

Article

An Analytical Approach to Gear Mesh Dynamics for the Sustainable Design of Agricultural Machinery Drive Systems

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Abstract: This research aims to advance the understanding and application of dynamic models for gears within agricultural machinery drive trains by developing analytical solutions. Despite the significant advancements in vibration analysis, there is a notable scarcity of comprehensive research that addresses the analytical modeling of gear dynamics, particularly using advanced mathematical techniques such as the multiple scale method (MSM). A new approach to modeling gear meshing is introduced, where the Fourier expansion of a rectangular signal is utilised to simulate the time-varying mesh stiffness (TVMS). Such an approach allows the use of the MSM as an efficient tool to obtain solutions for parametrically induced vibrations. A dynamic model of a simple helical gear system is introduced in the form of Hill's equation, and a sensitivity analysis is conducted for the main parameters of the system based on the solution obtained with the MSM. The results show the high credibility of the provided method when compared with a well-known state-of-the-art model and that the model reacts to parameter variations as expected. Additionally, an analysis of the energy harvesting possibilities is presented, which shows that, for the default parameter values, the harvester should be tuned to 5694.31 Hz to generate the maximum energy. It is concluded that the proposed model and MSM approach can serve as suitable tools for gear analysis, and the future paths for research are defined.

Keywords: gear mesh; vibration analysis; analytical solution; numerical modeling; multiple scale method (MSM); agricultural machinery; agricultural sustainability



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1. Introduction

This theoretical research aims to advance the understanding and application of dynamic models for gears within agricultural machinery drive trains by developing analytical solutions and focuses on a model of a helical gear system (provided as a Hill's equation) with time-variable mesh stiffness (TVMS) as a source of parametric vibration. Utilising the TVMS allows us to obtain solutions based on the multiple scale method (MSM) [1]. Despite the significant advancements in vibration analysis, there is a notable scarcity of comprehensive research that addresses the analytical modeling of gear dynamics, particularly using advanced mathematical techniques such as the MSM [2–4]. To bridge this gap, the simplest gear model was selected as a starting point to provide a substantial basis for further scaling to more complex systems. The MSM was employed to derive analytical solutions that enable the determination of vibration frequency ranges and the identification of spectral components. This mathematical approach offers a generalised framework for the understanding of vibrations generated by drive trains and gearboxes and can significantly

enhance the precision of vibration management strategies. By leveraging these insights, this research addresses the improvement of the design of agricultural machinery drive trains, emphasising vibration reduction, efficient propagation control, and the ability to utilise vibrations as an energy source. Ultimately, this work seeks to enhance the machinery's durability and the operator's comfort and reduce the environmental impact, fostering safer and more effective agricultural practices as part of sustainable development.

The operators of agricultural machinery (AM) are subjected to many harmful vibroacoustic (VA) signals. Large unsprung masses, high chassis stiffness, large-displacement compression ignition engines, and a wide variety of geared mechanisms and types of ground excitation create an extremely hazardous work environment, as the only isolation is provided by the seat, its suspension, and the cabin [5–9]. However, the non-direct influence on the environment and surroundings should not be omitted [10,11]. Formerly rural areas close to large cities are being urbanised at a growing pace; hence, neighbourhoods consisting of apartments or single-family houses with farmland and farms are not an unusual sight anymore. Recently, the reduction of engine noise in AM, whether by dampening internal combustion engines or by switching to electric motors, has introduced new challenges, as the noise generated by transmissions and gears has become more evident among operators and their surroundings [12–15]. The vibration energy (VE) can be investigated in terms of sustainable development in two ways. The first approach focuses on reducing the amount of VE to develop drive trains and gearboxes that will last longer and produce less noise pollution, diminishing the impact on people and the environment. However, thanks to technological advances, it is possible to harvest VE and convert it into electrical energy. This opens up new perspectives for sustainable development, enabling a reduction in the consumption of conventional energy and limiting the emission of harmful substances [14,16].

To complete the picture, the influence of the vibration on the machine itself should also be highlighted [7,17]. The presence of vibration can indicate both the normal operation of the drive system [18,19] and the beginning of a failure [20,21]. Therefore, it is crucial to recognise and interpret the characteristics of the generated vibrations. One of the main sources of vibration in agricultural machinery is gearboxes [14]. Their operation is associated with the occurrence of parametric vibrations [19,22–26], characterised by time-varying system parameters such as stiffness and damping [19,22,25,26]. Parametric vibrations differ significantly from other types of vibration because their characteristics depend not only on the properties of the system but also on its operating conditions. Understanding the mechanisms involved in the generation and propagation of these vibrations is crucial in ensuring the reliability and durability of gearboxes. Parametric vibrations can lead to resonances, which in turn can cause damage to structural components [27,28]. The analysis of the vibrations generated by the drive system of an agricultural machine allows for the early detection of potential failures and the prevention of more serious damage [28–30]. Thanks to appropriate sensors and analyses (supported by analytical solutions), it is possible to track changes in vibration characteristics and, based on this, make decisions regarding the need for inspection or repair [18,31], hence extending the expected service life.

Although numerical methods provide a very broad set of approaches to analysing the problem in question and are widely used, analytical solutions, where applicable, are advantageous in many aspects [32,33]. Firstly, properly defined and assessed formulas ensure the repeatability of the obtained results. Secondly, although the analytical solution may be initially more time-consuming than its numerical counterparts, it shortens the overall analysis time as the simple substitution of different parameter values is sufficient to obtain new results. In contrast, each change in the parameters with a numerical approach requires the simulations to be run again. Given this, a demand arises for precise compu-

tational methods based on the determination of analytical solutions also in the context of gear meshing analysis [18,28,34]. Unfortunately, obtaining such a solution for the complex process of gear meshing is difficult, especially in the case of multiple-degrees-of-freedom transmission with high-speed gears, which are subjected to broad changes in rotational speeds, heavy loads, and numerous excitation sources, as each pair of working gears has a time-varying mesh stiffness. Given this, numerical methods are far more popular, but approaches that obtain hybrid analytical–numerical and fully analytical solutions are of interest [35–37]. Various examples introduce the kinematics and the dynamical effects of centrifugal forces for high-speed transmissions [38]; time-varying backlash [39]; models of the meshing stiffness, load sharing, and transmission errors [18,40–42]; and the load sharing and friction torque of involute spurs and helical gears considering a non-uniform line stiffness and line load [43]. The variable stiffness has elicited interest not only in the case of analysing the dynamics of mating gears [44,45]. It also serves as a tool for the analysis and prediction of defects in gears, such as spalling, pitting, and cracking, where analytical approaches are also utilised in this respect [46–48], or when focusing on the gear shaping process [49]. It should be noted that, as most of the solutions introduced over the years have been derived from the basic cantilever beam approach, major modifications of often different sources are needed to properly represent a specific behaviour.

The problem of vibration reduction (VR) and simultaneous energy harvesting (EH) is becoming increasingly important in the context of technological development and is the subject of intensive research [36,50]. There are many vibration reduction methods, such as passive (vibration dampers, vibration isolation, high damping materials) [40], active (control systems that generate forces opposite to the forces exciting vibrations) [28], and hybrid (a combination of passive and active methods) [51]. The choice of a suitable method depends on many factors, such as the vibration frequency, vibration amplitude, operating environment, and cost and mass requirements. Energy harvesting from vibrations involves converting the mechanical energy of vibrations into electrical energy. A few types of phenomena can be useful to achieve this goal: piezoelectricity (the use of materials that generate an electric voltage when deformed), electromagnetism (the use of the electromagnetic induction phenomenon), or electrostatics (the use of electric charges). Analytical solutions can significantly facilitate the process of selecting vibration reduction and energy harvesting methods [50]. Thanks to them, it is possible to quickly assess the effectiveness of different solutions, optimise the system parameters, or minimise the number of costly experiments [52]. It can be stated that vibration reduction and energy harvesting are very complex subjects and require interdisciplinary knowledge. Thanks to the development of new materials, computational methods, and sensor technologies, it is possible to create increasingly advanced solutions [53]. Analytical solutions play a key role in this process, enabling the rapid and efficient design of vibration reduction and energy harvesting systems.

2. Dynamic Model of Gear Meshing and Intertooth Force

A thorough study of a gear meshing problem requires the consideration of mesh stiffness changes in time. Therefore, a dynamic model of gear meshing (Figure 1) that considered time-varying mesh stiffness (TVMS) was used. Gears in a normal mesh (where contact ratio ε_α falls between 1 and 2) experience both single-tooth and double-tooth contact along the path of contact, as shown in Figure 2.

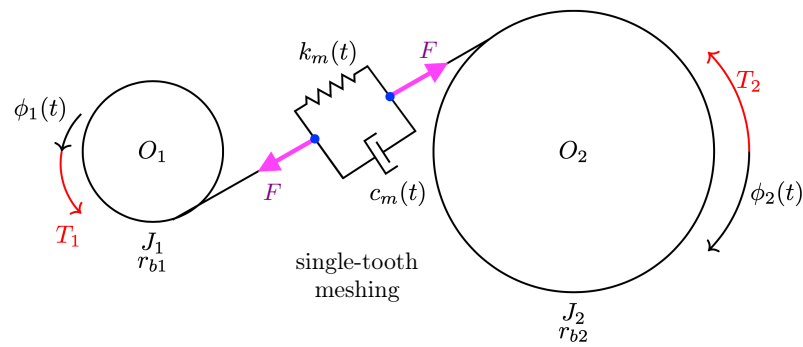


Figure 1. Dynamic model of single-tooth meshing with forces along the line of action.

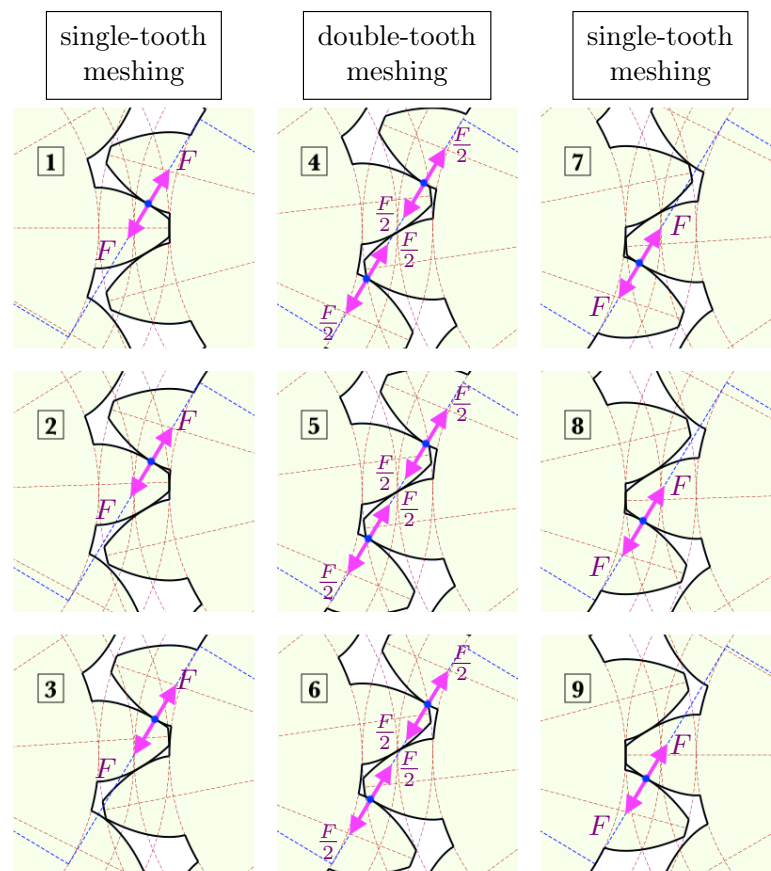


Figure 2. Force diagram along the path of contact (in line of action) during gear meshing: nine distinctive positions of tooth contact.

To understand the overall stiffness of a normal mesh, it is first necessary to analyse the stiffness of the individual teeth during engagement. This can be performed analytically for a single pair of teeth (denoted as $k_p(t)$). Method B from the ISO 6336 standard [54] can be used to find a single pair's maximum stiffness (k'_{th}). Equation (1) is then applied for complete depiction, as described in detail in reference [55]:

$$k_p = \frac{20k'_{th}}{17\varepsilon_\alpha} \left(0.55 + \frac{1.8t}{\varepsilon_\alpha t_z} - \frac{1.8t^2}{\varepsilon_\alpha^2 t_z^2} \right) \quad (1)$$

where

k_p —stiffness of a single pair of teeth;

k'_{th} —maximum stiffness;
 ε_α —contact ratio;
 t_z —period of mesh stiffness variation;
 t —time.

It is important to note that the comprehensive mesh stiffness of the gear is the elastic response of the comprehensive meshing effect of each pair of teeth along the path of contact (in the whole meshing area). It is directly related to the elastic deflection of a single tooth, the comprehensive elastic deflection of a single pair of teeth, and the degree of engagement of the gear system. For a helical gear system with a contact ratio greater than 1 and less than 2, the single-tooth meshing and double-tooth meshing areas occur in one meshing period. A specific function shape defines the time course of mesh stiffness in the single-tooth meshing region (a single tooth pair's mesh stiffness). For example, it is well known that one can approximate the course of mesh stiffness using the shape of a parabolic function. However, in the double-tooth meshing area, the stiffness is calculated through superposition, treating both pairs as parallel-connected springs with varying characteristics. Calculating the combined mesh stiffness when two pairs of teeth engage along the path of contact (double-tooth meshing) involves summing the mesh stiffness of each pair of teeth, as in Formula (3). The comprehensive time-varying mesh stiffness causes changes with a constant period, as shown in Figure 3. A higher stiffness value corresponds to double-tooth meshing, while a lower one corresponds to single-tooth meshing.

$$k_{pi} = \frac{20k'_{th} \left(0.55 + \frac{1.8(-it_z + t)}{\varepsilon_\alpha t_z} - \frac{1.8(-it_z + t)^2}{\varepsilon_\alpha^2 t_z^2} \right)}{17\varepsilon_\alpha} \quad (2)$$

$$k_{TVMS} = \sum_{i=-\infty}^{+\infty} k_{pi} \quad (3)$$

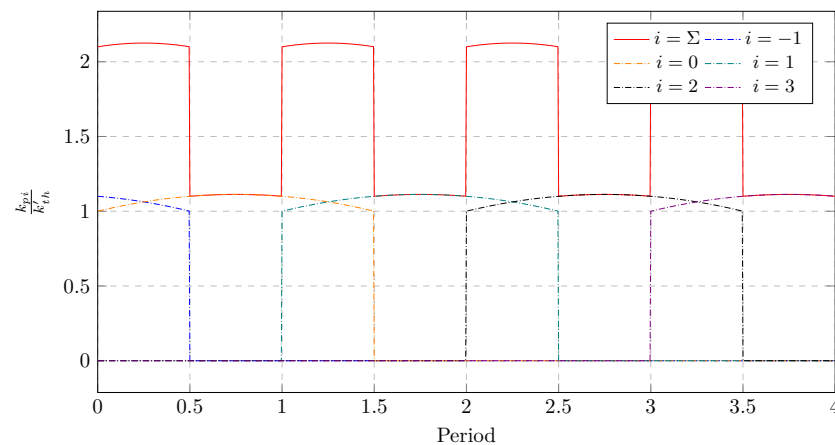


Figure 3. Superposition of mesh stiffness with the shape of a parabolic function.

Formula (1) includes the parameter t_z , which represents the period of mesh stiffness change corresponding to the time of meshing of gear teeth. The pair of teeth are in contact during this time, and the point of contact of the meshing gear teeth traces a specific path. The contact points of the teeth are aligned with the line of action (pressure line) during gear rotation. The tooth-to-tooth force is generally directed along the same line for involute gears. In other words, the line of action is constant, and, along it, the force vector between two meshing gear teeth is directed (Figures 1 and 2).

The equation of motion of a pair of gears (Figure 1) reduced to a single degree of freedom is given by Formula (4) (Hill's equation):

$$-F + c_g \dot{z} + m_{eq} \ddot{z} + k_g (\kappa_{mesh} \varepsilon + 1) z = 0 \quad (4)$$

where

F —excitation force;

c_g —mesh damping coefficient;

m_{eq} —equivalent mass;

k_g —mesh stiffness coefficient;

κ_{mesh} —formula representing stiffness variation;

ε —small parameter;

z —gear vibration displacement;

$\dot{z}$ —gear vibration velocity;

$\ddot{z}$ —gear vibration acceleration.

Several models have been proposed based on the obtained relationship that defines the variable mesh stiffness. Figure 4 shows a comparison of four different mesh stiffness models that are the most efficient approximations from a simulational point of view:

- reference—typical parabolic shape;
- wave-shaped—commonly used to represent the parabolic nature of teeth meshing due to the sufficient representation of physical properties;
- rectangular—rather less popular for gear meshing approximation, with poorer accuracy in comparison with the wave-shaped case;
- rectangular Fourier approximation—the approach proposed in this study.

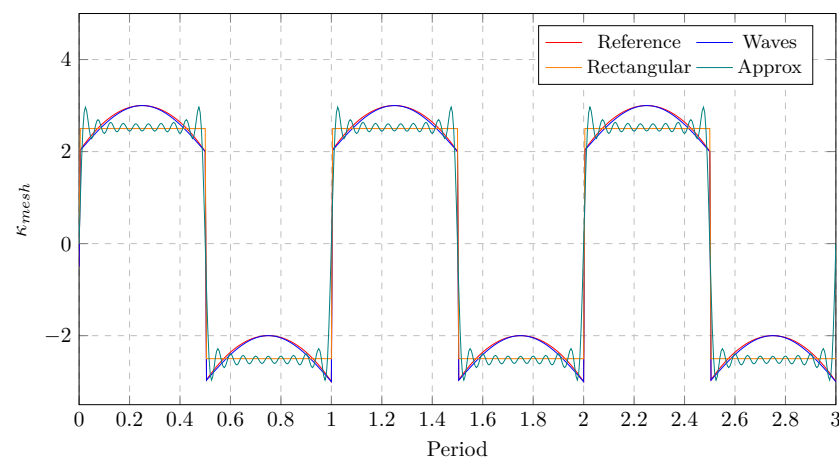


Figure 4. Comparison of different mesh stiffness signals.

The proposal to expand the rectangular signal with the utilisation of the Fourier transform was driven by the need to obtain a formulation of variable stiffness dependency, which would allow us to exploit the MSM approach. The Fast Fourier Transform (FFT) was involved as an efficient tool to determine the influence of specific harmonics on the signal amplitude. Formulas regarding the expansion coefficients are presented below:

$$\hat{k}_{TVMS} = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \left(\frac{\pi n k_{TVMS}}{T} \right) + b_n \sin \left(\frac{\pi n k_{TVMS}}{T} \right) \right) \quad (5)$$

where

$$a_0 = \frac{\int_{-T}^T k_{TVMS} dt}{T} \quad (6)$$

$$a_n = \frac{2 \int_{-T}^T k_{TVMS} \cos\left(\frac{\pi n t}{T}\right) dt}{T} \quad (7)$$

$$b_n = \frac{2 \int_{-T}^T k_{TVMS} \sin\left(\frac{\pi n t}{T}\right) dt}{T} \quad (8)$$

From the properties of the analysed function, it can be observed that only odd terms of the series will be non-zero. Additionally, from the theorem about the integration of odd functions in the symmetrical domain, which is the case here, it occurs that only cosine terms will be present. In such a case, a large number of terms would be required for satisfactory approximation, so the FFT algorithm is used to analyze the signal in the frequency domain to obtain the amplitudes of particular harmonics.

The frequency spectra based on the time courses (Figure 4) of different mesh stiffness signals are shown in Figures 5 and 6. A detailed description of the determination of the overall time-varying mesh stiffness course has been provided in [56]. All calculations were performed for the data presented in Table 1, based on the research in [57].

Table 1. Basic parameters of helical gear.

	Symbol	Value	Unit
Number of teeth of the pinion	z_1	26	-
Number of teeth of the gear	z_2	53	-
Module (gear system parameter)	m_n	2	mm
Center distance of the pinion	d_1	52	mm
Center distance of the gear	d_2	106	mm
Gear ratio	u	2.04	-
Pressure angle	α_{wt}	20	°
Pitch diameter of the pinion	d_{b1}	48.864	mm
Pitch diameter of the gear	d_{b2}	99.607	mm
Face width ratio	ϵ_α	1.412	-
Length of the contact line	g_α	8.338	mm
Rotational speed of the pinion	n_1	1460	rpm
Input torque	T_1	98.12	Nm

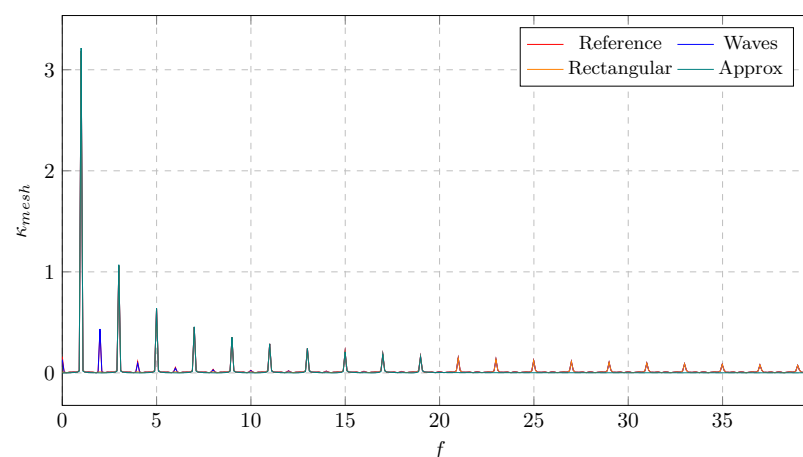


Figure 5. Comparison of FFT spectra for different stiffness models.

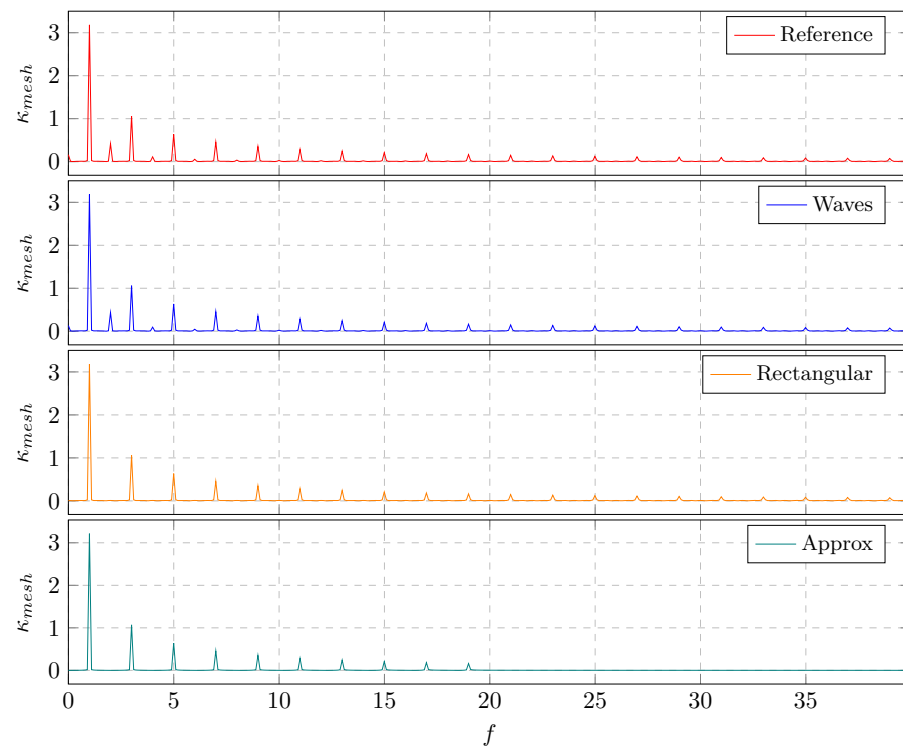


Figure 6. Comparison of FFT spectra for different stiffness models—separate view based on Figure 5.

A series of numerical simulations using Equation (4) was conducted to evaluate the effectiveness of the adopted gear stiffness models (Figure 4). These simulations were performed to initially check the correctness of the meshing gears' displacements (Figures 7–9) for each stiffness model.

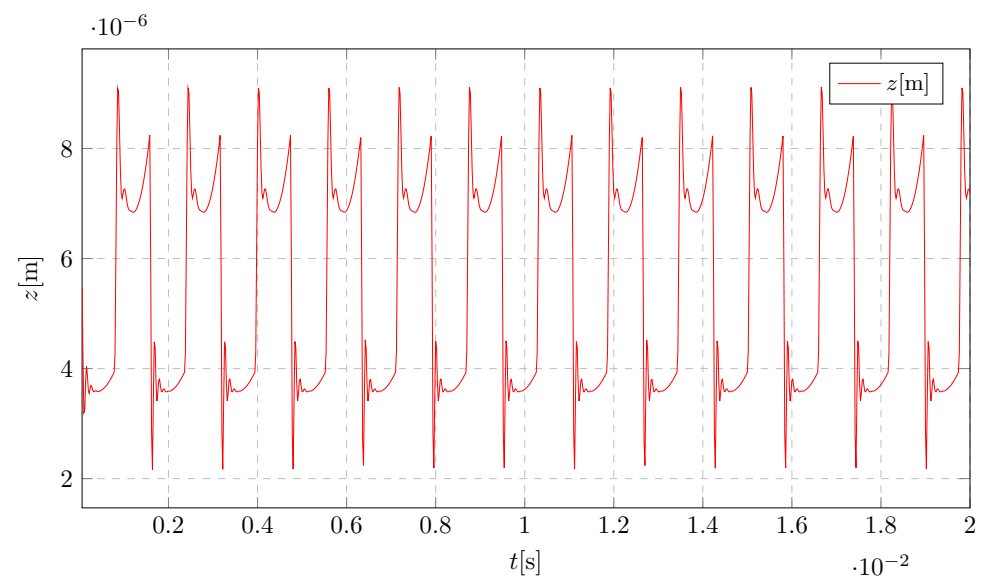


Figure 7. Displacement of system vibrations with time-varying mesh stiffness as wave signal.

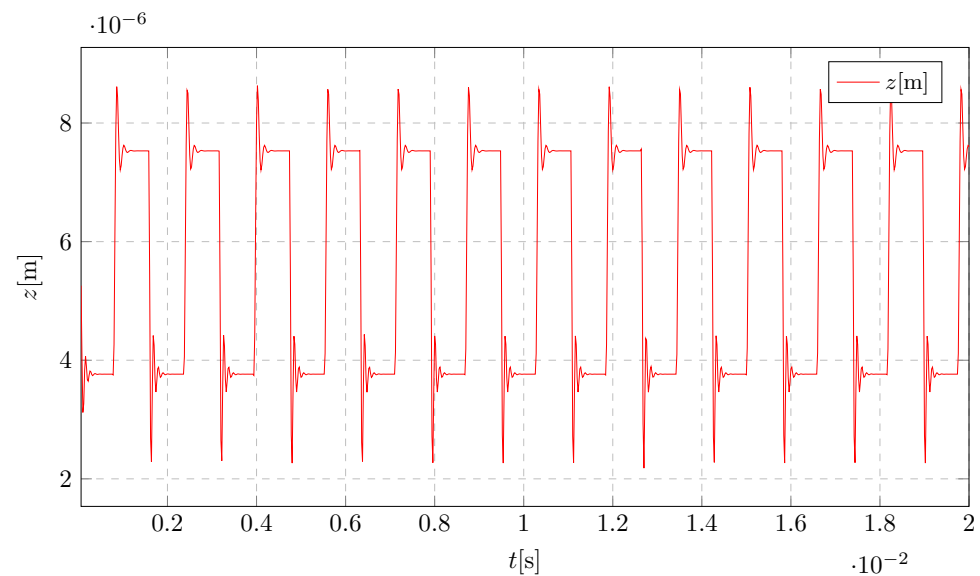


Figure 8. Displacement of system vibrations with time-varying mesh stiffness as rectangular signal.

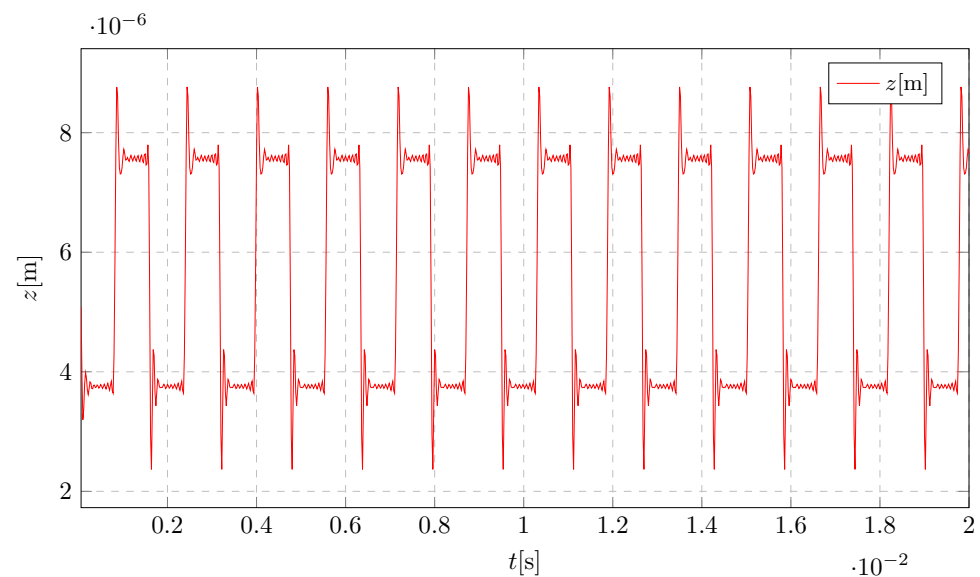


Figure 9. Displacement of system vibrations with time-varying mesh stiffness as Fourier expansion signal.

Observations of the graphs in Figures 7–9 allow for a comparison of the simulation results obtained based on the solution of the gear system, where the mesh stiffness is given consecutively as a wave signal (WS), a rectangular signal (RS), and its Fourier approximation (AP). This comparison aims to identify the differences in the obtained waveforms. All simulations were conducted for the same period to avoid doubts about the frequencies of the obtained signals and allow for a pure comparison of the waveform shape and amplitude. In general, the amplitudes corresponding to single-tooth and two-teeth contact reach similar values, but larger discrepancies are visible when referring to the shape of the plots. The parabolic model, as the classic mesh stiffness model known in the literature, is a well-established mathematical representation of the real course of the mesh stiffness along the line of action. The WS model approximates the parabolic model with high accuracy—the nature of the displacement is well represented in the time domain, and the frequency spectra confirm this observation. The RS case spectra, although generally providing almost the same amount of information, lack some of the frequency components due to the further simplification—the absence of parabolic parts in the time

domain (Figure 8). Nevertheless, it can still be regarded as a sufficient representation of the gear meshing process. The Fourier series-approximated signal introduces additional differences. Above 20 Hz, no spectral components are visible, in comparison with other models, where small amplitudes appear. The rectangular signal, as stated previously, is less accurate than wave approximation, but the use of Fourier series expansion results in smoother stiffness variations. Locally, the displacement values can be artificially overstated or underestimated. In the case of the WS model (Figure 7), sudden changes result in longer transients, which are evident when the number of pairs of teeth in the contact line changes. It is impossible to remove excess stiffness in the Fourier-approximated model due to the Gibbs phenomenon, which is an inherent feature of the Fourier series of a periodic function with low smoothness (C0 class).

The equivalent meshing force for the pair of gears reduced to a single degree of freedom is given by Formula (9):

$$q_{eq} = \frac{F_c}{m_{eq}} = \frac{c_g \dot{z}}{m_{eq}} + \frac{k_g z}{m_{eq}} \quad (9)$$

where

q_{eq} —equivalent meshing force;

F_c —meshing force.

Further transformation of Formula (9) allows us to obtain the following form of the analysed equation:

$$q_{eq} = \omega_0^2 z + 2h\dot{z} \quad (10)$$

where

ω_0 —natural frequency;

h —damping coefficient.

The time-varying meshing force waveform based on the Fourier-approximated model and including damping along the line of action is shown in Figure 10.

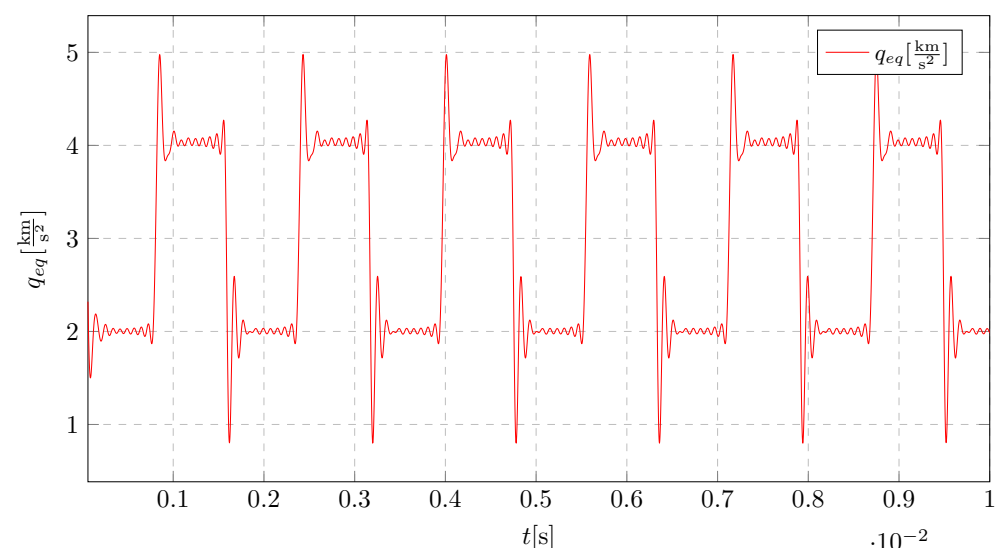


Figure 10. Time-varying meshing force along the line of action.

The apparent transients associated with the stabilisation of the inter-tooth force values are due to the damping occurring in the mesh system. The time course of the meshing force changes with a constant period, which correlates with the mesh stiffness (Figure 9). The distribution of the meshing force is characterised by sudden increments when one pair of

teeth starts to mesh. This corresponds to the temporal moment at which the meshing force takes its maximum value. Lower meshing force values occur when two pairs of teeth are in contact along the line of action—specifically, from entering the line of action to exiting it.

3. Analytical Solution

The general formulation of the dynamic model provided in Equation (4) was further simplified for analytical investigation. Equation (11) is more convenient for this analysis, as the natural frequency of the system is substituted with the meshing frequency. To ensure that the natural frequency will be much higher than the meshing frequency, a relationship between these two frequencies was selected, namely $\omega_0 = 16\omega$, leading to the appearance of numerical values of 256 and 582 in consecutive parts of the equation. The small parameter in the considered system indicates the elements of the equation that influence the main part of the solution and the perturbative components. The multiple scale method is a technique in perturbation theory based on introducing additional rescaled variables (usually time variables), formally considered as independent variables and each describing a different time scale. The method comprises techniques used to construct uniformly valid approximations to the solutions of perturbation problems, both for small and large values of the independent variables. This is achieved by introducing fast-scale and slow-scale variables for an independent variable and subsequently treating these variables as if they are independent. Due to the introduction of the new independent variables, secular terms appear in the solution process, which are later removed. This places constraints on the approximate solution, called solvability conditions. In fact, as a result of applying the MSM, a recursive perturbational system of equations is obtained. It is solved in a sequential way by looking for solutions to successive approximations:

$$256\omega^2 z + 256\delta\omega^2 u_s \varepsilon + 256\delta\omega^2 \varepsilon z + 582.0\omega^2 u_s \varepsilon \sin(\omega t) + 582.0\omega^2 \varepsilon z \sin(\omega t) + \ddot{z} = 0 \quad (11)$$

where

u_s —static deflection;

δ —detuning parameter;

ω —meshing frequency.

The solution is approximated by a Taylor polynomial. For the initial conditions, it is assumed that the solution can be approximated by a sum of functions, each depending on a small parameter. This provides a recursive structure for the solution of the equations [58]:

$$z = \varepsilon z_1 + \varepsilon^2 z_2 + \varepsilon^3 z_3 + z_0 \quad (12)$$

$$z = \varepsilon z_1(t_0, t_1, t_2) + \varepsilon^2 z_2(t_0, t_1, t_2) + \varepsilon^3 z_3(t_0, t_1, t_2) + z_0(t_0, t_1, t_2) \quad (13)$$

where

z_0, z_1, z_2, z_3 —sequence of solution approximations.

Formulas that represent arguments of approximation (time scales) are as follows:

$$t_0 = t \quad (14)$$

$$t_1 = t\varepsilon \quad (15)$$

$$t_2 = t\varepsilon^2 \quad (16)$$

Consecutive derivatives of the functions are calculated and substituted into Equation (11). By grouping the elements with respect to the small parameter, a set of equations can be

formulated. Their solution estimates the approximation functions. At this stage, the solution was approximated for different orders of ε . These approximations provide increasing accuracy concerning the original nonlinear equation:

$$\dot{z} = \varepsilon \dot{z}_1 + \varepsilon^2 \dot{z}_2 + \dot{z}_0 \quad (17)$$

$$\ddot{z} = \varepsilon \ddot{z}_1 + \varepsilon^2 \ddot{z}_2 + \ddot{z}_0 \quad (18)$$

The function is approximated in regard to different time scales (denoted as $t_0, t_1, t_2, \dots, t_n$). Each time scale represents a different rate of change in the system's dynamics. The resulting partial functions are substituted back into the original function and derivatives are calculated, which later can be substituted into the equation of motion.

$$z = \varepsilon z_1(t_0, t_1, t_2) + \varepsilon^2 z_2(t_0, t_1, t_2) + z_0(t_0, t_1, t_2) \quad (19)$$

$$\begin{aligned} \dot{z} = & \dot{t}_0 \frac{d}{dt_0} z_0(t_0, t_1, t_2) + \dot{t}_1 \frac{d}{dt_1} z_0(t_0, t_1, t_2) + \dot{t}_2 \frac{d}{dt_2} z_0(t_0, t_1, t_2) + \varepsilon \dot{t}_0 \frac{d}{dt_0} z_1(t_0, t_1, t_2) + \varepsilon \dot{t}_1 \frac{d}{dt_1} z_1(t_0, t_1, t_2) \\ & + \varepsilon \dot{t}_2 \frac{d}{dt_2} z_1(t_0, t_1, t_2) + \varepsilon^2 \dot{t}_0 \frac{d}{dt_0} z_2(t_0, t_1, t_2) + \varepsilon^2 \dot{t}_1 \frac{d}{dt_1} z_2(t_0, t_1, t_2) + \varepsilon^2 \dot{t}_2 \frac{d}{dt_2} z_2(t_0, t_1, t_2) \end{aligned} \quad (20)$$

$$\begin{aligned} \ddot{z} = & \ddot{t}_0 \frac{d^2}{dt_0^2} z_0(t_0, t_1, t_2) + \ddot{t}_1 \frac{d^2}{dt_1^2} z_0(t_0, t_1, t_2) + \ddot{t}_2 \frac{d^2}{dt_2^2} z_0(t_0, t_1, t_2) + \ddot{t}_0 \frac{d}{dt_0} z_0(t_0, t_1, t_2) + \ddot{t}_1 \frac{d}{dt_1} z_0(t_0, t_1, t_2) + \ddot{t}_2 \frac{d}{dt_2} z_0(t_0, t_1, t_2) + \varepsilon \ddot{t}_0 \frac{d^2}{dt_0^2} z_1(t_0, t_1, t_2) \\ & + \varepsilon \ddot{t}_1 \frac{d^2}{dt_1^2} z_1(t_0, t_1, t_2) + \varepsilon \ddot{t}_2 \frac{d^2}{dt_2^2} z_1(t_0, t_1, t_2) + \varepsilon \ddot{t}_0 \frac{d}{dt_0} z_1(t_0, t_1, t_2) + \varepsilon \ddot{t}_1 \frac{d}{dt_1} z_1(t_0, t_1, t_2) + \varepsilon \ddot{t}_2 \frac{d}{dt_2} z_1(t_0, t_1, t_2) + \varepsilon^2 \ddot{t}_0 \frac{d^2}{dt_0^2} z_2(t_0, t_1, t_2) \\ & + \varepsilon^2 \ddot{t}_1 \frac{d^2}{dt_1^2} z_2(t_0, t_1, t_2) + \varepsilon^2 \ddot{t}_2 \frac{d^2}{dt_2^2} z_2(t_0, t_1, t_2) + \varepsilon^2 \ddot{t}_0 \frac{d}{dt_0} z_2(t_0, t_1, t_2) + \varepsilon^2 \ddot{t}_1 \frac{d}{dt_1} z_2(t_0, t_1, t_2) + \varepsilon^2 \ddot{t}_2 \frac{d}{dt_2} z_2(t_0, t_1, t_2) + 2\dot{t}_0 \dot{t}_1 \frac{d^2}{dt_1 dt_0} z_0(t_0, t_1, t_2) \\ & + 2\dot{t}_0 \dot{t}_2 \frac{d^2}{dt_2 dt_0} z_0(t_0, t_1, t_2) + 2\dot{t}_1 \dot{t}_2 \frac{d^2}{dt_2 dt_1} z_0(t_0, t_1, t_2) + 2\varepsilon \dot{t}_0 \dot{t}_1 \frac{d^2}{dt_1 dt_0} z_1(t_0, t_1, t_2) + 2\varepsilon \dot{t}_0 \dot{t}_2 \frac{d^2}{dt_2 dt_0} z_1(t_0, t_1, t_2) + 2\varepsilon \dot{t}_1 \dot{t}_2 \frac{d^2}{dt_2 dt_1} z_1(t_0, t_1, t_2) \\ & + 2\varepsilon^2 \dot{t}_0 \dot{t}_1 \frac{d^2}{dt_1 dt_0} z_2(t_0, t_1, t_2) + 2\varepsilon^2 \dot{t}_0 \dot{t}_2 \frac{d^2}{dt_2 dt_0} z_2(t_0, t_1, t_2) + 2\varepsilon^2 \dot{t}_1 \dot{t}_2 \frac{d^2}{dt_2 dt_1} z_2(t_0, t_1, t_2) \end{aligned} \quad (21)$$

where

$\dot{t}_0, \dot{t}_1, \dot{t}_2$ —first time derivatives of time scales;

$\ddot{t}_0, \ddot{t}_1, \ddot{t}_2$ —second time derivatives of time scales.

The resulting approximated equation of motion is as follows:

$$\begin{aligned} & \varepsilon \frac{d^2}{dt_0^2} z_1(t_0, t_1, t_2) + \varepsilon^2 \frac{d^2}{dt_1^2} z_0(t_0, t_1, t_2) + \varepsilon^2 \frac{d^2}{dt_0^2} z_2(t_0, t_1, t_2) + \varepsilon^3 \frac{d^2}{dt_1^2} z_1(t_0, t_1, t_2) + \varepsilon^4 \frac{d^2}{dt_2^2} z_0(t_0, t_1, t_2) + \varepsilon^4 \frac{d^2}{dt_1^2} z_2(t_0, t_1, t_2) + \varepsilon^5 \frac{d^2}{dt_2^2} z_1(t_0, t_1, t_2) \\ & + \varepsilon^6 \frac{d^2}{dt_2^2} z_2(t_0, t_1, t_2) + 2\varepsilon \frac{d^2}{dt_1 dt_0} z_0(t_0, t_1, t_2) + 2\varepsilon^2 \frac{d^2}{dt_2 dt_0} z_0(t_0, t_1, t_2) + 2\varepsilon^2 \frac{d^2}{dt_1 dt_0} z_1(t_0, t_1, t_2) + 2\varepsilon^3 \frac{d^2}{dt_2 dt_1} z_0(t_0, t_1, t_2) \\ & + 2\varepsilon^3 \frac{d^2}{dt_2 dt_0} z_1(t_0, t_1, t_2) + 2\varepsilon^3 \frac{d^2}{dt_1 dt_0} z_2(t_0, t_1, t_2) + 2\varepsilon^4 \frac{d^2}{dt_2 dt_1} z_1(t_0, t_1, t_2) + 2\varepsilon^4 \frac{d^2}{dt_2 dt_0} z_2(t_0, t_1, t_2) + 2\varepsilon^5 \frac{d^2}{dt_2 dt_1} z_2(t_0, t_1, t_2) \\ & + 256\omega^2 z_0(t_0, t_1, t_2) + 256\omega^2 \varepsilon z_1(t_0, t_1, t_2) + 256\omega^2 \varepsilon^2 z_2(t_0, t_1, t_2) + 256\delta\omega^2 u_s \varepsilon + 256\delta\omega^2 \varepsilon z_0(t_0, t_1, t_2) + 256\delta\omega^2 \varepsilon^2 z_1(t_0, t_1, t_2) \\ & + 256\delta\omega^2 \varepsilon^3 z_2(t_0, t_1, t_2) + 582.0\omega^2 u_s \varepsilon \sin(\omega t_0) + 582.0\omega^2 \varepsilon z_0(t_0, t_1, t_2) \sin(\omega t_0) + 582.0\omega^2 \varepsilon^2 z_1(t_0, t_1, t_2) \sin(\omega t_0) \\ & + 582.0\omega^2 \varepsilon^3 z_2(t_0, t_1, t_2) \sin(\omega t_0) + \frac{d^2}{dt_0^2} z_0(t_0, t_1, t_2) = 0 \end{aligned} \quad (22)$$

The ordering and separation of Equation (22) in terms of the power of a small parameter ε leads to a recursive sequence of linear equations of motion, leading to the solution of the nonlinear equation. The zeroth-order approximate linear equation is given in (23):

$$256\omega^2 z_0(t_0, t_1, t_2) + \frac{d^2}{dt_0^2} z_0(t_0, t_1, t_2) = 0 \quad (23)$$

The next component of a recursive sequence of linear equations of motion leading to the solution of the considered nonlinear one is given in (24):

$$\begin{aligned}
& -\frac{d^2}{dt_0^2}z_1(t_0, t_1, t_2) - 2\frac{d^2}{dt_1 dt_0}z_0(t_0, t_1, t_2) - 256\omega^2 z_1(t_0, t_1, t_2) - 256\delta\omega^2 u_s \\
& - 256\delta\omega^2 z_0(t_0, t_1, t_2) - 582.0\omega^2 u_s \sin(\omega t_0) \\
& - 582.0\omega^2 z_0(t_0, t_1, t_2) \sin(\omega t_0) = 0
\end{aligned} \quad (24)$$

The system was solved and an important issue was to remove the secular terms. When the secular terms were removed, the following solution was obtained in the form of a power series of a small parameter:

$$\begin{aligned}
z = & 2.59u_s \varepsilon^2 + C_1 \cos(16\omega t) \cos(8\delta\omega t\varepsilon) + C_2 \sin(8\delta\omega t\varepsilon) \cos(16\omega t) + C_2 \sin(16\omega t) \cos(8\delta\omega t\varepsilon) \\
& - C_1 \sin(16\omega t) \sin(8\delta\omega t\varepsilon) - \delta u_s \varepsilon + 1.0\delta^2 u_s \varepsilon^2 - 2.28u_s \varepsilon \sin(\omega t) - 2.64u_s \varepsilon^2 \cos(2\omega t) \\
& + 9.39C_1 \varepsilon \sin(8\delta\omega t\varepsilon) \cos(15\omega t) + 9.39C_1 \varepsilon \sin(15\omega t) \cos(8\delta\omega t\varepsilon) + 9.39C_2 \varepsilon \sin(15\omega t) \sin(8\delta\omega t\varepsilon) \\
& + 37.7C_1 \varepsilon^2 \sin(18\omega t) \sin(8\delta\omega t\varepsilon) + 8.82C_1 \varepsilon \sin(8\delta\omega t\varepsilon) \cos(17\omega t) + 8.82C_1 \varepsilon \sin(17\omega t) \cos(8\delta\omega t\varepsilon) \\
& + 8.82C_2 \varepsilon \sin(17\omega t) \sin(8\delta\omega t\varepsilon) + 4.57\delta u_s \varepsilon^2 \sin(\omega t) + 45.5C_1 \varepsilon^2 \sin(14\omega t) \sin(8\delta\omega t\varepsilon) \\
& - 9.39C_2 \varepsilon \cos(15\omega t) \cos(8\delta\omega t\varepsilon) - 37.7C_1 \varepsilon^2 \cos(18\omega t) \cos(8\delta\omega t\varepsilon) - 37.7C_2 \varepsilon^2 \sin(8\delta\omega t\varepsilon) \cos(18\omega t) \\
& - 37.7C_2 \varepsilon^2 \sin(18\omega t) \cos(8\delta\omega t\varepsilon) - 8.82C_2 \varepsilon \cos(17\omega t) \cos(8\delta\omega t\varepsilon) - 45.5C_1 \varepsilon^2 \cos(14\omega t) \cos(8\delta\omega t\varepsilon) \\
& - 45.5C_2 \varepsilon^2 \sin(8\delta\omega t\varepsilon) \cos(14\omega t) - 45.5C_2 \varepsilon^2 \sin(14\omega t) \cos(8\delta\omega t\varepsilon) + 4.28C_2 \delta \varepsilon^2 \cos(17\omega t) \cos(8\delta\omega t\varepsilon) \\
& + 4.84C_2 \delta \varepsilon^2 \cos(15\omega t) \cos(8\delta\omega t\varepsilon) - 4.28C_1 \delta \varepsilon^2 \sin(8\delta\omega t\varepsilon) \cos(17\omega t) - 4.28C_1 \delta \varepsilon^2 \sin(17\omega t) \cos(8\delta\omega t\varepsilon) \\
& - 4.28C_2 \delta \varepsilon^2 \sin(17\omega t) \sin(8\delta\omega t\varepsilon) - 4.84C_1 \delta \varepsilon^2 \sin(8\delta\omega t\varepsilon) \cos(15\omega t) - 4.84C_1 \delta \varepsilon^2 \sin(15\omega t) \cos(8\delta\omega t\varepsilon) \\
& - 4.84C_2 \delta \varepsilon^2 \sin(15\omega t) \sin(8\delta\omega t\varepsilon)
\end{aligned} \quad (25)$$

where

C_1, C_2 —integration constants.

It was necessary to determine the relative velocity from the first derivative for further calculations:

$$\begin{aligned}
\dot{z} = & 16.0C_2\omega \cos(16\omega t) \cos(8\delta\omega t\varepsilon) + 5.27\omega u_s \varepsilon^2 \sin(2\omega t) \\
& - 16.0C_1\omega \sin(8\delta\omega t\varepsilon) \cos(16\omega t) - 16.0C_1\omega \sin(16\omega t) \cos(8\delta\omega t\varepsilon) \\
& - 16.0C_2\omega \sin(16\omega t) \sin(8\delta\omega t\varepsilon) - 2.28\omega u_s \varepsilon \cos(\omega t) \\
& + 141.0C_1\omega \varepsilon \cos(15\omega t) \cos(8\delta\omega t\varepsilon) + 141.0C_2\omega \varepsilon \sin(8\delta\omega t\varepsilon) \cos(15\omega t) \\
& + 141.0C_2\omega \varepsilon \sin(15\omega t) \cos(8\delta\omega t\varepsilon) + 679.0C_1\omega \varepsilon^2 \sin(8\delta\omega t\varepsilon) \cos(18\omega t) \\
& + 679.0C_1\omega \varepsilon^2 \sin(18\omega t) \cos(8\delta\omega t\varepsilon) + 679.0C_2\omega \varepsilon^2 \sin(18\omega t) \sin(8\delta\omega t\varepsilon) \\
& + 4.57\delta\omega u_s \varepsilon^2 \cos(\omega t) + 150.0C_1\omega \varepsilon \cos(17\omega t) \cos(8\delta\omega t\varepsilon) \\
& + 150.0C_2\omega \varepsilon \sin(8\delta\omega t\varepsilon) \cos(17\omega t) + 150.0C_2\omega \varepsilon \sin(17\omega t) \cos(8\delta\omega t\varepsilon) \\
& + 637.0C_1\omega \varepsilon^2 \sin(8\delta\omega t\varepsilon) \cos(14\omega t) + 637.0C_1\omega \varepsilon^2 \sin(14\omega t) \cos(8\delta\omega t\varepsilon) \\
& + 637.0C_2\omega \varepsilon^2 \sin(14\omega t) \sin(8\delta\omega t\varepsilon) - 141.0C_1\omega \varepsilon \sin(15\omega t) \sin(8\delta\omega t\varepsilon) \\
& - 679.0C_2\omega \varepsilon^2 \cos(18\omega t) \cos(8\delta\omega t\varepsilon) - 150.0C_1\omega \varepsilon \sin(17\omega t) \sin(8\delta\omega t\varepsilon) \\
& - 637.0C_2\omega \varepsilon^2 \cos(14\omega t) \cos(8\delta\omega t\varepsilon) + 72.8C_1\delta\omega \varepsilon^2 \sin(15\omega t) \sin(8\delta\omega t\varepsilon) \\
& + 72.8C_1\delta\omega \varepsilon^2 \sin(17\omega t) \sin(8\delta\omega t\varepsilon) - 72.8C_1\delta\omega \varepsilon^2 \cos(15\omega t) \cos(8\delta\omega t\varepsilon) \\
& - 72.8C_1\delta\omega \varepsilon^2 \cos(17\omega t) \cos(8\delta\omega t\varepsilon) - 72.8C_2\delta\omega \varepsilon^2 \sin(8\delta\omega t\varepsilon) \cos(15\omega t) \\
& - 72.8C_2\delta\omega \varepsilon^2 \sin(8\delta\omega t\varepsilon) \cos(17\omega t) - 72.8C_2\delta\omega \varepsilon^2 \sin(15\omega t) \cos(8\delta\omega t\varepsilon) \\
& - 72.8C_2\delta\omega \varepsilon^2 \sin(17\omega t) \cos(8\delta\omega t\varepsilon)
\end{aligned} \quad (26)$$

The MSM made it possible to obtain approximate relationships describing the velocity and relative position of the gear (measured along the line of action). The obtained dependencies have the characteristics of a series of small parameters. This makes it possible

to estimate the size of the error occurring in the approximate solution. In the case under consideration, it is at the level of ε^2 .

The obtained formulas allow an analytical study of the dynamics of the considered system despite their approximate nature. In particular, they allow us to analyze the forces and slip rates occurring in the system.

4. Sensitivity Analysis

In this section, an analysis of the sensitivity of the system to changes in the values of the main parameters (namely the mesh stiffness k_g , mesh damping c_g , and equivalent mass m_{eq}) is conducted. The time-domain results are presented for displacement z signals, while frequency spectra show vibrational accelerations $\ddot{z}$. Such an approach was selected due to the fact that, in the time domain, displacements give more intuitive results than other dynamic quantities, i.e., the velocity and acceleration. On the other hand, the analysis in the frequency domain is more suitable for accelerations, since the mean value that dominates the spectrum of the displacement (frequency equal to zero) is eliminated. Reference values for the system's parameters are presented in Table 2.

Table 2. Reference values of the parameters for the simulations.

	Symbol	Value	Unit
Stiffness	k_g	8.0×10^8	N/m
Force	F	4.0×10^3	N
Damping	c_g	9.0×10^3	Ns/m
Period	T	0.0016	s
Mass	m_{eq}	0.28	kg
Small parameter	ε	0.13	-

4.1. Mesh Stiffness— k_g

The numerical investigation of the approximated analytical solutions began with a stiffness analysis. Figure 11 shows the time-varying displacement for different mesh stiffness values. Observations of the vibrations' displacement based on the time-domain signals in Figures 11 and 12 indicate the strong dependence of the amplitudes on the mesh stiffness. The displacement amplitudes decrease consistently as the gear mesh stiffness increases. A periodic pattern is evident in the displacement, related to the parametric nature of the gear system (the time-varying number of tooth pairs in contact along the line of action). A smaller displacement occurs when two pairs of teeth are in contact, resulting in higher mesh stiffness. Conversely, a larger displacement occurs when only one pair of teeth is in contact, leading to lower mesh stiffness. In Figure 13, consecutive meshing stages (double-tooth, single-tooth, and double-tooth, referred to as d-s-d) are presented to visualise the correlation in reference to the plotted time courses. The occurrence of periodicity confirms the inherent time-varying nature of gear meshing and the associated occurrence of the parametric vibration of the gear system. Figures 14 and 15 show the system's vibration acceleration waveforms in the frequency domain.

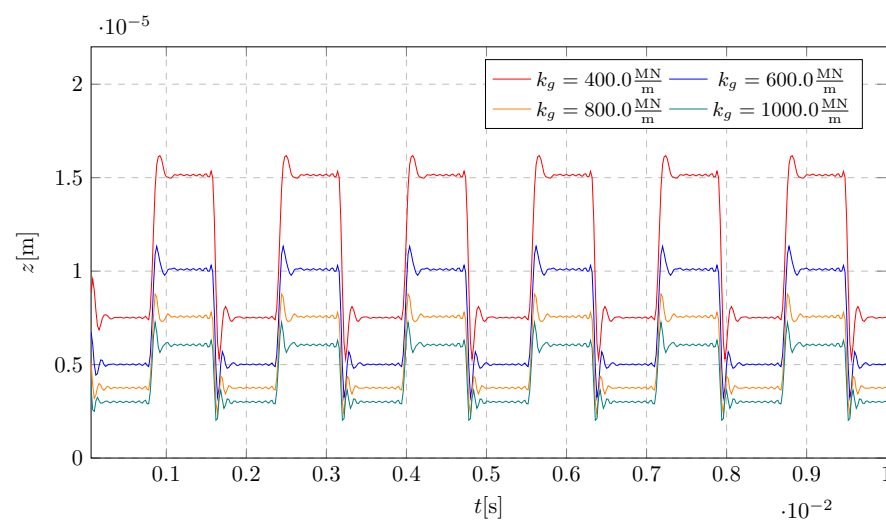


Figure 11. Displacement of the system vibrations: time-domain signals for different mesh stiffness values.

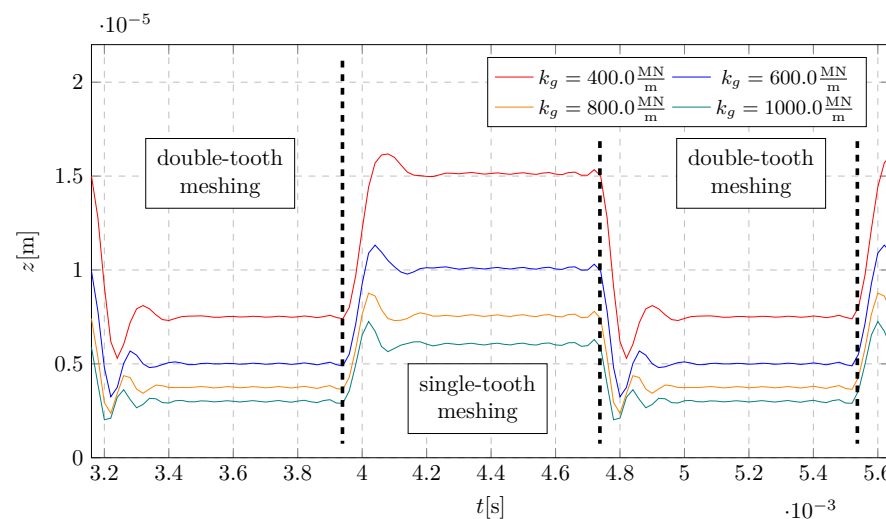


Figure 12. Displacement of the system vibrations for different mesh stiffness values: narrowing the time domain (qualitative magnification of the signals in Figure 11).

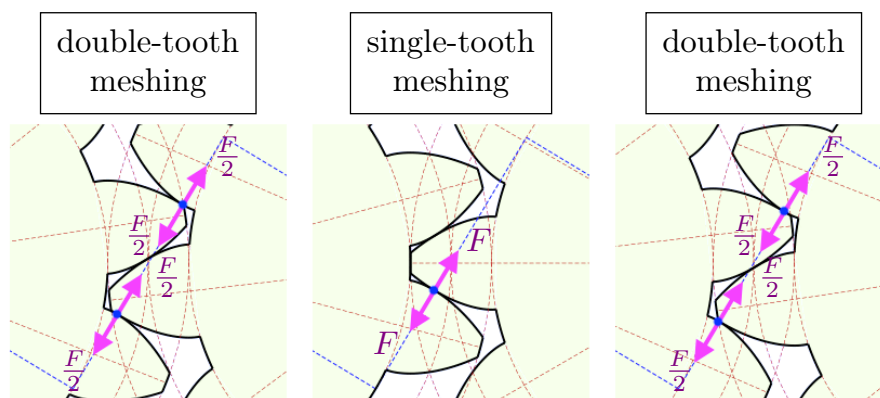


Figure 13. Force diagram along the path of contact: [d-s-d] gear meshing with reference to Figures 12, 17 and 21.

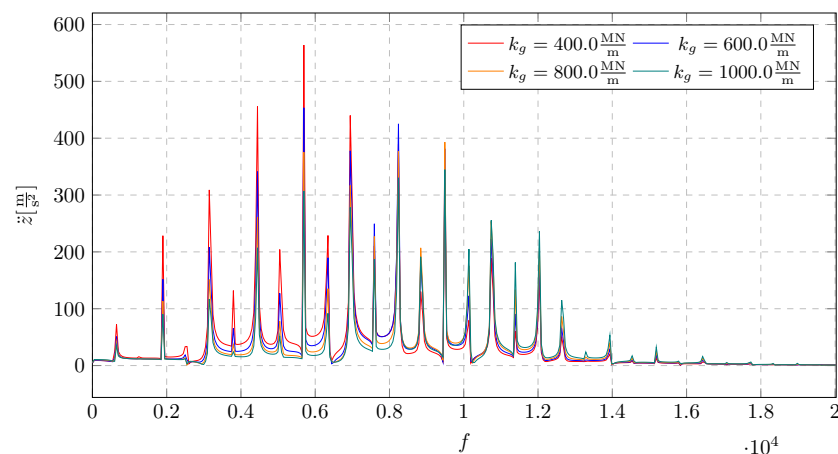


Figure 14. Acceleration of the system vibrations: frequency-domain signals for different mesh stiffness values.

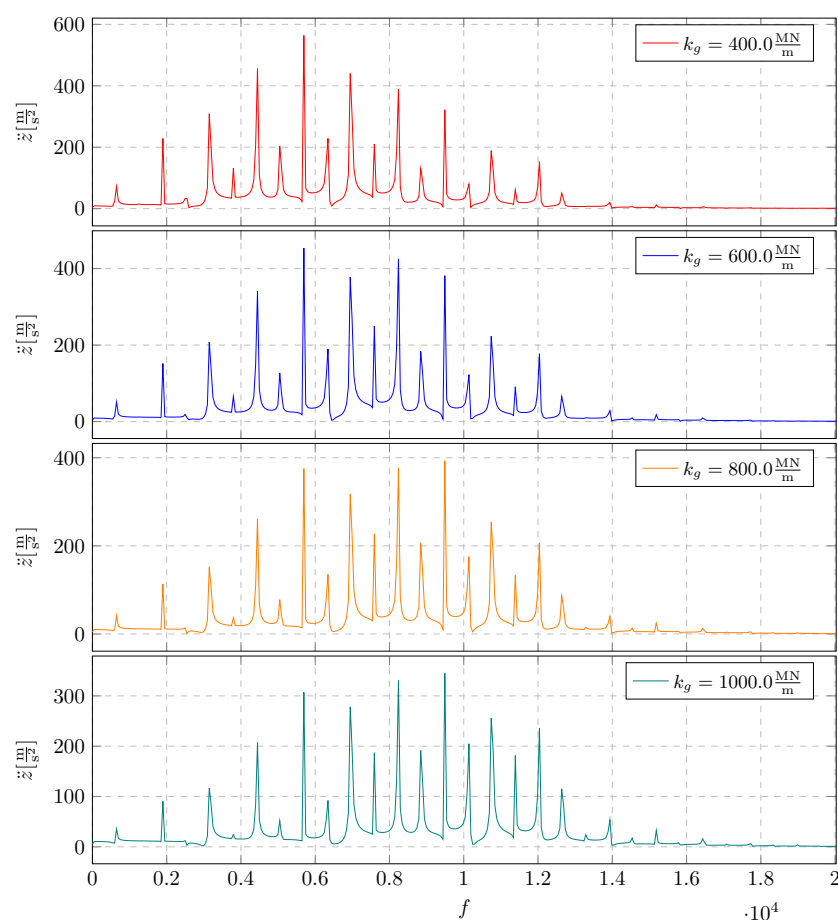


Figure 15. Acceleration of the system vibrations: frequency-domain signals for different mesh stiffness values—separated view based on Figure 14.

The peak acceleration for $k_g = 4.0e + 8 \text{ N m}^{-1}$ was estimated for 563.906 m s^{-2} at 5694.31 Hz . For k_g value $6.0e + 8 \text{ N m}^{-1}$, the peak acceleration of 453.587 m s^{-2} was observed at 5694.31 Hz . Based on numerical research carried out for $k_g = 8.0e + 8 \text{ N m}^{-1}$, the peak acceleration took a value of 393.234 m s^{-2} at 9490.51 Hz . An analysis of the curve for k_g value $1.0e + 9 \text{ N m}^{-1}$ showed that the peak of acceleration was not higher than 344.763 m s^{-2} and was registered at 9490.51 Hz .

4.2. Reduced Mass— m_{eq}

Subsequently, a study was carried out on the system's dynamic behaviour for different values of the reduced mass of the gears. Overall, Figures 16 and 17 demonstrate that larger mass values result in more pronounced dynamic responses. For an assumed constant mesh stiffness, an increase in mass results in less favourable system dynamics—the vibrations need more time to stabilise. As can easily be seen, the greater the mass for an assumed gear stiffness, the greater the vibration amplitude. The displacement amplitudes increase consistently as the equivalent mass (m_{eq}) increases from 0.2 kg to 3 kg. The amplitude increase with increasing mass suggests a direct correlation between the mass and the system's ability to suppress vibrational responses. It is also easy to see that the values of the equivalent mass affect the instability of the oscillation period (sweeping). The mass affects both the amplitude and frequency of the oscillation. For smaller masses, the displacement signals show concentrated energy at lower amplitudes, reflecting a system prone to stronger oscillatory behaviour. In Figures 18 and 19, the system's acceleration response in the frequency domain is shown.

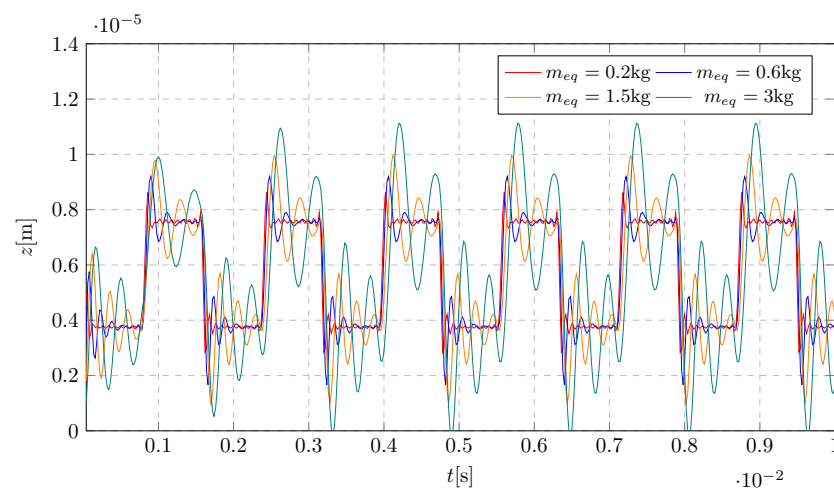


Figure 16. Displacement of the system vibrations: the time-domain signals for different equivalent masses.

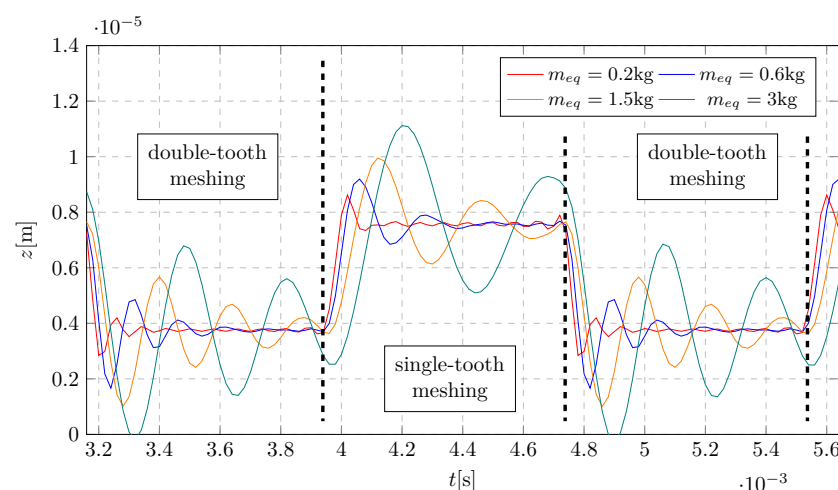


Figure 17. Displacement of the system vibrations for different equivalent masses: narrowing the time domain (qualitative magnification of the signals in Figure 16).

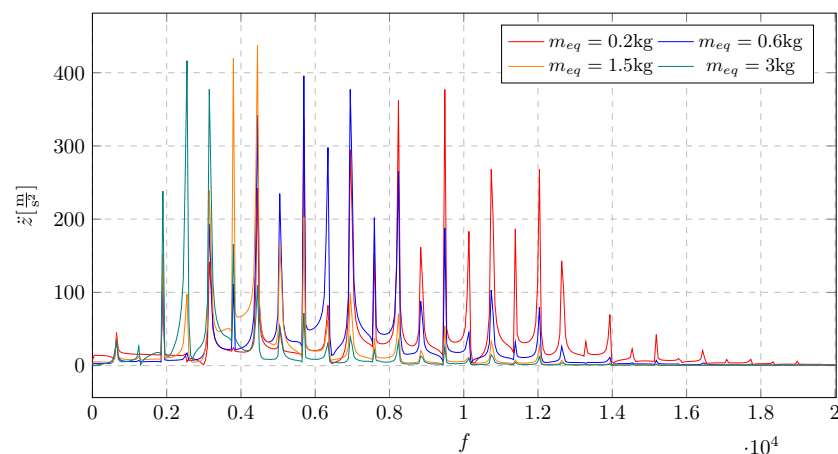


Figure 18. Acceleration of the system vibrations: the frequency-domain signals for different equivalent masses.

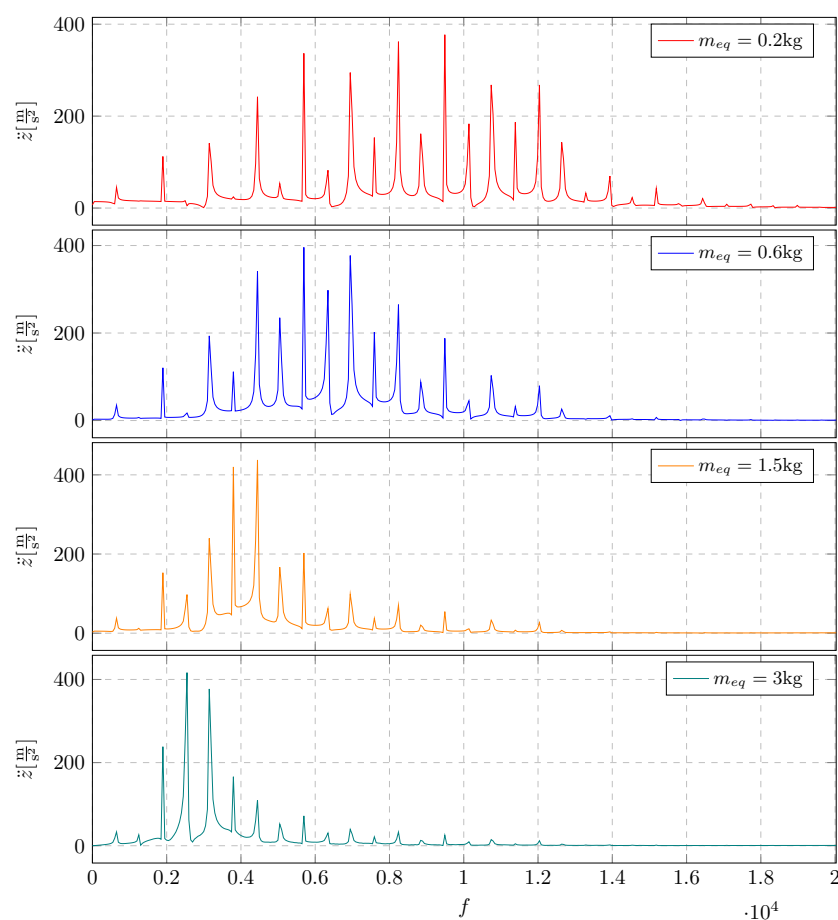


Figure 19. Acceleration of the system vibrations for different equivalent masses: separate frequency-domain signals based on Figure 18.

The analysis of an acceleration curve representing the system's response in the frequency domain for m_{eq} value 0.2 kg resulted in a peak of 377.165 m s^{-2} at 9490.51 Hz . The analysis of the curve for m_{eq} value 0.6 kg showed that the peak of acceleration was not higher than 395.888 m s^{-2} and was registered at 5694.31 Hz . Simulations for $m_{eq} = 1.5 \text{ kg}$ resulted in a 437.626 m s^{-2} peak acceleration value appearing at 4445.55 Hz . It was observed that, for m_{eq} value 3 kg, the peak appeared at 2547.45 Hz , with a value of 416.525 m s^{-2} .

4.3. Mesh Damping— c_g

The graphs in Figures 20 and 21 show the system's displacement response over time for different damping coefficients. The displacement amplitudes decrease as the mesh damping coefficient c_g increases from 3000 to 40,000 N s m⁻¹, indicating that the logarithmic decrement increases; hence, stabilisation occurs faster and the oscillations decrease. Lower damping values allow for larger displacement oscillations. As the mesh damping coefficient (c_g) increases, the subsequent amplitudes are much smaller, indicating that the system is being damped more rapidly. For lower damping ($c_g = 3000$ N s m⁻¹ and $c_g = 6000$ N s m⁻¹), the displacement amplitudes are significantly higher and exhibit slower decay. For higher damping ($c_g = 20,000$ N s m⁻¹, and $c_g = 40,000$ N s m⁻¹), the displacement signals rapidly approach minimal amplitudes, reflecting efficient energy dissipation. In particular, the zoomed-in view in Figure 21 shows that higher damping coefficients suppress the peaks more effectively, confirming the role of c_g in moderating the displacement magnitudes. Increasing the mesh damping coefficient reduces the displacement amplitudes and accelerates vibrational energy dissipation. Higher values stabilise the system more effectively, which is indicated by the absence of oscillations at the transient zones. In turn, Figures 22 and 23 show the acceleration of the system vibrations in the frequency domain for different mesh damping coefficients.

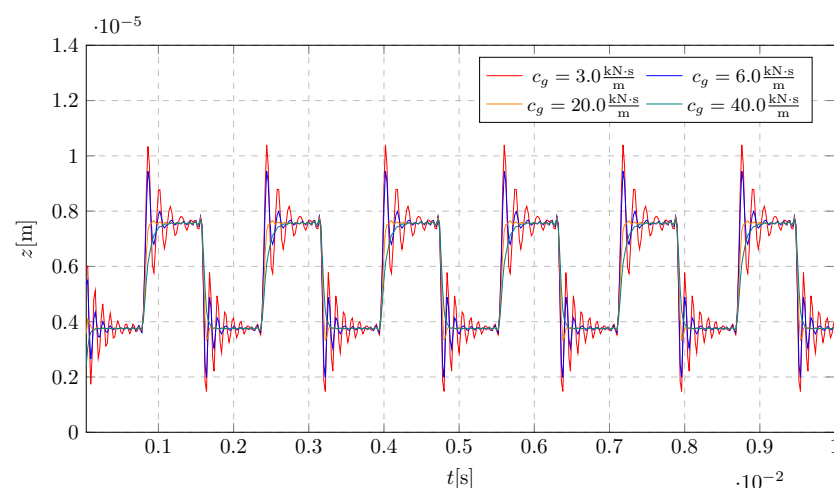


Figure 20. Displacement of the system vibrations: the time-domain signals for different mesh damping coefficients.

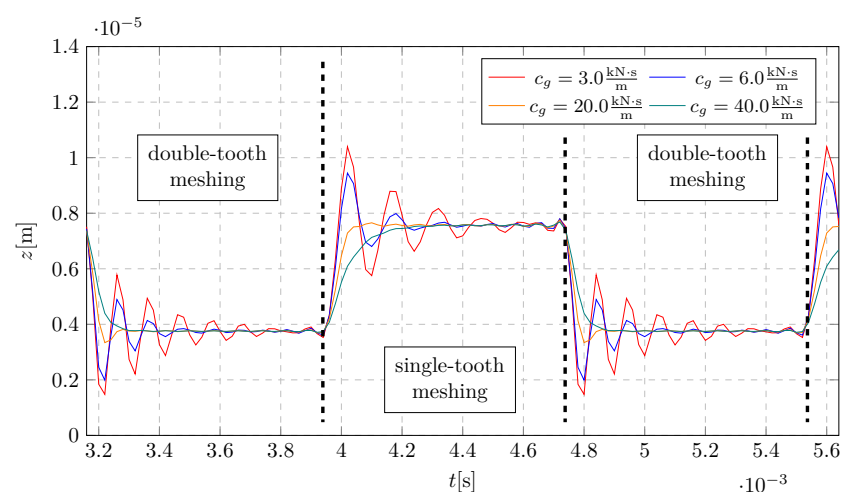


Figure 21. Displacement of the system vibrations for different mesh damping coefficients: narrowing the time domain (qualitative magnification of the signals in Figure 20).

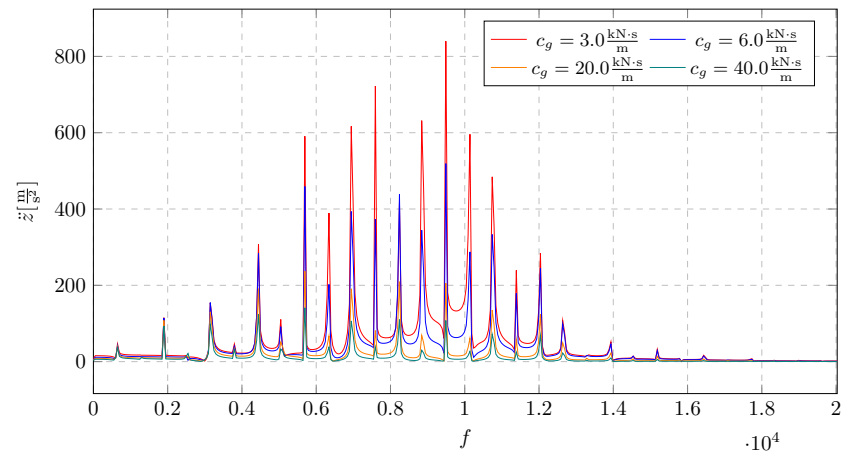


Figure 22. Acceleration of the system vibrations: the frequency-domain signals for different mesh damping coefficients.

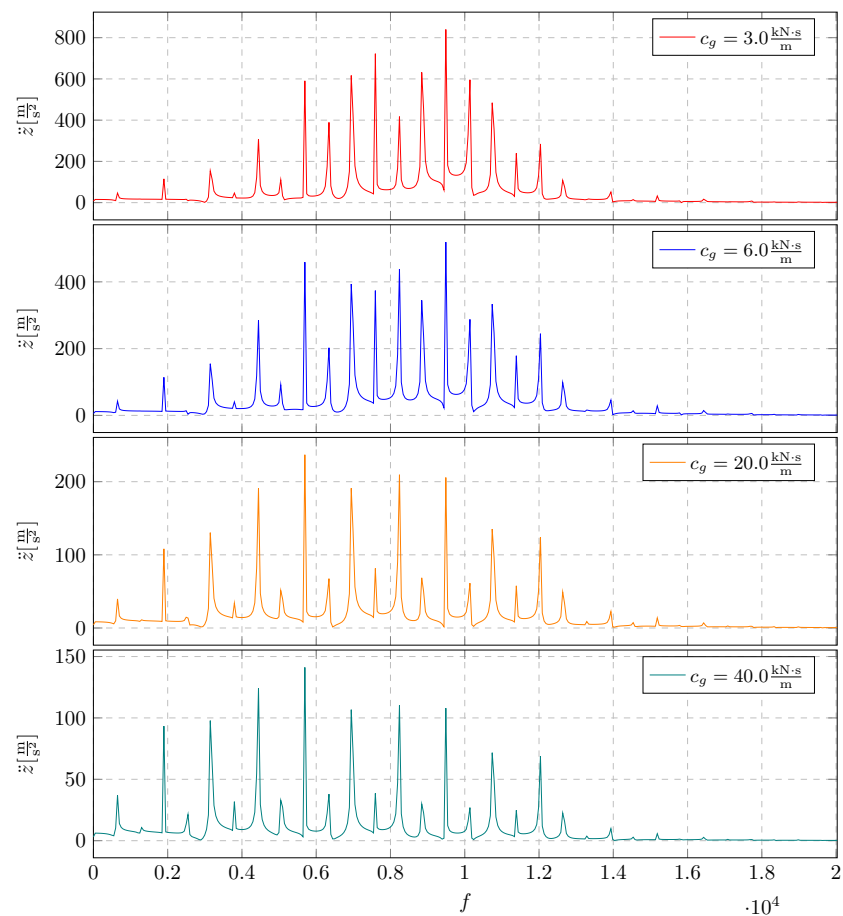


Figure 23. Acceleration of the system vibrations for different mesh damping coefficients: separate frequency-domain signals based on Figure 22.

The peak acceleration for $c_g = 3000 \text{ N s m}^{-1}$ was estimated for 839.901 m s^{-2} at 9490.51 Hz . Based on numerical research carried out for $c_g = 6000 \text{ N s m}^{-1}$, the peak acceleration took a value of 519.113 m s^{-2} at 9490.51 Hz . The simulations for $c_g = 20,000 \text{ N s m}^{-1}$ resulted in a 236.845 m s^{-2} peak acceleration value appearing at 5694.31 Hz . The peak acceleration for $c_g = 40,000 \text{ N s m}^{-1}$ was estimated for 141.188 m s^{-2} at 5694.31 Hz .

5. Discussion

5.1. General Observations

Based on the results of the conducted simulations, the following observations can be made. Starting with the mesh stiffness, it has a direct effect on the displacement of the gear system. Lower stiffness leads to higher amplitudes, indicative of weaker resistance to deformation. On the other hand, higher stiffness corresponds to more rapid stabilisation and smaller oscillatory amplitudes. The signals show periodic vibrations with consistent periods across different stiffness values, indicating that the gear system frequency remains largely unaffected by the stiffness within the examined range.

Observations of the frequency-domain signals in Figures 14 and 15 allow us to conclude that the acceleration waveforms demonstrate significant peaks during transitions between single-pair and double-pair tooth meshing. However, the transient behaviour remains consistent, as can also be seen in Figure 12. The frequency associated with the gear meshing ($f_z = 632.67$ Hz) is dominant for lower values of the mesh stiffness. This peak's frequency value remains visible, regardless of the stiffness, indicating fixed excitation, although the acceleration spectra show concentrated energy at higher frequencies as the mesh stiffness increases. While the peak magnitudes decrease as the mesh stiffness increases, the high-frequency concentration is more pronounced at higher stiffness levels. This is because an increase in mesh stiffness causes an increase in the transient vibration frequency, i.e., during transitions from two-pair tooth meshing to one-pair tooth meshing. When a single pair of teeth is in contact at the line of action, the lower mesh stiffness for single-tooth meshing enhances the frequency effects, increasing the risk of fatigue and damage. In general, the energy distribution expands across the spectrum as the stiffness increases, which amplifies the damping-like effects, as previously observed. This reflects the enhanced system strength under high-stiffness conditions. It is also important to remember that acceleration is a vital metric in assessing material degradation and fatigue.

The general observations regarding the mesh stiffness are as follows. The analysis demonstrates the critical role of the mesh stiffness in gear vibrational behaviour. Increased mesh stiffness improves the overall vibrational stability by reducing the displacement, velocity, and acceleration amplitudes, indicating better vibrational control. The fundamental frequencies remain stable across different stiffness levels, while the energy redistribution shifts towards higher frequencies as the stiffness increases. Higher mesh stiffness values lead to broader energy dispersion across the frequency spectra, suggesting enhanced mechanical stability. The energy redistribution across higher frequencies implies reduced wear and tear from parametric resonance. Reducing resonance effects, especially low-frequency resonance risks, promotes a longer machinery lifespan.

An important observation is that transitions between single-tooth and double-tooth meshing produce noticeable peaks in all vibratory metrics. These transient effects are more pronounced in systems with lower stiffness, highlighting the need for designs that are robust under such dynamic behaviour. The sharpness of the transitions in the approximated waveforms might be influenced by the Gibbs phenomenon, an inherent characteristic of Fourier series approximations of periodic functions with low smoothness.

Moving on to the equivalent mass analysis, observations of the frequency-domain signals in Figures 18 and 19 indicate that lower masses ($m_{eq} < 0.6$ kg) exhibit significantly higher acceleration peaks (greater susceptibility to vibration). Additionally, for smaller masses, the acceleration spectra exhibit the distribution of vibrational energy over a broader range of frequencies, with the predominance of higher-frequency peaks. The increase in mass narrows the range of vibrational energy distribution towards lower frequencies. Higher masses ($m_{eq} = 1.5$ kg and greater) show a reduction in peak acceleration.

However, it should be noted that the observed trends are specific to the analyzed gear system and operating conditions. The sensitivity of the gear system to an increase in mass was examined using a particular configuration (Table 1) and reference values for the simulation parameters (Table 2). Normally, it is the case that an increase in mass is correlated with an increase in the geometric dimensions of the gearbox, accompanied by an increase in stiffness and damping. Thus, an increased equivalent mass significantly enhances the vibrational stability, reducing the displacement and acceleration magnitudes under dynamic conditions, acting as a natural damper.

In this simulation study, the increase in mass did not affect the assumed values of the stiffness and damping, which determine the dynamic behaviour of the transmission system. Above a certain value of equivalent mass, the stability of the solution of the system's mathematical model is lost. Figure 17 shows that, in the case of a mass of $m_{eq} = 3$ kg, the displacement periodically obtains small negative values, which should be interpreted as a loss of contact between the teeth (a lack of meshing). Meanwhile, the model does not consider backlash. Therefore, the mass of 3 kg is its application limit. The sensitivity analysis reveals that the identification of the system's parameters is essential, and not all of the data can be selected arbitrarily. The limit values of the parameters determine the range of model applicability in the physical sense.

Nevertheless, the analysis shows the system's sensitivity to a change in mass and its importance in managing the vibration dynamics. This should be considered in the sustainable design of drive system gears.

Lastly, higher mesh damping coefficients enhance the system stability by minimising the displacement and acceleration responses to dynamic loads. The acceleration amplitudes decrease sharply with higher damping coefficients. The fundamental frequencies remain stable across all damping coefficients, while a higher value of c_g redistributes the vibrational energy across a broader spectrum. Energy redistribution across frequencies improves the mechanical stability and mitigates resonance risks. Effective damping reduces the vibrational energy more rapidly, promoting efficient system operation and reducing mechanical fatigue. Optimising the damping coefficients can mitigate vibrational forces, critically improving the operational reliability and lifespan of machinery. The predictable behaviour of the highest-amplitude peaks is observed, where, for increasing damping, the accelerations decrease.

5.2. Application of Obtained Results for Energy Harvesting System

The operating principle of piezoelectric energy harvesters is based on converting vibration displacements into an electrical voltage. Therefore, these devices should be selected depending on the frequency of the vibrations occurring in the system. The resonance frequency of the energy harvester unit corresponds to the point at which the largest amount of electrical energy can be generated. Therefore, to achieve the maximum efficiency of the energy harvester, a piezoelectric material with a resonance frequency close to the major vibration frequency in the system should be selected to ensure the resonance vibrations of the device, providing the largest amplitude at a given frequency.

To effectively select a piezoelectric material, it is necessary to thoroughly examine the vibration characteristics in the system. The frequencies at which the largest amplitudes are generated should be identified. For this purpose, the following can be used: modal analysis, measurement methods, or numerical or analytical simulations. In the case under consideration, a series of analyses was carried out based on a formula:

$$E_i = 2\pi^2 A_i^2 f_i^2 \quad (27)$$

where

E_i —energy of i -th frequency stamp;
 A_i —amplitude of i -th frequency stamp;
 f_i — i -th frequency stamp.

An analysis of the plots (Figures 24–26) allows us to conclude that the changes in the values of the subsequent parameters affect the energy generation. In the case of the stiffness, the shape of the plots remains constant, with the only difference being in the amplitudes of the energy—the higher the stiffness, the less energy is generated. Changes in the mass affect the energy waveforms both qualitatively and quantitatively. With the increase in mass, the overall energy also increases, but the span in which visible peaks appear shortens rapidly. When considering damping, with an increase in its value, the generated energy decreases and larger discrepancies in the plot shape are visible—the range of frequencies in which the highest peaks of energy appear gradually shortens and shifts towards lower values.

In Figure 27, energy levels for reference data (Table 2) are shown. The highest energy peak appears at 5694.31 Hz, which constitutes the optimal working point of such a pair of gears from the point of view of energy harvesting. Hence, as stated above, the piezoelectric device should be chosen in such a manner as to respond in the optimal manner for said frequency value. If, for any reason, such an application is not possible, further investigation of the spectral plot shows at least five different points at which the amount of generated energy would not be as high but would still be reasonable in comparison with the highest peak. The application of a piezoelectric device responding to frequencies higher than 10,000 Hz would not provide satisfactory effects for such a configuration of the system.

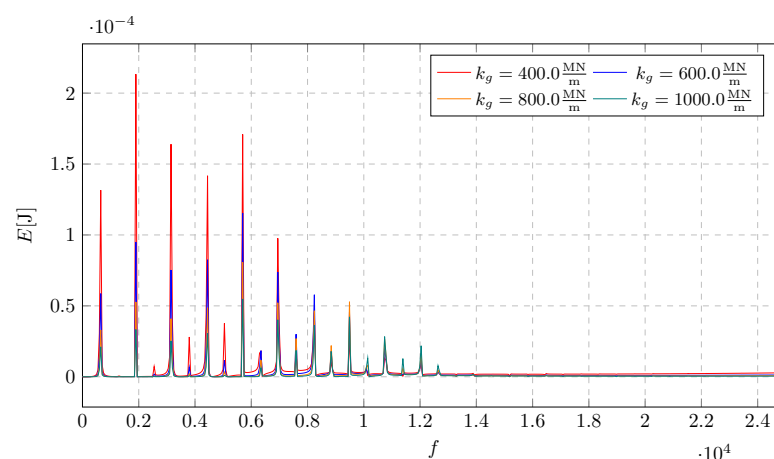


Figure 24. Energy generated by displacement vibrational signal in the frequency domain for different mesh stiffness coefficients.

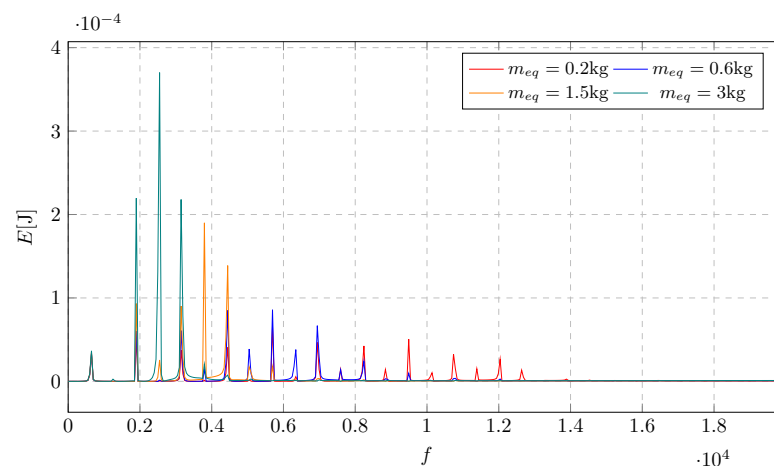


Figure 25. Energy generated by displacement vibrational signal in the frequency domain for different equivalent masses.

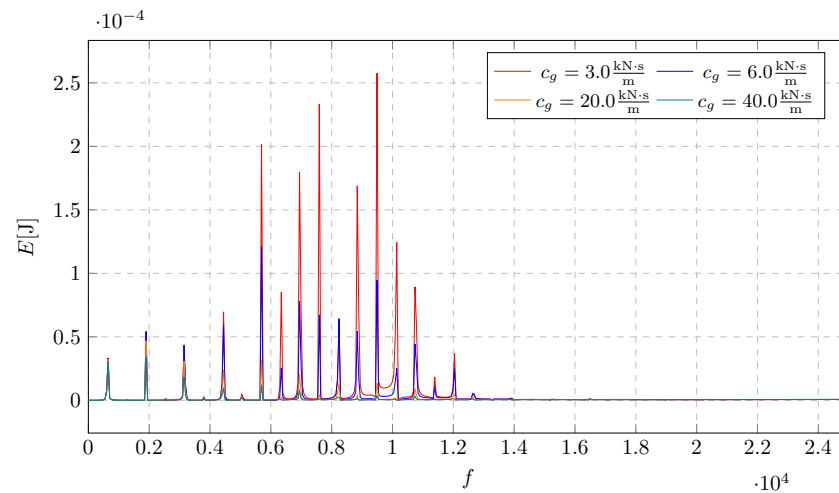


Figure 26. Energy generated by displacement vibrational signal in the frequency domain for different mesh damping coefficients.

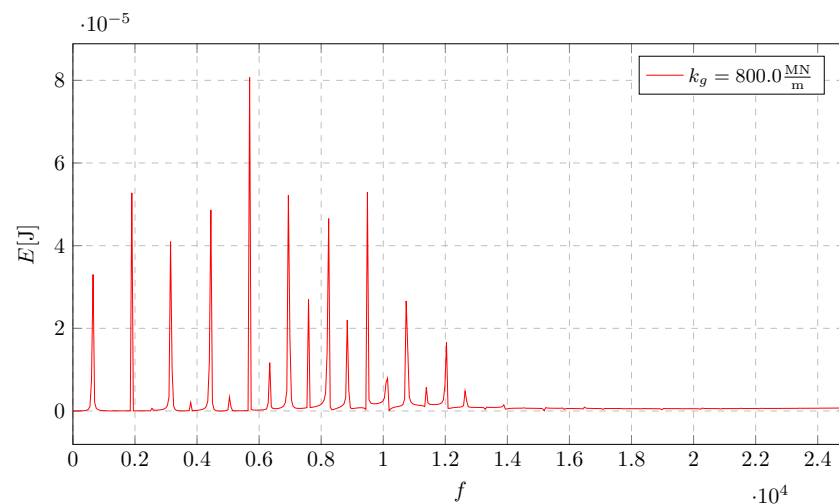


Figure 27. Energy generated by displacement vibrational signal in the frequency domain for reference data.

The harvester's operating point is strongly related to the system parameters, so, depending on their configuration, a different material and piezoelectric system solution should be selected. The analysis indicates that using analytical solutions allows the preliminary identification of the problem and the estimation of the harvester's performance. It should be emphasised that the presented analysis can be performed before conducting expensive experimental studies or at the design stage, which are the benefits of applying analytical solutions. Mathematical models allow for an initial estimation of the range of resonant frequencies and initial calculations of the profitability of using the energy harvester.

5.3. Future Works

The promising results obtained in the current theoretical research allow us to define the future paths of development of the proposed approach. Firstly, the results were implicitly compared with the well-known, state-of-the-art model and supported by the experiments analysed in [56]. We plan to conduct independent experimental research. The objective will be the acquisition of real-life data for the simple helical gear system (generated frequencies and acceleration amplitudes). The comparison of the presented analytical model with empirical signals will allow us to determine the actual accuracy of the model (instead of

focusing on strictly theoretical errors based on the power of the small parameter) and the possibilities of its tuning.

The positive verification of the model with experimental data would enable theoretical works, including various aspects omitted in the current iteration, such as backlash. Furthermore, the applicability of the model to represent the dynamic behaviour of different types of gears and their configurations is foreseen, aiming to assess whether and how advanced modifications to its current form should be provided.

6. Conclusions

This study provides an analytical framework for the understanding and optimisation of gear mesh dynamics in machinery drive systems. The research bridges critical gaps in vibration analysis by employing the Fourier expansion of rectangular signals and Hill's equation to model the time-varying mesh stiffness and parametric vibrations. The multiple scale method was utilised to solve the resulting equation of motion. The main objective was to develop and validate analytical solutions and advance the understanding of the dynamic models of gears within drive trains. The MSM was used to derive analytical solutions to determine the vibration frequency ranges and identify spectral components. We argue for the need to integrate such methods to optimise gearbox design, reduce vibrations, and improve the operational reliability.

Using the ISO 6336 standards and the Fourier series, we formalised the stiffness variations during single-tooth and double-tooth meshing cycles. The model captures the periodic and parametric nature of gear vibrations, providing a solid mathematical foundation for analysis. Based on this model, we derived a series of approximations using the MSM to solve nonlinear equations of motion for the gear system. These solutions offer insights into the displacement and acceleration dynamics, emphasising the role of damping and stiffness in modulating the vibration amplitudes.

Based on the conducted simulations, the investigation of the gear system, and the analysis of the results, the following can be concluded.

1. The proposed TVMS model (Figure 4) is more efficient from a simulational point of view and has very similar frequency spectra to state-of-the-art models, as shown in Figures 5 and 6. This allows the use of the models interchangeably depending on the assumed computational technique.
2. Applying the presented TVMS model to the dynamic model under consideration results in highly convergent results. The time plots of the displacement (Figures 7–9) and meshing force (Figure 10) obtained align with practice, confirming the methodology's effectiveness for the initial gear calculations. Thus, the amplitudes and frequencies of the gear mechanism can be correctly identified.
3. The analytical solution ensures the reliability and repeatability of the results. A single run of the solving process allows results to be obtained for different parameters by substituting new values into the already given solution, which significantly reduces the calculation time.
4. The obtained analytical solution reveals the mathematical structure of the gear vibration signal. The form of the harmonic signal indicates the crucial frequencies of the gear system's dynamic behaviour due to the vibrational meaning of the result. It allows for a quick and effective analysis of the vibroacoustic properties of drive systems, including identifying the operating points of machinery.
5. The possibility of the energy-based determination of the operating point is an important factor in the sustainable design of agricultural machinery. The use of the analytical solution allows for the faster and more accurate selection of single gear

- (mass) and transmission (meshing stiffness, damping) physical parameters, so the overall vibroacoustic influence on operators and the environment can be diminished.
6. The energy analysis showed that, when considering different equivalent mass, mesh stiffness, and mesh damping values, changes in these parameters influence the peak amplitude values and their positions in the spectrum. Moreover, multiple operating points could be chosen, allowing for optimisation regarding energy harvesting and other technical requirements for the system.
 7. Analyses of the energy generated by the displacement vibrational signals in the frequency domain for selected parameters led to the following conclusions:
 - For a mesh stiffness coefficient $k_g = 400,000,000 \text{ N m}^{-1}$, the highest energy peak $E = 2.132e - 4 \text{ J}$ appears at $f_z = 1896.08 \text{ Hz}$;
 - For an equivalent mass $m_{eq} = 3 \text{ kg}$, the highest energy peak $E = 3.696e - 4 \text{ J}$ appears at $f_z = 2543.48 \text{ Hz}$;
 - For a damping stiffness coefficient $c_g = 3000 \text{ N s m}^{-1}$, the highest energy peak $E = 2.575e - 4 \text{ J}$ appears at $f_z = 9491.35 \text{ Hz}$.

The general conclusion is that the multiple scale method provides a robust analytical framework for the prediction and management of parametric vibrations in gear systems. Increasing the gear mesh stiffness effectively reduces the vibrational amplitudes and shifts the energy distribution to higher frequencies, enhancing the stability. Optimised damping coefficients minimise transient oscillations, which contributes to improved machine durability and operator comfort.

The conducted analyses show that the energy generation strongly depends on the system parameters. The presented results confirm that the model can be applied for the assessment of energy harvesting parameters, as described in the Introduction. The conclusion can be drawn that, for the selected gear parameters, the amount of harvested energy and the operating point of the machine can be preliminarily determined.

The utility of the proposed approach in enhancing helical gear (especially the dynamic of gear meshing) configurations to minimise vibrations and improve durability and the early detection of failures through vibration signal analysis has to be highlighted. The application of the MSM in gear mesh dynamics offers precision in predicting the vibration behaviour. The presented research results could be helpful in the sustainable design of gear meshing to optimise the drive system's dynamics, especially in agricultural machinery, where vibration control is important for the comfort and health of the operator. It can be concluded that detailed mathematical modeling and a focus on the computing performance could contribute to fostering safer and more efficient agricultural practices.

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