\mathbf{p}^T \mathbf{p} = T_{p,in}$ inside the contour of Θ . For any $\mathbf{p}$ inside the sphere, $P_{HML,q,p} < P_{HML,q,IR}$ must be guaranteed. Given a sphere $\mathbf{p}^T \mathbf{p} = T_{p,in}$, we have:

$$P_{HML,q,p} \mid_{\mathbf{p}^T \mathbf{p} = T_{p,in}} = \left(\sum_{j=0}^h P_{HML,q,j,p} \right) \Big|_{\mathbf{p}^T \mathbf{p} = T_{p,in}} \leq \sum_{j=0}^h \max \left(P_{HML,q,j,p} \mid_{\mathbf{p}^T \mathbf{p} = T_{p,in}} \right) \quad (18)$$

The sphere defined by $\sum_{j=0}^h \max \left(P_{HML,q,j,p} \mid_{\mathbf{p}^T \mathbf{p} = T_{p,in}} \right) = P_{HML,q,IR}$ can serve as the expected sphere inside the contour of Θ . Then, we have:

$$P_{FA,q} = \int_{P_{HML,q,p} \geq P_{HML,q,IR}} f(\mathbf{p}) d\mathbf{p} \leq 1 - \int_{\mathbf{p}^T \mathbf{p} \leq T_{p,in}} f(\mathbf{p}) d\mathbf{p} \quad (19)$$

where $f(\mathbf{p}) = \frac{1}{(2\pi)^{\frac{n-m}{2}}} \exp\left(-\frac{\mathbf{p}^T \mathbf{p}}{2}\right)$ refers to the probability density function of $\mathbf{p}$ under the fault-free situation.

Equation (19) provides an upper bound for $P_{FA,q}$ in the posterior integrity monitoring. Similarly, a lower bound for the $P_{FA,q}$ in posterior integrity monitoring would be evaluated by a sphere $\mathbf{p}^T \mathbf{p} = T_{p,out}$ outside the contour of Θ . For any $\mathbf{p}$ outside the sphere, $P_{HML,q,p} >$

$P_{HML,q,IR}$. Such a sphere can be defined by $\sum_{j=0}^h \min(P_{HML,q,j,p} | \mathbf{p}^T \mathbf{p} = T_{p,out}) = P_{HML,q,IR}$, which leads to:

$$P_{FA,q} = \int_{P_{HML,q,p} \geq P_{HML,q,IR}} f(\mathbf{p}) d\mathbf{p} \geq 1 - \int_{\mathbf{p}^T \mathbf{p} \leq T_{p,out}} f(\mathbf{p}) d\mathbf{p} \quad (20)$$

In the prior integrity monitoring, $P_{FA} = 1 - \int_{\mathbf{p}^T \mathbf{p} \leq T_{th}} f(\mathbf{p}) d\mathbf{p}$ under the fault-free condition. The contour of Θ in the posterior integrity monitoring exhibits symmetry with respect to the origin and is convex [12]. We assume that the P_{FA} in the prior integrity monitoring satisfies $P_{FA,lower} < P_{FA} < P_{FA,upper}$. The relationships among the spheres $\mathbf{p}^T \mathbf{p} = T_{p,in}$, $\mathbf{p}^T \mathbf{p} = T_{p,out}$, and $\mathbf{p}^T \mathbf{p} = T_{th}$ can be categorized into three cases, as shown in Figure 1.

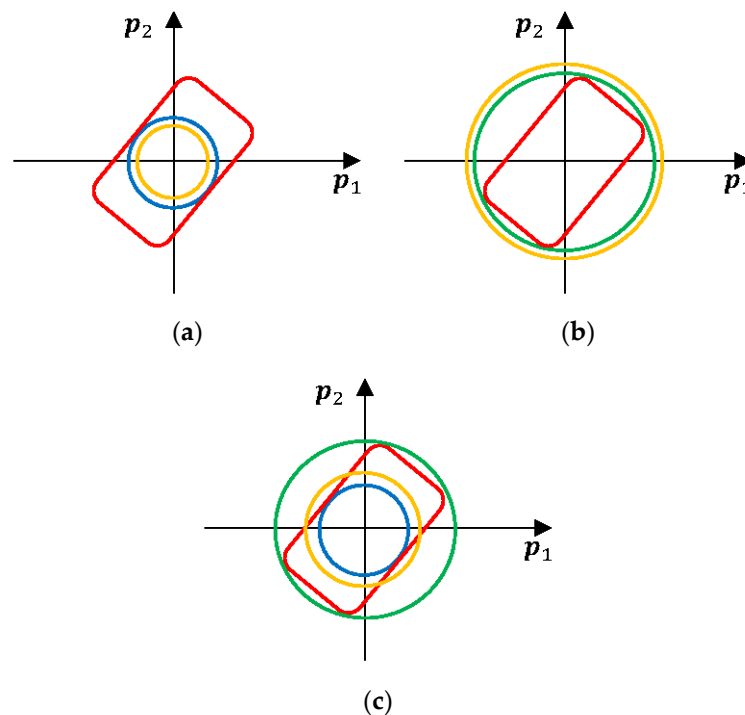


Figure 1. An example of the two-dimensional contours of Θ (red), $\mathbf{p}^T \mathbf{p} = T_{p,in}$ (blue), $\mathbf{p}^T \mathbf{p} = T_{p,out}$ (green), and $\mathbf{p}^T \mathbf{p} = T_{th}$ (yellow). (a) $T_{p,in} \geq T_{th}$. (b) $T_{p,out} \leq T_{th}$. (c) Otherwise.

The continuity relationship between the prior integrity monitoring and the posterior integrity monitoring can be determined accordingly.

(a) $T_{p,in} \geq T_{th}$

When $T_{p,in} \geq T_{th}$, the sphere $\mathbf{p}^T \mathbf{p} = T_{p,in}$ is larger than the sphere $\mathbf{p}^T \mathbf{p} = T_{th}$. Then, for any hypothesis $h_j (0 \leq j \leq h)$, we have:

$$P^+ \{ \mathbf{p} \notin \Theta | h_j \} \leq 1 - \int_{\mathbf{p}^T \mathbf{p} \leq T_{p,in}} f(\mathbf{p}) d\mathbf{p} \leq P^- \{ \mathbf{p} \notin \Theta | h_j \} = 1 - \int_{\mathbf{p}^T \mathbf{p} \leq T_{th}} f(\mathbf{p}) d\mathbf{p} \quad (21)$$

where $P^- \{ \mathbf{p} \notin \Theta | h_j \}$ and $P^+ \{ \mathbf{p} \notin \Theta | h_j \}$ represent the $P \{ \mathbf{p} \notin \Theta | h_j \}$ of the prior integrity monitoring and the posterior integrity monitoring, respectively. Substituting $P^- \{ \mathbf{p} \notin \Theta | h_j \}$ and $P^+ \{ \mathbf{p} \notin \Theta | h_j \}$ into (15), we have:

$$P_{CONT,q}^+ \leq P_{CONT,q}^- \quad (22)$$

where $P_{CONT,q}^-$ and $P_{CONT,q}^+$ represent the continuity risks of the prior integrity monitoring and the posterior integrity monitoring, respectively.

$$(b) \quad T_{p,out} \leq T_{th}$$

When $T_{p,out} \leq T_{th}$, for any hypothesis $h_j (0 \leq j \leq h)$, we have:

$$P^+ \{ \mathbf{p} \notin \Theta | h_j \} \geq 1 - \int_{\mathbf{p}^T \mathbf{p} \leq T_{p,out}} f(\mathbf{p}) d\mathbf{p} \geq P^- \{ \mathbf{p} \notin \Theta | h_j \} = 1 - \int_{\mathbf{p}^T \mathbf{p} \leq T_{th}} f(\mathbf{p}) d\mathbf{p} \quad (23)$$

Which leads to:

$$P_{CONT,q}^+ \geq P_{CONT,q}^- \quad (24)$$

- (c) Otherwise, the relationship between $P_{CONT,q}^+$ and $P_{CONT,q}^-$ is unclear. The continuity risk under the fault-free scenario can be evaluated through Monte Carlo simulation.

3.2. Impact on Accuracy

The accuracy performance of the prior integrity monitoring and the posterior integrity monitoring is compared under the same hypothesis $h_j (0 \leq j \leq h)$ and the same probability $P \{ \mathbf{p} \notin \Theta | h_j \}$. Since $P \{ \mathbf{p} \notin \Theta | h_j \} (1 \leq j \leq h)$ in the posterior integrity monitoring highly depends on the contour shape of Θ and the fault vector $\mathbf{f}$, here, we only analyze the accuracy performance under the fault-free scenario.

Without loss of generality, we assume that $\mathbf{p}$ is two-dimensional. Figure 2 shows an example of the two-dimensional contours of Θ .

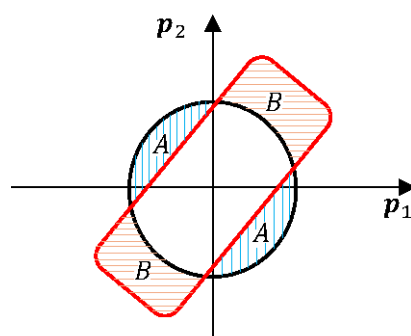


Figure 2. An example of the two-dimensional contours of Θ (red: posterior integrity monitoring, black: Chi-squared residual-based prior integrity monitoring).

The cumulative probabilities that $\mathbf{p}$ falls outside or inside the two contours under fault-free condition are identical. For every $\mathbf{p}$ on the red contour, $P_{HML,q,p} = P_{HML,q,IR}$. Since $P_{HML,q,p} = P \{ |\Delta x_q| > AL_q | \mathbf{p} \}$, we have $P \{ |\Delta x_q| > AL_q | \mathbf{p} \in A \} > P_{HML,q,IR}$ and $P \{ |\Delta x_q| > AL_q | \mathbf{p} \in B \} < P_{HML,q,IR}$. This means that, when $\mathbf{p}$ is located in region A rather than region B, it is more likely that the position error Δx_q will be greater than AL_q . Hence, it can be concluded that the posterior integrity monitoring could theoretically outperform the prior integrity monitoring in positioning accuracy under the fault-free scenario.

4. Simulations and Method Validation

In addition to theoretical impact analysis, the aforementioned methods are validated and analyzed based on simulated cases in this section. Simulations can ensure that the generated GNSS data strictly adhere to the assumed distributions, thereby enhancing the reliability of the conclusions drawn from performance analysis. The cases examined employ the broadcast GPS ephemeris, available at <https://cddis.nasa.gov/archive/gnss/data/daily/2023/001/23p/> (13 October 2023), to create the GPS WLS positioning geometries for 1 January 2023, at 23:30:00. It is assumed that the receiver is stationed at a static position with an altitude of 1000 m and coordinates of 0° latitude and 0° longitude. Figure 3 displays the GPS satellites that are visible. It is assumed that the pseudoranges are measured by L1+L5C dual-frequency technology, and

the pseudorange measurement error model widely used in integrity monitoring in aviation applications is embraced here for performance analysis [9].

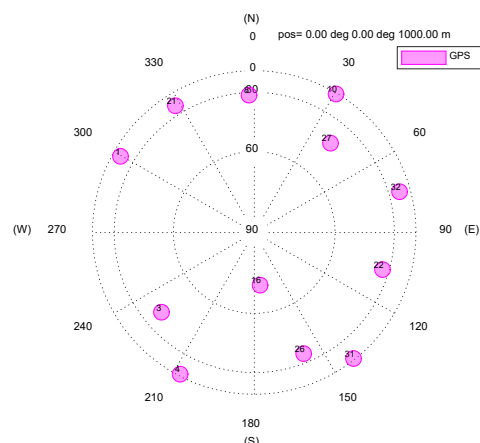


Figure 3. Skyplot of visible GPS satellites.

In order to visualize the test results in the parity space, we only use six of the visible satellites for WSL positioning. Therefore, the parity space is two-dimensional. The fault-free hypothesis and all the fault event hypotheses with one satellite fault are considered in the integrity monitoring. The prior probability of each satellite fault P_{sat} is set to 1×10^{-4} . For simplicity, only the integrity monitoring for the direction $q = E$ is investigated. The results for the west and up directions are similar to that in the east direction. In the tests, MATLAB 2023a on a PC equipped with an intel CPU i7-9700 is utilized to perform the calculations. The computation of the integrity risk associated with a two-dimensional parity vector takes approximately 0.01 s.

4.1. Case 1

First, we use the six satellites (PRN 1, 3, 8, 16, 31, and 32) for positioning. To clearly visualize the test results, we set the alert limit AL_E to 5.6 m. When $P_{HML,q,IR}$, the integrity risk budget allocated to direction $q = E$, is set to different values in the range of $10^{-[5:0.1:8]}$, the probability of false alarm estimated by 10^7 Monte-Carlo runs and its upper bound and lower bound, respectively, evaluated by the sphere $\mathbf{p}^T \mathbf{p} = T_{p,in}$ inscribing and the sphere $\mathbf{p}^T \mathbf{p} = T_{p,out}$ circumscribing the contour of Θ in the posterior integrity monitoring are shown in Figure 4.

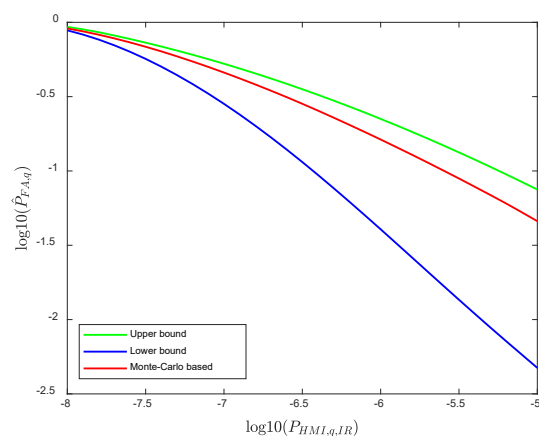


Figure 4. Monte Carlo-based estimate and the upper bound and lower bound of $P_{FA,q}$ in the posterior integrity monitoring of case 1.

Figure 4 suggests that the evaluated upper bound and lower bound of $P_{FA,q}$ can bound the Monte Carlo-based $P_{FA,q}$ estimate in the posterior integrity monitoring. In the sub-figures of Figure 5, the magenta circle $\mathbf{p}^T \mathbf{p} = T_{th}$ results in a probability of false alarm $1 - \int_{\mathbf{p}^T \mathbf{p} \leq T_{th}} f(\mathbf{p}) d\mathbf{p}$ equal to the Monte Carlo-based $P_{FA,q}$ estimate in the posterior integrity monitoring. Because the six satellites used form a good satellite geometry, the contour of Θ in the posterior integrity monitoring is close to a circle, as shown in Figure 5. Consequently, the evaluated upper bound and lower bound of $P_{FA,q}$ are close to each other. And the distributions of the parity samples meeting the two distinct sets of conditions, specifically, $P_{HMI,q} > P_{HMI,q,IR}$ and $\mathbf{p}^T \mathbf{p} < T_{th}$ and $P_{HMI,q} < P_{HMI,q,IR}$ and $\mathbf{p}^T \mathbf{p} > T_{th}$ in the Monte Carlo simulation are nearly identical for both scenarios where $P_{HMI,E,IR} = 10^{-8}$ and $P_{HMI,E,IR} = 10^{-5}$, as illustrated in Figure 6. Note that the third color displayed in Figure 6 results from the overlap of the two colors defined by the figure legends. Figure 7 presents both the means and the standard deviations of the absolute position errors of the two sets of parity samples. It can be observed that the statistical properties of the two sets of parity samples remain closely aligned as $P_{HMI,E,IR}$ varies from 10^{-8} to 10^{-5} .

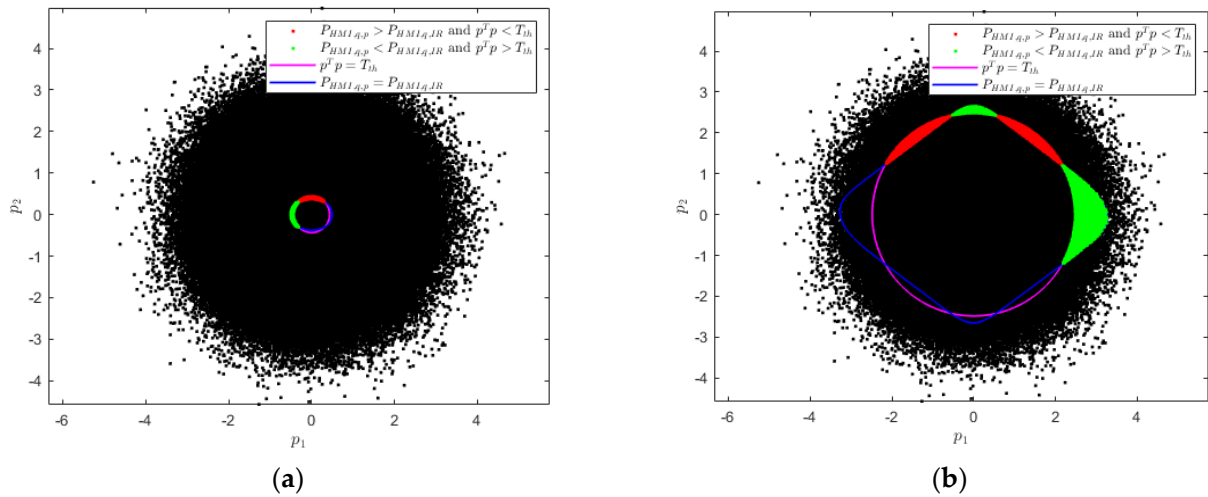


Figure 5. The parity samples, contour $P_{HMI,q} = P_{HMI,q,IR}$, and circle $\mathbf{p}^T \mathbf{p} = T_{th}$ in the Monte Carlo simulation of case 1. (a) $P_{HMI,q,IR} = 10^{-8}$. (b) $P_{HMI,q,IR} = 10^{-5}$.

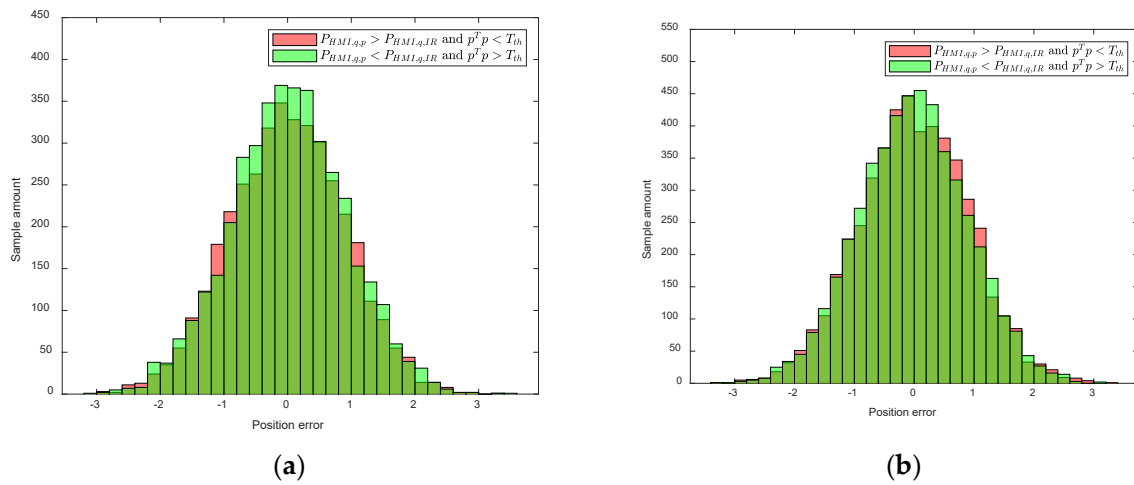


Figure 6. Histograms of the parity samples respectively satisfying $P_{HMI,q} > P_{HMI,q,IR}$ and $\mathbf{p}^T \mathbf{p} < T_{th}$ and $P_{HMI,q} < P_{HMI,q,IR}$ and $\mathbf{p}^T \mathbf{p} > T_{th}$ in the Monte Carlo simulation of case 1. (a) $P_{HMI,E,IR} = 10^{-8}$. (b) $P_{HMI,E,IR} = 10^{-5}$.

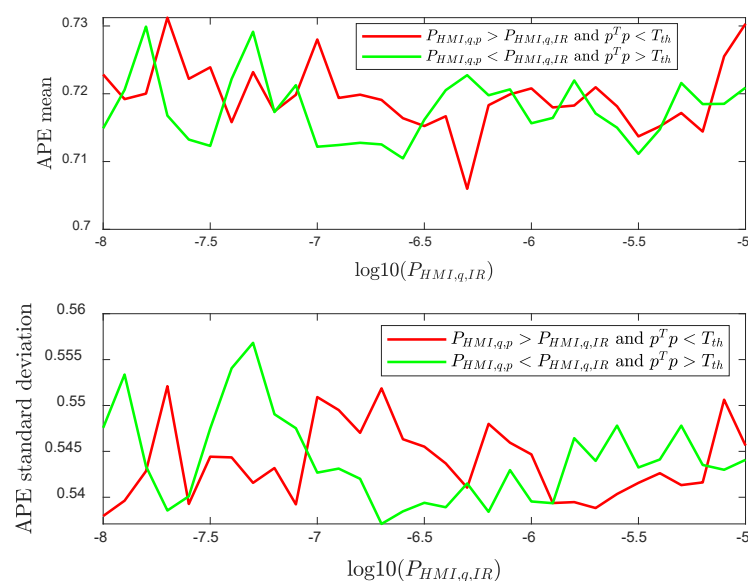


Figure 7. Means and standard deviations of the absolute position errors of the two sets of parity samples in the Monte Carlo simulation of case 1.

4.2. Case 2

When we use the six satellites (PRN 3, 8, 16, 22, 26, and 27) for positioning, and set the alert limit AL_E to 7.8 m, the results computed from the Monte Carlo runs are shown in Figures 8–11. In comparison with the satellites used in the above case, the six satellites used in this case form a relatively poor satellite geometry. As illustrated in Figure 9, the contour of Θ in the posterior integrity monitoring exhibits a significant deviation from a circular shape. The deviation is particularly significant when $P_{HMI,E,IR} = 10^{-8}$. This leads to the evaluated upper bound and lower bound of $P_{FA,q}$ providing a loose constraint on the Monte Carlo-based estimate of $P_{FA,q}$ in the posterior integrity monitoring.

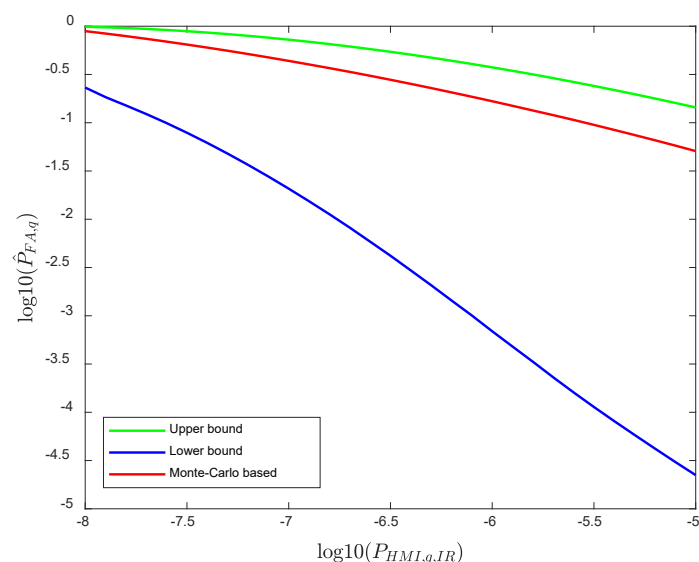


Figure 8. Monte Carlo-based estimate and the upper bound and lower bound of $P_{FA,q}$ in the posterior integrity monitoring of case 2.

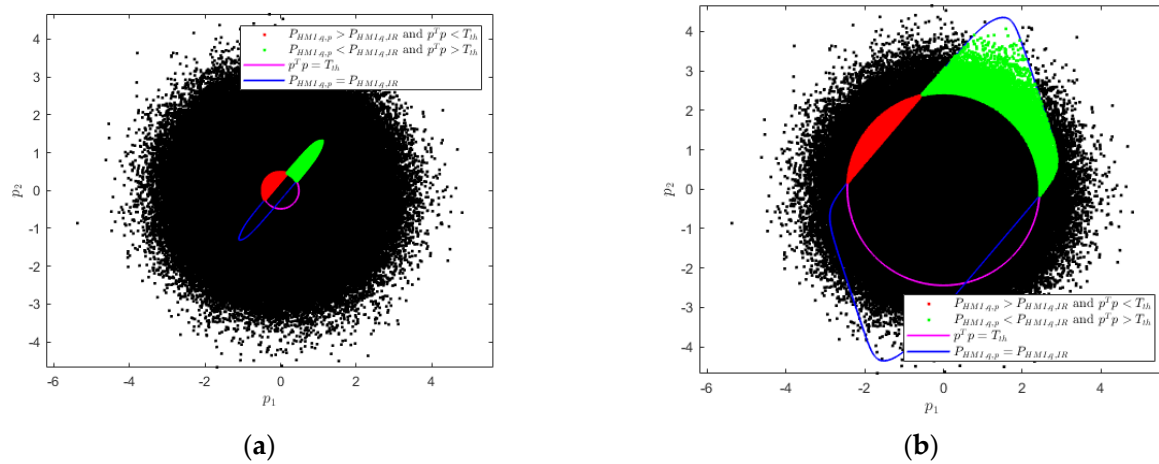


Figure 9. Parity samples, contour $P_{HMI,q} = P_{HMI,q,IR}$, and circle $\mathbf{p}^T \mathbf{p} = T_{th}$ in the Monte Carlo simulation of case 2. (a) $P_{HMI,E,IR} = 10^{-8}$. (b) $P_{HMI,E,IR} = 10^{-5}$.

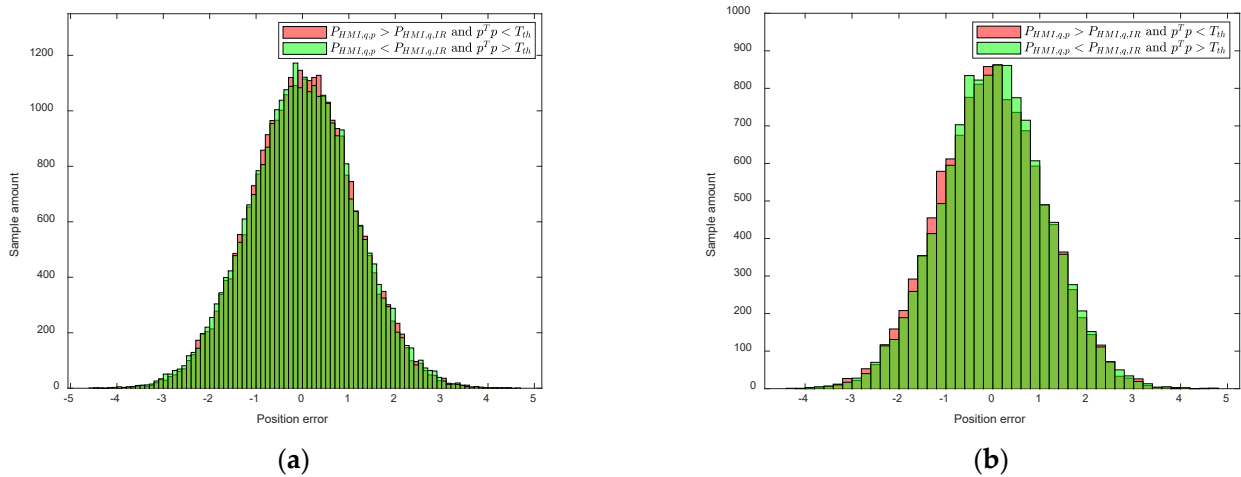


Figure 10. Histograms of the parity samples respectively satisfying $P_{HMI,q} > P_{HMI,q,IR}$ and $\mathbf{p}^T \mathbf{p} < T_{th}$ and $P_{HMI,q} < P_{HMI,q,IR}$ and $\mathbf{p}^T \mathbf{p} > T_{th}$ in the Monte Carlo simulation of case 2. (a) $P_{HMI,E,IR} = 10^{-8}$. (b) $P_{HMI,E,IR} = 10^{-5}$.

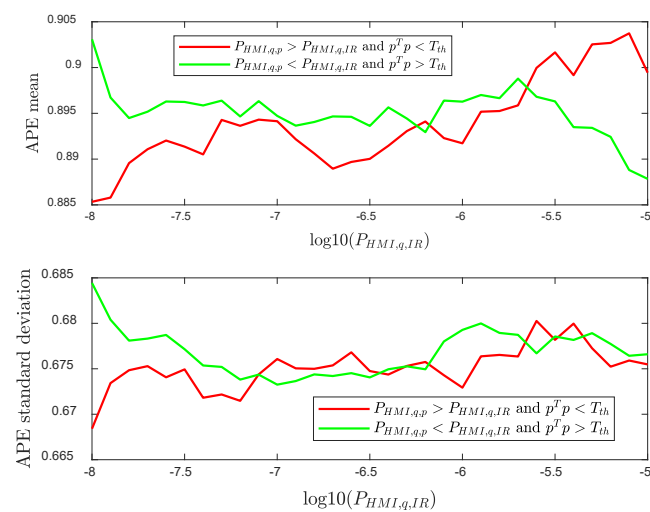


Figure 11. Means and standard deviations of the absolute position errors of the two sets of parity samples in the Monte Carlo simulation of case 2.

As shown in Figure 10, the distributions of the parity samples respectively satisfying $P_{\text{HML},q} > P_{\text{HML},q,\text{IR}}$ and $\mathbf{p}^T \mathbf{p} < T_{\text{th}}$ and $P_{\text{HML},q} < P_{\text{HML},q,\text{IR}}$ and $\mathbf{p}^T \mathbf{p} > T_{\text{th}}$ in the Monte Carlo simulation are still almost the same when $P_{\text{HML},E,\text{IR}} = 10^{-8}$ and $P_{\text{HML},E,\text{IR}} = 10^{-5}$. This observation holds true despite the significant difference in the number of satisfied parity samples between these two values of $P_{\text{HML},E,\text{IR}}$. And the statistical means and the standard deviations of the absolute position errors of the two sets of parity samples are close to each other when $P_{\text{HML},E,\text{IR}}$ ranges from 10^{-8} to 10^{-5} .

The test results presented validate the method for evaluating the probability of false alarm in posterior integrity monitoring by constructing spheres both inside and outside the contour of Θ . Furthermore, the distributions and statistical properties of the two sets of parity samples respectively satisfying $P_{\text{HML},q} > P_{\text{HML},q,\text{IR}}$ and $\mathbf{p}^T \mathbf{p} < T_{\text{th}}$ and $P_{\text{HML},q} < P_{\text{HML},q,\text{IR}}$ and $\mathbf{p}^T \mathbf{p} > T_{\text{th}}$ in the Monte Carlo simulation support the correctness of computing posterior integrity risk and defining the contour of Θ . Notably, although the parity samples constrained by $P_{\text{HML},q} < P_{\text{HML},q,\text{IR}}$ and $\mathbf{p}^T \mathbf{p} > T_{\text{th}}$ are situated far from the origin, their induced position errors remain comparable to those of nearer parity samples that satisfy $P_{\text{HML},q} > P_{\text{HML},q,\text{IR}}$ and $\mathbf{p}^T \mathbf{p} < T_{\text{th}}$.

By comparing Case 1 and Case 2, we observe that, when the contour of Θ , which represents the boundary of the acceptable parity set associated with a specified integrity risk requirement, does not approximate a perfect sphere or circle, the evaluation accuracy of the probability of false alarm in posterior integrity monitoring will be compromised. This issue becomes particularly pronounced when few satellites are used for positioning. As the number of satellites utilized increases, the contour of Θ tends to approach a more spherical shape, leading to an improved evaluation accuracy. Therefore, to enhance the performance, it is advisable to utilize multi-constellation, multi-frequency GNSS satellites for positioning. Another potential approach to improve evaluation accuracy is to construct the largest possible ellipsoid within the contour of Θ and the smallest possible ellipsoid outside this contour, rather than relying solely on spheres.

Figures 4 and 8 indicate that the evaluated upper bound and lower bound of $P_{\text{FA},q}$ in the posterior integrity monitoring decrease as the integrity risk requirement $P_{\text{HML},q,\text{IR}}$ increases. In contrast, the $P_{\text{FA},q}$ in the prior integrity monitoring is a preset constant parameter. To mitigate the continuity risk in integrity monitoring, one can choose prior and posterior integrity monitoring based on the continuity relationship analyzed in Section 3.1. Additionally, the accuracy relationship discussed in Section 3.2 and demonstrated in the test results indicates that GNSS fault detection and exclusion based solely on parity magnitude do not ensure a better positioning accuracy.

It is important to note that our method and the analysis assume that GNSS measurements follow established models, such as noise distributions and fault occurrence probabilities. In practice, GNSS measurements can be affected by various failures, including constant bias and ionospheric scintillation. The continuity risk can be correctly estimated as long as the GNSS measurement models are properly established. Otherwise, unmodeled measurement noises and faults may adversely impact the continuity risk calculated by the proposed method.

5. Conclusions

In conclusion, the distinct probabilistic nature of prior and posterior integrity risks results in different impacts on the continuity and accuracy of GNSS positioning. Prior integrity monitoring and posterior integrity monitoring each influence these critical performance metrics in unique ways. One of the challenges faced in posterior integrity monitoring is accurately computing the probability of false alarm in fault-free scenarios. To overcome this challenge, we propose a novel method involving the definition of spheres inside and

outside the contour of the parity set, where the integrity risk requirements are satisfied. This approach enables us to establish the lower and upper bounds for the probability of false alarm and the continuity risk in posterior integrity monitoring.

Our finding reveals that the continuity risks associated with both prior and posterior integrity monitoring can be effectively compared using the proposed method. Specifically, the continuity risks are comparable when the parity sphere defined by the fault detection test of the Chi-squared residual-based prior integrity monitoring is either larger than the outside sphere or smaller than the inside sphere. This comparative framework provides valuable insights into the relative performance and reliability of these monitoring approaches.

Moreover, our analysis shows that, in a fault-free scenario, posterior integrity monitoring generally offers superior accuracy compared with the Chi-squared residual-based prior integrity monitoring. This enhanced accuracy underscores the potential advantages of posterior integrity monitoring in applications where precision is paramount. These insights contribute to the optimization of GNSS integrity monitoring strategies, supporting the development of more reliable and accurate GNSS-based solutions for safety-critical applications.

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