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A New Distribution for Modeling Data with Increasing Hazard Rate: A Case of COVID-19 Pandemic and Vinyl Chloride Data

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Abstract: A novel lifetime distribution has been defined and examined in this study. The odd Lindley–Pareto (OLiP) distribution is the name we give to the new distribution. The new density function can be written as an odd Lindley-G distribution with Pareto amplification. The moment-generating function and characteristic function, entropy and asymptotic behavior, order statistics and moments, mode, variance, skewness, and kurtosis are some of the aspects of the OLiP distribution that are discovered. Seven non-Bayesian estimation techniques and Bayesian estimation utilizing Markov chain Monte Carlo were compared for performance. Additionally, when the lifetime test is truncated after a predetermined period, single acceptance sampling plans (SASPs) are created for the newly suggested, OLiP distribution. The median lifetime of the OLiP distribution with pre-specified factors is taken as the truncation time. To guarantee that the specific life test is obtained at the defined risk to the user, the minimum sample size is required. For a particular consumer's risk, the OLiP distribution's parameters, and the truncation time, numerical results are obtained. The new distribution is illustrated using mortality rates of COVID-19 patients in Canada and vinyl chloride data in (g/L) from ground-water monitoring wells that are located in clean-up-gradient areas.

Keywords: acceptance sampling; estimation; odd Lindley-G family of distributions; odd Lindley–Pareto distribution

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1. Introduction

Due to their flexibility and stability in use, new distributions created from newly developing families of distributions and other probability distribution generators have drawn a lot of interest in the literature. Accordingly, John, Kotz, and Balakrishnan [1,2] provide a thorough analysis of hundreds of continuous univariate distributions. According to Gomes-Silva et al. [3], recent research has instead concentrated on the definition of new families of distributions that expand well-known distributions while also offering a great deal of modeling flexibility for lifetime data. The work of Taketomi et al. [4] provides a thorough analysis of parametric distributions for lifetime models.

Furthermore, Shakil and Kibria [5] and Shakil et al. [6] have developed a family of lifetime distributions based on a generalized Pearson differential equation. The Lindley distribution was originally proposed by Lindley [7] as a counter-example of fiducial statistics. Ghitany, Atieh, and Nadarajah [8] showed through a numerical example that the hazard function of the Lindley distribution does not exhibit a constant hazard rate, indicating its improved flexibility over the exponential distribution. The Lindley distribution has recently received considerable attention as an appropriate model to analyze lifetime data, for instance, in Zakerzadeh and Dolati [9], Mazucheli and Achcar [10], Gupta and Singh [11], Warahena-Liyanage and Pararai [12], Onyekwere and Obulezi [13], and very recently Anabike et al. [14], Shakil et al. [15], among others. An expanded generalized cosine family of distributions was investigated for use in dependability modeling and modeling the COVID-19 mortality rate with a new flexible modification of the log-logistic distribution by Muse et al. [16]. Several other authors including Sankaran [17], Nadarajah et al. [18], and Asgharzedah, Bakouch, and Esmaeli [19] developed some structural properties of various generalized Lindley distributions. Furthermore, Ramadan et al. [20] discussed the generalized alpha power Akshaya distribution and some statistical inferences for the unknown parameters. An acceptance sample plan is used when testing is damaging, 100% examination would be highly expensive, or 100% examination would take too long, according to Lu et al. [21]. Gui and Zhang [22] argued that it is typical to end a lifetime test by the predetermined period in order to save money and time because the predicted lifetime of a product is relatively high and it may take a long time to wait until all the items fail. Another argument is that the goal of these tests is to determine a given mean life, t_0, with a probability of at least $P*$, which is the amount of trust the consumer has in it. Al-Omari [23] stated that the procedure begins with collecting the smallest sample size required to highlight a particular average life when the life test is terminated after a predetermined period; hence, such tests are known as truncated lifetime tests.

The primary objective of this article is to present the odd Lindley–Pareto distribution, a novel lifetime distribution, and some of its characteristics. We will also develop a single acceptance sampling strategy using the new distribution, specify its operating characteristic function, and provide the important decision rule. Usually, modification of existing distributions yields new distributions that have a better goodness of fit with respect to some scenarios. However, the superiority in performance of modified distributions has some trade-offs in usage. Hence, the lifetime distribution proposed in this article has the potential of modeling data strictly with increasing hazard rates. In addition, the advantages of deploying the single acceptance sampling plans include being economical and easy to understand, the comparatively small computation work involved, and that scheduling and delivery times are improved due to the quick inspection process. The distribution is capable of being used in survival analysis of patients suffering from common diseases such as COVID-19, chickenpox, tuberculosis, etc., and can also be used to model the volatility of organic molecules which are relevant to environmental studies, such as vinyl chloride, but we cannot guarantee its applicability in situations where the hazard function is either a bathtub or reversed bathtub shape. Essentially, a major motivation for this study is to demonstrate that the proposed distribution has the potential to model the COVID-19 data just as Yiu et al. [24] and Lü et al. [25] demonstrated. In terms of application, the scope of this study is non-censored data, and a simulation study is also involved.

The following is the order of this article. We introduce the odd Lindley–Pareto (OLiP) distribution in Section 2. The statistical properties of the OLiP distribution are described in Section 3. The non-Bayesian parameter estimation of the OLiP distribution is covered in Section 4. The Bayesian estimation of the OLiP distribution's parameters is shown in Section 5. Section 6 investigates the single acceptance sampling strategies. Numerical computations and real data analysis are analyzed in Section 7. The conclusions are provided in Section 8.

2. The New Distribution

Let X be a random variable with the cumulative distribution function (cdf) $F(x,\lambda,\theta)$, and probability density function (pdf) $f(x,\lambda,\theta)$, from the odd Lindley-G family of distributions. Gomes-Silva et al. [3] defined the cdf and pdf of the odd Lindley-G family of distributions as

$$F(x;\lambda,\vartheta) = 1 - \frac{\lambda + \bar{G}(x;\vartheta)}{(1+\lambda)\bar{G}(x;\vartheta)} e^{-\frac{\lambda G(x;\vartheta)}{\bar{G}(x;\vartheta)}},\ x \geq 0. \quad (1)$$

$$f(x;\lambda,\vartheta) = \frac{\lambda^2 g(x;\vartheta)}{(1+\lambda)\bar{G}(x;\vartheta)^3} e^{-\frac{\lambda G(x;\vartheta)}{\bar{G}(x;\vartheta)}}, \; x \geq 0. \tag{2}$$

$G(x)$ and $g(x)$, respectively, denote the cdf and pdf of the Pareto distribution and are due to Vilfredo [26]:

$$g(x) = \frac{k\theta^k}{x^{k+1}}, \quad x \geq \theta, \quad k > 0 \tag{3}$$

$$G(x) = 1 - \left(\frac{x}{\theta}\right)^{-k}. \tag{4}$$

Noting that $\bar{G}(.) = 1 - G(.)$, the pdf of the OLiP distribution is obtained by substituting (3) and (4) into (2). Simplifying, we obtain

$$f_{OLiP}(x;\lambda,\theta,k) = \frac{\lambda^2}{1+\lambda}\left(\frac{k}{x}\right)\left(\frac{x}{\theta}\right)^{2k} e^{-\lambda\left\{\left(\frac{x}{\theta}\right)^k - 1\right\}}, \quad x \geq \theta, \quad (\theta,\lambda,k) > 0, \quad \theta \neq 0 \tag{5}$$

By applying a little algebra to (5), we obtain the cdf of the OLiP distribution. Thus,

$$F_{OLiP}(x;\lambda,\theta,k) = 1 - \frac{\lambda\left(\frac{x}{\theta}\right)^k + 1}{(1+\lambda)} e^{-\lambda\left\{\left(\frac{x}{\theta}\right)^k - 1\right\}} \tag{6}$$

To show that (5) is a proper pdf, recall that

$$\int_\theta^\infty f_{OLiP}(x;\lambda,\theta,k)dx = 1. \tag{7}$$

Proof.

$$\int_\theta^\infty f_{OLiP}(x;\lambda,\theta,k)dx = \int_\theta^\infty \frac{\lambda^2}{1+\lambda}\left(\frac{k}{x}\right)\left(\frac{x}{\theta}\right)^{2k} e^{-\lambda\left\{\left(\frac{x}{\theta}\right)^k - 1\right\}} dx,$$

$$= \frac{\lambda^2 k e^\lambda}{(1+\lambda)\theta^{2k}} \int_\theta^\infty x^{2k-1} e^{-\lambda\left(\frac{x}{\theta}\right)^k} dx,$$

$$\text{Let } u = \lambda\left(\frac{x}{\theta}\right)^k \Rightarrow x = \theta\sqrt[k]{\frac{u}{\lambda}}; dx = \frac{\theta}{k\sqrt[k]{\lambda}} u^{\frac{1}{k}-1} du$$

$$\Rightarrow \int_\theta^\infty f_{OLiP}(x;\lambda,\theta,k)dx = \frac{e^\lambda}{(1+\lambda)} \int_\lambda^\infty u e^{-u} du.$$

Note that for upper incomplete Gamma function

$$\int_z^\infty x^{\alpha-1} e^{-x} dx = \Gamma(\alpha,z)$$

$$\Rightarrow \int_\theta^\infty f_{OLiP}(x;\lambda,\theta,k)dx = \frac{e^\lambda}{(1+\lambda)}\Gamma(2,\lambda) = \frac{e^\lambda\Gamma(2,\lambda)}{(1+\lambda)}$$

$$\Gamma(n,z) = \Gamma(n)e^{-z}\sum_{m=0}^{n-1}\frac{z^m}{m!} \Rightarrow \Gamma(2,\lambda) = \Gamma(2)e^{-\lambda}\sum_{m=0}^{n-1}\frac{\lambda^m}{m!} = (1+\lambda)e^{-\lambda}$$

$$\int_\theta^\infty f_{OLiP}(x;\lambda,\theta,k)dx = \frac{e^\lambda}{(1+\lambda)}(1+\lambda)e^{-\lambda} = 1$$

□

The following methods are used to determine the OLiP distribution's reliability and hazard rate functions:

$$R(x) = 1 - F(x) = \frac{\lambda\left(\frac{x}{\theta}\right)^k + 1}{(1+\lambda)} e^{-\lambda\left\{\left(\frac{x}{\theta}\right)^k - 1\right\}}, \tag{8}$$

and

$$h(x) = \frac{f(x)}{R(x)} = \frac{\lambda^2 \left(\frac{k}{x}\right)\left(\frac{x}{\theta}\right)^k}{1 + \left(\frac{\theta}{x}\right)^k}. \tag{9}$$

Figures 1 and 2 illustrate, respectively, the pdf and cdf of the OLiP distribution for various values of the parameters. For varying values of λ, k, and θ, Figures 3 and 4 show the shapes of the plots of $h(x)$ and $S(x)$, respectively. Furthermore, Figure 3 shows that the OLiP distribution has an increasing hazard rate function.

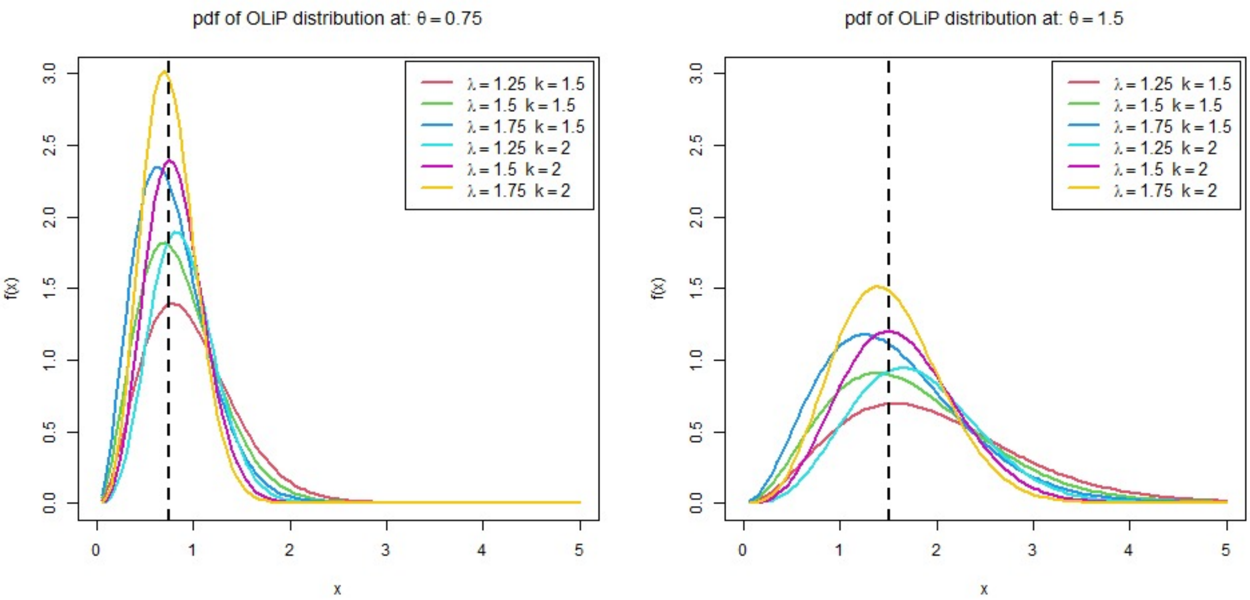

Figure 1. The pdf of the OLiP distribution for various values of the parameters.

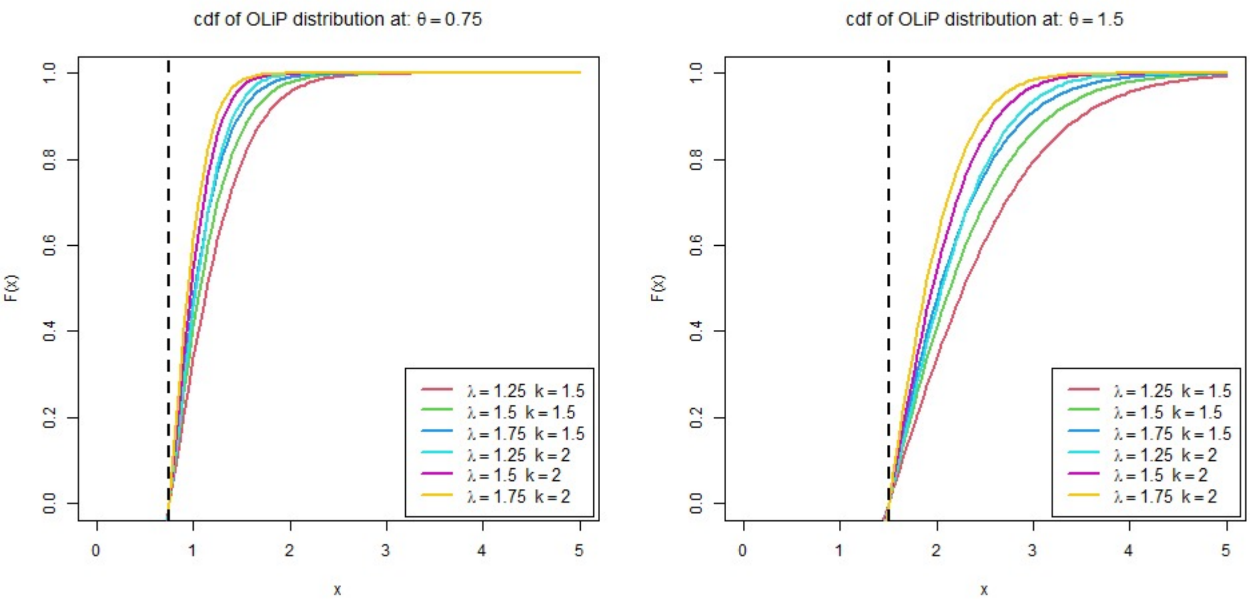

Figure 2. The cdf of the OLiP distribution for various values of the parameters.

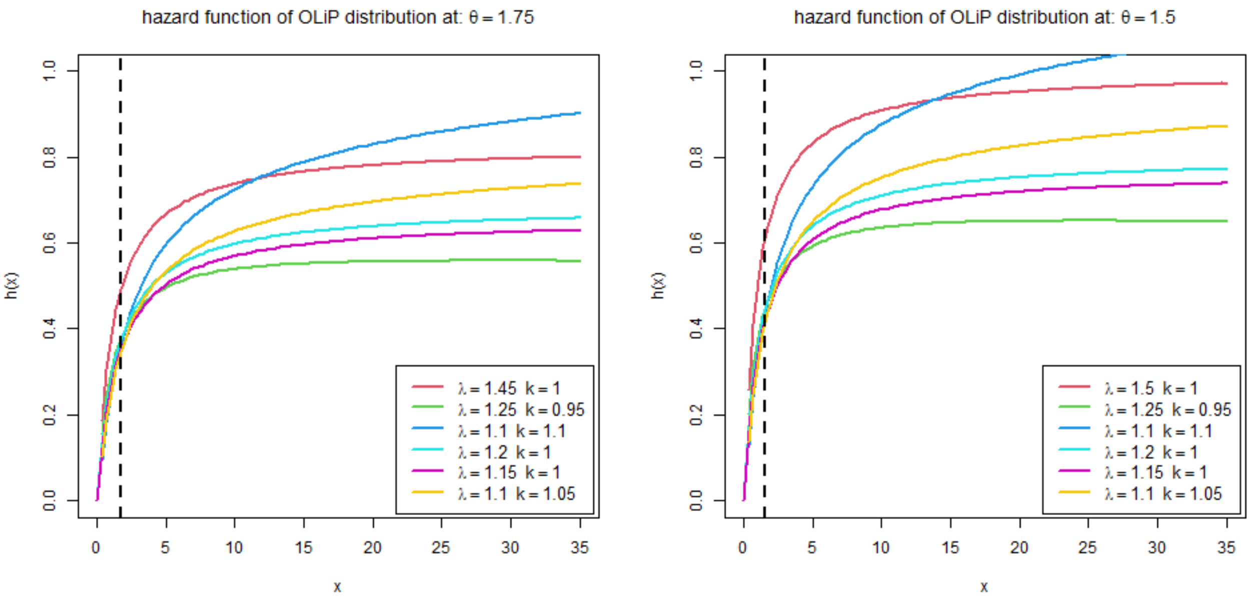

Figure 3. The hazard rate function of the OLiP distribution for various values of the parameters.

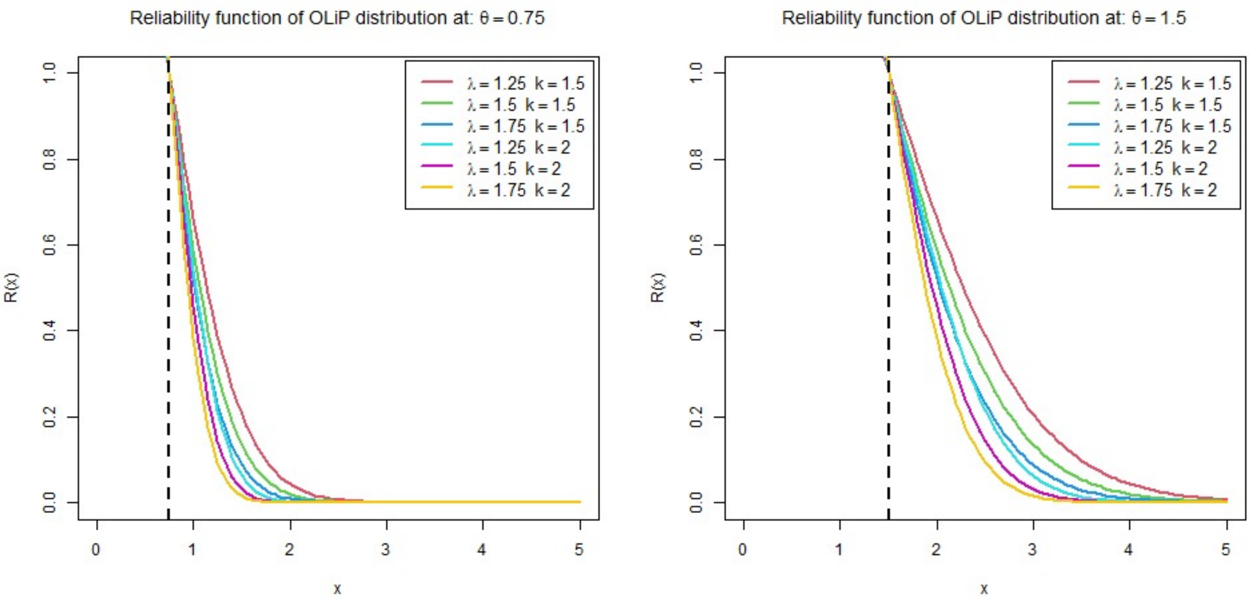

Figure 4. The reliability function of the OLiP distribution for various values of the parameters.

3. Statistical Properties of OLiP Distribution

We go through a few of the important statistical properties of the OLiP distribution in this section.

3.1. The OLiP Distribution's Moment

The r^{th} moment of the OLiP distribution is given by

$$\mu_r' = E(X^r) = \int_\theta^\infty x^r \frac{\lambda^2}{1+\lambda}\left(\frac{k}{x}\right)\left(\frac{x}{\theta}\right)^{2k} \mathrm{e}^{-\lambda\left\{\left(\frac{x}{\theta}\right)^k - 1\right\}} dx = \frac{\theta^r \mathrm{e}^\lambda}{(1+\lambda)\lambda^{\frac{r}{k}}}\Gamma(\frac{r+2k}{k},\theta), r = 1,2,3,\ldots \quad (10)$$

We obtain the first four moments of the OLiP distribution for r = 1, 2, 3, and 4. As a result, we have

$$\mu_1^{'} = \frac{\theta e^{\lambda}}{(1+\lambda)\lambda^{\frac{1}{k}}}\Gamma(\frac{1+2k}{k},\theta), \tag{11}$$

$$\mu_2^{'} = \frac{\theta^2 e^{\lambda}}{(1+\lambda)\lambda^{\frac{2}{k}}}\Gamma(\frac{2+2k}{k},\theta), \tag{12}$$

$$\mu_3^{'} = \frac{\theta^3 e^{\lambda}}{(1+\lambda)\lambda^{\frac{3}{k}}}\Gamma(\frac{3+2k}{k},\theta), \tag{13}$$

and

$$\mu_4^{'} = \frac{\theta^4 e^{\lambda}}{(1+\lambda)\lambda^{\frac{4}{k}}}\Gamma(\frac{4+2k}{k},\theta). \tag{14}$$

The mode of the OLiP distribution can be obtained by taking the first derivative of (5) and solving the nonlinear equation. Thus,

$$X_{mode} = \theta\left\{\frac{2k-1}{\lambda k}\right\}^{\frac{1}{k}}. \tag{15}$$

Using (11) and (12), and employing the relationship between moments about zero and central moments, we obtain the variance of (5) as follows

$$\sigma^2 = \mu_2^{'} - (\mu_1^{'})^2 = \frac{\theta^2 e^{\lambda}}{(1+\lambda)\lambda^{\frac{2}{k}}}\Gamma(\frac{2+2k}{k},\theta) - \left\{\frac{\theta e^{\lambda}}{(1+\lambda)\lambda^{\frac{1}{k}}}\Gamma(\frac{1+2k}{k},\theta)\right\}^2. \tag{16}$$

3.2. Skewness and Kurtosis Coefficients for the OLiP Distribution

We will calculate the coefficient of skewness and kurtosis below in order to determine the degree of asymmetry and peakedness of the OLiP distribution. The following methods are used to determine the OLiP distribution's coefficient of skewness:

$$\begin{aligned}\beta_1 &= \frac{\mu_3^{'} - 3\mu_1^{'}\mu_2^{'} + 2(\mu_1^{'})^2}{\sigma^{\frac{3}{2}}} \\ &= \frac{\frac{\theta^3 e^{\lambda}}{(1+\lambda)\lambda^{\frac{3}{k}}}\Gamma(\frac{3+2k}{k},\theta) - 3\frac{\theta e^{\lambda}}{(1+\lambda)\lambda^{\frac{1}{k}}}\Gamma(\frac{1+2k}{k},\theta)\frac{\theta^2 e^{\lambda}}{(1+\lambda)\lambda^{\frac{2}{k}}}\Gamma(\frac{2+2k}{k},\theta) + 2\left\{\frac{\theta e^{\lambda}}{(1+\lambda)\lambda^{\frac{1}{k}}}\Gamma(\frac{1+2k}{k},\theta)\right\}^2}{\left\{\sqrt{\frac{\theta^2 e^{\lambda}}{(1+\lambda)\lambda^{\frac{2}{k}}}\Gamma(\frac{2+2k}{k},\theta) - \left\{\frac{\theta e^{\lambda}}{(1+\lambda)\lambda^{\frac{1}{k}}}\Gamma(\frac{1+2k}{k},\theta)\right\}^2}\right\}^{\frac{3}{2}}}.\end{aligned} \tag{17}$$

Similarly, the OLiP distribution's coefficient of kurtosis is calculated as follows;

$$\begin{aligned}\beta_3 &= \frac{\mu_4^{'} - 4\mu_1^{'}\mu_3^{'} + 6(\mu_1^{'})^2\mu_2^{'} - 4(\mu_1^{'})^4}{\sigma^4} \\ &= \frac{\begin{array}{c}\frac{\theta^4 e^{\lambda}}{(1+\lambda)\lambda^{\frac{4}{k}}}\Gamma(\frac{4+2k}{k},\theta) - 4\frac{\theta e^{\lambda}}{(1+\lambda)\lambda^{\frac{1}{k}}}\Gamma(\frac{1+2k}{k},\theta)\frac{\theta^3 e^{\lambda}}{(1+\lambda)\lambda^{\frac{3}{k}}}\Gamma(\frac{3+2k}{k},\theta) \\ +6\left\{\frac{\theta e^{\lambda}}{(1+\lambda)\lambda^{\frac{1}{k}}}\Gamma(\frac{1+2k}{k},\theta)\right\}^2\frac{\theta^2 e^{\lambda}}{(1+\lambda)\lambda^{\frac{2}{k}}}\Gamma(\frac{2+2k}{k},\theta) - 4\left\{\frac{\theta e^{\lambda}}{(1+\lambda)\lambda^{\frac{1}{k}}}\Gamma(\frac{1+2k}{k},\theta)\right\}^4\end{array}}{\left\{\sqrt{\frac{\theta^2 e^{\lambda}}{(1+\lambda)\lambda^{\frac{2}{k}}}\Gamma(\frac{2+2k}{k},\theta) - \left\{\frac{\theta e^{\lambda}}{(1+\lambda)\lambda^{\frac{1}{k}}}\Gamma(\frac{1+2k}{k},\theta)\right\}^2}\right\}^2}.\end{aligned} \tag{18}$$

3.3. The OLiP Distribution's Moment-Generating and Characteristic Functions

The moment-generating function $M_X(t)$ of the OLiP distribution is obtained as follows

$$
\begin{aligned}
M_X(t) = E(\mathrm{e}^{tx}) = \int_{\theta}^{\infty} \mathrm{e}^{tx} f(x)dx &= \int_{\theta}^{\infty} \mathrm{e}^{tx} \frac{\lambda^2}{1+\lambda}\left(\frac{k}{x}\right)\left(\frac{x}{\theta}\right)^{2k} \mathrm{e}^{-\lambda\left\{\left(\frac{x}{\theta}\right)^k - 1\right\}} dx \\
&= \frac{\lambda^2 k \mathrm{e}^{\lambda}}{(1+\lambda)\theta^{2k}} \sum_{n=0}^{\infty} \frac{t^n}{n!} \int_{\lambda}^{\infty} x^{n+2k-1} \mathrm{e}^{-\lambda\left\{\left(\frac{x}{\theta}\right)^k - 1\right\}} dx \\
&= \frac{\mathrm{e}^{\lambda}}{(1+\lambda)} \sum_{n=0}^{\infty} \frac{(\theta t)^n}{n! \lambda^{\frac{n}{k}}} \Gamma\left(\frac{n}{k} + 2, \theta\right).
\end{aligned} \tag{19}
$$

The characteristic function of the OLiP distribution has also been discovered in a similar manner. As a result, we obtain

$$
\phi_x(it) = \frac{\mathrm{e}^{\lambda}}{(1+\lambda)} \sum_{n=0}^{\infty} \frac{(\theta i t)^n}{n! \lambda^{\frac{n}{k}}} \Gamma\left(\frac{n}{k} + 2, \theta\right). \tag{20}
$$

3.4. Order Statistics of the OLiP Distribution

Let $X_1, X_2, \ldots, X_n$ denote a random sample of size n from the OLiP distribution with parameters λ, θ, k and $X_r.(r = 1, 2, \ldots, n)$ are the r^{th}-order statistics obtained by arranging X_r in ascending order, then the pdf of the r^{th}-order statistics is obtained as follows:

$$
f_{r:n}(x) = \frac{n!}{(r-1)!(n-r)!} f_{OLiP}(x) [F_{OLiP}(x)]^{r-1} [1 - F_{OLiP}(x)]^{n-r}, \tag{21}
$$

where $f_{OLiP}(x)$ and $F_{OLiP}(x)$ are, respectively, the pdf and cdf of the OLiP distribution given in (5) and (6). Using the binomial series expansion of $[1 - F_{OLiP}(x)]^{n-r}$ in (21), we obtain

$$
f_{r:n}(x) = \sum_{t=0}^{n-r} \frac{n!(-1)^t}{(r-1)!(n-r+t)!} f_{OLiP}(x) [F_{OLiP}(x)]^{r+t-1}. \tag{22}
$$

Therefore, the pdf of the r^{th}-order statistics for the OLiP distribution is given by

$$
f_{r:n}(x;\lambda,\theta,k) = \sum_{t=0}^{n-r} \frac{n!(-1)^t}{(r-1)!(n-r+t)!} \frac{\lambda^2}{1+\lambda}\left(\frac{k}{x}\right)\left(\frac{x}{\theta}\right)^{2k} \mathrm{e}^{-\lambda\left\{\left(\frac{x}{\theta}\right)^k - 1\right\}} \left\{1 - \frac{\lambda\left(\frac{x}{\theta}\right)^k + 1}{(1+\lambda)} \mathrm{e}^{-\lambda\left\{\left(\frac{x}{\theta}\right)^k - 1\right\}}\right\}^{r+t-1}. \tag{23}
$$

The pdf of the largest-order statistics of the OLiP distribution is obtained by substituting n for r in (23)

$$
f_{n:n}(x;\lambda,\theta,k) = \frac{n!(-1)^t}{(n-1)!t!} \frac{\lambda^2}{1+\lambda}\left(\frac{k}{x}\right)\left(\frac{x}{\theta}\right)^{2k} \mathrm{e}^{-\lambda\left\{\left(\frac{x}{\theta}\right)^k - 1\right\}} \left\{1 - \frac{\lambda\left(\frac{x}{\theta}\right)^k + 1}{(1+\lambda)} \mathrm{e}^{-\lambda\left\{\left(\frac{x}{\theta}\right)^k - 1\right\}}\right\}^{n+t-1}. \tag{24}
$$

While the pdf of the smallest-order statistics of the OLiP distribution is obtained by substituting 1 for r in (23)

$$
f_{1:n}(x;\lambda,\theta,k) = \frac{n!(-1)^t}{(n+t-1)!} \frac{\lambda^2}{1+\lambda}\left(\frac{k}{x}\right)\left(\frac{x}{\theta}\right)^{2k} \mathrm{e}^{-\lambda\left\{\left(\frac{x}{\theta}\right)^k - 1\right\}} \left\{1 - \frac{\lambda\left(\frac{x}{\theta}\right)^k + 1}{(1+\lambda)} \mathrm{e}^{-\lambda\left\{\left(\frac{x}{\theta}\right)^k - 1\right\}}\right\}^{t}. \tag{25}
$$

3.5. Entropy and Asymptotic Behavior of the OLiP Distribution

Entropy is a measure of how much uncertainty or randomness there is in a system. It belongs to the class of non-negative $\omega \neq 1$ information measures. For the OLiP distributed random variable X, the Rényi entropy is

$$
R_{\omega}(x) = \lim_{n\to\infty} (I_{\omega}(f_n) - \log n) = \frac{1}{1-\omega} \log \int_{\theta}^{\infty} f^{\omega}(x) dx, \tag{26}
$$

where $f(x)$ is the pdf of the OLiP distribution in (5). Therefore,

$$R_\omega(x) = \left(\frac{\lambda k}{\theta}\right)^{\omega-1} \frac{e^{\lambda\omega}}{\omega^{\frac{2\omega k-\omega+1}{k}}} \frac{\Gamma\left(\frac{2\omega k-\omega+1}{k},\theta\right)}{(1-\omega)(1+\lambda)^\omega}. \tag{27}$$

By taking the limit of the pdf in (5) as $x \to \infty$ and as $x \to 0$, the asymptotic behavior of the OLiP distributed random variable is examined.

$$\lim_{x\to\infty} f(x;\lambda,\theta,k) = \lim_{n\to\infty}\left\{\frac{\lambda^2}{1+\lambda}\left(\frac{k}{x}\right)\left(\frac{x}{\theta}\right)^{2k} e^{-\lambda\left\{\left(\frac{x}{\theta}\right)^k-1\right\}}\right\} = 0. \tag{28}$$

Additionally, it is simple to see that

$$\lim_{x\to 0} f(x;\lambda,\theta,k) = \lim_{x\to 0}\left\{\frac{\lambda^2}{1+\lambda}\left(\frac{k}{x}\right)\left(\frac{x}{\theta}\right)^{2k} e^{-\lambda\left\{\left(\frac{x}{\theta}\right)^k-1\right\}}\right\} = 0. \tag{29}$$

The OLiP distribution's unimodality is thus justified (29).

3.6. Stochastic Ordering of OLiP Distribution

An important tool for evaluating the behavior of system components is the stochastic ordering of a non-negative continuous random variable. It is stated that a random variable X is smaller than a random variable Y in the following equation:

(i) Stochastic order $(X \le_{st} Y)$ if $F_X(x) \ge F_Y(x)\ \forall x$.
(ii) Hazard rate order $(X \le_{hr} Y)$ if $h_X(x) \ge h_Y(x)\ \forall x$.
(iii) Mean residual life order $(X \le_{mrl} Y)$ if $m_X(x) \ge m_Y(x)\ \forall x$.
(iv) Likelihood ratio order $(X \le_{lr} Y)$ if $\frac{f_X(x)}{F_Y(x)}$ decreases in x.

This implies that

$$X \le_{lr} Y \Rightarrow X \le_{hr} Y \Rightarrow X \le_{st} Y \Rightarrow X \le_{mrl} Y.$$

Here, we establish the theorem below, which states that the OLiP distribution is ordered in accordance with the strongest "likelihood ratio".

Theorem 1. *Let $X \sim OLiP(\lambda_1, k_1, \theta_1)$ and $Y \sim OLiP(\lambda_2, k_2, \theta_2)$. If any of $\lambda_1 > \lambda_2$, $k_1 > k_2$, or $\theta_1 > \theta_2$, then $X \le_{lr} Y$, hence $X \le_{hr} Y$, $X \le_{mrl} Y$, and $X \le_{st} Y$.*

Proof.

$$\frac{f_X(x)}{f_Y(x)} = \left(\frac{\lambda_1}{\lambda_2}\right)^2\left(\frac{1+\lambda_2}{1+\lambda_1}\right)\frac{k_1\theta_2}{k_2\theta_1}x^{2(k_1-k_2)}e^{-\lambda_1\left(\frac{x}{\theta_1}\right)^{k_1}+\lambda_2\left(\frac{x}{\theta_2}\right)^{k_2}+\lambda_1-\lambda_2}.$$

Taking natural log of the ratio will yield

$$\ln\frac{f_X(x)}{f_Y(x)} = 2\ln\frac{\lambda_1}{\lambda_2} + \ln\left(\frac{1+\lambda_2}{1+\lambda_1}\right) + \ln\left(\frac{k_1\theta_2}{k_2\theta_1}\right) + 2(k_1-k_2)\ln x + 2(k_1-k_2)\ln\left(\frac{\theta_2}{\theta_1}\right) - \lambda_1\left(\frac{x}{\theta_1}\right)^{k_1}$$
$$+ \lambda_2\left(\frac{x}{\theta_2}\right)^{k_2} + (\lambda_1-\lambda_2).$$

Differentiating the natural log of the ratio wrt x will yield

$$\frac{d}{dx}\ln\frac{f_X(x)}{f_Y(x)} = \frac{2(k_1-k_2)}{x} - \frac{\lambda_1 k_1}{\theta_1^{k_1}}x^{k_1-1} + \frac{\lambda_2 k_2}{\theta_2^{k_2}}x^{k_2-1}$$

If $k_2 > k_1$, $\frac{d}{dx}\ln\frac{f_X(x)}{f_Y(x)} < 0$, and $\frac{f_X(x;\lambda_1,k_1,\theta_1)}{f_Y(x;\lambda_2,k_2,\theta_2)}$ is decreasing in x.

That is, $X \le_{lr} Y$ and hence, $X \le_{hr} Y$, $X \le_{mrl} Y$ and $X \le_{st} Y$.

□

4. Non-Bayesian Estimation of OLiP Distribution Parameters

Different approaches to parameter estimates will be discussed in this section. For a similar study refer to [27]. Numerical computation using the R program and the **optim()** function will be used in this section.

4.1. Maximum Likelihood Estimation (MLE)

Let $(x_1, x_2, \ldots, x_n)$ be n random samples drawn from the OLiP distribution, then the likelihood function, as studied extensively by [28], is given as

$$\ell(\phi) = \prod_{i=1}^{n} f(x;\lambda,\theta,k) = \prod_{i=1}^{n} \frac{\lambda^2}{1+\lambda}\left(\frac{k}{x}\right)\left(\frac{x}{\theta}\right)^{2k} e^{-\lambda\left\{\left(\frac{x}{\theta}\right)^{k}-1\right\}} = \left(\frac{\lambda^2 k}{(1+\lambda)^n \theta^{2nk}}\right)^{n} \prod_{i=1}^{n} x^{2k-1} e^{-\lambda\left\{\left(\frac{x}{\theta}\right)^{k}-1\right\}}. \quad (30)$$

Taking the natural logarithm of (30) yields

$$\begin{aligned}\ln \ell(\phi) &= \ln\left(\frac{\lambda^{2n}k^n}{(1+\lambda)^n\theta^{2nk}}\right) - \lambda\sum_{i=1}^{n}\left(\frac{x}{\theta}\right)^{k} + n\lambda + \sum_{i=1}^{n}\ln x^{2k-1} \\ &= 2n\ln\lambda + n\ln k - n\ln(1+\lambda) - 2nk\ln\theta - \lambda\sum_{i=1}^{n}\left(\frac{x}{\theta}\right)^{k} + n\lambda + (2k-1)\sum_{i=1}^{n}\ln x_i.\end{aligned} \quad (31)$$

We consider the estimated value of θ to be equal to the minimum value of the random variable X in case the maximum likelihood approach for estimation is used in this lifetime OLiP distribution, where there is a relation between the random variable X and the parameter θ $(x > \theta)$. Additionally, we estimate the other two parameters, k and λ, using the projected value of θ as the minimum value of X. Equations (32) and (33) are produced by differentiating (31) with respect to λ and k and equating the results to zero.

$$\frac{2n}{\hat{\lambda}} - \frac{n}{1+\lambda} = \frac{\sum x}{\theta^k} - n, \quad (32)$$

$$\frac{n}{k} - \lambda\sum_{i=1}^{n}\left(\frac{x}{\theta}\right)^{k}\ln\left(\frac{x}{\theta}\right) = 2\left(n\ln\theta + \sum \ln x\right). \quad (33)$$

Since there is no closed-form solution for Equations (32) and (33), estimations of λ and k are determined iteratively using the Newton–Raphson method (Albert [29] and Gelman [30]).

4.2. Maximum Product Space Estimators (MPSE)

The maximum product spacing method, which approaches the Kullback–Leibler information measure, is an acceptable substitute for the highest likelihood strategy. Assume for a moment that the data are now arranged in ascending order. Following that, the OLiP's maximum product spacing is provided as follows.

$$Gs(\lambda,k,\theta|data) = \left(\prod_{i=1}^{n+1} D_l(x_i,\lambda,k,\theta)\right)^{\frac{1}{n+1}}, \quad (34)$$

where $D_l(x_i,\lambda,k,\theta) = F(x_i;\lambda,k,\theta) - F(x_{i-1};\lambda,k,\theta)$, $i = 1,2,3,\ldots,n$

In a similar manner, one may decide to increase the function.

$$H(\lambda,k,\theta) = \frac{1}{n+1}\sum_{i=1}^{n+1}\ln D_i(\lambda,k,\theta). \quad (35)$$

The parameter estimates are determined by calculating the first derivative of the function $H(\vartheta)$ with respect to λ, k, and θ, and solving the resulting nonlinear equations at $\frac{\partial H(\phi)}{\partial\lambda} = 0$, $\frac{\partial H(\phi)}{\partial k} = 0$, and $\frac{\partial H(\phi)}{\partial\theta} = 0$, where $\phi = (\lambda,k,\theta)$,

4.3. Least Squares Estimation (LSE)

Swain et al. [31] suggested using the least squares estimation to estimate the Beta distribution's parameters. Using the inferences from the study of Swain et al. [31], we write

$$E[F(x_{i:n}|\lambda,k,\theta)] = \frac{i}{n+1}.$$

$$V[F(x_{i:n}|\lambda,k,\theta)] = \frac{i(n-i+1)}{(n+1)^2(n+2)}.$$

The least squares estimates $\hat{\lambda}_{LSE}$, $\hat{k}_{LSE}$, and $\hat{\theta}_{LSE}$ of the parameters λ, k, and θ are obtained by minimizing the function $L(\lambda,k,\theta)$ with respect to λ, k, and θ:

$$L(\lambda,k,\theta) = \underset{(\lambda,k,\theta)}{\arg\min} \sum_{i=1}^{n} \left[F(x_{i:n}|\lambda,k,\theta) - \frac{i}{n+1}\right]^2. \tag{36}$$

The estimates are obtained by solving the following nonlinear equations:

$$\begin{aligned}
&\sum_{i=1}^{n} \left[F(x_{i:n}|\lambda,k,\theta) - \frac{i}{n+1}\right]^2 \Delta_1(x_{i:n}|\lambda,k,\theta) = 0 \\
&\sum_{i=1}^{n} \left[F(x_{i:n}|\lambda,k,\theta) - \frac{i}{n+1}\right]^2 \Delta_2(x_{i:n}|\lambda,k,\theta) = 0 \\
&\sum_{i=1}^{n} \left[F(x_{i:n}|\lambda,k,\theta) - \frac{i}{n+1}\right]^2 \Delta_3(x_{i:n}|\lambda,k,\theta) = 0,
\end{aligned} \tag{37}$$

where

$$\begin{aligned}
\Delta_1(x_{i:n}|\lambda,k,\theta) &= \mathrm{e}^{-\lambda\left\{\left(\frac{x}{\theta}\right)^k - 1\right\}} \frac{\left(\lambda\left(\frac{x}{\theta}\right)^k + 1\right)\left(\frac{x}{\theta}\right)^k - \left(\frac{x}{\theta}\right)^k}{(1+\lambda)^2}, \\
\Delta_2(x_{i:n}|\lambda,k,\theta) &= \frac{\lambda^2\left(\frac{x}{\theta}\right)^{2k} \ln\left(\frac{x}{\theta}\right)}{1+\lambda} \mathrm{e}^{-\lambda\left\{\left(\frac{x}{\theta}\right)^k - 1\right\}}, \\
\Delta_3(x_{i:n}|\lambda,k,\theta) &= -\left(\frac{k}{\theta}\right)\left(\frac{\lambda^2}{1+\lambda}\right)\left(\frac{x}{\theta}\right)^{2k} \mathrm{e}^{-\lambda\left\{\left(\frac{x}{\theta}\right)^k - 1\right\}}.
\end{aligned} \tag{38}$$

4.4. Weighted Least Squares Estimation (WLSE)

The weighted least squares estimates $\hat{\lambda}_{WLSE}$, $\hat{k}_{WLSE}$, and $\hat{\theta}_{WLSE}$ of the OLiP distribution parameters λ, k, and θ are obtained by minimizing the function $W(\lambda,k,\theta)$ with respect to λ, k, and θ:

$$W(\lambda,k,\theta) = \underset{(\lambda,k,\theta)}{\arg\min} \sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \left[F(x_{i:n}|\lambda,k,\theta) - \frac{i}{n+1}\right]^2. \tag{39}$$

Solving the following nonlinear equations yields the estimates

$$\begin{aligned}
&\sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \left[F(x_{i:n}|\lambda,k,\theta) - \frac{i}{n+1}\right] \Delta_1(x_{i:n}|\lambda,k,\theta) = 0, \\
&\sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \left[F(x_{i:n}|\lambda,k,\theta) - \frac{i}{n+1}\right] \Delta_2(x_{i:n}|\lambda,k,\theta) = 0, \\
&\sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \left[F(x_{i:n}|\lambda,k,\theta) - \frac{i}{n+1}\right] \Delta_3(x_{i:n}|\lambda,k,\theta) = 0,
\end{aligned} \tag{40}$$

where $\Delta_1(x.|\lambda,k,\theta)$, $\Delta_2(.|\lambda,k,\theta)$, and $\Delta_3(.|\lambda,k,\theta)$ are as defined in (38).

4.5. Cramér–von Mises Estimation (CVME)

The Cramér–von Mises estimates $\hat{\lambda}_{CVME}$, $\hat{k}_{CVME}$, and $\hat{\theta}_{CVME}$ of the OLiP distribution parameters λ, k, and θ are obtained by minimizing the function $C(\lambda, k, \theta)$ with respect to λ, k, and θ

$$C(\lambda, k, \theta) = \underset{(\lambda,k,\theta)}{\arg\min} \left\{ \frac{1}{12n} + \sum_{i=1}^{n} \left[F(x_{i:n}|\lambda, k, \theta) - \frac{2i-1}{2n} \right]^2 \right\}. \quad (41)$$

The estimates are obtained by solving the following nonlinear equations:

$$\begin{aligned} \sum_{i=1}^{n} \left(F(x_{i:n}|\lambda, k, \theta) - \frac{2i-1}{2n} \right) \Delta_1(x_{i:n}|\lambda, k, \theta) &= 0 \\ \sum_{i=1}^{n} \left(F(x_{i:n}|\lambda, k, \theta) - \frac{2i-1}{2n} \right) \Delta_2(x_{i:n}|\lambda, k, \theta) &= 0 \\ \sum_{i=1}^{n} \left(F(x_{i:n}|\lambda, k, \theta) - \frac{2i-1}{2n} \right) \Delta_3(x_{i:n}|\lambda, k, \theta) &= 0. \end{aligned} \quad (42)$$

where $\Delta_1(.|\lambda, k, \theta)$, $\Delta_2(.|\lambda, k, \theta)$, and $\Delta_3(.|\lambda, k, \theta)$ are as defined in (38).

4.6. Anderson–Darling Estimation (ADE)

The Anderson–Darling estimates $\hat{\lambda}_{ADE}$, $\hat{k}_{ADE}$, and $\hat{\theta}_{ADE}$ of the OLiP distribution parameters λ, k, and θ are obtained by minimizing the function $A(\lambda, k, \theta)$ with respect to λ, k, and θ

$$A(\lambda, k, \theta) = \underset{(\lambda,k,\theta)}{\arg\min} \sum_{i=1}^{n} (2i-1) \left\{ \ln F(x_{i:n}|\lambda, k, \theta) + \ln \left[1 - F(x_{n+1-i:n}|\lambda, k, \theta) \right] \right\}. \quad (43)$$

The estimates are obtained by solving the following sets of nonlinear equations:

$$\begin{aligned} \sum_{i=1}^{n} (2i-1) \left[\frac{\Delta_1(x_{i:n}|\lambda, k, \theta)}{F(x_{i:n}|\lambda, k, \theta)} - \frac{\Delta_1(x_{n+1-i:n}|\lambda, k, \theta)}{1 - F(x_{n+1-i:n}|\lambda, k, \theta)} \right] &= 0 \\ \sum_{i=1}^{n} (2i-1) \left[\frac{\Delta_2(x_{i:n}|\lambda, k, \theta)}{F(x_{i:n}|\lambda, k, \theta)} - \frac{\Delta_2(x_{n+1-i:n}|\lambda, k, \theta)}{1 - F(x_{n+1-i:n}|\lambda, k, \theta)} \right] &= 0 \\ \sum_{i=1}^{n} (2i-1) \left[\frac{\Delta_3(x_{i:n}|\lambda, k, \theta)}{F(x_{i:n}|\lambda, k, \theta)} - \frac{\Delta_3(x_{n+1-i:n}|\lambda, k, \theta)}{1 - F(x_{n+1-i:n}|\lambda, k, \theta)} \right] &= 0, \end{aligned} \quad (44)$$

where $\Delta_1(.|\lambda, k, \theta)$, $\Delta_2(.|\lambda, k, \theta)$, and $\Delta_3(.|\lambda, k, \theta)$ are as defined in (38).

4.7. Right-Tailed Anderson–Darling Estimation (RTADE)

The right-tailed Anderson–Darling estimates $\hat{\lambda}_{RTADE}$, $\hat{k}_{RTADE}$, and $\hat{\theta}_{RTADE}$ of the OLiP distribution parameters λ, k, and θ are obtained by minimizing the function $R(\lambda, k, \theta)$ with respect to λ, k, and θ

$$R(\lambda, k, \theta) = \underset{(\lambda,k,\theta)}{\arg\min} \left\{ \frac{n}{2} - 2\sum_{i=1}^{n} F(x_{i:n}|\lambda, k, \theta) - \frac{1}{n} \sum_{i=1}^{n} (2i-1) \ln \left[1 - F(x_{n+1-i:n}|\lambda, k, \theta) \right] \right\}. \quad (45)$$

The following set of nonlinear equations can be solved to obtain the estimates:

$$
\begin{aligned}
-2\sum_{i=1}^{n}\frac{\Delta_1(x_{i:n}|\lambda,k,\theta)}{F(x_{i:n}|\lambda,k,\theta)}+\frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\frac{\Delta_1(x_{n+1-i:n}|\lambda,k,\theta)}{1-F(x_{n+1-i:n}|\lambda,k,\theta)}\right]&=0\\
-2\sum_{i=1}^{n}\frac{\Delta_2(x_{i:n}|\lambda,k,\theta)}{F(x_{i:n}|\lambda,k,\theta)}+\frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\frac{\Delta_2(x_{n+1-i:n}|\lambda,k,\theta)}{1-F(x_{n+1-i:n}|\lambda,k,\theta)}\right]&=0,\\
-2\sum_{i=1}^{n}\frac{\Delta_3(x_{i:n}|\lambda,k,\theta)}{F(x_{i:n}|\lambda,k,\theta)}+\frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\frac{\Delta_3(x_{n+1-i:n}|\lambda,k,\theta)}{1-F(x_{n+1-i:n}|\lambda,k,\theta)}\right]&=0,
\end{aligned}
\tag{46}
$$

where $\Delta_1(.|\lambda,k,\theta)$, $\Delta_2(.|\lambda,k,\theta)$, and $\Delta_3(x.|\lambda,k,\theta)$ are as defined in (38). The estimates given in (33), (9), (37), (40), (42), (44), and (46) are obtained using the **optim()** function in R with the Newton–Raphson iterative algorithm.

5. Bayesian Estimation of OLiP Distribution Parameters

This section deals with the Bayesian estimate (BE) of the unknown parameters of the OLiP distribution. For Bayesian parameter estimation, many loss functions, including squared error, LINEX, and generalized entropy loss functions, can be taken into consideration by Albert [29] and Mood [32]. We can consider applying independent gamma priors for the variables λ, k, and θ with pdfs in the parameter prior distributions of OLiP.

$$
\begin{aligned}
\pi_1(\lambda) &\propto \lambda^{s_1-1}e^{-q_1\lambda} && \lambda>0, s_1>0, q_1>0,\\
\pi_2(k) &\propto k^{s_2-1}e^{-q_2k} && k>0, s_2>0, q_2>0,\\
\pi_3(\theta) &\propto \theta^{s_3-1}e^{-q_3\theta} && \theta>0, s_3>0, q_3>0,
\end{aligned}
\tag{47}
$$

where the hyper-parameters $s_j, q_j, j=1,2,3$ are selected to reflect the prior knowledge about the unknown parameters. The joint prior for $\phi=(\lambda,k,\theta)$ is given by

$$
\begin{aligned}
\pi(\phi) &= \pi_1(\lambda)\pi_2(k)\pi_3(\theta)\\
\pi(\phi) &\propto \lambda^{s_1-1}k^{s_2-1}\theta^{s_3-1}e^{\{-q_1\lambda-q_2k-q_3\theta\}}.
\end{aligned}
\tag{48}
$$

The corresponding posterior density given the observed data $\mathbf{x}=(x_1,x_2,\ldots,x_n)$ is given by:

$$
\pi(\phi\mid\mathbf{x})=\frac{\pi(\phi)\ell(\phi)}{\int_\phi \pi(\phi)\ell(\phi)d\phi},
$$

Consequently, the posterior density function is denoted by:

$$
\pi(\phi\mid\mathbf{x}) \quad\propto\quad \left(\frac{\lambda^{(2n+s_1-1)}k^{(n+s_2-1)}}{(1+\lambda)^n\theta^{(2nk-s_3+1)}}\right)e^{\{-q_1\lambda-q_3k-q_3\theta\}}\prod_{i=1}^{n}x^{2k-1}e^{-\lambda\left\{\left(\frac{x}{\theta}\right)^k-1\right\}}. \tag{49}
$$

Given any function, such as $l(\phi)$ under the squared error loss (SEL) function, the Bayes estimator is given by

$$
\hat{\phi}_{BE_{SEL}}=E[l(\phi)|\mathbf{x}]=\int_\phi l(\phi)\pi(\phi|x)d\phi. \tag{50}
$$

The SEL impacts underestimation and overestimation equally because it has an asymmetric loss function. In many actual situations, both underestimation and overestimation can have serious implications. A proposed LINEX loss can be made in certain instances as an alternative to the SE loss given by

$$
\left(l(\phi),\hat{l}(\phi)\right)=e^{\{\hat{l}(\phi)-l(\phi)\}}-v\left(\hat{l}(\phi)-l(\phi)\right)-1.
$$

where $v \neq 0$ is a shape parameter. Here, $v > 1$ suggests that an overestimation is more serious than an underestimation, and vice versa for $v < 0$. Further, v approaching zero replicates the SE loss function itself. One may refer to Varian [33] and Doostparast et al. [34] for more details in this regard. The BE of $l(\phi)$ under this loss can be derived as

$$\hat{\phi}_{BE_{LINEX}} = E\left[e^{\{-vl(\phi)\}}|\mathbf{x}\right] = -\frac{1}{v}\log\left[\int_{\phi} e^{\{-vl(\phi)\}}\pi(\phi|x)d\phi\right]. \tag{51}$$

Additionally, we take into account the general entropy loss (GEL) function suggested by Calabria and Pulcini [35], which is defined as follows:

$$\left(l(\phi),\hat{l}(\phi)\right) = \left(\frac{\hat{l}(\phi)}{l(\phi)}\right)^{\tau} - \tau\log\left(\frac{\hat{l}(\phi)}{l(\phi)}\right) - 1,$$

where the shape parameter $\tau \neq 0$ denotes a departure from symmetry. It views overestimation as more significant than underestimation when $\tau > 0$ and the opposite is true when $\tau < 0$. Given below is the Bayes estimator with regard to the GE loss function.

$$\hat{\phi}_{BE_{GEL}} = \left[E\left((l(\phi))^{-\tau}|\mathbf{x}\right)\right]^{-1/\tau} = \left[\int_{\phi}(l(\phi))^{-\tau}\pi(\phi|x)d\phi\right]^{-1/\tau}. \tag{52}$$

The estimations produced by (50), (51), and (52) can be seen to not be able to be transformed into closed-form expressions. We then use the Markov chain Monte Carlo (MCMC) approach to generate posterior samples and arrive at suitable BEs. A general simulation technique for computing posterior quantities of interest and sampling from posterior distributions is the MCMC technique (read Ravenzwaaij et al. [36] and Albert [29] for further details on MCMC). In fact, using a kernel estimate of the posterior distribution and the MCMC samples, it is possible to properly quantify the posterior uncertainty with regard to the parameter ϕ.

Finally, part of the initial samples can be eliminated (burn-in) from the random samples of size M derived from the posterior density, and the remaining samples can then be used to calculate Bayes estimates. Using MCMC under the SEL, LINEX, and GEL functions, the BEs of $\phi^{(i)} = \left(\lambda^{(i)},k^{(i)},\theta^{(i)}\right)$ can be calculated as follows:

$$\hat{\phi}_{BE_{SEL}} = \frac{1}{M-l_B}\sum_{i=l_B}^{M}\phi^{(i)}, \tag{53}$$

$$\hat{\phi}_{BE_{LINEX}} = -\frac{1}{v}\log\left[\frac{1}{M-l_B}\sum_{i=l_B}^{M}e^{\left\{-v\phi^{(i)}\right\}}\right], \tag{54}$$

$$\hat{\phi}_{BE_{GEL}} = \left[\frac{1}{M-l_B}\sum_{i=l_B}^{M}\left(\phi^{(i)}\right)^{-\tau}\right]^{-1/\tau}, \tag{55}$$

where l_B represents the number of burn-in samples.

6. Single Acceptance Sampling Plans

Assume that a product's lifetime is based on the OLiP distribution, which has the parameters (λ,k,θ) stated in Equation (6), and that the producer's claimed industry standard for the lifetime of units is represented by M_0. The main goal is to determine if the proposed lot should be accepted or rejected based on the fact that the actual median life cycle of the units, m, is longer than the recommended lifetime, M_0. It is important to remember that it is standard procedure in lifetime testing to end the test by the time indicated by T_0 and count the number of failures.

Singh et al. [37] provided us with some guidelines on how to accept the proposed lot based on the evidence that $M \geq M_0$, given a probability of at least α^* (consumer's risk), using a single acceptance sampling plan. The experiment is run for $T_0 = M_0$ units of time, a multiple of the claimed median lifetime with any positive constant a. These are the actions:

1. Take n units at random from the proposed lot as a sample.
2. Run the following test for T_0 units of time.
   Accept the entire lot if c or fewer units (the acceptance number) fail throughout the experiment; else, reject the entire lot.

Be aware that the proposed sampling plan is given by the equation below and that the chance of accepting a lot considers suitably large lots to help with the application of the binomial distribution.

$$L(p) = \sum_{i=0}^{c} \binom{n}{i} p^i (1-p)^{n-i}, \quad i = 1, 2, \ldots, n, \tag{56}$$

where p is defined as $p = F_{OLiP}(T_0; \lambda, k, \theta)$, according to Equation (6). The sampling plan's operating characteristic function, or the acceptance probability of the lot as a function of the failure probability, is represented by the function $L(p)$. Using $T_0 = aM_0$, further, p_0 can be written as follows:

$$p_0 = F_{OLiP}(T_0 = aM_0; \lambda, k, \theta) = 1 - \frac{\lambda \left(\frac{T_0}{\theta}\right)^k + 1}{(1+\lambda)} \mathrm{e}^{-\lambda\left\{\left(\frac{T_0}{\theta}\right)^k - 1\right\}}. \tag{57}$$

Now, the problem is to determine for given values of $\alpha^*(0 < \alpha^* < 1), kM_0$, and c, the smallest positive integer n such that

$$L(p_0) = \sum_{i=0}^{c} \binom{n}{i} p_0^i (1-p_0)^{n-i} \leq 1 - \alpha^*, \tag{58}$$

where p_0 is given by Equation (57). The minimum values of n satisfying the inequality (58) and its corresponding operating characteristic probability are determined and shown in Tables 1–4 for the following assumed parameters:

1. The consumer's risk α^* is given as $0.30, 0.60$, and 0.95.
2. The acceptance number c is given as $0, 2, 4, 8$, and 10.
3. The constant a is assumed to be $0.10, 0.25, 0.50$, and 0.75. If $a = 1$, thus T_0 is the median life time $M_0 = 0.5 \quad \forall \quad (\lambda, k, \theta)$.
4. The parameters (λ, k, θ) of the OLiP distribution are assumed as:

$$\lambda = (0.15, 0.25, 0.30, 0.50) \quad \& \quad k = (0.20, 0.30, 0.40, 0.50) \quad \& \quad \theta = 0.5.$$

From the results obtained in Tables 1–3, we notice that:

- With increasing α^* and c, the required sample size n increases and $L(p_0)$ decreases.
- With increasing a, the required sample size n decreases and $L(p_0)$ increases.
- With increasing λ and fixed k, the required sample size n increases and $L(p_0)$ decreases.
- With increasing k and fixed λ, the required sample size n increases and $L(p_0)$ decreases.

Finally, for all results we have obtained, we checked that $L(p_0) \leq 1 - \alpha^*$. Moreover, when $a = 1$, we have $p_0 = 0.5$ as $T_0 = M_0$ and hence all results $(n, L(p_0))$ for any vector of parameter (λ, k, θ) are the same.

Table 1. SASPs for OLiP distribution with parameters $k = 0.20$ and $\theta = 0.5$ and different values of λ.

α^*	c	a = 0.10		a = 0.25		a = 0.50		a = 0.75		a = 1	
		n	$L(p_0)$	n	$L(p_0)$	n	$L(p_0)$	n	$L(p_0)$	n	$L(p_0)$
$\lambda = 0.15$											
0.3	0	1	1.00000	1	1.00000	1	1.00000	1	1.00000	1	1.00000
	2	7	0.77621	6	0.77887	5	0.81397	4	0.89188	4	0.87500
	4	13	0.76870	11	0.77123	9	0.81711	8	0.81031	8	0.77344
	8	25	0.78632	21	0.78889	18	0.78665	16	0.75800	15	0.78802
	10	32	0.76146	27	0.75451	23	0.75429	19	0.81809	19	0.75966
0.6	0	5	0.26519	4	0.28639	3	0.35299	3	0.27416	2	0.50000
	2	13	0.29732	11	0.28186	9	0.30325	8	0.26709	7	0.34375
	4	22	0.25067	18	0.25979	15	0.26432	12	0.33005	12	0.27441
	8	37	0.27468	31	0.25731	26	0.25397	22	0.25670	21	0.25172
	10	45	0.26454	37	0.27098	31	0.26938	26	0.28741	25	0.27063
0.95	0	10	0.05047	8	0.05406	6	0.07403	5	0.07517	5	0.06250
	2	21	0.05022	17	0.05253	14	0.05319	11	0.07334	11	0.05469
	4	30	0.05742	25	0.05100	20	0.06303	17	0.05694	16	0.05923
	8	48	0.05605	39	0.05989	33	0.05022	27	0.06216	26	0.05388
	10	57	0.05281	46	0.06056	38	0.06282	32	0.06123	30	0.06802
$\lambda = 0.30$											
0.3	0	1	1.00000	1	1.00000	1	1.00000	1	1.00000	1	1.00000
	2	7	0.79279	6	0.78878	5	0.81826	4	0.89246	4	0.87500
	4	13	0.79167	11	0.78502	9	0.82291	8	0.81158	8	0.77344
	8	26	0.77873	22	0.75731	18	0.79573	16	0.76010	15	0.78802
	10	33	0.76316	27	0.77690	23	0.76541	19	0.82002	19	0.75966
0.6	0	5	0.27954	4	0.29509	3	0.35737	3	0.27506	2	0.50000
	2	14	0.26784	11	0.29703	9	0.31082	8	0.26862	7	0.34375
	4	22	0.28229	18	0.27875	15	0.27361	12	0.33215	12	0.27441
	8	39	0.25313	31	0.28247	26	0.26611	22	0.25925	21	0.25172
	10	47	0.25310	38	0.26165	31	0.28309	26	0.29036	25	0.27063
0.95	0	10	0.05682	8	0.05797	6	0.07635	5	0.07566	5	0.06250
	2	21	0.06019	17	0.05851	14	0.05610	11	0.07410	11	0.05469
	4	31	0.05865	25	0.05829	20	0.06709	17	0.05772	16	0.05923
	8	50	0.05397	40	0.05778	33	0.05467	27	0.06323	26	0.05388
	10	59	0.05351	47	0.06018	39	0.05417	32	0.06239	30	0.06802

Table 2. SASPs for OLiP distribution with parameters $k = 0.30$ and $\theta = 0.5$ and different values of λ.

α^*	c	a = 0.10		a = 0.25		a = 0.50		a = 0.75		a = 1	
		n	$L(p_0)$	n	$L(p_0)$	n	$L(p_0)$	n	$L(p_0)$	n	$L(p_0)$
$\lambda = 0.15$											
0.3	0	2	0.79910	1	1.00000	1	1.00000	1	1.00000	1	1.00000
	2	9	0.79493	7	0.79149	5	0.86157	4	0.89963	4	0.87500
	4	18	0.75514	13	0.78988	10	0.80705	8	0.82714	8	0.77344
	8	35	0.76897	26	0.77609	20	0.78240	16	0.78580	15	0.78802
	10	44	0.76647	33	0.76008	25	0.78077	20	0.77903	19	0.75966
0.6	0	7	0.26037	5	0.27837	4	0.25946	3	0.28651	3	0.25000
	2	19	0.26824	14	0.26584	10	0.30950	8	0.28839	7	0.34375
	4	31	0.25136	22	0.27968	17	0.25543	13	0.26907	12	0.27441
	8	53	0.25658	38	0.28106	29	0.26327	22	0.29266	21	0.25172
	10	64	0.25448	46	0.27784	35	0.26216	27	0.26821	25	0.27063
0.95	0	14	0.05417	10	0.05629	7	0.06732	5	0.08209	5	0.06250
	2	30	0.05085	21	0.05934	16	0.05100	12	0.05375	11	0.05469
	4	43	0.05643	31	0.05761	23	0.05708	17	0.06839	16	0.05923
	8	69	0.05283	50	0.05273	37	0.05411	28	0.05753	26	0.05388
	10	81	0.05430	59	0.05216	44	0.05038	33	0.05923	30	0.06802

Table 2. *Cont.*

α^*	c	a = 0.10		a = 0.25		a = 0.50		a = 0.75		a = 1	
		n	$L(p_0)$	n	$L(p_0)$	n	$L(p_0)$	n	$L(p_0)$	n	$L(p_0)$
$\lambda = 0.30$											
0.3	0	2	0.81308	1	1.00000	1	1.00000	1	1.00000	1	1.00000
	2	10	0.77215	7	0.80838	6	0.75452	4	0.90046	4	0.87500
	4	19	0.76429	14	0.75839	10	0.81697	8	0.82895	8	0.77344
	8	38	0.75562	27	0.77279	20	0.79750	16	0.78877	15	0.78802
	10	47	0.76987	34	0.76604	25	0.79771	20	0.78241	19	0.75966
0.6	0	7	0.28894	5	0.29398	4	0.26636	3	0.28790	2	0.50000
	2	20	0.28253	14	0.29287	10	0.32212	8	0.29080	7	0.34375
	4	33	0.25997	23	0.27314	17	0.27066	13	0.27209	12	0.27441
	8	57	0.25660	40	0.26437	29	0.28379	22	0.29679	21	0.25172
	10	69	0.25151	48	0.27113	35	0.28475	27	0.27261	25	0.27063
0.95	0	15	0.05519	10	0.06364	7	0.07095	5	0.08288	5	0.06250
	2	32	0.05354	22	0.05730	16	0.05573	12	0.05469	11	0.05469
	4	47	0.05191	32	0.06025	23	0.06343	17	0.06976	16	0.05923
	8	74	0.05475	52	0.05264	37	0.06205	28	0.05909	26	0.05388
	10	88	0.05058	61	0.05475	44	0.05864	33	0.06097	30	0.06802

Table 3. SASPs for OLiP distribution with parameters $k = 0.40$ and $\theta = 0.5$ and different values of λ.

α^*	c	a = 0.10		a = 0.25		a = 0.50		a = 0.75		a = 1	
		n	$L(p_0)$	n	$L(p_0)$	n	$L(p_0)$	n	$L(p_0)$	n	$L(p_0)$
$\lambda = 0.25$											
0.3	0	3	0.76119	2	0.79199	1	1.00000	1	1.00000	1	1.00000
	2	14	0.77509	9	0.77901	6	0.81414	5	0.75567	4	0.87500
	4	27	0.76882	17	0.77361	12	0.75153	8	0.84429	8	0.77344
	8	55	0.75395	34	0.76504	23	0.76293	16	0.81375	15	0.78802
	10	69	0.75482	43	0.75508	29	0.75058	20	0.81082	19	0.75966
0.6	0	11	0.25554	6	0.31159	4	0.31903	3	0.30016	3	0.25000
	2	30	0.26615	18	0.28173	12	0.27158	8	0.31231	7	0.34375
	4	49	0.25054	29	0.27902	19	0.27906	13	0.29919	12	0.27441
	8	84	0.25338	51	0.26042	33	0.27249	23	0.26968	21	0.25172
	10	101	0.25630	62	0.25028	40	0.26645	28	0.25611	25	0.27063
0.95	0	22	0.05697	13	0.06090	8	0.06955	5	0.09010	5	0.06250
	2	48	0.05082	29	0.05024	18	0.05878	12	0.06346	11	0.05469
	4	70	0.05054	42	0.05232	27	0.05169	18	0.05754	16	0.05923
	8	110	0.05308	67	0.05016	43	0.05115	29	0.05521	26	0.05388
	10	130	0.05136	78	0.05512	50	0.05771	34	0.05941	30	0.06802
$\lambda = 0.50$											
0.3	0	4	0.77214	2	0.82502	1	1.00000	1	1.00000	1	1.00000
	2	22	0.75144	11	0.75169	6	0.83976	5	0.76140	4	0.87500
	4	42	0.75192	20	0.77122	12	0.79442	8	0.84987	8	0.77344
	8	84	0.75434	40	0.76710	24	0.77786	16	0.82273	15	0.78802
	10	106	0.75085	51	0.75011	30	0.77865	21	0.75693	19	0.75966
0.6	0	17	0.25179	8	0.26016	4	0.34618	3	0.30488	2	0.50000
	2	47	0.25659	22	0.26239	13	0.25802	8	0.32066	7	0.34375
	4	75	0.25871	35	0.26637	20	0.28916	13	0.30979	12	0.27441
	8	130	0.25307	61	0.25515	35	0.27671	23	0.28328	21	0.25172
	10	157	0.25084	73	0.26411	43	0.25314	28	0.27079	25	0.27063
0.95	0	35	0.05336	16	0.05584	9	0.05908	6	0.05132	5	0.06250
	2	75	0.05016	34	0.05560	19	0.06224	12	0.06706	11	0.05469
	4	109	0.05058	50	0.05395	28	0.06191	18	0.06171	16	0.05923
	8	172	0.05112	80	0.05074	46	0.05019	29	0.06044	26	0.05388
	10	202	0.05173	94	0.05146	54	0.05186	34	0.06544	30	0.06802

Table 4. SASPs for OLiP distribution with parameters $k = 0.50$ and $\theta = 0.5$ and different values of λ.

α^*	c	a = 0.10		a = 0.25		a = 0.50		a = 0.75		a = 1	
		n	$L(p_0)$	n	$L(p_0)$	n	$L(p_0)$	n	$L(p_0)$	n	$L(p_0)$
$\lambda = 0.25$											
0.3	0	4	0.77994	2	0.84197	1	1.00000	1	1.00000	1	1.00000
	2	22	0.76905	12	0.75407	7	0.78462	5	0.77116	4	0.87500
	4	43	0.76064	22	0.77088	13	0.78039	9	0.75762	8	0.77344
	8	87	0.75630	44	0.76927	26	0.76214	17	0.76874	15	0.78802
	10	110	0.75118	56	0.75546	32	0.77997	21	0.77768	19	0.75966
0.6	0	17	0.26565	9	0.25256	5	0.27234	3	0.31313	2	0.50000
	2	49	0.25425	24	0.27226	14	0.25556	8	0.33533	7	0.34375
	4	78	0.25776	39	0.26133	22	0.26630	13	0.32855	12	0.27441
	8	135	0.25344	67	0.26496	38	0.26332	23	0.30761	21	0.25172
	10	163	0.25154	81	0.26237	46	0.25842	28	0.29721	25	0.27063
0.95	0	37	0.05066	18	0.05370	10	0.05358	6	0.05487	5	0.06250
	2	78	0.05001	38	0.05407	21	0.05505	12	0.07366	11	0.05469
	4	113	0.05132	56	0.05141	31	0.05246	18	0.06948	16	0.05923
	8	179	0.05058	89	0.05005	49	0.05490	30	0.05285	26	0.05388
	10	210	0.05157	104	0.05302	58	0.05308	35	0.05924	30	0.06802
$\lambda = 0.50$											
0.3	0	11	0.76188	3	0.77837	1	1.00000	1	1.00000	1	1.00000
	2	65	0.75374	15	0.77713	7	0.82436	5	0.77815	4	0.87500
	4	126	0.75438	29	0.77171	14	0.78406	9	0.76750	8	0.77344
	8	256	0.75151	59	0.76083	28	0.77160	17	0.78207	15	0.78802
	10	322	0.75297	74	0.76305	35	0.77353	21	0.79217	19	0.75966
0.6	0	51	0.25671	12	0.25208	5	0.30989	3	0.31921	2	0.50000
	2	146	0.25064	33	0.25609	15	0.27005	8	0.34621	7	0.34375
	4	233	0.25238	53	0.25194	24	0.26901	14	0.26245	12	0.27441
	8	402	0.25055	91	0.25367	42	0.25194	24	0.26569	21	0.25172
	10	484	0.25174	110	0.25032	50	0.26764	29	0.26295	25	0.27063
0.95	0	111	0.05021	24	0.05606	11	0.05346	6	0.05757	5	0.06250
	2	233	0.05050	52	0.05127	23	0.05605	13	0.05223	11	0.05469
	4	339	0.05049	76	0.05040	34	0.05311	19	0.05346	16	0.05923
	8	535	0.05053	120	0.05094	54	0.05341	30	0.05940	26	0.05388
	10	629	0.05045	141	0.05145	64	0.05084	36	0.05154	30	0.06802

7. Numerical Computations and Real Data Analysis

We introduce the numerical calculations for the underlying distribution in this section, including simulation research and the use of real data sets.

7.1. Simulation Study

In order to evaluate the efficacy of the estimation methods (non-BEs and BEs) outlined in the previous part, we simulate data for the OLiP in this subsection. From the OLiP distribution, we produce 1000 data by using the initial parameter values:

- $\lambda = 0.50, k = 0.75$, and $\theta = 1.5$.
- $\lambda = 0.75, k = 0.75$, and $\theta = 1.5$.
- $\lambda = 0.50, k = 1.25$ and $\theta = 1.5$.
- $\lambda = 0.75, k = 1.25$ and $\theta = 1.5$.
- $\lambda = 1.25, k = 1.50$ and $\theta = 2.5$.
- $\lambda = 1.50, k = 1.50$ and $\theta = 2.5$.
- $\lambda = 1.25, k = 2.00$ and $\theta = 2.5$.
- $\lambda = 1.50, k = 2.00$, and $\theta = 2.5$,

and sample sizes $n = 25, 50, 75, 100$. For each estimate $\hat{\phi} = (\hat{\lambda}, \hat{k}, \hat{\theta})$, we compute the bias and root mean squared error (RMSE), respectively, as

$$Bias(\hat{\phi}) = \frac{1}{B}\sum_{i=1}^{B}(\hat{\phi}_i - \phi),$$

and

$$RMSE(\hat{\phi}) = \sqrt{\frac{1}{B}\sum_{i=1}^{B}(\hat{\phi}_i - \phi)^2}.$$

We used the Newton–Raphson algorithm to find the desired estimates for the non-Bayesian procedure. For the Bayesian method, BEs were generated using MCMC and the MH algorithm with an informative prior. We made the assumption that all gamma distribution hyperparameters were equal to twice the parameter values while calculating the informative prior. The desired estimations were then calculated using these data as inputs. The MH algorithm was applied by the MLEs while taking into account the initial estimate values. Out of the total 10,000 samples created from the posterior density and subsequently obtained BEs under various loss functions, SEL, LINEX at $v = -1.5, 1.5$, and finally GEL at $\tau = -0.5, 0.5$, 2000 burn-in samples were ultimately deleted. We calculated the bias and RMSE for each strategy.

The following conclusions can be drawn from the simulation results in Tables 5–12:

1. In most cases, as the sample size increases, all estimators' bias and RMSE values fall, demonstrating improved accuracy in the model parameter estimation.
2. The least biased parameters across all the parameters and various sample sizes are LSE, WLSE, ADE, and RTADE.
3. For all sample sizes, the estimators' biases are positive.

Table 5. Average estimated biases and RMSEs of different estimation methods for OLiP distribution at different sample sizes n and parameter values of $\lambda = 0.5,\ k = 0.75$, and $\theta = 1.5$.

Method		n = 25		n = 50		n = 75		n = 100	
		Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
MLE	λ	0.26398	0.47037	0.13337	0.25212	0.08191	0.17391	0.06277	0.13892
	k	0.01850	0.17848	0.01406	0.12237	0.00650	0.09745	0.00564	0.08215
	θ	0.47292	0.64996	0.24692	0.34590	0.16615	0.22949	0.13022	0.18184
MPSE	λ	0.19417	0.52644	0.09979	0.26472	0.06173	0.18035	0.04958	0.14316
	k	0.06363	0.19233	0.04307	0.13196	0.02958	0.10327	0.02520	0.08667
	θ	0.01996	0.57775	0.01237	0.27534	0.01030	0.17209	0.01400	0.13451
LSE	λ	0.09609	0.87123	0.01985	0.35208	0.00710	0.26960	0.01013	0.22639
	k	0.01156	0.21845	0.00551	0.15867	0.00291	0.12859	0.00324	0.11018
	θ	0.30644	0.88078	0.20736	0.67475	0.17959	0.55052	0.14416	0.46822
WLSE	λ	0.06947	0.56218	0.02303	0.27159	0.00510	0.19071	0.00423	0.15130
	k	0.00217	0.20713	0.00170	0.14488	0.00574	0.11281	0.00437	0.09451
	θ	0.23341	0.75304	0.10867	0.45168	0.06934	0.29304	0.04051	0.20738
CVME	λ	0.08777	0.76741	0.02601	0.34048	0.00161	0.26352	0.00508	0.22106
	k	0.03266	0.23319	0.01437	0.16288	0.01569	0.13139	0.01255	0.11182
	θ	0.15252	0.85650	0.11638	0.64930	0.11700	0.53057	0.09399	0.44559
ADE	λ	0.09905	0.54377	0.03088	0.26605	0.00436	0.18757	0.00061	0.15072
	k	0.00458	0.19564	0.00401	0.13762	0.00902	0.10994	0.00750	0.09233
	θ	0.06788	0.69838	0.05975	0.45300	0.05503	0.30632	0.03977	0.23216
RTADE	λ	0.09553	0.62632	0.03758	0.37271	0.00079	0.30128	0.00644	0.26335
	k	0.01442	0.21273	0.00102	0.14698	0.00754	0.12180	0.00711	0.10560
	θ	0.15678	0.98024	0.12840	0.77904	0.14923	0.67435	0.13323	0.60210
BE_{SEL}	λ	0.01219	0.22120	0.05615	0.15999	0.07738	0.13774	0.07678	0.12748
	k	0.41285	0.43089	0.34868	0.35925	0.30080	0.30940	0.26236	0.26939
	θ	1.41729	1.41936	1.35751	1.35953	1.29651	1.29944	1.22700	1.23079
BE_{Linex1}	λ	0.00520	0.22991	0.05372	0.16076	0.07592	0.13750	0.07567	0.12715
	k	0.40646	0.42512	0.34411	0.35492	0.29747	0.30621	0.25994	0.26708
	θ	1.41289	1.41533	1.34968	1.35204	1.28556	1.28886	1.21326	1.21749
BE_{Linex2}	λ	0.01933	0.21294	0.05861	0.15926	0.07885	0.13800	0.07790	0.12782
	k	0.41924	0.43669	0.35326	0.36359	0.30413	0.31260	0.26476	0.27171
	θ	1.42138	1.42314	1.36482	1.36656	1.30679	1.30940	1.23999	1.24339

Table 5. *Cont.*

Method		n = 25		n = 50		n = 75		n = 100	
		Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
BE_{GEL1}	λ	0.02535	0.21449	0.06148	0.16042	0.08079	0.13914	0.07941	0.12881
	k	0.43776	0.45643	0.36153	0.37225	0.30864	0.31725	0.26743	0.27442
	θ	1.45398	1.45517	1.40953	1.41093	1.35236	1.35495	1.27997	1.28356
BE_{GEL2}	λ	0.05546	0.20576	0.07285	0.16230	0.08780	0.14235	0.08475	0.13169
	k	0.49858	0.52018	0.38955	0.40115	0.32482	0.33356	0.27772	0.28465
	θ	1.49499	1.49522	1.48519	1.48555	1.44908	1.45086	1.38057	1.38362

Table 6. Average estimated biases and RMSEs of different estimation methods for OLiP distribution at different sample sizes n and parameter values of $\lambda = 0.75,\ k = 0.75$, and $\theta = 1.5$.

Method		n = 25		n = 50		n = 75		n = 100	
		Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
MLE	λ	0.30859	0.66631	0.15248	0.35629	0.08929	0.24383	0.06825	0.19576
	k	0.00910	0.20923	0.00995	0.14385	0.00280	0.11508	0.00319	0.09692
	θ	0.25305	0.35129	0.13060	0.18439	0.08737	0.12137	0.06857	0.09619
MPSE	λ	0.32137	0.88629	0.15782	0.40676	0.09542	0.26853	0.07594	0.21298
	k	0.06598	0.22893	0.04978	0.15544	0.03418	0.12199	0.02954	0.10252
	θ	0.01055	0.30994	0.00941	0.13937	0.00616	0.08887	0.00801	0.07019
LSE	λ	0.27046	1.78544	0.07844	0.52681	0.02593	0.35781	0.01843	0.28720
	k	0.00151	0.26813	0.00251	0.19467	0.00534	0.15663	0.00297	0.13133
	θ	0.22696	0.61159	0.11629	0.40696	0.08729	0.30084	0.05844	0.22251
WLSE	λ	0.20526	1.41174	0.06583	0.39817	0.02558	0.26672	0.01868	0.21310
	k	0.00408	0.25362	0.00200	0.17367	0.00632	0.13406	0.00413	0.11161
	θ	0.14112	0.46891	0.04445	0.22437	0.02792	0.13562	0.01616	0.10120
CVME	λ	0.20576	1.47393	0.06949	0.49703	0.02263	0.34752	0.01645	0.28078
	k	0.04837	0.28548	0.02015	0.19881	0.02006	0.15925	0.01381	0.13274
	θ	0.12842	0.57645	0.06286	0.38309	0.05160	0.28440	0.03186	0.20919
ADE	λ	0.20115	1.22871	0.06686	0.38554	0.02115	0.25905	0.01303	0.20851
	k	0.01057	0.23646	0.00518	0.16459	0.01034	0.13020	0.00782	0.10869
	θ	0.04564	0.42518	0.02278	0.23095	0.02174	0.14885	0.01518	0.10793
RTADE	λ	0.18113	1.11262	0.06812	0.49333	0.02427	0.36782	0.01494	0.31141
	k	0.02522	0.25384	0.00622	0.17620	0.01141	0.14481	0.00920	0.12416
	θ	0.16651	0.69090	0.09766	0.50944	0.07695	0.39031	0.06047	0.32979
BE_{SEL}	λ	0.03735	0.35146	0.07638	0.25787	0.09845	0.20857	0.09333	0.18711
	k	0.34915	0.37949	0.27869	0.29641	0.22741	0.24281	0.19032	0.20277
	θ	1.38094	1.38363	1.26767	1.27096	1.15022	1.15550	1.02966	1.03661
BE_{Linex1}	λ	0.02573	0.36991	0.07282	0.25976	0.09636	0.20851	0.09173	0.18681
	k	0.34385	0.37508	0.27538	0.29350	0.22516	0.24080	0.18872	0.20135
	θ	1.37557	1.37858	1.25660	1.26023	1.13555	1.14117	1.01329	1.02053
BE_{Linex2}	λ	0.04937	0.33345	0.07998	0.25598	0.10055	0.20867	0.09492	0.18743
	k	0.35445	0.38394	0.28200	0.29933	0.22966	0.24483	0.19192	0.20419
	θ	1.38601	1.38843	1.27817	1.28117	1.16427	1.16924	1.04547	1.05215
BE_{GEL1}	λ	0.05063	0.34179	0.08123	0.25760	0.10155	0.20951	0.09573	0.18810
	k	0.36524	0.39569	0.28610	0.30352	0.23184	0.24703	0.19320	0.20545
	θ	1.42149	1.42357	1.31943	1.32260	1.19656	1.20212	1.06751	1.07490
BE_{GEL2}	λ	0.08049	0.32484	0.09126	0.25744	0.10785	0.21157	0.10059	0.19020
	k	0.40232	0.43448	0.30147	0.31852	0.24082	0.25569	0.19901	0.21089
	θ	1.47952	1.48046	1.41655	1.41920	1.28975	1.29596	1.14435	1.15276

Table 7. Average estimated biases and RMSEs of different estimation methods for OLiP distribution at different sample sizes n and parameter values of $\lambda = 0.5,\ k = 1.25$, and $\theta = 1.5$.

Method		n = 25		n = 50		n = 75		n = 100	
		Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
MLE	λ	0.26404	0.47046	0.13339	0.25213	0.08193	0.17390	0.06277	0.13891
	k	0.03085	0.29748	0.02345	0.20396	0.01085	0.16241	0.00941	0.13691
	θ	0.25856	0.34363	0.14018	0.19185	0.09598	0.13043	0.07575	0.10440
MPSE	λ	0.18686	0.52595	0.09982	0.26471	0.06173	0.18034	0.04957	0.14316
	k	0.10068	0.32129	0.07182	0.21992	0.04931	0.17211	0.04199	0.14444
	θ	0.05871	0.40042	0.00168	0.16190	0.00397	0.10028	0.00704	0.07845
LSE	λ	0.10126	0.89955	0.02297	0.34788	0.00509	0.26607	0.00894	0.22393
	k	0.01928	0.36399	0.00923	0.26442	0.00482	0.21428	0.00537	0.18359
	θ	0.27519	0.64874	0.17297	0.48574	0.13825	0.38808	0.10779	0.32156
WLSE	λ	0.07163	0.56028	0.02381	0.27022	0.00528	0.19026	0.00423	0.15129
	k	0.00325	0.34559	0.00273	0.24117	0.00956	0.18798	0.00728	0.15751
	θ	0.20228	0.53784	0.08463	0.30644	0.04931	0.19077	0.02801	0.12824
CVME	λ	0.08719	0.68506	0.02855	0.33699	0.00047	0.25979	0.00403	0.21885
	k	0.05433	0.38829	0.02381	0.27128	0.02608	0.21887	0.02083	0.18621
	θ	0.16369	0.59232	0.11239	0.45562	0.09527	0.36225	0.07431	0.29882
ADE	λ	0.10092	0.54222	0.03179	0.26442	0.00467	0.18678	0.00071	0.15040
	k	0.00772	0.32610	0.00661	0.22928	0.01502	0.18322	0.01252	0.15387
	θ	0.08857	0.47764	0.05587	0.31065	0.04211	0.20712	0.02884	0.15157
RTADE	λ	0.10092	0.62229	0.04242	0.36658	0.00468	0.29519	0.00409	0.25896
	k	0.02411	0.35468	0.00175	0.24502	0.01247	0.20292	0.01183	0.17588
	θ	0.18675	0.67454	0.13382	0.53964	0.13287	0.47115	0.11673	0.42370
BE_{SEL}	λ	0.14700	0.34451	0.05650	0.19897	0.02014	0.13858	0.01025	0.11608
	k	0.17767	0.31947	0.10442	0.20871	0.04844	0.15744	0.00790	0.12460
	θ	0.81740	0.84586	0.59798	0.62017	0.45358	0.47524	0.34033	0.36394
BE_{Linex1}	λ	0.14813	0.34658	0.05697	0.19943	0.02045	0.13877	0.01051	0.11619
	k	0.17078	0.31693	0.09954	0.20710	0.04465	0.15666	0.00492	0.12478
	θ	0.79910	0.82838	0.58159	0.60398	0.43992	0.46157	0.32899	0.35274
BE_{Linex2}	λ	0.14587	0.34242	0.05603	0.19851	0.01984	0.13839	0.00999	0.11596
	k	0.18456	0.32216	0.10929	0.21045	0.05222	0.15832	0.01088	0.12451
	θ	0.83522	0.86293	0.61413	0.63614	0.46709	0.48878	0.35157	0.37509
BE_{GEL1}	λ	0.14572	0.34306	0.05573	0.19853	0.01958	0.13838	0.00975	0.11597
	k	0.18424	0.32334	0.10869	0.21079	0.05162	0.15854	0.01030	0.12475
	θ	0.84786	0.87751	0.61751	0.64045	0.46734	0.48962	0.35048	0.37442
BE_{GEL2}	λ	0.14314	0.34015	0.05417	0.19765	0.01845	0.13799	0.00875	0.11576
	k	0.19754	0.33155	0.11727	0.21524	0.05798	0.16097	0.01511	0.12521
	θ	0.91023	0.94266	0.65714	0.68179	0.49511	0.51875	0.37092	0.39557

Table 8. Average estimated biases and RMSEs of different estimation methods for OLiP distribution at different sample sizes n and parameter values of $\lambda = 0.75,\ k = 1.25$, and $\theta = 1.5$.

Method		n = 25		n = 50		n = 75		n = 100	
		Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
MLE	λ	0.30867	0.66640	0.15237	0.35608	0.08931	0.24383	0.06757	0.19579
	k	0.01522	0.34873	0.01653	0.23966	0.00469	0.19181	0.00472	0.16183
	θ	0.14361	0.19488	0.07592	0.10562	0.05133	0.07061	0.04029	0.05610
MPSE	λ	0.32135	0.88587	0.15786	0.40673	0.09543	0.26855	0.07518	0.21283
	k	0.10998	0.38149	0.08300	0.25907	0.05697	0.20330	0.04872	0.17090
	θ	0.01467	0.19373	0.00420	0.08140	0.00309	0.05228	0.00420	0.04124
LSE	λ	0.28259	1.92663	0.07965	0.52520	0.02637	0.35694	0.01648	0.28720
	k	0.00277	0.44657	0.00427	0.32425	0.00887	0.26098	0.00663	0.21978
	θ	0.17338	0.43391	0.08507	0.27833	0.06079	0.20185	0.04033	0.14214

Table 8. *Cont.*

Method		***n* = 25**		***n* = 50**		***n* = 75**		***n* = 100**	
		Bias	**RMSE**	**Bias**	**RMSE**	**Bias**	**RMSE**	**Bias**	**RMSE**
WLSE	λ	0.20108	1.29820	0.06619	0.39764	0.02558	0.26672	0.01744	0.21339
	k	0.00672	0.42274	0.00319	0.28915	0.01052	0.22343	0.00792	0.18672
	θ	0.10689	0.32561	0.03020	0.13965	0.01829	0.08296	0.01102	0.06099
CVME	λ	0.20674	1.45911	0.07007	0.49600	0.02313	0.34653	0.01445	0.28102
	k	0.08106	0.47758	0.03374	0.33164	0.03333	0.26522	0.02471	0.22230
	θ	0.10853	0.39525	0.05132	0.25719	0.03766	0.18552	0.02397	0.13283
ADE	λ	0.19176	0.99906	0.06698	0.38530	0.02122	0.25888	0.01165	0.20889
	k	0.01771	0.39394	0.00861	0.27429	0.01722	0.21696	0.01415	0.18189
	θ	0.04384	0.28652	0.01857	0.15049	0.01494	0.09504	0.01066	0.06615
RTADE	λ	0.18746	1.12300	0.07087	0.48936	0.02554	0.36532	0.01462	0.30947
	k	0.04184	0.42291	0.01024	0.29340	0.01893	0.24119	0.01616	0.20703
	θ	0.14297	0.47809	0.08139	0.34880	0.05941	0.26151	0.04588	0.21819
BE_{SEL}	λ	0.08648	0.44789	0.01738	0.26812	0.05889	0.19683	0.06988	0.17186
	k	0.07153	0.31505	0.00297	0.20744	0.06856	0.18539	0.10770	0.17958
	θ	0.75874	0.78027	0.53114	0.54626	0.38176	0.39758	0.27991	0.29675
BE_{Linex1}	λ	0.08805	0.45069	0.01677	0.26848	0.05850	0.19689	0.06956	0.17183
	k	0.06583	0.31535	0.00678	0.20840	0.07135	0.18683	0.10977	0.18111
	θ	0.74597	0.76754	0.52102	0.53602	0.37427	0.38994	0.27426	0.29101
BE_{Linex2}	λ	0.08491	0.44506	0.01799	0.26775	0.05928	0.19677	0.07020	0.17188
	k	0.07722	0.31488	0.00084	0.20656	0.06578	0.18399	0.10562	0.17807
	θ	0.77129	0.79277	0.54116	0.55641	0.38920	0.40518	0.28553	0.30247
BE_{GEL1}	λ	0.08515	0.44657	0.01814	0.26797	0.05943	0.19689	0.07034	0.17199
	k	0.07638	0.31607	0.00005	0.20720	0.06645	0.18457	0.10618	0.17859
	θ	0.77804	0.80067	0.54212	0.55773	0.38871	0.40488	0.28467	0.30172
BE_{GEL2}	λ	0.08248	0.44393	0.01968	0.26768	0.06051	0.19702	0.07125	0.17227
	k	0.08617	0.31841	0.00612	0.20691	0.06222	0.18302	0.10313	0.17665
	θ	0.81723	0.84235	0.56425	0.58094	0.40265	0.41958	0.29418	0.31169

Table 9. Average estimated biases and RMSEs of different estimation methods for OLiP distribution at different sample sizes n and parameter values of $\lambda = 1.25,\ k = 1.50$, and $\theta = 2.50$.

Method		***n* = 25**		***n* = 50**		***n* = 75**		***n* = 100**	
		Bias	**RMSE**	**Bias**	**RMSE**	**Bias**	**RMSE**	**Bias**	**RMSE**
MLE	λ	0.58392	2.04825	0.23919	0.75051	0.12663	0.45110	0.09306	0.35804
	k	0.01888	0.55553	0.00402	0.38037	0.00859	0.30527	0.00512	0.25783
	θ	0.09587	0.13187	0.04992	0.07013	0.03351	0.04641	0.02623	0.03672
MPSE	λ	1.52837	7.21457	0.38888	1.38987	0.20265	0.54731	0.15406	0.41841
	k	0.16383	0.60572	0.12872	0.41227	0.08863	0.32371	0.07617	0.27260
	θ	0.00261	0.11543	0.00344	0.05231	0.00225	0.03366	0.00282	0.02666
LSE	λ	1.16645	5.02036	0.37339	1.79880	0.14489	0.85579	0.08858	0.59572
	k	0.02083	0.74108	0.00686	0.53354	0.01203	0.42470	0.00820	0.35530
	θ	0.11858	0.34597	0.04132	0.16209	0.02792	0.10183	0.01967	0.07647
WLSE	λ	0.98640	4.68454	0.25422	1.32488	0.09050	0.54032	0.05886	0.41886
	k	0.02102	0.69798	0.00238	0.46835	0.01515	0.36094	0.01182	0.30197
	θ	0.06430	0.24204	0.01382	0.07710	0.00945	0.04923	0.00597	0.03822
CVME	λ	0.90379	4.44090	0.31350	1.77509	0.11306	0.76396	0.07056	0.57665
	k	0.15107	0.78952	0.05394	0.54358	0.05197	0.43054	0.03774	0.35888
	θ	0.06949	0.30603	0.01975	0.14955	0.01393	0.09547	0.00944	0.07294
ADE	λ	0.81006	3.95265	0.24357	1.36289	0.07618	0.51766	0.04630	0.40851
	k	0.03583	0.63995	0.01079	0.44110	0.02566	0.34816	0.02141	0.29227
	θ	0.02499	0.21408	0.00702	0.08560	0.00716	0.05188	0.00583	0.04032

Table 9. *Cont.*

Method		n = 25		n = 50		n = 75		n = 100	
		Bias	**RMSE**	**Bias**	**RMSE**	**Bias**	**RMSE**	**Bias**	**RMSE**
RTADE	λ	0.80808	4.07546	0.22503	1.14825	0.09604	0.61756	0.06385	0.49744
	k	0.08021	0.67930	0.01955	0.46669	0.02827	0.37528	0.02338	0.32030
	θ	0.10239	0.38687	0.03906	0.22604	0.02288	0.14975	0.01603	0.10906
BE_{SEL}	λ	0.33732	1.43260	0.12454	0.69167	0.04443	0.42614	0.02949	0.34645
	k	0.21356	0.51830	0.16473	0.36074	0.10642	0.28928	0.07955	0.24346
BE_{Linex1}	λ	0.34460	1.46243	0.12581	0.69300	0.04511	0.42650	0.02997	0.34663
	k	0.21012	0.51847	0.16296	0.36070	0.10534	0.28931	0.07886	0.24350
	θ	1.06782	1.09677	0.67368	0.69065	0.46071	0.47819	0.32865	0.34461
BE_{Linex2}	λ	0.32968	1.40087	0.12326	0.69031	0.04375	0.42578	0.02902	0.34628
	k	0.21699	0.51817	0.16651	0.36079	0.10750	0.28926	0.08025	0.24341
	θ	1.10922	1.13990	0.69690	0.71464	0.47403	0.49210	0.33646	0.35285
BE_{GEL1}	λ	0.33523	1.42904	0.12371	0.69132	0.04392	0.42603	0.02913	0.34641
	k	0.21608	0.51894	0.16599	0.36101	0.10716	0.28939	0.08002	0.24349
	θ	1.10518	1.13654	0.69196	0.70967	0.47075	0.48873	0.33440	0.35070
BE_{GEL2}	λ	0.33099	1.42183	0.12204	0.69062	0.04291	0.42582	0.02839	0.34633
	k	0.22115	0.52029	0.16850	0.36157	0.10865	0.28962	0.08096	0.24356
	θ	1.13826	1.17307	0.70519	0.72366	0.47746	0.49587	0.33807	0.35462

Table 10. Average estimated biases and RMSEs of different estimation methods for OLiP distribution at different sample sizes n and parameter values of $\lambda = 1.5$, $k = 1.5$, and $\theta = 2.5$.

Method		n = 25		n = 50		n = 75		n = 100	
		Bias	**RMSE**	**Bias**	**RMSE**	**Bias**	**RMSE**	**Bias**	**RMSE**
MLE	λ	1.46136	10.5203	0.32656	1.28209	0.15673	0.59491	0.11324	0.46504
	k	0.03825	0.62805	0.00429	0.42890	0.01597	0.34434	0.01063	0.29080
	θ	0.07475	0.10326	0.03875	0.05458	0.02596	0.03602	0.02030	0.02847
MPSE	λ	3.17054	13.7576	0.72242	4.21258	0.28583	0.76704	0.21216	0.56417
	k	0.17815	0.67787	0.14380	0.46474	0.09933	0.36526	0.08559	0.30777
	θ	0.00130	0.08998	0.00276	0.04042	0.00178	0.02599	0.00219	0.02061
LSE	λ	1.80335	6.81846	0.66134	2.92522	0.26961	1.41089	0.15492	0.91673
	k	0.03537	0.83982	0.00786	0.60520	0.01237	0.48385	0.00842	0.40473
	θ	0.08930	0.27313	0.02855	0.11445	0.01990	0.07409	0.01436	0.05771
WLSE	λ	1.73642	7.29051	0.48155	2.66739	0.15000	0.78717	0.09482	0.58928
	k	0.02912	0.79033	0.00212	0.53160	0.01658	0.41070	0.01315	0.34359
	θ	0.04623	0.18444	0.00981	0.05739	0.00693	0.03757	0.00441	0.02940
CVME	λ	1.37261	5.82908	0.56335	2.77843	0.21258	1.27291	0.12691	0.92837
	k	0.17928	0.89383	0.06118	0.61738	0.05812	0.49034	0.04225	0.40872
	θ	0.05256	0.24572	0.01196	0.10544	0.00917	0.06996	0.00645	0.05524
ADE	λ	1.34734	5.81064	0.42521	2.32886	0.12813	0.75561	0.07802	0.58107
	k	0.04416	0.71981	0.01144	0.49823	0.02819	0.39493	0.02372	0.33168
	θ	0.01503	0.15658	0.00444	0.06035	0.00521	0.03915	0.00436	0.03088
RTADE	λ	1.25583	5.63175	0.38182	1.97515	0.15329	0.84204	0.10047	0.64540
	k	0.09519	0.76283	0.02124	0.52275	0.03051	0.42001	0.02510	0.35779
	θ	0.07372	0.30156	0.02408	0.16198	0.01469	0.10957	0.01029	0.08001
BE_{SEL}	λ	0.42900	1.73098	0.21168	1.20528	0.08085	0.57230	0.05786	0.45524
	k	0.16148	0.56712	0.12371	0.39200	0.06861	0.31846	0.04931	0.27250
	θ	0.96003	0.99128	0.55461	0.57286	0.35374	0.36998	0.24060	0.25598
BE_{Linex1}	λ	0.43709	1.75647	0.21332	1.20953	0.08155	0.57271	0.05833	0.45547
	k	0.15863	0.56788	0.12237	0.39234	0.06786	0.31868	0.04884	0.27264
	θ	0.94179	0.97192	0.54611	0.56398	0.34951	0.36552	0.23831	0.25353
BE_{Linex2}	λ	0.44054	1.82075	0.20999	1.20030	0.08014	0.57189	0.05738	0.45500
	k	0.16433	0.56639	0.12506	0.39167	0.06935	0.31825	0.04978	0.27235
	θ	0.97791	1.01030	0.56305	0.58167	0.35796	0.37443	0.24288	0.25843

Table 10. *Cont.*

Method		n = 25		n = 50		n = 75		n = 100	
		Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
BE_{GEL1}	λ	0.42695	1.72782	0.21090	1.20486	0.08041	0.57220	0.05755	0.45518
	k	0.16343	0.56724	0.12461	0.39198	0.06909	0.31842	0.04962	0.27245
	θ	0.97335	1.00642	0.55910	0.57763	0.35575	0.37213	0.24163	0.25710
BE_{GEL2}	λ	0.42281	1.72143	0.20933	1.20403	0.07954	0.57201	0.05694	0.45508
	k	0.16735	0.56751	0.12640	0.39195	0.07007	0.31832	0.05023	0.27237
	θ	1.00017	1.03736	0.56805	0.58719	0.35976	0.37642	0.24369	0.25933

Table 11. Average estimated biases and RMSEs of different estimation methods for OLiP distribution at different sample sizes n and parameter values of $\lambda = 1.25,\ k = 2$, and $\theta = 2.50$.

Method		n = 25		n = 50		n = 75		n = 100	
		Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
MLE	λ	0.58474	2.06245	0.23918	0.75078	0.12648	0.45084	0.09301	0.35797
	k	0.02520	0.74072	0.00532	0.50719	0.01155	0.40700	0.00684	0.34375
	θ	0.07127	0.09767	0.03725	0.05221	0.02504	0.03464	0.01961	0.02743
MPSE	λ	1.63110	8.34237	0.39093	1.42884	0.20258	0.54730	0.15405	0.41844
	k	0.21850	0.80791	0.17161	0.54979	0.11813	0.43161	0.10154	0.36354
	θ	0.00242	0.08665	0.00246	0.03906	0.00163	0.02517	0.00208	0.01994
LSE	λ	1.17517	5.10722	0.37463	1.79044	0.14112	0.80036	0.08948	0.60991
	k	0.02767	0.98786	0.00921	0.71160	0.01614	0.56605	0.01094	0.47384
	θ	0.09202	0.26752	0.03174	0.12264	0.02136	0.07733	0.01498	0.05760
WLSE	λ	1.00971	4.93957	0.25672	1.36062	0.09045	0.54027	0.05881	0.41881
	k	0.02815	0.93110	0.00318	0.62454	0.02024	0.48125	0.01579	0.40262
	θ	0.04920	0.18417	0.01060	0.05808	0.00719	0.03696	0.00454	0.02867
CVME	λ	0.92823	4.91894	0.30896	1.70535	0.11458	0.78671	0.07053	0.57658
	k	0.20174	1.05136	0.07191	0.72456	0.06928	0.57415	0.05034	0.47851
	θ	0.05611	0.24336	0.01548	0.11345	0.01080	0.07227	0.00729	0.05485
ADE	λ	0.85680	4.43572	0.23986	1.31289	0.07613	0.51760	0.04625	0.40851
	k	0.04759	0.85351	0.01445	0.58802	0.03425	0.46421	0.02858	0.38970
	θ	0.02030	0.16792	0.00556	0.06518	0.00548	0.03906	0.00445	0.03029
RTADE	λ	0.80978	4.09285	0.22578	1.15575	0.09604	0.61759	0.06384	0.49744
	k	0.10703	0.90597	0.02602	0.62215	0.03770	0.50039	0.03118	0.42707
	θ	0.08175	0.30176	0.03113	0.17514	0.01810	0.11613	0.01249	0.08249
BE_{SEL}	λ	0.09022	1.01062	0.10060	0.50661	0.17534	0.36285	0.19576	0.32268
	k	0.37867	0.79431	0.46044	0.65990	0.54310	0.66601	0.58096	0.66762
	θ	0.48733	0.50976	0.25900	0.27529	0.14878	0.16520	0.08245	0.10337
BE_{Linex1}	λ	0.09144	1.01412	0.10024	0.50685	0.17513	0.36282	0.19560	0.32264
	k	0.38302	0.79816	0.46298	0.66245	0.54485	0.66777	0.58224	0.66896
	θ	0.48071	0.50282	0.25588	0.27204	0.14705	0.16341	0.08139	0.10228
BE_{Linex2}	λ	0.08900	1.00721	0.10096	0.50636	0.17555	0.36288	0.19592	0.32272
	k	0.37432	0.79049	0.45790	0.65737	0.54135	0.66425	0.57967	0.66629
	θ	0.49389	0.51665	0.26210	0.27852	0.15051	0.16699	0.08351	0.10445
BE_{GEL1}	λ	0.08968	1.01008	0.10088	0.50660	0.17553	0.36293	0.19591	0.32275
	k	0.37695	0.79317	0.45946	0.65903	0.54243	0.66539	0.58047	0.66715
	θ	0.49070	0.51337	0.26041	0.27677	0.14952	0.16598	0.08290	0.10382
BE_{GEL2}	λ	0.08860	1.00900	0.10145	0.50659	0.17592	0.36309	0.19620	0.32291
	k	0.37350	0.79088	0.45748	0.65731	0.54109	0.66414	0.57950	0.66620
	θ	0.49740	0.52059	0.26321	0.27973	0.15100	0.16753	0.08378	0.10474

Table 12. Average estimated biases and RMSEs of different estimation methods for OLiP distribution at different sample sizes n and parameter values of $\lambda = 1.50,\ k = 2.00$, and $\theta = 2.50$.

Method		n = 25		n = 50		n = 75		n = 100	
		Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
MLE	λ	1.39264	10.7766	0.32673	1.28416	0.15674	0.59487	0.11323	0.46502
	k	0.05107	0.83721	0.00571	0.57188	0.02127	0.45911	0.01417	0.38777
	θ	0.05568	0.07667	0.02895	0.04071	0.01942	0.02692	0.01520	0.02129
MPSE	λ	3.10580	13.07001	0.68125	3.44042	0.28577	0.76667	0.21227	0.56427
	k	0.23753	0.90388	0.19180	0.61961	0.13245	0.48698	0.11419	0.41037
	θ	0.00125	0.06779	0.00201	0.03020	0.00131	0.01944	0.00163	0.01542
LSE	λ	1.74388	6.49361	0.64582	2.77969	0.26322	1.33119	0.15857	0.98361
	k	0.04740	1.11875	0.01044	0.80657	0.01654	0.64490	0.01116	0.53977
	θ	0.06929	0.21163	0.02191	0.08743	0.01514	0.05592	0.01090	0.04340
WLSE	λ	1.69690	6.84416	0.46930	2.55752	0.15003	0.78719	0.09484	0.58927
	k	0.03886	1.05419	0.00291	0.70852	0.02209	0.54760	0.01751	0.45812
	θ	0.03558	0.14138	0.00748	0.04310	0.00525	0.02818	0.00334	0.02205
CVME	λ	1.39491	5.97465	0.53376	2.61637	0.20754	1.19819	0.12448	0.88223
	k	0.23715	1.18478	0.08195	0.82220	0.07751	0.65364	0.05634	0.54486
	θ	0.04136	0.19360	0.00945	0.08070	0.00707	0.05272	0.00495	0.04150
ADE	λ	1.37331	5.69657	0.43195	2.42288	0.12817	0.75569	0.07804	0.58111
	k	0.05870	0.96120	0.01521	0.66435	0.03756	0.52659	0.03160	0.44224
	θ	0.01310	0.12552	0.00347	0.04542	0.00396	0.02943	0.00331	0.02318
RTADE	λ	1.18004	5.10334	0.38381	1.98377	0.15329	0.84213	0.10046	0.64540
	k	0.12715	1.01647	0.02822	0.69685	0.04069	0.56003	0.03347	0.47706
	θ	0.05768	0.23146	0.01870	0.12258	0.01155	0.08480	0.00796	0.06030
BE_{SEL}	λ	0.07901	1.38259	0.16185	0.69725	0.25676	0.46216	0.28176	0.41645
	k	0.51620	0.94651	0.59347	0.80286	0.67673	0.80657	0.71067	0.80268
	θ	0.41952	0.43936	0.20968	0.22520	0.11367	0.12856	0.05619	0.07608
BE_{Linex1}	λ	0.08088	1.39115	0.16146	0.69773	0.25654	0.46212	0.28161	0.41639
	k	0.52007	0.95034	0.59554	0.80503	0.67814	0.80804	0.71170	0.80378
	θ	0.41449	0.43409	0.20755	0.22298	0.11255	0.12740	0.05555	0.07543
BE_{Linex2}	λ	0.07720	1.37462	0.16224	0.69677	0.25698	0.46221	0.28192	0.41652
	k	0.51235	0.94271	0.59141	0.80070	0.67532	0.80510	0.70964	0.80157
	θ	0.42450	0.44459	0.21180	0.22741	0.11478	0.12972	0.05684	0.07672
BE_{GEL1}	λ	0.07846	1.38185	0.16211	0.69725	0.25693	0.46225	0.28189	0.41653
	k	0.51479	0.94541	0.59272	0.80216	0.67622	0.80608	0.71031	0.80231
	θ	0.42198	0.44199	0.21062	0.22619	0.11414	0.12906	0.05646	0.07635
BE_{GEL2}	λ	0.07736	1.38037	0.16263	0.69725	0.25728	0.46242	0.28214	0.41670
	k	0.51196	0.94322	0.59122	0.80076	0.67521	0.80509	0.70958	0.80156
	θ	0.42687	0.44724	0.21249	0.22817	0.11508	0.13005	0.05699	0.07689

7.2. *Applications to Real Data Set*

This section uses two real data sets to show how well the OLiP distribution models fit data sets.

7.2.1. Vinyl Chloride Data Set

The first data set, as reported by [38], displays vinyl chloride data (g/L) from groundwater monitoring wells that are located in clean-up-gradient areas. Given that vinyl chloride is both anthropogenic and carcinogenic, it is believed to be a volatile organic molecule, a property that is relevant to environmental studies. In order to compare the performance of the OLiP distribution with other distributions, the vinyl chloride data must be fitted to the OLiP distribution.

5.1	1.2	1.3	0.6	0.5	2.4	0.5	1.1	8.0	0.8	0.4	0.6
0.9	0.4	2.0	0.5	5.3	3.2	2.7	2.9	2.5	2.3	1.0	0.2
0.1	0.1	1.8	0.9	2.0	4.0	6.8	1.2	0.4	0.2		

The results in Table 13 are derived from the vinyl chloride data fitted to the OLiP distribution and compared with those from the Pareto distribution (PD), Lindley distribution (LD), odds generalized exponential (OGE), odds generalized exponential-power Lomax (OG-EPL), and Weibull (WE) distributions. The criteria log-likelihood, Akaike information criterion, Bayesian information criterion, and Kolmogorov–Smirnov statistic were employed to distinguish between the models, as shown below.

$$AIC = -2\ell(\hat{\Phi}) + 2q$$

$$BIC = -2\ell(\hat{\Phi}) + q \ln n$$

where $\ell(\hat{\Phi})$ denotes the log-likelihood at maximum likelihood estimates, q is the number of parameters, while n is sample size. Given an ordered random sample $X_1, X_2, ..., X_n$ from OLiP$(\hat{\lambda}, \hat{k}, \hat{\theta})$, the Kolmogorov–Smirnov (KS) statistic is

$$KS = \max_i \left[\frac{1}{n} - F(x_i, \hat{\lambda}, \hat{k}, \hat{\theta}), F(x_i, \hat{\lambda}, \hat{k}, \hat{\theta}) - \frac{i-1}{n}\right]$$

The examination of the real data set I yield the results in Table 13, which demonstrate that the OLiP distribution has the lowest values for all the criteria LL, AIC, BIC, and KS for the data set when compared to other distributions. Consequently, the suggested model presents a desirable alternative. Figure 5 compares the OLiP, OG-EPL, OGE, LD, PD, and WE distributions using the density, CDF, empirical reliability, and TTT plots of the real data set. Figure 6 also includes the PP plot for this date. To be more specific, the first real data set is better suited to the OLiP distribution. The findings are shown in Table 14 and include the estimates and standard error (std. err) for the unknown parameters of the OLiP distribution based on the underlying estimation techniques covered in earlier sections. The p-value is compared to an α level of 0.05. The criterion is that the p-value of any of the fitted distributions must be greater than 0.05 in order to fit the data, and the distribution whose p-values meet the first condition and is also greater than the rest has the best goodness of fit. It is obvious from both Tables 13 and 15 that the proposed OLiP distribution best fits the two data sets.

Table 13. MLEs, LL, AIC, CAIC, BIC, HQIC, KS, the p-value of the real data set I.

Model	MLEs	LL	AIC	CAIC	BIC	HQIC	KS	p-Value
OLiP	$\lambda = 0.41717$ $k = 0.59434$ $\theta = 0.10000$	−53.35211	110.69382	111.08090	113.74651	111.73481	0.07506	0.99091
OGE	$\lambda = 2.82592$ $\beta = 0.57813$ $\gamma = 1.23910$ $\theta = 2.30869$	−55.14256	118.37101	119.75024	124.47631	120.45310	0.09767	0.90190
OEPL	$\lambda = 3.40603$ $\beta = 0.40108$ $\delta = 1.33945$ $\theta = 2.32600$	−51.40121	112.83102	114.20711	118.93334	114.91315	0.10948	0.80981
PD	$\theta = 54.0098$ $\alpha = 29.7229$	−55.44012	114.87893	115.26611	117.93162	115.92214	0.08138	0.97791
WE	$\alpha = 0.9739$ $\beta = 1.8341$	−56.54913	117.0983	117.4854	120.1510	118.1393	0.12215	0.69071
LD	$\theta = 0.82383$	−56.32051	114.60647	114.73232	116.13360	115.12781	0.13262	0.58823

Table 14. Estimates and standard errors of various estimation procedures of the OLiP distribution for the real data set I.

Method	λ		k		θ	
	Estimate	**Std. Err**	**Estimate**	**Std. Err**	**Estimate**	**Std. Err**
MLE	0.41717	0.08710	0.59434	0.09626	0.10000	—
MPS	0.30715	0.23391	0.57851	0.10591	0.05908	0.04616
LSE	0.30355	2.27254	0.57633	0.64212	0.05732	0.54144
WLSE	0.25304	0.10603	0.59156	0.03471	0.04691	0.02469
CVME	0.33629	2.31720	0.59318	0.68862	0.07319	0.58988
ADE	0.26275	0.56124	0.61308	0.21023	0.05551	0.13789
RTADE	0.29124	1.40565	0.60553	0.32673	0.06245	0.39334
BE$_{SEL}$	0.31990	0.03932	0.65006	0.04827	0.09620	0.00202

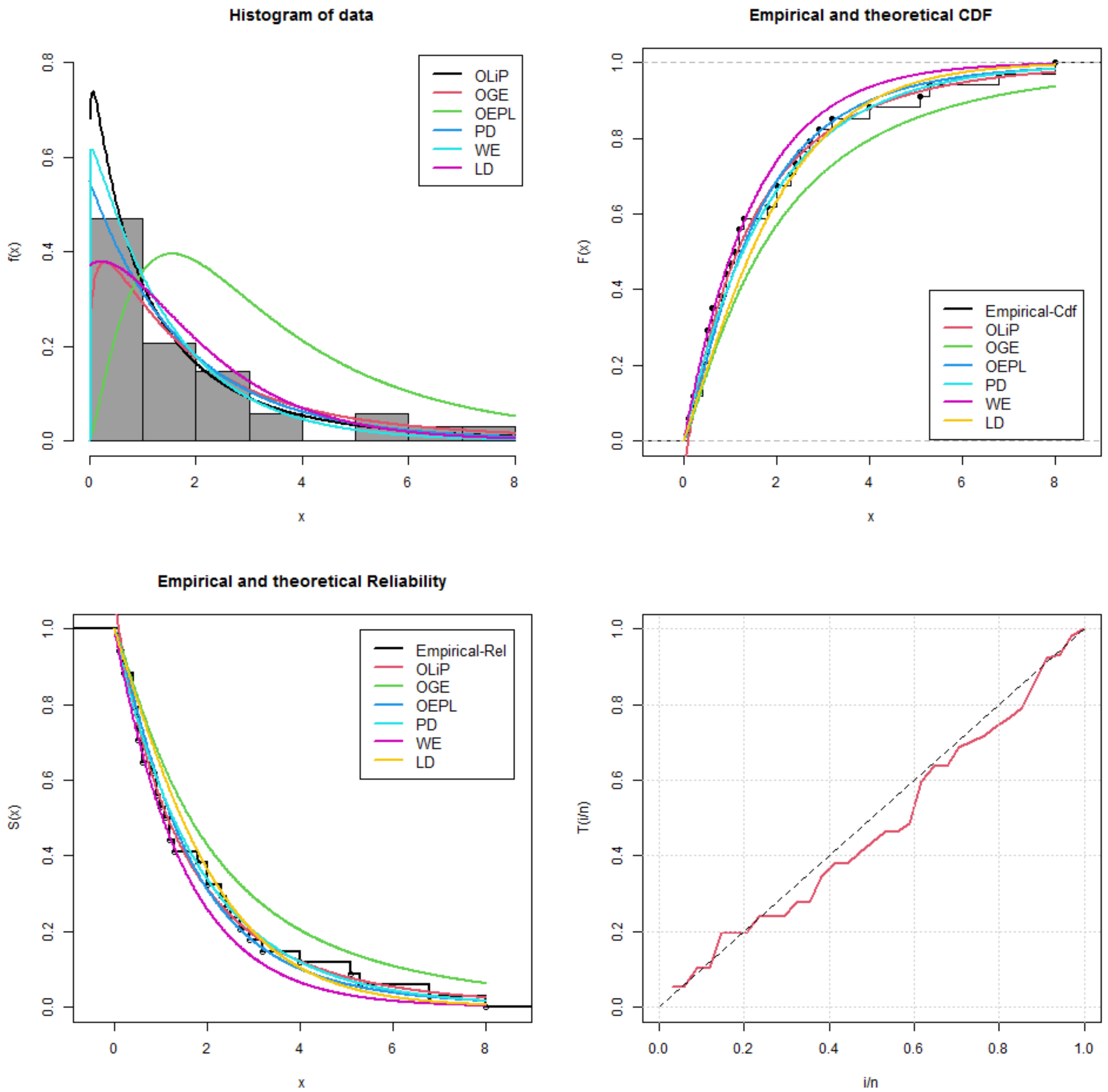

Figure 5. The density, CDF, empirical reliability, and TTT plots for the first real data set.

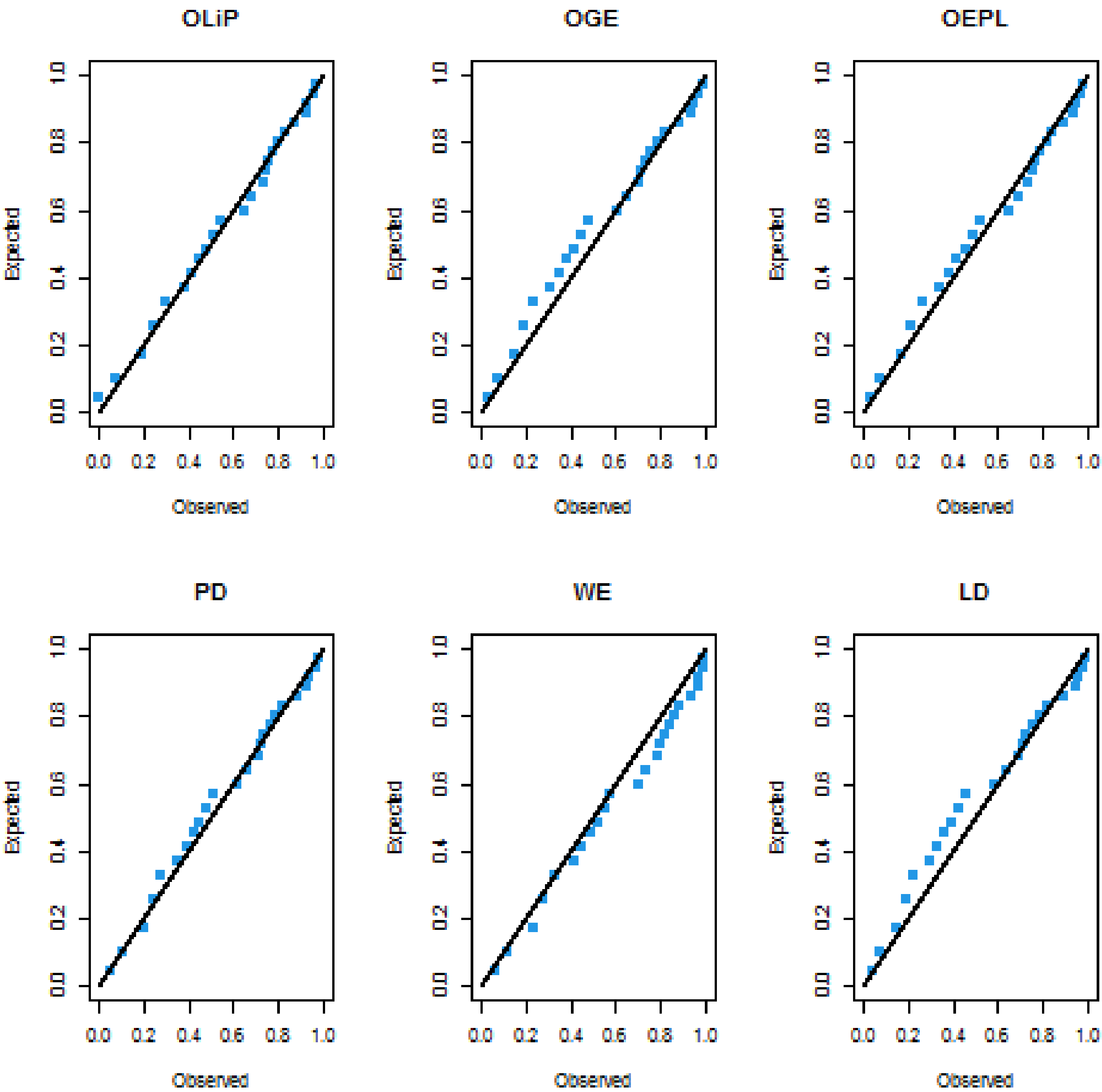

Figure 6. The PP plot for the first real data set.

7.2.2. COVID-19 Data Set

The second data set, which consists of 36 observations and shows the mortality rates of COVID-19 patients in Canada, is presented in this subsection. Access to this information can be found at [https://covid19.who.int/ (accessed on 27 July 2023)]. Additionally, the information is offered below and may be found in the article authored by [39]:

3.1091	3.3825	3.1444	3.2135	2.4946	3.5146	4.9274	3.3769	6.8686	3.0914	4.9378
3.1091	3.2823	3.8594	4.0480	4.1685	3.6426,	3.2110	2.8636	3.2218	2.907	3.6346
2.7957	4.2781	4.2202	1.5157	2.6029	3.3592	2.8349	3.1348	2.5261	1.5806	
2.7704	2.1901	2.4141	1.9048							

The COVID-19 data are fitted to the OLiP distribution in order to assess its suitability, and the results are shown in Table 15 with comparison to some other models such as the generalized exponential (Gen-Ex), inverse gamma (Inv-Ga), OEPL, and WE distributions. The same criteria in the data set will be used to compare these models.

Table 15. MLEs, LL, AIC, CAIC, BIC, HQIC, KS, the p-value of the COVID-19 data set.

Model	MLEs	LL	AIC	CAIC	BIC	HQIC	KS	p-Value
OLiP	$\lambda = 0.35251$ $k = 2.16421$ $\theta = 1.51570$	-48.29042	100.58230	100.94591	103.74931	101.68772	0.12066	0.67101
Gen-Ex	$\alpha = 29.07834$ $\lambda = 0.83437$	-48.51342	101.02681	101.39050	104.19387	102.13220	0.12358	0.64156
Inv-Ga	$\theta = 54.0098$ $\alpha = 29.7229$	-48.93963	101.87931	102.24286	105.04631	102.98461	0.13783	0.50092
OEPL	$\lambda = 2.33980$ $\beta = 0.22465$ $\delta = 5.28708$ $\theta = 171.9655$	-57.38508	122.77016	124.06048	129.10424	124.98092	0.15500	0.35266
WE	$\alpha = 3.31387$ $\lambda = 3.63702$	-51.47427	106.94852	107.31224	110.115603	108.05385	0.14997	0.39296

The examination of the real data set II yields the results in Table 15, which demonstrate that the OLiP distribution has the least values for all the criteria LL, AIC, BIC, and KS for the data set when compared to the other distributions taken into consideration. Consequently, the suggested model presents a desirable alternative. Figure 7 compares the OLiP, Gen-Ex, Inv-Ga, OEPL, and WE distributions using the density, CDF, empirical reliability, and TTT plots of the real data set. Figure 8 also includes the PP plot for this date. To be more specific, the second real data set is better suited to the OLiP distribution.

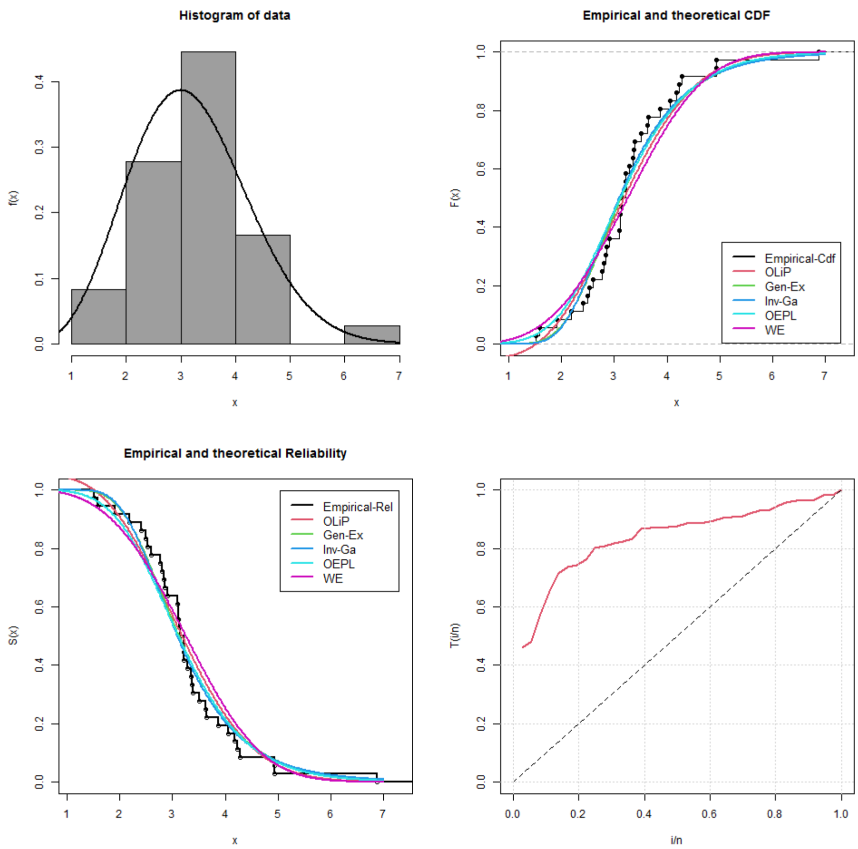

Figure 7. The density, CDF, empirical reliability, and TTT plots for the second real data set.

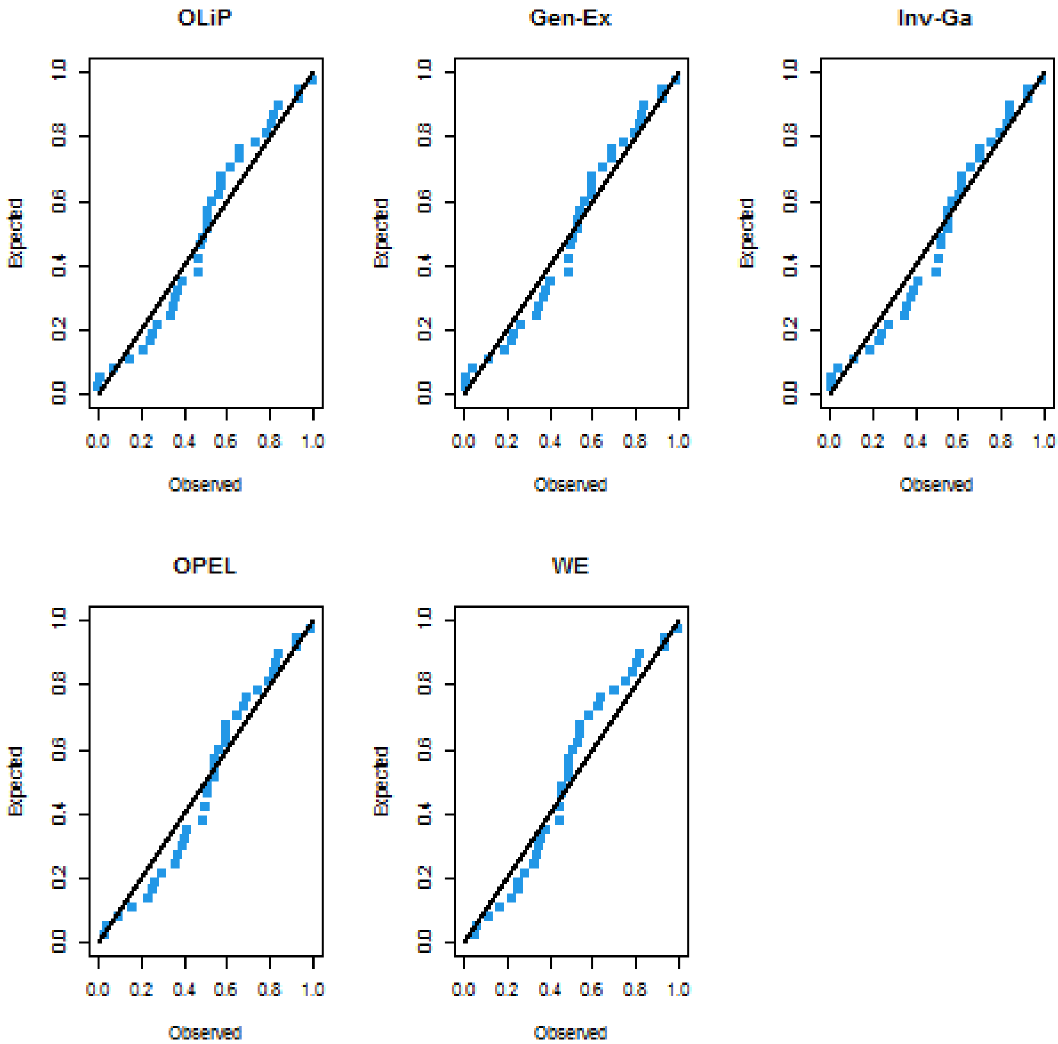

Figure 8. The P-P plot for the second real data set.

The findings are shown in Table 16 and include the estimates and standard error (std. err) obtained using the underlying estimating techniques covered in earlier sections for the unknown parameters of the OLiP distribution.

Table 16. Estimates and standard errors of various estimation procedures of the OLiP distribution for COVID-19 data set.

Method	λ		k		θ	
	Estimate	**Std. Err**	**Estimate**	**Std. Err**	**Estimate**	**Std. Err**
MLE	0.35251	0.11773	2.16421	0.32332	1.51570	—
MPS	0.25126	0.22176	2.06242	0.33171	1.26042	0.37469
LSE	0.05321	0.48064	3.11215	2.60681	1.05261	3.14173
WLSE	0.02459	0.00736	2.90658	0.14649	0.74861	0.09298
CVME	0.06561	0.74547	3.27890	2.76180	1.18667	4.10497
ADE	0.10695	0.58364	2.71285	0.75442	1.15982	2.14264
RTADE	0.61831	1.20303	2.22982	1.21607	1.91067	1.00573
BE$_{SEL}$	0.38171	0.05449	2.10173	0.22464	1.51337	0.06459

8. Conclusions

A new model has been introduced in this article. Several properties are derived and thoroughly examined, including moments and their measures, the moment-generating function, the characteristic function, the hazard rate, Rényi entropy, order statistics and stochastic ordering. To estimate and study the parameters, eight methods are considered and comparisons are made. The techniques looked at include maximum likelihood estimation, maximum product spacing, least squares, weighted least squares, Cramer–von Mises, right-tailed Anderson–Darling, and Bayesian techniques. We ran a simulated study to contrast the different approaches. In terms of bias and mean squared error, we have compared the estimators. According to the simulation's findings, the most competitive method is maximum product spacing. Applications to vinyl chloride data from clean-up-gradient ground-water monitoring wells in g/L, and COVID-19 data from Canada also complement the findings by demonstrating that the OLiP distribution strongly fits the data sets. From the two analytical measures of fitness and performance (LL, AD, CVM, AIC, CAIC, BIC, HQIC, K-S, and p-value), the proposed OLiP distribution is preferred to the following fitted distributions, Pareto (PD), Lindley (LD), odds generalized exponential (OGE), odds generalized exponential-power Lomax (OG-EPL), and Weibull (WE) distributions. The OLiP distribution outperformed the competing distributions based on the AIC, BIC, CAIC, HQIC, LL, KS, and probability values. When the lifetime test is terminated at the median life of the proposed distribution, the SASPs have also been computed based on the OLiP distribution. The required sample size was calculated using numerous truncation times at various parameter values and degrees of consumer risk. Additionally, the probability of acceptance was assessed for various values of n, the collected sample sizes, to make sure that it is less than or equal to the complement of the consumer's risk α^*.

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