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Solution Procedure for Fractional Casson Fluid Model Considered with Heat Generation and Chemical Reaction

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Abstract: In this work, the objective is to get the exact analytical solution of a generalized Casson fluid model with heat generation and chemical reaction described by the Caputo fractional operator, using the approach that the Laplace transform method includes the Laplace transform of the Caputo derivative. After the exact solution, it will be studied the impact of the order of the fractional derivative and the most essential parameters included in the modeling like the Prandtl number, the thermal Grashof number, the mass Grashof number, the Schmidt number, the heat generation parameter, and the chemical reaction parameter. The physical points of view of the influence will be discussed and analyzed. The findings of the paper will be illustrated by several graphics. The development in industry and engineering science, it makes important to study the flow behavior of non-Newtonian fluids. The domains of applications of the flow behavior of non-Newtonian fluids are diverse such as geophysics, biorheology, and chemical and petroleum industries.

Keywords: fractional casson fluid; grashof numbers; schmidt numbers; Prandtl number; Caputo fractional derivative

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1. Introduction

Fractional calculus has gained much interest these last decade and continues to be used and discussed in the literature. It is recognized that fractional calculus has many applications in modeling biological models [1,2], in modeling physical phenomena [3,4], in mathematical physics madelings [5,6] like modeling fluid and nanofluid models, the fundamental mathematics [7] and many others applications, see the following papers to complete the studies [8,9]. Fractional calculus is also discussed in the literature and some researchers asked questions about the validities of the fractional operators and the applications of fractional calculus in real-life problems. Note that, it has been found many types of fractional derivatives in Fractional calculus, there are that the Caputo and the Riemann-Liouville derivative [10,11]. They were the first fractional operators discovered in the literature on fractional calculus. The first type of derivative was the Riemann-Liouville derivative but its inconvenience was that at first the derivative of the constant function is not zero, secondly, the initial conditions for each problem should be in integral forms which are not realistic. And then to solve these problems the Caputo derivative [10,11] has been proposed and answers the previously listed inconveniences. Recently since 2015, other news forms of fractional operators without singular kernels have been proposed in the literature. It has been discovered that the Caputo-Fabriozio derivative [8] and the Mittag-Leffler fractional operator [7] named in the literature by the Atangana-Baleanu derivative.

This paper will be destined to model fluid under the Caputo derivative. In physics, due to its excellent properties, it will be used this derivative in the present investigations. The novelties of the present investigations can be summarized in the following sentences. The Tzou algorithm to propose the numerical solutions of the considered Casson fluid model were used in the present investigations. The second point in the investigations is the use of the Caputo derivative, which will permit us to see the influence of the order of the derivative on the dynamics of the fluid, the influence generated by the order of the

Caputo derivative will be explained in physical viewpoints. The last point of the new information will be the analysis of the influence of the parameters of the model on the dynamics of the considered fluid model type according to the variation of the order of the fractional operator.

The literature on the modeling fluid and nanofluid is vast, but in this paper was assigned the works which are found interesting for the investigations. In fractional calculus many of them use the Caputo derivative, there also exist works that used the new fractional derivatives, and here they have been recalled. In [12], the reference works in a fluid model with fractional order derivative, the subject of the investigation was to give a method to obtain the analytical solution via the Laplace transform. In [13], they proposed the model with Mittag-Leffler derivative and model with Caputo-Fabrizio derive applied to the second-grade fluids models. In this investigation, the authors try to propose a comparative study between the two fractional fluid models. They also used Laplace transform to get the solutions of the fluid models. In [14], the authors proposed work on the influence of magnetic field on the double convection problem of viscous fluid over an exponentially moving vertical plate, the proposed model has been described by the Caputo time-fractional derivative. In [9], the authors have proposed some semi-analytical solutions using the semi-analytical method for the fractional fluid model. In [15], the authors investigate the works related to the subject tilted as MHD Casson Fluid Flow Over an Oscillating Plate with Thermal Radiation described by the fractional operator known as the most used and named the Caputo fractional derivative. In [16], the authors used the Atangana-Baleanu and the Caputo Fabrizio derivatives to model the convective flow of a generalized Casson fluid, the authors have proposed the Laplace transform method and give a comparative study between the models presented in this paper. In [17], the authors have used the Caputo derivative in modeling the free convection flow of Brinkman-type fluid by using the Laplace transform method to get the analytical solutions, the authors also explained the influence of some parameters used in the model and give their physical interpretations as well. In [18], in the same direction the authors have exploited in this paper via the Laplace transform the analytical solution of the second-grade fluid with Newtonian heating and have used the Caputo fractional derivative operator to do the investigations. In [19], the authors combined the Fourier transform and the Laplace transform for proposing the analytical solutions of a class of fluid models. In [20], the authors have used also the Laplace transform method to propose the analytical solutions and depict the figures of the dynamics of the solutions associated with the convection flow of an incompressible viscous fluid under the Newtonian heating and mass diffusion represented by the Caputo derivative. The authors also have analyzed the influence of the order of the fractional operator in the obtained dynamics and have explained the behaviors from physical views points. In [21], the authors have considered a second-grade fluid model and used the Caputo derivative in the modelings, using the standard method known as the application of the Laplace transform including the Laplace transform of the fractional operator to get the behavior of the considered model, the influence of the order of the fractional operator and the parameter used in the considered model in this paper have been analyzed and explained physically. For more informative recent literature on the subject of MHD Casson fluid flow [22], on unsteady MHD Casson fluid flow through a parallel plate with hall current [23], the conducting Casson fluid flow past a stretching cylinder [24], and others [25].

The aims of the present investigation consist to determine the exact analytical solution for a generalized Casson fluid model with heat generation and chemical reaction described by the Caputo fractional derivative. The main information related to the approach used is the utilization of the Laplace transform. The secondary results associated with the present work are the studies of the influence of the order of the Caputo derivative, the Prandtl number, the thermal Grashof number, the mass Grashof number, the Schmidt number, the heat generation parameter, and the chemical reaction parameter, it has been tried to explain the cause of these influences in physical views points. The motivation for using the Casson fluid model is that it is to illustrate the utilization of the Laplace transforms for getting the solution in case of a differential equation described by a fractional diffusion equation with perturbations. Note that the Casson fluid model and Navier-Stokes equations are in the same classes of fractional differential equations because all of them, the fractional diffusion equations are utilized to describe the diffusion processes.

2. Preliminaries

Modeling with fractional operators has attracted many researchers and many fractional operators have been proposed in the literature. In the present paper, it has been recalled some fractional operators necessary for the investigations in modeling fluid models. It will be proposed the Riemann-Liouville integral, the Caputo derivative, and the Laplace transform of the associated Caputo derivative. Some necessary functions as the Mittag-Leffler functions and associated functions will be recalled as well to complete this section. The beginning of this part will be a review of the fractional operators with the Riemann-Liouville integral which will play an important role in the investigations, they notably will be used to determine the analytical solutions to the considered problem. The following definitions have been recalled.

Definition 1 ([10,11])**.** *Let that the function* $h : [0, +\infty[\longrightarrow \mathbb{R}$. *And then the Liouville-Riemann integral of the function h is represented in the literature by the following description that*

$$I^{\alpha} h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds, \tag{1}$$

with the time verifying the condition that $t > 0$, $\alpha \in (0,1)$ *is the order of the fractional integral, and the* $\Gamma(\dots)$ *denotes the Gamma Euler function well known in the literature of mathematics.*

The second operator which will be recalled in this section concerns the Caputo operator, due to the memory effect discussed in the literature, this derivative will be used to model the fluid in this investigation. The definition is recalled in the next lines.

Definition 2 ([10,11])**.** *Let that the function* $h : [0, +\infty[\longrightarrow \mathbb{R}$. *And then the Liouville-Riemann integral of the function h is represented in the literature by the following description that*

$$(D_c^{\alpha} h)(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} h'(s) ds, \tag{2}$$

with the time verifying the condition that $t > 0$, $\alpha \in (0,1)$ *is the order of the fractional integral, and the* $\Gamma(\dots)$ *denotes the Gamma Euler function well known in the literature of mathematics.*

The Laplace transform is used in fluid modeling. The Laplace transform will be applied in the differential equation of the fluid model. In many other types of differential equations, the Laplace transform is used and its inverse. In this part, the Laplace transform of the Caputo derivative has been considered. The transformation [10,11] is as follows

$$\mathcal{L}\{(D_c^{\alpha} f)(t)\} = s^{\alpha} \mathcal{L}\{f(t)\} - s^{\alpha-1} f(0). \tag{3}$$

The Mittag-Leffler function is utilized in many domains to express the solutions of differential equations. This function also received many attractions and many generalized forms have been proposed. There exist in the literature many versions of the Mittag-Leffler function and generalization, the present form version is used

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \tag{4}$$

where the parameters satisfies the conditions that $\alpha > 0$, $\beta \in \mathbb{R}$ and the variable t is into the set $\mathbb{C}$. The Gaussian error function and the exponential function are obtained when the parameters satisfy the condition that $\alpha = \beta = 1$. The inverse of the Laplace transform is very important in this present paper and to this end, the Tzou algorithm is represented by the following form

$$h(x,t) = \frac{e^{4.7}}{t}\left[\frac{1}{2}\bar{h}(x,\frac{4.7}{t}) + Re\left\{\sum_{k=1}^{N}(-1)^k\bar{h}(x,\frac{4.7+k\pi i}{t})\right\}\right]. \tag{5}$$

where $Re(\ldots)$ represents the so-called real part of the complex number, i is the imaginary unit, N is a natural number, and $\bar{h}$ is the Laplace transform any continuous function $h : [0,+\infty[\longrightarrow \mathbb{R}$. The detail on this formula can be found in the literature, for example in [26].

3. Fractional Modeling

In the present modeling, a Casson fluid model with heat and mass transfer over an infinite vertical flat plate is under consideration. As in the paper in the literature, a flow along the x-direction and the y-axis normal to the plate are considered. At initial, the fluid and plate are at rest with uniform temperature and concentration denoted respectively by T_∞ and C_∞ are considered. Note that at $t = 0^+$, it is supposed that the plate starts motion in its plane with constant velocity represented by U. Furthermore the temperature and concentration of the considered Casson fluid to be maintained at constant as T_w and C_w. As reported in the literature, for the present Casson model the constructive equations can be obtained after the application of the usual Boussinesq's approximation, see in [27], and then the model for the free convection flow of Casson fluid along with heat and mass transfer is described by the following differential equations with Caputo derivative and dimensionless variables

$$D_t^\alpha u = \left(1+\frac{1}{\beta}\right)\frac{\partial^2 u}{\partial x^2} + Grv + Gmw, \tag{6}$$

$$D_t^\alpha v = \frac{1}{Pr}\frac{\partial^2 v}{\partial x^2} + \eta v, \tag{7}$$

$$D_t^\alpha w = \frac{1}{Sc}\frac{\partial^2 w}{\partial x^2} - Kw. \tag{8}$$

The dimensionless initial and boundary conditions considered for the velocity u, the temperature v, and the concentration w in the investigation are described in the following forms

$$u(x,0) = v(x,0) = w(x,0) = 0, \tag{9}$$

$$u(0,t) = 0,\ v(0,t) = w(0,t) = t. \tag{10}$$

The Prandtl number Pr, the thermal Grashof number Gr, the mass Grashof number Gm, the Schmidt number Sc, the heat generation parameter η, and the chemical reaction parameter K are represented respectively as the following descriptions, see in [27]

$$Pr = \frac{\mu C_p}{k}, \quad Gr = \frac{\nu g\beta_T(T_w - T_\infty)}{U^3}, \tag{11}$$

$$\eta = \frac{\nu Q_0}{U^2\rho C_p}, \quad K = \frac{\nu k_1}{U^2}, \tag{12}$$

$$Sc = \frac{\nu}{d}, \quad Gm = \frac{\gamma g\nu(C_w - C_\infty)}{U^3}. \tag{13}$$

For simplifications of the read of the present investigation, the nomenclature of the parameters used in the previous modeling is summarized in the following Tables 1 and 2.

Table 1. First Nomenclature Table.

Parameters	***Descriptions***
ρ	Density
T_w	Constant temperature
T_∞	Ambient fluid Temperature
β_T	Thermal expansion coefficient
C_w	Constant concentration level at the plate
C_∞	Ambient fluid concentration
ν	Kinematic viscosity
μ	Dynamic viscosity of the fluid
k	Thermal conductivity
β	Material parameter of Casson fluid
γ	Coefficient of concentration

Table 2. Second Nomenclature Table.

Parameters	***Descriptions***
k_1	Chemical reaction parameter
ρ	Density of the fluid
C_p	Specific heat capacity of fluids
Q_0	Heat generation term
d	Mass diffusivity
Gm	Mass Grashof number
Gr	Thermal Grashof number
Pr	Prandtl number
Sc	Schmidt number
g	Acceleration due to gravity
K	The chemical reaction parameter
η	The heat generation parameter

The interest of the present paper is the initial and boundary conditions for the velocity, the temperature, and the concentration represented in Equations (9) and (10) which are not similar to the initial condition represented in the papers in the literature in this field. In this paper, it will be observed that the initial and boundary conditions have a significant impact on the behavior the dynamics of the Casson fluid model. In other words, the fluid model is sensitive to the changes in the initial and boundary conditions. Another importance of the present model is the use of the Caputo derivative which introduce the memory in the model. The importance of the fractional operator in modeling real-world problems can be found in the literature. The initial and boundaries conditions are motivated by the fact for the velocity at the initial condition, in real problems, there is no fluid, and at a certain time, for a notably long time, the fluid is used, and then it lost the velocity and thus here also, it is zero in the conditions. For the temperature and the concentration, it is natural, to begin with, certain values for the temperature and the concentration because at the beginning of the diffusion processes the fluid has a concentration and fluid temperature. Important notices are that the variable time "t" is obtained after the application of the Boussinesq approximation. The constructive equation obtained in Equations (6)–(8) are obtained after the application of Boussinesq approximation and considering that

$$\begin{aligned} v &= \frac{C - C_\infty}{C_w - C_\infty}, \\ w &= \frac{T - T_\infty}{T_w - T_\infty}. \end{aligned}$$

4. Solutions Procedures for the Constructive Equations

In this section, it is provided that the solutions to the constructive equations considered in models (6)–(8). The determination of the solutions of the fractional differential Equations (6)–(8) by considering Equation (8) under the initial and boundary conditions defined in Equations (9) and (10), has been started. Firstly, it is applied that the Laplace transform to both sides of Equation (8), and it got the following representations

$$\begin{aligned} s^\alpha \bar{w} - s^{\alpha-1} w(0) &= \frac{1}{Sc}\frac{\partial^2 \bar{w}}{\partial x^2} - K\bar{w}, \\ s^\alpha \bar{w} &= \frac{1}{Sc}\frac{\partial^2 \bar{w}}{\partial x^2} - K\bar{w}, \\ \frac{\partial^2 \bar{w}}{\partial x^2} - Scs^\alpha \bar{w} - ScK\bar{w} &= 0. \end{aligned} \tag{14}$$

Solving the second order differential equation represented by the Equation (14) under the Laplace initial condition defined by $w(0,s) = 1/s^2$, it is got the following form

$$\bar{w}(x,s) = \frac{\exp\left[-x\sqrt{Sc(s^\alpha + K)}\right]}{s^2}. \tag{15}$$

The algorithm to get the inverse of the Laplace transform has been proposed by Tzou and Puri, 1997, thus the first author's name is associated with the method. The process is described in the following. For the inverse of the Laplace transform of Equation (15), using the Tzou formula, it is obtained in the following form

$$w(x,t) = \frac{e^{4.7}}{t}\left[\frac{1}{2}\bar{w}(x, \frac{4.7}{t}) + Re\left\{\sum_{k=1}^{N}(-1)^k \bar{w}(x, \frac{4.7 + k\pi i}{t})\right\}\right]. \tag{16}$$

where $Re(\dots)$ represents the so-called real part of the complex number, i is the imaginary unit, and N is a natural number. The solutions of some special cases can be determined using the Laplace transform method, the first case is when the order of the Caputo operator converges to 1. It is obtained by the following Laplace transform

$$\begin{aligned} s\bar{w} - w(0) &= \frac{1}{Sc}\frac{\partial^2 \bar{w}}{\partial x^2} - K\bar{w}, \\ s\bar{w} &= \frac{1}{Sc}\frac{\partial^2 \bar{w}}{\partial x^2} - K\bar{w}, \\ \frac{\partial^2 \bar{w}}{\partial x^2} - Scs\bar{w} - ScK\bar{w} &= 0. \end{aligned} \tag{17}$$

Using the Laplace transform of the initial condition as previously represented, it is got a solution of the second-order differential Equation (17) described by the following

$$\bar{w}(x,s) = \frac{\exp\left[-x\sqrt{Sc(s + K)}\right]}{s^2}. \tag{18}$$

Here again, the Tzou algorithm can be applied to get the solution of the considered case (17), it has been obtained in the following form

$$w(x,t) = \frac{e^{4.7}}{t}\left[\frac{1}{2}\bar{w}(x,\frac{4.7}{t}) + Re\left\{\sum_{k=1}^{N}(-1)^k\bar{w}(x,\frac{4.7+k\pi i}{t})\right\}\right]. \quad (19)$$

The second method is using the classical procedure of inverting the Laplace transform of Equation (18) for getting an exact solution, it has been obtained that the following representation for the solution

$$\begin{aligned} w(x,t) &= \frac{1}{2}\left[\left(t - \frac{xSc}{2\sqrt{K}}\right)\exp\left(-x\sqrt{KSc}\right)ercf\left(\frac{x\sqrt{Sc}}{2\sqrt{t}} - \sqrt{Kt}\right)\right] \\ &+ \frac{1}{2}\left[\left(t + \frac{xSc}{2\sqrt{K}}\right)\exp\left(x\sqrt{KSc}\right)ercf\left(\frac{x\sqrt{Sc}}{2\sqrt{t}} + \sqrt{Kt}\right)\right]. \end{aligned} \quad (20)$$

Note that for the case where the chemical reaction parameter is null, it has been obtained that a solution for the second differential equation represented in (17), it has been obtained that the following form

$$\bar{w}(x,s) = \frac{\exp\left[-x\sqrt{Scs^{\alpha}}\right]}{s^2}. \quad (21)$$

The same procedure previously adopted in the Laplace transform inverse can be used. As previously mentioned, it has been used that the Tzou procedure and it has been obtained that the following form for the solution, that

$$w(x,t) = \frac{e^{4.7}}{t}\left[\frac{1}{2}\bar{w}(x,\frac{4.7}{t}) + Re\left\{\sum_{k=1}^{N}(-1)^k\bar{w}(x,\frac{4.7+k\pi i}{t})\right\}\right]. \quad (22)$$

Note that when the order of the Caputo operator converges to 1 in Equation (8) with $K = 0$, it has been recovered that the classical solution mentioned in the literature, it has been obtained that

$$w(x,t) = \left(\frac{x^2Sc}{2} + t\right)erfc\left(\frac{x\sqrt{Sc}}{2\sqrt{t}}\right) - \frac{x\sqrt{Sct}}{2\sqrt{\pi}}\exp\left(-\frac{x^2Sc}{4t}\right). \quad (23)$$

It has been continued that now with the fractional differential equation described by the second equation of the considered model, it is mean that Equation (7) under the initial and boundary conditions mentioned in Equations (9) and (10). After the application of the Laplace transform including the Laplace transform of the Caputo derivative, it has been obtained that the following transformations

$$\begin{aligned} s^{\alpha}\bar{v} - s^{\alpha-1}v(0) &= \frac{1}{Pr}\frac{\partial^2\bar{v}}{\partial x^2} + \eta\bar{v}, \\ s^{\alpha}\bar{v} &= \frac{1}{Pr}\frac{\partial^2\bar{v}}{\partial x^2} + \eta\bar{v}, \\ \frac{\partial^2\bar{v}}{\partial x^2} - Prs^{\alpha}\bar{v} + Pr\eta\bar{v} &= 0. \end{aligned} \quad (24)$$

The resolution of the second order differential equation represented in Equation (24) with the Laplace transform of the initial condition given by $v(0,s) = 1/s^2$, it has been obtained that the following equation

$$\bar{v}(x,s) = \frac{\exp\left[-x\sqrt{Pr(s^{\alpha}-\eta)}\right]}{s^2}. \quad (25)$$

It has been repeated that the same procedure of inversion using the Tzou method and then it has been obtained that the following form of a solution, that is

$$v(x,t)=\frac{e^{4.7}}{t}\left[\frac{1}{2}\bar{v}(x,\frac{4.7}{t})+Re\left\{\sum_{k=1}^{N}(-1)^{k}\bar{v}(x,\frac{4.7+k\pi i}{t})\right\}\right]. \tag{26}$$

The special case can be obtained when the order of the Caputo derivative converges to 1. it has been obtained that the following procedure of solution by applying the Laplace transform on Equation (7)

$$\begin{aligned} s\bar{v}-v(0) &= \frac{1}{Pr}\frac{\partial^2\bar{v}}{\partial x^2}+\eta\bar{v}, \\ s\bar{v} &= \frac{1}{Pr}\frac{\partial^2\bar{v}}{\partial x^2}+\eta\bar{v}, \\ \frac{\partial^2\bar{v}}{\partial x^2}-Prs\bar{v}+Pr\eta\bar{v} &= 0. \end{aligned} \tag{27}$$

The solution of the second order differential equation represented by Equation (27) is represented as the following form

$$\bar{w}(x,s)=\frac{\exp\left[-x\sqrt{Pr(s-\eta)}\right]}{s^2}. \tag{28}$$

For inverting Equation (28), it is used that the Tzou procedure, it has been obtained that as a solution the form defined by the form

$$v(x,t)=\frac{e^{4.7}}{t}\left[\frac{1}{2}\bar{w}(x,\frac{4.7}{t})+Re\left\{\sum_{k=1}^{N}(-1)^{k}\bar{w}(x,\frac{4.7+k\pi i}{t})\right\}\right]. \tag{29}$$

It is finished by determining the solution of the first Equation (6) of the model by using the Tzou procedure. At first, it is applied the Laplace transform to both sides of Equation (6) and with the initial and boundary condition, and then it has been obtained that

$$\begin{aligned} s^{\alpha}\bar{u}-s^{\alpha-1}\bar{u}(0) &= \kappa\frac{\partial^2\bar{u}}{\partial x^2}+Gr\bar{v}+Gm\bar{w}, \\ s^{\alpha}\bar{u} &= \kappa\frac{\partial^2\bar{u}}{\partial x^2}+Gr\bar{v}+Gm\bar{w}, \\ \frac{\partial^2\bar{u}}{\partial x^2}-\frac{s^{\alpha}}{\kappa}\bar{u} &= -\frac{Gr}{\kappa}\bar{v}-\frac{Gm}{\kappa}\bar{w}. \end{aligned} \tag{30}$$

where $\kappa = 1+\frac{1}{\beta}$. Replacing the results found in Equation (15) and the form got in Equation (25), it is to solve the following differential equation

$$\frac{\partial^2\bar{u}}{\partial x^2}-\frac{s^{\alpha}}{\kappa}\bar{u}=-\frac{Gr}{\kappa}\frac{\exp\left[-x\sqrt{Pr(s^{\alpha}-\eta)}\right]}{s^2}-\frac{Gm}{\kappa}\frac{\exp\left[-x\sqrt{Sc(s^{\alpha}+K)}\right]}{s^2}. \tag{31}$$

The resolution of this equation will be decomposed into two parts, it is begun by solving the differential equation described in the form

$$\frac{\partial^2\bar{u}}{\partial x^2}-\frac{s^{\alpha}}{\kappa}\bar{u}=-\frac{Gr}{\kappa}\frac{\exp\left[-x\sqrt{Pr(s^{\alpha}-\eta)}\right]}{s^2}. \tag{32}$$

Solving the previous differential Equation (32) using the classical procedure to solve the second-order differential equation, and considering the Laplace transform of the initial and boundary conditions, it has been obtained that the following form

$$\bar{u}_1(x,s) = A \exp\left[-x\sqrt{s^\alpha/\kappa}\right] + C \exp\left[-x\sqrt{Pr(s^\alpha - \eta)}\right] \tag{33}$$

Considering the Laplace transform of the boundary condition, it has been obtained that $A = -C$ where it has been obtained that

$$C = -\frac{Gr}{s^2[s^\alpha(Pr\kappa - 1) - Pr\eta\kappa]}. \tag{34}$$

It is continued with the resolution of the second differential equation described in the following equation, it has been obtained that

$$\frac{\partial^2 \bar{u}}{\partial x^2} - \frac{s^\alpha}{\kappa}\bar{u} = -\frac{Gm}{\kappa}\frac{\exp\left[-x\sqrt{Sc(s^\alpha + K)}\right]}{s^2}. \tag{35}$$

Adopting the previous procedure of resolution, the form of the solution taking into account the boundary condition gives the form

$$\bar{u}_2(x,s) = B \exp\left[-x\sqrt{s^\alpha/\kappa}\right] + D \exp\left[-x\sqrt{Sc(s^\alpha + K)}\right] \tag{36}$$

Considering the Laplace transform of the boundaries conditions, it has been obtained that $B = -D$ where it has been obtained that

$$D = -\frac{Gm}{s^2[s^\alpha(Sc\kappa - 1) + ScK\kappa]}. \tag{37}$$

The solution of the differential Equation (31) can be summarized by summing Equations (33) and (36) and then it has been obtained that the following form

$$\bar{u}(x,s) = \bar{u}_1(x,s) + \bar{u}_2(x,s) \tag{38}$$

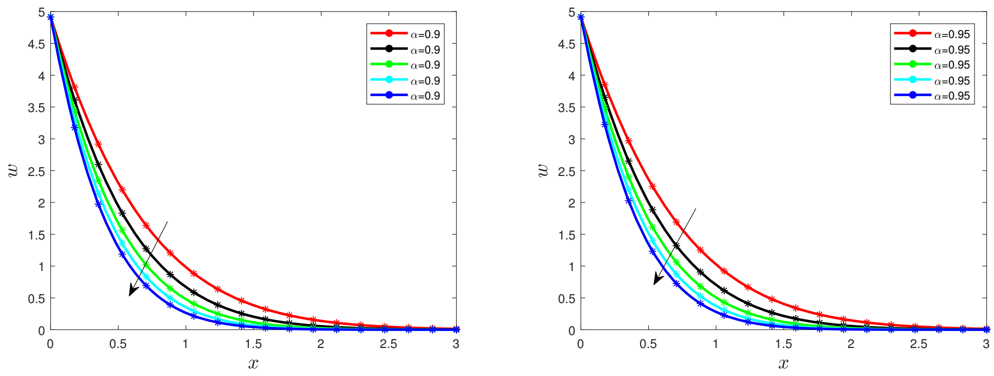
The analytical solution of Equation (6) uses the Tzou algorithm utilized for the inversion of the Laplace transform of Equation (38), and then it has been obtained that the form

$$u(x,t) = \frac{e^{4.7}}{t}\left[\frac{1}{2}\bar{u}(x,\frac{4.7}{t}) + Re\left\{\sum_{k=1}^{N}(-1)^k\bar{u}(x,\frac{4.7+k\pi i}{t})\right\}\right]. \tag{39}$$

5. Results and Interpretation

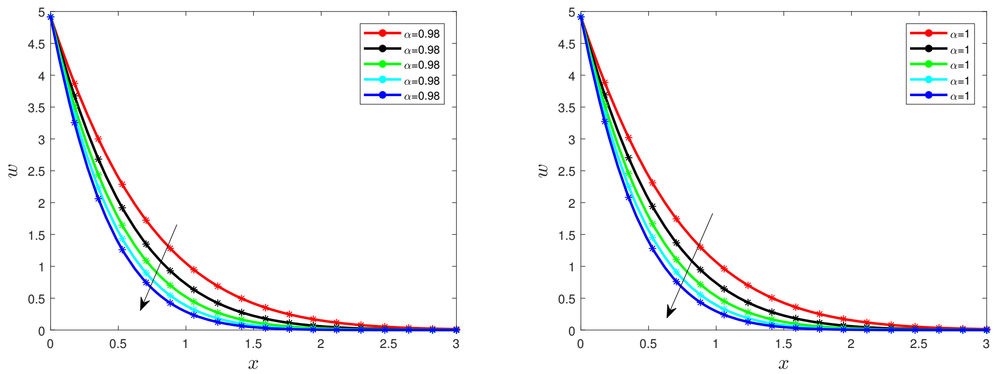
This section, it is given the graphics of the dynamics of the solutions considering the Caputo fractional derivative in the model (6)–(8). As mentioned in the previous section, it will be analyzed the influence of the parameters of the model utilized in the modeling will give the physical interpretations. The testing values are selected from the recent investigations in the literature, some of the values can be found in [12], and some of them in reference [27]. The motivation for using these values is to observe the behaviors of the solutions in this paper and to compare them with the behaviors observed in the literature, it does not notice a difference.

This part begins with the graphics of the concentration distribution represented in Equation (8) and it is used the analytical solution described in Equation (16) to give the curves obtained with Matlab. It is fixed that respectively $Sc = 6.5$; $Sc = 9.5$; $Sc = 12.5$; $Sc = 15.5$; and $Sc = 18.5$; and it is considered different values of the order of the fractional operator. In all the graphics $K = 0.04$ and $t = 5$. it has been obtained that the following graphical representations Figures 1a,b and 2a,b.



(**a**) (**b**)

Figure 1. Concentration distribution for $\alpha = 0.9$ (**a**). Concentration distribution for $\alpha = 0.95$ (**b**).



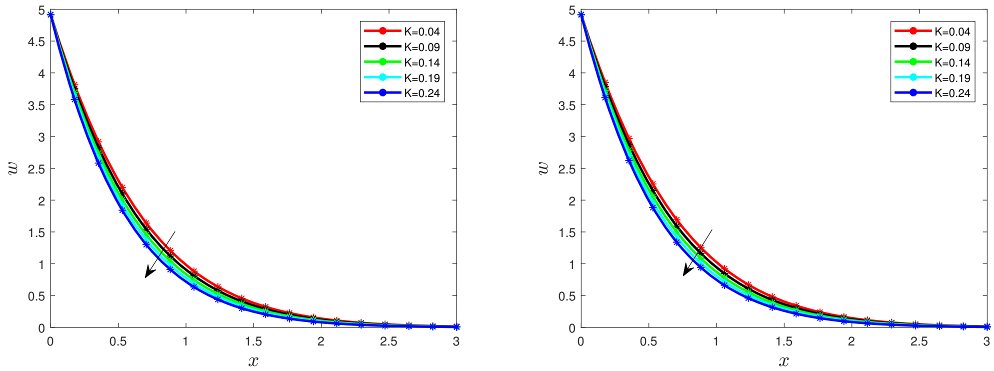
(**a**) (**b**)

Figure 2. Concentration distribution for $\alpha = 0.98$ (**a**). Concentration distribution for $\alpha = 1$ (**b**).

In these Figures 1a,b and 2a,b, it has been illustrated the impact of the variation of the Schmidt number. in the present case, its increase according to the variation of the order of the fractional order does not change the behaviors. That is the increase of the Schmidt number implies a decrease in the concentration distribution, this behavior has a physical interpretation. This behavior is because the increase in the Schmidt number enhances the diffusivity, which in turn generates a reduction of the concentration of the considered fluid model. In the second analysis, it is fixed the Schmidt number to $Sc = 6.5$, it is considered different orders of the Caputo derivative and by increasing the values of the Thermal conductivity K. It has been obtained that the following graphical representations Figures 3a,b and 4a,b.

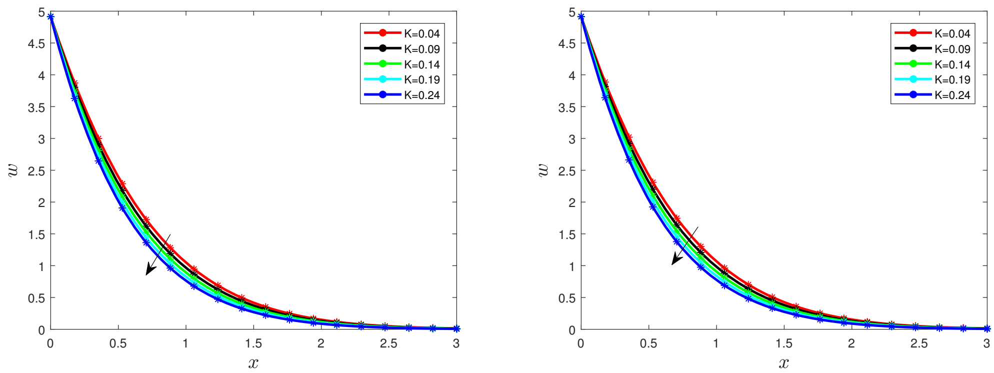
In the previous Figures 3a,b and 4a,b, it is noticed that increasing the values of Thermal conductivity K generates according to the variation of the values of the fractional operator a decrease in the profile of the concentration. This behavior is due to the fall of the diffusivity when the Thermal conductivity K increases. Finally, it is concluded that the Thermal conductivity K and the Schmidt number Sc influence the value of the diffusivity of the concentration. It is finished by focusing on the influence of the order of the Caputo derivative in the dynamics of the fluid model considered in this paper. Let that $Sc = 6.5$, $Sc = 9.5$, $Sc = 12.5$ and $Sc = 15.5$, the thermal conductivity $K = 0.04$, it is considered that the order of the fractional operator increases into the interval $(0, 1)$. It has been obtained that the following Figures 5a,b and 6a,b.

By the previous Figures 5a,b and 6a,b, it is noticed that after a certain time, it is noticed that the increase in the order of the Caputo derivative influence the diffusivity, making an increase in the diffusivity and then generating an increase in the value of the concentration distribution. It is concluded that the order of the fractional operator influences the diffusivity. In other words, the memory effect affects the diffusivity of the considered concentration distribution.



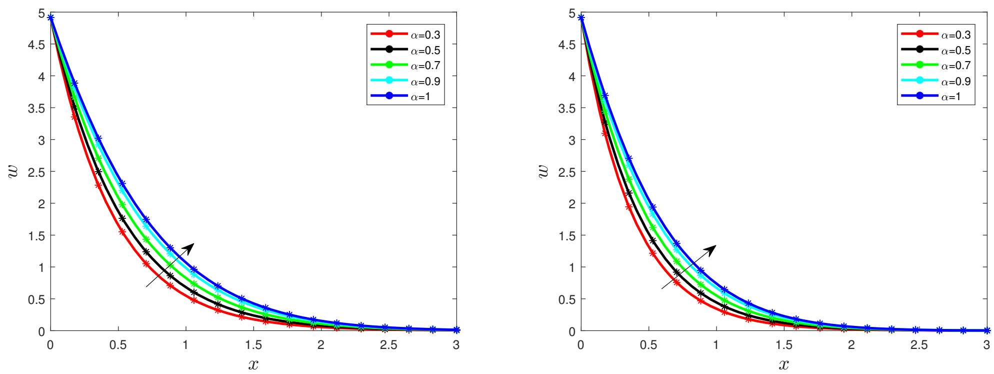
(**a**) (**b**)

Figure 3. Concentration distribution for $\alpha = 0.9$ for different K (**a**). Concentration distribution for $\alpha = 0.95$ for different K (**b**).



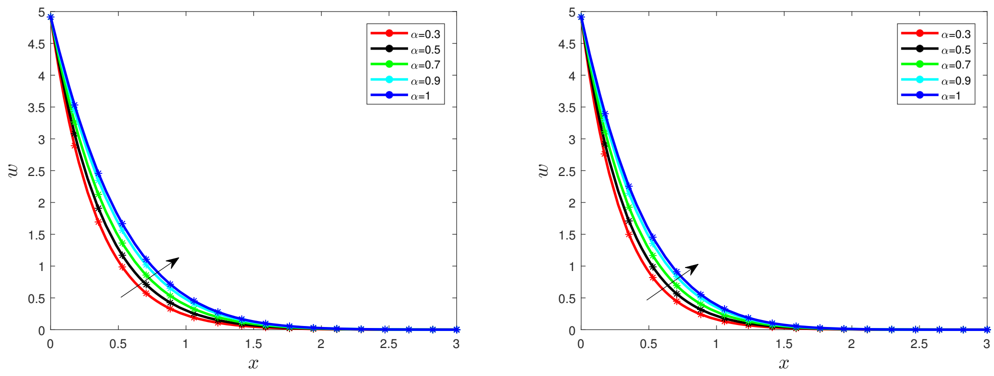
(**a**) (**b**)

Figure 4. Concentration distribution for $\alpha = 0.98$ for different K (**a**). Concentration distribution for $\alpha = 1$ for different K (**b**).



(**a**) (**b**)

Figure 5. Concentration distribution for $Sc = 6.5$ (**a**). Concentration distribution for $Sc = 9.5$ (**b**).

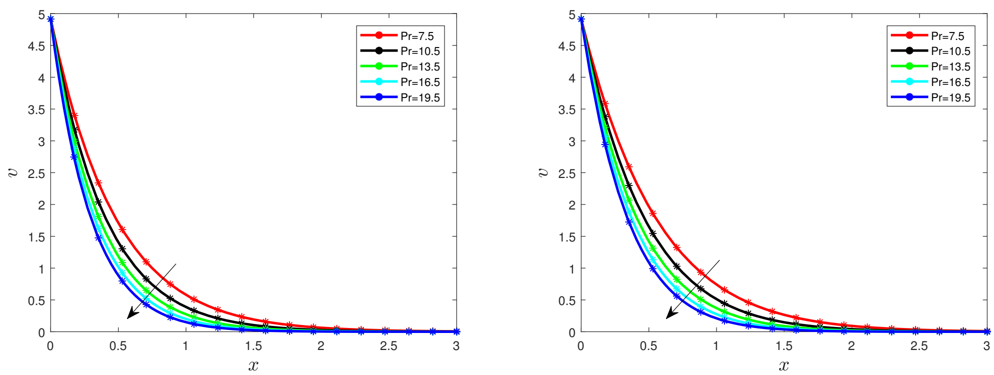


(**a**) (**b**)

Figure 6. Concentration distribution for $Sc = 12.5$ (**a**). Concentration distribution for $Sc = 15.5$ (**b**).

Let's analyze the influence of the parameters of the model and the order of the fractional operator in the dynamics of the temperature distribution of the considered fluid. It is begun by analyzing the influence of the Prandtl number Pr in the dynamics to arrive at the end it is given the information in the following Figures 7a,b and 8a,b represented by using $\eta = 0.1$.

It is observed that the same behavior as in the influence of the Schmidt number, it is noticed that the increase in the values of the Prandtl number causes the decrease of the temperature which is explained by the fall of the diffusivity in the temperature distribution. It is continued with the influence of the parameter η and $Pr = 6.5$. It has been obtained the following Figures 9a,b and 10a,b for illustrative examples.

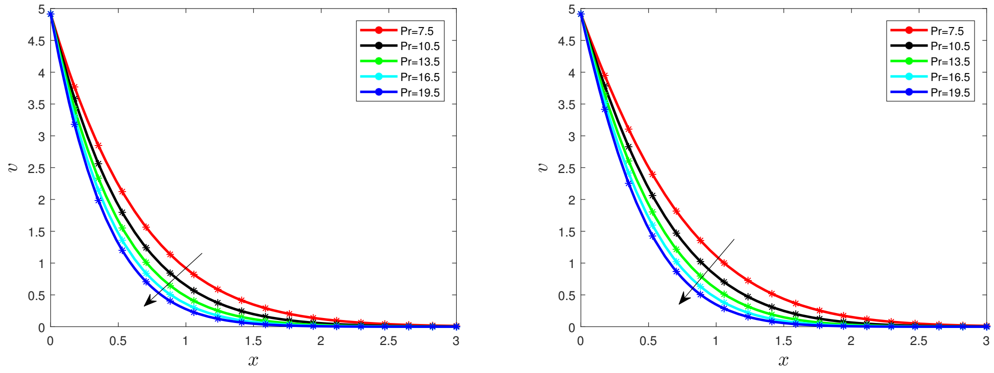


(**a**) (**b**)

Figure 7. Temperature distribution for $\alpha = 0.3$ (**a**). Temperature distribution for $\alpha = 0.5$ (**b**).

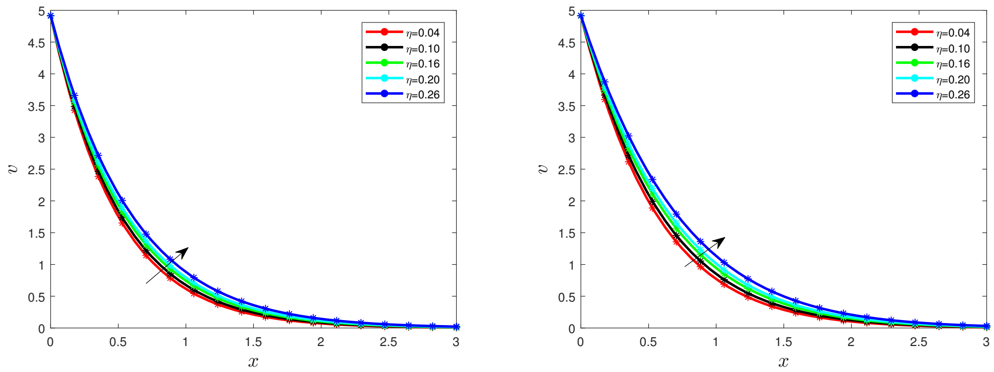
It is remarked that the increase of the value of the η generates an increase in the values of the temperature distribution, and the second remark is the influence of the order of the fractional derivative does not changes. In other words for every considered order of the Caputo derivative, the concentration distribution increase versus the increase of the parameter η. It is noticed that the parameter η has the same influence as the order of the operator, and then it concludes that the η influence the diffusivity of the temperature distribution too, increasing its values and generating an increase in the temperature.

It is to finish this paper with the graphical representations of the velocity distribution, it is also to analyze in detail the influence of the parameter used in the modeling of the considered dynamics. It is started with the order of the fractional operator, it is to explain how the order influences the dynamics of the velocity. It is considered that $Pr = 26$, $Gr = 12$, $Sc = 20$, $\eta = 0.4$, $\beta = 1.8$ and then it has been obtained that the following graphics Figures 11a,b and 12a,b.



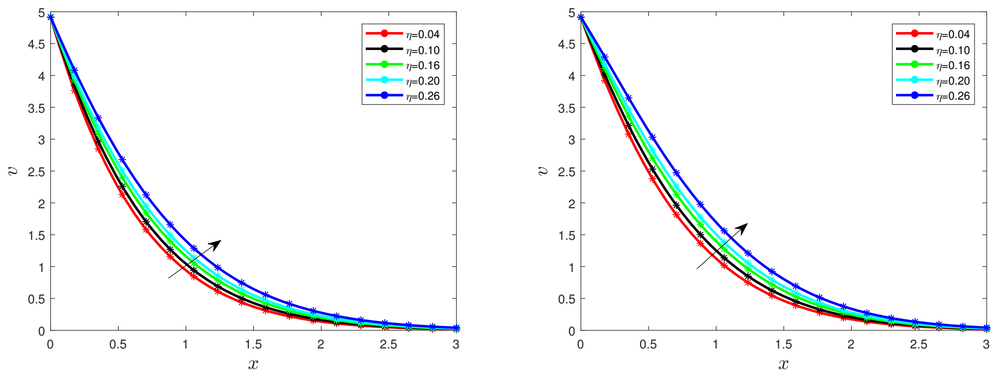
(**a**) (**b**)

Figure 8. Temperature distribution for $\alpha = 0.7$ (**a**). Temperature distribution for $\alpha = 0.9$ (**b**).



(**a**) (**b**)

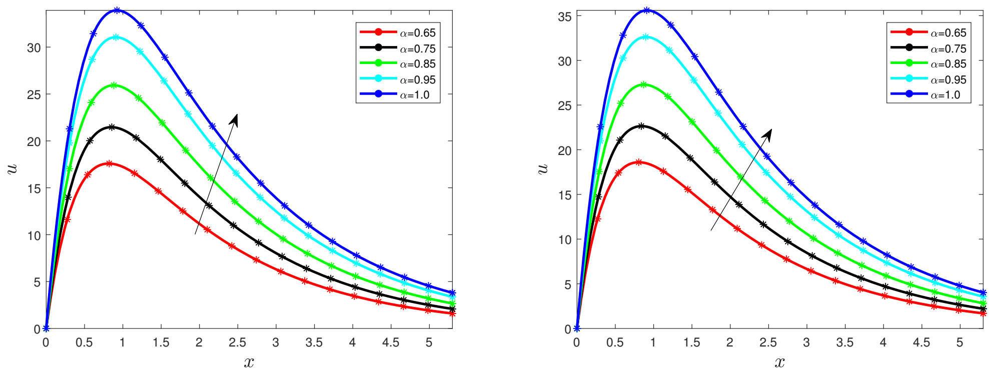
Figure 9. Temperature distribution for $\alpha = 0.3$ for different η (**a**). Temperature distribution for $\alpha = 0.5$ for different η (**b**).



(**a**) (**b**)

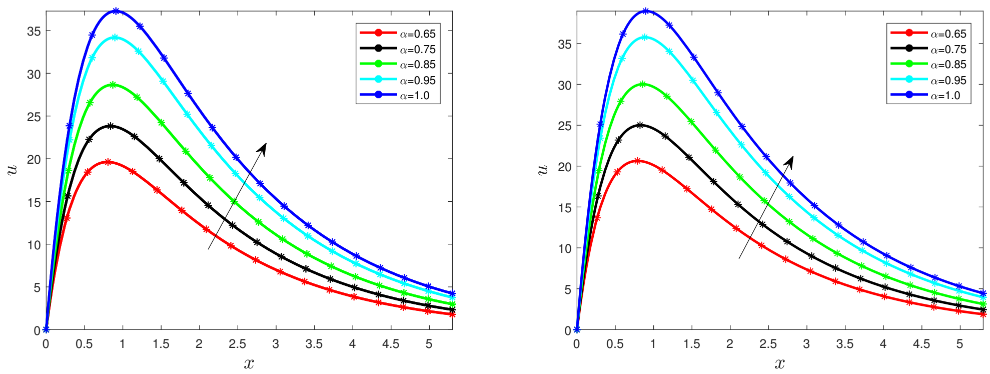
Figure 10. Temperature distribution for $\alpha = 0.7$ for different η (**a**). Temperature distribution for $\alpha = 0.9$ for different η (**b**).

It is observed that the increase in the order of the fractional operator caused an increase in the velocity distribution. We notice the same increase in behavior when the Mass Grahof number Gm increases. These behaviors are generated by the fact that the increase of buoyancy forces is generated by the increase of the Mass Grahof number and in turn, it causes the increase of the velocity as observed in the previous figures. At the start of the new section, it is fixed the following values $Pr = 26$, $Gm = 12$, $Sc = 20$, $\eta = 0.4$, $\beta = 1.8$, it is now considered taking different values of the Thermal Grashof number Gr, it has been obtained that the following Figures 13a,b and 14a,b for illustrative examples.



(**a**) (**b**)

Figure 11. Velocity distribution for $Gm = 10$ (**a**). Velocity distribution for $Gm = 12$ (**b**).

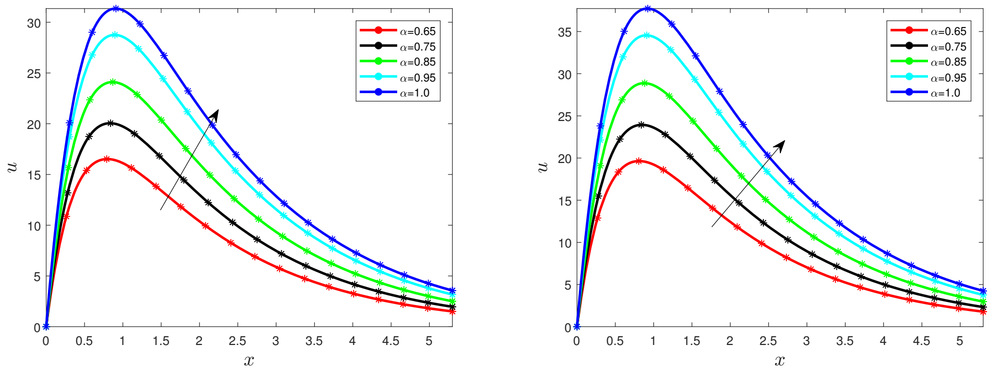


(**a**) (**b**)

Figure 12. Velocity distribution for $Gm = 14$ (**a**). Velocity distribution for $Gm = 16$ (**b**).

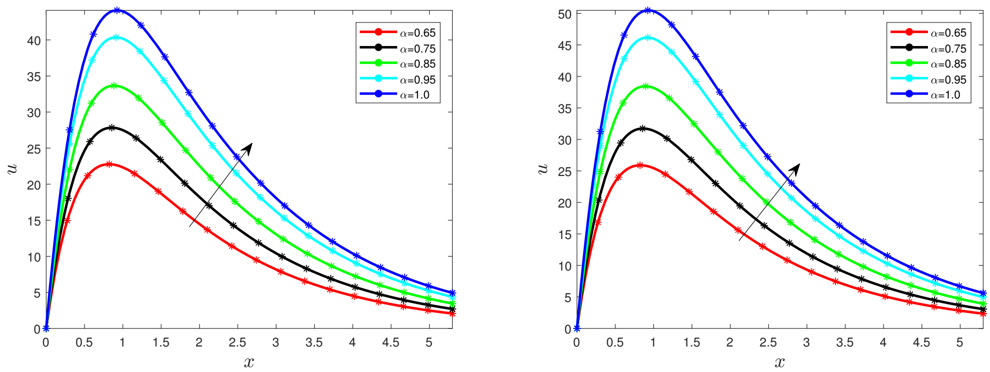
The behavior noticed with the Mass Grahof number Gm causing an increase of the velocity is noticed with the Thermal Grahof number Gr. Thus the Thermal Grahof number also impacts the buoyancy force. Due to the buoyancy forces, note that the increase of the Mass Grahof number Gm and the increase of the Thermal Grahof number directly cause the increase of the temperature and the concentration. It can be noticed by the constructive equations of the velocity, the temperature, and the concentration influence the equation satisfied by the velocity, and then their increase causes the increase of the velocity. It is continued by influencing the diffusivity of the temperature and the concentration, in other words, it is given that the graphics with different values of the Prandtl number Pr and the Schmidt number Sc. It is fixed the following values $Gr = 10$, $Gm = 12$, $Sc = 20$, $\eta = 0.4$, $\beta = 1.8$. It has been obtained that the following illustrative examples Figures 15a,b and 16a,b.

It is noticed that the increase in the Prandtl number Pr reduces the diffusivity of the temperature which reduces the velocity and then it is observed the confirmation in the different Figures 15a,b and 16a,b, it is noticed that the velocity decrease as well as with the increase of the Prandtl number Pr. It is focused now on the Schmidt number Sc, it has been got that the following graphics Figures 17a,b and 18a,b.



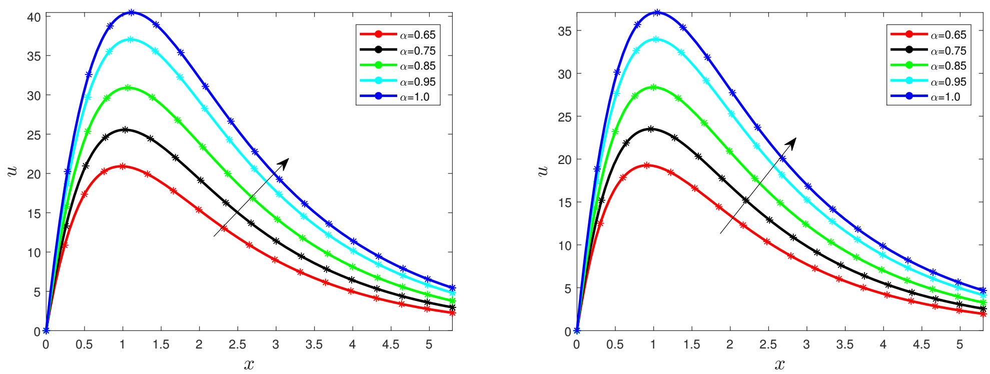
(**a**) (**b**)

Figure 13. Velocity distribution for $Gr = 10$ (**a**). Velocity distribution for $Gr = 13$ (**b**).



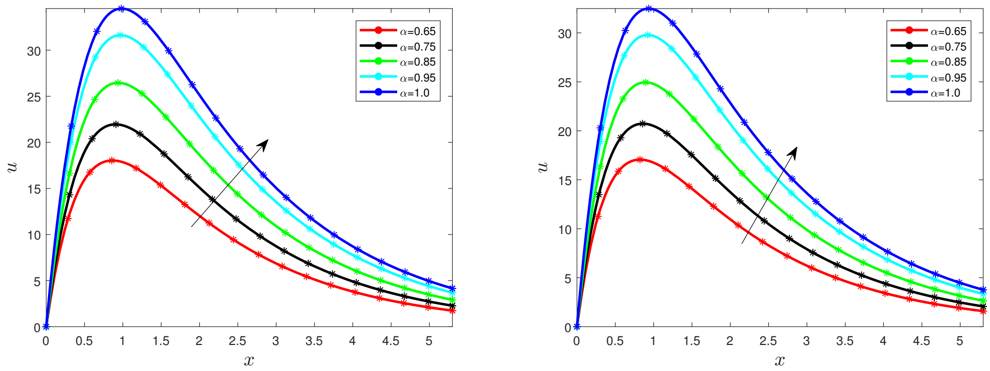
(**a**) (**b**)

Figure 14. Velocity distribution for $Gr = 16$ (**a**). Velocity distribution for $Gr = 19$ (**b**).



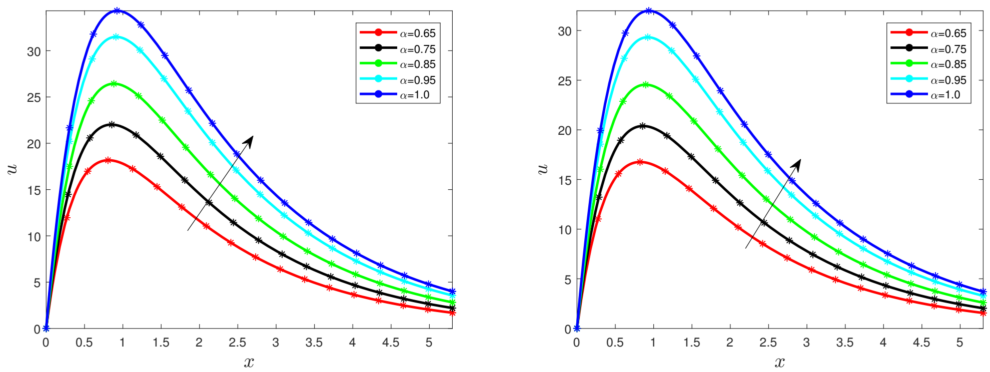
(**a**) (**b**)

Figure 15. Velocity distribution for $Pr = 15$ (**a**). Velocity distribution for $Pr = 18$ (**b**).



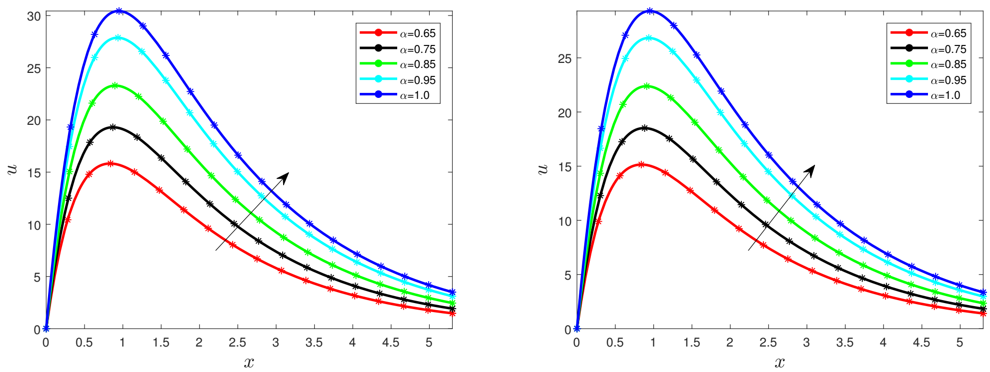
(**a**) (**b**)

Figure 16. Velocity distribution for $Pr = 21$ (**a**). Velocity distribution for $Pr = 24$ (**b**).



(**a**) (**b**)

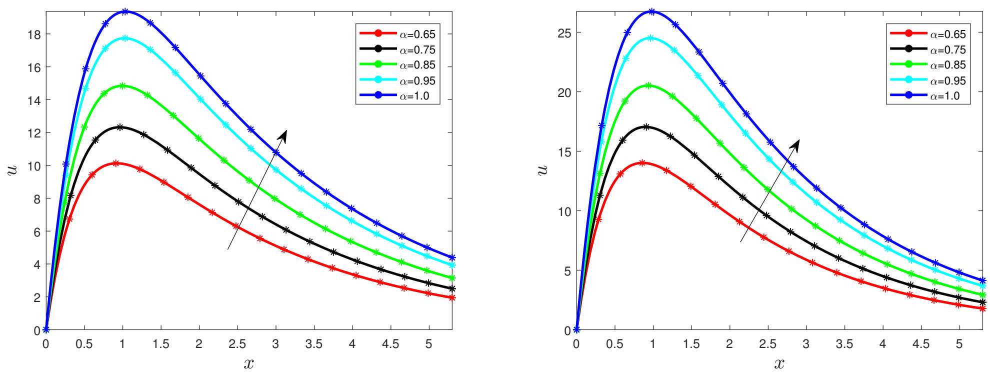
Figure 17. Velocity distribution for $Sc = 17$ (**a**). Velocity distribution for $Sc = 21$ (**b**).



(**a**) (**b**)

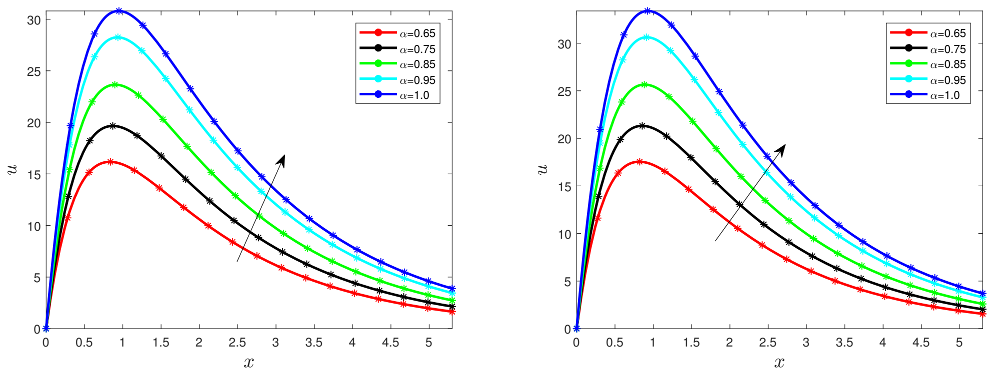
Figure 18. Velocity distribution for $Sc = 25$ (**a**). Velocity distribution for $Sc = 30$ (**b**).

It is noticed that the increase of the Schmidt number Sc impacts the velocity by reducing its values, it is because the Schmidt number Sc reduces the diffusivity of the concentration which in turn reduces the velocity because the concentration influence as well the constructive equations of the velocity. It is now focused on the impact on the velocity when it is considered different values for the Casson fluid parameter, it is fixed that $Pr = 24$, $Gr = 10$, $Gm = 12$, $Sc = 20$, $\eta = 0.4$. It has been obtained that the following graphics Figures 19a,b and 20a,b.



(**a**) (**b**)

Figure 19. Velocity distribution for $\beta = 0.5$ (**a**). Velocity distribution for $\beta = 1.0$ (**b**).



(**a**) (**b**)

Figure 20. Velocity distribution for $\beta = 1.5$ (**a**). Velocity distribution for $\beta = 2.0$ (**b**).

It is noticed in this section remarking that the increase in the values of the Casson parameter increases the values of the velocity, the simple explanation is that increasing the Casson fluid causes directly the reduction of the boundary layer thickness which generates the increase of the velocity according to the increase of the order of the fractional operator, as well.

Before closing this investigation, it is important to compare the present results with some results in the literature, on the same subject. As previously mentioned this paper provides a new method for getting an analytical solution, thus the testing values used for the graphics came from the papers [12,27]. as it can be observed in these papers using a different method of inversion to get their analytical solution, we observe the behaviors for the velocity, the temperature distribution and the concentration are in good agreements with the present investigations.

6. Conclusions

In this paper, it is used the Tzou algorithm to determine the values of the numerical inverse of the Laplace transforms. For findings, the fluid model constructive equations are constituted by fractional diffusion equations all with reaction terms that make the calculations complex, and for the alternative to get the analytical solutions it is proposed the Tzou algorithm to get solutions. The influence of the parameters of the model like the Prandtl number and Schmidt number Sc reduces the diffusivity and causes a decrease in the values of the concentration, temperature, and velocity. It is also focussed on the impact of the Grahof numbers which impact the buoyancy forces and cause an increase in the velocity distributions. It is also focussed on the influence of the Casson fluid parameter which impacts the boundary layer thickness and causes an increase in the velocity, too. The memory effect plays an acceleration effect in the present paper as can be observed by the arrows in the present paper. The future direction of investigation can interest the use of the non-singular fractional operator, and focus on the behavior of the algorithm used in the present paper when the Caputo operator is replaced by the fractional operator as the Caputo-Fabrizio derivative or the Atangana-Baleanu derivative.

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References

1. Qureshi, S.; Yusuf, A.; Shaikh, A.A.; Mustafa Inc. Transmission dynamics of varicella zoster virus modeled by classical and novel fractional operators using real statistical data. *Phys. A Stat. Mech. Its Appl.* **2019**, *534*, 122149. [CrossRef]
2. Wang, X.; Wang, Z. Dynamic Analysis of a Delayed Fractional-Order SIR Model with Saturated Incidence and Treatment Functio. *Int. J. Bifurc. Chaos* **2018**, *28*, 1850180. [CrossRef]
3. Khan, I.; Shah, N.A.; Vieru, D. Unsteady flow of generalized Casson fluid with fractional derivative due to an infinite plate. *Eur. Phys. J. Plus* **2016**, *131*, 181. [CrossRef]
4. Narahari, M.; Dutta, B.K. Effects of Thermal Radiation and Mass Diffusion on Free Convection Flow Near a Vertical Plate with Newtonian Heating. *Chem. Eng. Commun.* **2012**, *199*, 628–643. [CrossRef]
5. Abro, K.A. A Fractional and Analytic Investigation of Thermo-Diffusion Process on Free Convection Flow: An Application to Surface Modification Technology. *Eur. Phys. J. Plus* **2019**, *135*, 31–45. [CrossRef]
6. Ali, F.; Saqib, M.; Khan, I.; Sheikh, N.A. Application of Caputo-Fabrizio derivatives to MHD free convection flow of generalized Walters'-B fluid model. *Eur. Phys. J. Plus* **2016**, *131*, 377. [CrossRef]
7. Atangana, A.; Baleanu, D. New fractional derivatives with nonlocal and non-singular kernel: Theory and application to heat transfer model. *Therm. Sci.* **2016**, *20*, 763–769. [CrossRef]
8. Caputo, M.; Fabrizio, M. A new definition of fractional derivative without singular kernel. *Prog. Fract. Differ. Appl.* **2015**, *1*, 1–15.
9. Imran, M.A.; Sarwar, S.; Abdullah, M.; Khan, I. An analysis of the semi-analytic solutions of a viscous fluid with old and new definitions of fractional derivatives. *Chin. J. Phys.* **2018**, *56*, 1853–1871. [CrossRef]
10. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; van Mill, J., Ed.; North-Holland Mathematics Studies; Elsevier: Amsterdam, The Netherlands, 2006; Volume 204.
11. Podlubny, I. *Fractional Differential Equations*; Mathematics in Science and Engineering; Academic Press: New York, NY, USA, 1999; Volume 198.
12. Vieru, D.; Fetecau, C.; Fetecau, C. Time-fractional free convection flow near a vertical plate with Newtonian heating and mass diffusion. *Therm. Sci.* **2015**, *19*, 85–98. [CrossRef]
13. Khan, A.; Abro, K.A.; Tassaddiq, A.; Khan, I. Atangana-Baleanu and Caputo Fabrizio Analysis of Fractional Derivatives for Heat and Mass Transfer of Second Grade Fluids over a Vertical Plate: A Comparative Study. *Entropy* **2017**, *19*, 279. [CrossRef]
14. Shah, N.A.; Khan, I.; Aleem, M.; Imran, M.A. Influence of magnetic field on double convection problem of fractional viscous fluid over an exponentially moving vertical plate: New trends of Caputo time-fractional derivative model. *Adv. Mech. Eng.* **2019**, *11*, 1–11. [CrossRef]
15. Reyaz, R.; Lim, Y.J.; Mohamad, A.Q.; Saqib, M.; Shafie, S. Caputo Fractional MHD Casson Fluid Flow Over an Oscillating Plate with Thermal Radiation. *J. Adv. Res. Fluid Mech. Therm. Sci.* **2021**, *85*, 145–158. [CrossRef]
16. Sheikh, N.A.; Ali, F.; Saqib, M.; Khan, I.; Jan, S.A.A. A comparative study of Atangana-Baleanu and Caputo-Fabrizio fractional derivatives to the convective flow of a generalized Casson fluid. *Eur. Phys. J. Plus* **2017**, *132*, 54. [CrossRef]
17. Sene, N. Analytical investigations of the fractional free convection flow of Brinkman type fluid described by the Caputo fractional derivative. *Results Phys.* **2022**, *37*, 105555. [CrossRef]
18. Sene, N. Second-grade fluid with Newtonian heating under Caputo fractional derivative: Analytical investigations via Laplace transforms. Mathematical. *Model. Numer. Simul. Appl.* **2022**, *2*, 13–25. [CrossRef]
19. Sene, N. Analytical Solutions of a Class of Fluids Models with the Caputo Fractional Derivative. *Fractal Fract.* **2022**, *6*, 35. [CrossRef]
20. Sene, N. Fractional Model and Exact Solutions of Convection Flow of an Incompressible Viscous Fluid under the Newtonian Heating and Mass Diffusion. *J. Math.* **2022**, *2022*, 8785197. [CrossRef]
21. Sene, N. Analysis of the Influences of Parameters in the Fractional Second-Grade Fluid Dynamics. *Mathematics* **2022**, *10*, 1125.

22. Alrehili, M.F.; Goud, B.S.; Reddy, Y.D.; Mishra, S.R.; Lashin, M.M.; Govindan, V.; Pimpunchat, B. Numerical computing of Soret and linear radiative effects on MHD Casson fluid flow toward a vertical surface through a porous medium: Finite element analysis. *Mod. Phys. Lett. B* **2023**, *36*, 2250170. [CrossRef]
23. Afikuzzaman, M.; Ferdows, M.; Alam, M.M. Unsteady MHD casson fluid flow through a parallel plate with hall current. *Procedia Eng.* **2015**, *105*, 287–293. [CrossRef]
24. Afikuzzaman, M.; Ferdows, M.; Quadir, R.A.; Alam, M.M. MHD Viscous incompressible Casson fluid flow with hall current. *J. Adv. Res. Fluid Mech. Therm. Sci.* **2019**, *60*, 270–282.
25. Jena, S.; Mishra, S.R.; Aghaeiboorkheili, M.; Pattnaik, P.K.; Muduli, K. Impact of Newtonian heating on the conducting Casson fluid flow past a stretching cylinder. *J. Interdiscip. Math.* **2022**, *25*, 2401–2416. [CrossRef]
26. Haque, E.U.; Awan, A.U.; Raza, N.; Abdullah, M.; Chaudhry, M.A. A computational approach for the unsteady flow of maxwell fluid with Caputo fractional derivatives. *Alex. Eng. J.* **2018**, *57*, 2601–2608. [CrossRef]
27. Sheikh, N.A.; Ali, F.; Saqib, M.; Khan, I.; Jan, S.A.A.; Alshomrani, A.S.; Alghamdi, M.S. Comparison and analysis of the Atangana–Baleanu and Caputo-Fabrizio fractional derivatives for generalized Casson fluid model with heat generation and chemical reaction. *Results Phys.* **2017**, *7*, 789–800. [CrossRef]