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A Sustainable Closed-Loop Supply Chains Inventory Model Considering Optimal Number of Remanufacturing Times

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Abstract: The mathematical modeling of reverse logistics inventory systems ignores the fact that returned items may arrive out of sequence, i.e., with different number of remanufacturing times. Moreover, such modeling assumes that the retuned items may retain the same quality upon recovery regardless of how many times they have been previously remanufactured. This paper develops a new mathematical expression of the percentage of retuned items that can be remanufactured a finite number of times. The proposed expression is modeled as a function of the expected number of times an item can be remanufactured in its lifecycle and the number of times an item can be technologically (or optimally) remanufactured based on its quality upon recovery. The model developed in this paper considers joint production and remanufacturing options. The return rate is a varying demand-dependent rate, which is a decision variable with demand, product deterioration, manufacturing, and remanufacturing rates being arbitrary functions of time. The model considers the initial inventory of returned items in the mathematical formulation, which enables decision-makers to adjust all functions and input parameters for subsequent cycles. Illustrative examples indicate that dependent purchasing price of recovery items and the incorporation of remanufacturing investment cost significantly impact the optimal remanufacturing policy.

Keywords: reverse logistics; number of remanufacturing times; first remanufacturing cycle; time-varying parameters; demand dependent return rate

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Citation: Alamri, A.A. A Sustainable Closed-Loop Supply Chains Inventory Model Considering Optimal Number of Remanufacturing Times. *Sustainability* **2023**, *15*, 9517. https://doi.org/10.3390/su15129517

Academic Editor: Shaojian Qu

Received: 9 May 2023
Revised: 26 May 2023
Accepted: 10 June 2023
Published: 13 June 2023

1. Introduction

Reverse logistics emerges as an opportunity beyond the traditional logistics role, with the main purpose being product returns from end customers for recapturing value or proper disposal [1]. Further, reverse logistics has been implemented to address economic drivers, government pressure/legislation, social interests, and environmental consciousness. The goal of reverse logistic is to effectively manage and control the flow of products returned from end customers to extend their useable lives, reduce solid waste disposal, and conserve natural resource consumption [2,3]. The importance of reverse logistics may vary among industries due to relevant costs or due to the dynamic nature of production and remanufacturing processes of the retuned items The reverse logistics process is a mirror image of the traditional forward supply chain one. It comprises activities such as the collection of returned items from end users, inspection, reprocessing, disassembly, and, finally, redistribution of returned items for recovery purposes [2,4], whereas a closed-loop supply chain is categorized by the combination of forward and reverse supply chain activities [4,5].

Inventory management in reverse logistics has received growing attention in recent years. Moreover, due to global competitiveness, there has been more focus among large companies to adopt joint production and remanufacturing options in their businesses [6,7]. For example, in Germany, about 10% of engines and starter engines are remanufactured [8]. In this regard, the remanufactured products save 80% of raw materials, require 33% of the labor force, and consume 50% of the energy and up to 50–70% less cost when compared with the newly manufactured products [9–12]. Several companies, including BMW and

Volkswagen, focus on accelerating the upgrading process of older cars and offer a fully warrantied service for remanufactured engines and other parts [10,13]. Therefore, reverse logistics can enhance productivity, reduce costs, improve profitability, satisfy demand, and meet customer loyalty [14].

Beyond the economic benefits, there exist a plethora of factors, such as social and environmental consciousness and government legislation, that may force manufacturers to include such product recovery systems in their businesses [15].

2. Literature Review

The first inventory model with returned items was conducted by Schrady [16]. He developed a deterministic economic order quantity (EOQ) model for repaired products assuming no disposal cost with instantaneous production and return rates. Nahmias and Rivera [17] generalized the model of Schrady [16] considering a finite repair rate. Richter [18–20], Richter and Dobos [21], and Dobos and Richter [22] carried out several investigations into the EOQ repair model, with the assumption that the return rate is a decision variable. Richter [18,19] generalized the model of Schrady [16] by investigating multiple repair and production cycles. Dobos and Richter [23] investigated a manufacturing/recycling system for noninstantaneous manufacturing and recycling rates.

Richter [20] investigated the model when the recovery rate is a decision variable. He showed that the optimal strategy occurs for no waste disposal (total repair) or for no repair (total waste disposal). Dobos and Richter [24] extended their previous work (Dobos and Richter [23]) assuming multiple repair and production cycles. They indicated that the case of a pure strategy is optimal. Dobos and Richter [25] assumed that some collected returned items are not always suitable for further recycling. There are numerous studies that relax different assumptions made so far. Examples of these works are cited in [26]. El Saadany and Jaber [27] considered the return rate to be dependent on the purchasing price of the retuned items and the use proportion of these items. They replicated the work of Dobos and Richter [23,24] and indicated that a mixed (production + remanufacturing) strategy is optimal, but not as suggested by Dobos and Richter [23,24]. Alamri [28] developed a general model and verified the examples given in Dobos and Richter [23,24] and in El Saadany and Jaber [27]. He showed that a mixed strategy dominates a pure strategy.

El Saadany and Jaber [29] pointed out that previous research accounted for an infinite planning horizon and ignored the effect of the first cycle as there are no returned items to be remanufactured. They rectified a minor error in the work of Richter [18,19] and, consequently, their model produces a lower cost because of the residual inventory assumed in Richter [18,19]. Kozlovskaya et al. [30] rectified the model of El Saadany and Jaber [29] and provided the optimal policy. They showed that the optimal policy depends on the disposal rate. Although El Saadany and Jaber [29] provided a closed-form formula for the first cycle, their mathematical formulation and the other studies in the literature are alike. Alamri [31] discussed this issue in detail and addressed this limitation by incorporating the initial inventory of returned items in the mathematical formulation. He showed that the optimal policy implies that the cumulative inventory for returned items vary for each cycle before the system plateaus. This is a key consideration that allows the adjustment of the input parameters for any given cycle.

The above cited contributions are directly relevant to this paper. Other researchers have developed models related to reverse logistics systems (see [32,33]).

3. Research Background and Contribution

Below, we specify some issues that are related to the number of times a product can be remanufactured, as advocated in El Saadany et al. [34], followed by some discussion that elaborates on our research contributions. Meanwhile, the work of Alamri [31] constitutes the base model of this research.

3.1. Theoretical Background and Motivation

El Saadany et al. [34] suggested a mathematical expression that indicates how many times a product can be remanufactured. They attempted to relax the general assumption that an item can be recovered for an indefinite number of times. They assumed that an item can be remanufactured for a finite (ξ) number of times. When the system plateaus, then for any ξ, a fraction β_ξ of a constant demand rate d is remanufactured and $(1-\beta_\xi)$ is produced, where $\beta_\xi = 1 - \frac{1-\beta}{1-\beta^{\xi+1}}$ and $\beta(0 < \beta < 1)$ is the fraction of used items returned for remanufacturing based on its recovery for an indefinite number of times. It is worth noting here that the mathematical expression used to derive β_ξ focused on the returns of what was produced in the previous period and ignored the rest of cumulative produced quantities that have been left or previously being remanufactured. Interested readers are referred to Table 1 in El Saadany et al. [34]. They stated that as $\xi \longrightarrow \infty$, $\beta_\xi = 1 - \frac{1-\beta}{1-\beta^\infty} = \beta$, which is identical to what the existing literature suggests. In the case of a pure production, i.e., $\xi = 0$, $\beta_\xi = 1 - \frac{1-\beta}{1-\beta^{0+1}} = 0$, though a pure production strategy implies that $\beta = 0$, i.e., there are no items returned for recovery purposes. Moreover, the mathematical expression used to derive β_ξ assumes no waste disposal (total repair) of the proportion β upon recovery. Then, they modified the work of Richter [20] and Teunter [35] by replacing β with β_ξ in Richter's [20] and Teunter's [35] models.

Table 1. The actual quality level of an item that recovers ξ number of times when $\tau = 1, 2, \ldots 8$.

1	2	3	4	5	6	7	8
0.368	0.607	0.717	0.779	0.819	0.846	0.867	0.882
	0.368	0.513	0.607	0.670	0.717	0.751	0.779
		0.368	0.472	0.549	0.607	0.651	0.687
			0.368	0.449	0.513	0.565	0.607
				0.368	0.435	0.490	0.535
					0.368	0.424	0.472
						0.368	0.417
							0.368

In their model, the produced quantity $(1-\beta_\xi)d$ is also disposed outside the system (e.g., Bazan et al. [36]) since they defined α as the disposal rate, where $\alpha(\alpha = 1-\beta_\xi)$. Moreover, the role of β in their model is somewhat ambiguous. Therefore, we can distinguish three cases: (1) As can be seen from Figure 1 in El Saadany et al. [34], $(\alpha + \beta_\xi)d$ enters the repairable stock from which $\beta_\xi d$ is remanufactured and $\alpha d = (1-\beta_\xi)d$ is disposed. This implies that the return rate is d; however, this contradicts what the existing literature suggests, i.e., the return rate is less than demand rate; (2) $\beta_\xi = 1 - \frac{1-\beta}{1-\beta^{\xi+1}}$, which is a function of β, and therefore, the value of β is used to compute β_ξ. In their examples, β is defined as the collection of used items and β_ξ is the effective proportion. In this case (Case 2), one can deduce that βd enters the repairable stock from which $\beta_\xi \beta d$ is remanufactured and $(1-\beta_\xi)\beta d$ is disposed. However, β represents the fraction of returned items that are recovered for an indefinite number of times and only β_ξ of β is remanufactured; (3) the return rate is $\beta_\xi d$, which enters the repairable stock and flows in the serviceable stock to be remanufactured with no waste disposal (total repair). That is, the purpose of β, which represents the fraction of returned items that are recovered for an indefinite number of times, is to compute β_ξ. In this case (Case 3), the system should collect β_ξ instead of β, which is reflected in their modified version of the work of Richter [20] and Teunter [35]. Hence, we can conclude that, in all cases, β is used to compute β_ξ, with Case 2 being the most appropriate scenario. However, β_ξ in all these cases has no relation with an item being recovered for a limited (ξ) number of times. In fact, ξ is an arbitrary integer value, which is implanted in β_ξ to minimize the total cost. Therefore, considering a fraction $\gamma(0 \le \gamma \le 1)$

of the return rate that meets the acceptance quality level to be remanufactured and that $(1-\gamma)$ is disposed outside the system is more practicable [24,27,28,31].

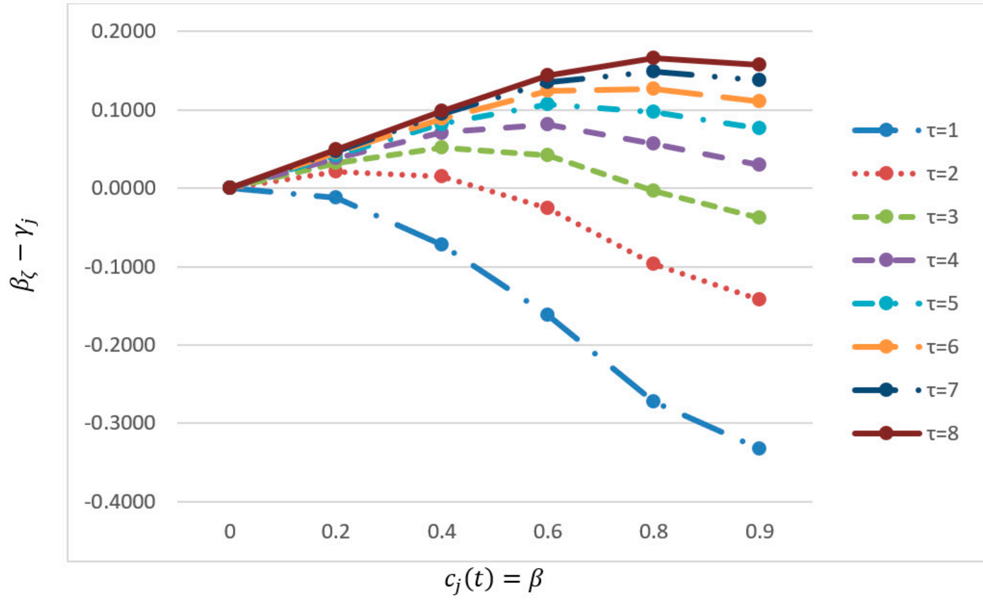

Figure 1. The difference between β_{ξ} and γ_j.

The difference between β and β_{ξ} represents about 50% when $\xi = 1$ and $\beta = 0.9$ (see El Saadany et al. [34]). Moreover, for a fixed value of β, this difference decreases with ξ (see El Saadany et al. [34]). This seems logical in their expression, since as $\xi \longrightarrow \infty$, $\beta_{\xi} \longrightarrow \beta$, because they assumed that all returned items have been remanufactured ξ number of times. On the contrary, however, this difference should increase, as a returned item with a greater number of remanufacturing times recovers with inferior quality. Furthermore, implementing β_{ξ}, as suggested by El Saadany et al. [34], would result in a large disposal quantity, especially for products that are associated with relatively small number of recovery times since β_{ξ} increases with ξ (see Figure 1). Finally, incorporating such β_{ξ} in the mathematical formulation entails that all returned items have been remanufactured ξ number of times. However, the fact remains that returned items may arrive out of sequence, i.e., with different number of remanufacturing times, assuming also that the previous number of remanufacturing times is labeled. It is true to say that considering such classification of returned items in the mathematical expression is not an easy task; however, this limitation will be discussed in the next section.

3.2. Mathematical Formulation of the Recovery Times

The comprehensive discussion in the previous section is necessary to position our contribution in the existing literature as well as to highlight its research impact. The aim of this paper is to enhance this line of research by developing a new mathematical expression that models the percentage of returns as a function of the number of times an item is recovered, the corresponding quality for the recovery item, and the expected number of times an item can be remanufactured in its lifecycle. In this paper, we assume that returned items are collected at a rate of $c_j(t)$ (decision variable), where j denotes the cycle index. Note that a pure production strategy occurs when $c_j(t) = 0$.

Only a fraction $\gamma_{\xi j}$ of these retuned items can be remanufactured. Namely, $\gamma_{\xi j} = e^{\frac{-\xi q_{\xi}}{\tau}}$, where $q_{\xi j}(0 < q_{\xi j} < 1)$ denotes the quality level of an item that has been recovered ξ number of times. We assume that $q_{\xi j} = e^{\frac{-\xi}{\tau}}$, where $q_{0j} = 1$, i.e., it refers to the quality of a newly manufactured item. In this paper, ξ refers to the maximum number of times an item can be technologically (or optimally) remanufactured, and $\tau \geq \xi$ denotes the expected number of times an item can be remanufactured in its lifecycle. Note that $q_{\xi j}$ decreases as ξ increases, and it attains a minimum value as $\xi \longrightarrow \tau$; in this case, $q_{\xi j} = q_\tau = q_{min} = e^{-1}$ (Table 1). Accordingly, a recovery item with a quality less than q_{min} is considered defective and incurs a disposal cost. Note that $\gamma_{\xi j}' < 0$ and $\gamma_{\xi j}'' > 0 \; \forall \xi > 0$, i.e., $\gamma_{\xi j}$ is a monotonically decreasing function over ξ, and as $\tau \longrightarrow \infty, \gamma_{\xi j} \longrightarrow 1$. The same arguments hold true for $q_{\xi j}$. This implies that $\gamma_{\xi j}$ is modeled as a function of the expected number of times an item can be remanufactured in its lifecycle and the number of times an item can be technologically remanufactured based on its quality upon recovery. Moreover, $\gamma_{\xi j}$ and $q_{\xi j}$ are free from any judgmental measurements.

In real-life settings, returned items may recover out of sequence. This can be attributed to the random number of times these items have been remanufactured. Let us define the returned amount for cycle j as R_j, where this amount undergoes a 100 percent inspection. Note that in an automated remanufacturing system, observing R_j appears to be realistic because all returned items are inspected. Therefore, returned items that are subjected to a 100 percent screening upon recovery to the repairable stock would imply that $\underline{R_j} = (r_{\xi j}, r_{\xi-1j}, \ldots, r_{0j})$. That is, r_{kj} is the collected used/returned items, with $k(k = \xi, \xi - 1, \ldots, 0)$ being the number of times these items have been previously remanufactured. Here, r_{0j} denotes returned items that have not yet been remanufactured, and $r_{\xi j}$ refers to defective (disposed) returned items that have been remanufactured ξ number of times or items that do not meet the minimum acceptance quality level, q_{min}.

It is worth noting here that the abovementioned classification seams realistic because the system can deal with items based on such classification upon recovery. Therefore, as the number of times an item can be recovered increases, its corresponding use proportion decreases. This finding, however, contradicts that of El Saadany et al. [34]. To justify this, suppose that among the returned quantity that can be remanufactured, say, five times, there exists a subquantity that arrived at the repairable stock for their first-time recovery. In this case, implementing $\gamma_{\xi j}$ for this subquantity would result in disposing of an equal fraction to that of items with a greater recovery time, though these items have not yet been remanufactured. Table 2 depicts the corresponding use proportion, where $\underline{\gamma_{ij}} = (\gamma_{1j}, \gamma_{2j}, \ldots \gamma_{\xi j})$. For instance, if there is a quantity of returned items that can be remanufactured, say, five times, then each subquantity is associated with its corresponding accepted fraction, i.e., $\gamma_{5j} = 0.692, \gamma_{4j} = 0.698, \gamma_{3j} = 0.719, \gamma_{2j} = 0.765$, and $\gamma_{1j} = 0.849$. That is, γ_{ij} represents the use proportion of the subquantity of the retuned items that can be remanufactured for its *ith* remanufacturing time. However, considering the abovementioned classification in the mathematical formulation emerges as a challenge in terms of the presentation of each subquantity in modeling. Therefore, to tackle this issue, we suggest that $\gamma_j = \frac{\sum_{i=1}^{\xi} \gamma_{ij}}{\xi}$, which constitutes an approximation of the average fraction (cumulative average up to ξ) that can be remanufactured in cycle j (Figure 2).

Table 2. The actual proportion of returned items that can be remanufactured ξ number of times when $\tau = 1, 2, \ldots 8$.

1	**2**	**3**	**4**	**5**	**6**	**7**	**8**
0.692	0.738	0.788	0.823	0.849	0.868	0.884	0.896
	0.692	0.710	0.738	0.765	0.788	0.807	0.823
		0.692	0.702	0.719	0.738	0.756	0.773
			0.692	0.698	0.710	0.724	0.738
				0.692	0.696	0.705	0.716
					0.692	0.695	0.702
						0.692	0.694
							0.692

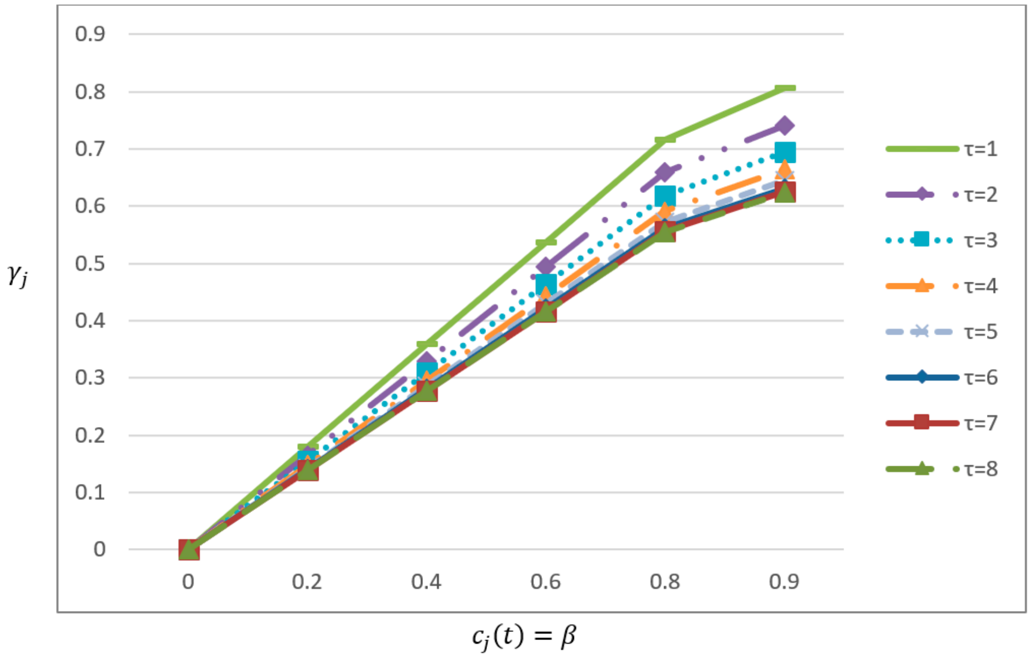

Figure 2. The behavior of γ_j that flows to be remanufactured in cycle j.

As can be seen from Table 2, assuming $\gamma_{\xi j}$ to be remanufactured in cycle j implies that the proportion $(1 - \gamma_{\xi j})c_j(t)$ is disposed outside the system. Conversely, $(1 - \gamma_j)c_j(t) \leq (1 - \gamma_{\xi j})c_j(t)$, i.e., γ_j considers the cumulative average up to time ξ of returned items (Table 4). This is a key consideration because it governs the behavior of returned items and ensures reducing the disposal of unnecessary amounts. In addition, returned items are coupled with distinct purchasing price $c_{prj} = c_{pm} e^{\frac{-1}{q_j}}$, where c_{pm} denotes unit purchasing price for new items.

3.3. Contribution and Organization of the Paper

This paper develops a new mathematical expression that specifies the number of times a product can be remanufactured. In particular, the proposed expression is modeled as a function of the expected number of times an item can be remanufactured in its lifecycle and the number of times an item can be technologically (or optimally) remanufactured based on its quality upon recovery.

In this paper, we present a general reverse-logistics inventory model with a single manufacturing cycle and a single remanufacturing cycle. Demand, deterioration, manufacturing, remanufacturing, and return rates are arbitrary functions of time. Therefore, a diverse range of time-varying forms can be disseminated from the general model. The mathematical formulation consists of serviceable and reparable stocks for joint manufacturing and remanufacturing options. New items are manufactured in the serviceable stock, while returned items are collected in the reparable stock to be remanufactured in the serviceable stock as good as new. Therefore, different holding costs and deterioration rates are considered for manufactured, remanufactured, and returned items (e.g., [28,35,37,38]).

Only a proportion of the returned items flows in the reverse direction, which specifies the number of times an item can be technologically (or optimally) remanufactured. In the first remanufacturing cycle, the initial inventory of retuned items is zero since there are no returned items to be remanufactured. Therefore, the accumulated amount of returned items (during the time gap of nonproduction and nonremanufacturing processes) represents the initial inventory of returns for the second cycle. This amount, indeed, should differ from that accumulated for subsequent cycles. This is key in our formulation, and therefore ensures that all optimal values vary for each cycle before the system plateaus. The proposed model accounts for setup changeover costs when switching from manufacturing phase to remanufacturing phase. The proposed model also considers an investment cost, which is a function of the number of times a product is remanufactured and its quality upon recovery. We assume that returned, manufactured, and remanufactured items deteriorate while they are effectively in stock. The return rate of the returned items is a decision variable, which is a function of the demand rate. The purchasing price of returned items is a function of the purchasing price of new items based on their quality upon recovery. All functions and input parameters can be adjusted for subsequent cycles.

The remainder of this paper is structured as follows. The joint manufacturing and remanufacturing model and the solution procedure are presented in Section 4. Illustrative examples and special cases are offered in Section 5. Managerial insights are given in Section 6, and the paper closes with concluding remarks provided in Section 7.

4. Mathematical Formulation of the General Model

4.1. Assumptions and Notations

The following notations are considered:

j	The cycle index;
z	$(z = gm, gr, r)$ gm denotes manufactured items, gr denotes remanufactured items and r denotes returned items;
$P_{mj}(t)$	The rate per unit time for manufactured items;
$P_{rj}(t)$	The rate per unit time for remanufactured items;
$c_j(t)$	The rate per unit time for returned items (decision variable), where $c_j(t) = \varnothing_j D_j(t)$ and $0 \leq \varnothing_j < 1$;
$D_j(t)$	The rate per unit time for demand items;
$I_{zj}(t)$	The inventory level at time t;
$\delta_{zj}(t)$	The deterioration rate per unit time;
d_{zj}	The deteriorated quantity for cycle j;
Q_{mj}	The manufactured quantity for cycle j;
Q_{rj}	The remanufactured quantity for cycle j;
R_j	The returned quantity for cycle j;
Δ_j	The accumulated quantity of returned items (during the time gap of nonproduction and nonremanufacturing processes);
ξ	The maximum number of times an item can be remanufactured;
τ	The expected number of times an item can be remanufactured in its lifecycle, where $\tau \geq \xi$;
$q_{\xi j}$	The actual quality level of an item that has been recovered ξ number of times in cycle j, where $q_{\xi j} = e^{\frac{-\xi}{\tau}}$ (Table 1);
q_j	The average fraction (cumulative average up to ξ) of the quality level of items that have been recovered for their i^{th} time in cycle j, where $q_j = \frac{\sum_{i=1}^{\xi} q_{ij}}{\xi}$ (Table 3);
$\gamma_{\xi j}$	The actual proportion of returned items that can be remanufactured in cycle j, where $\gamma_{\xi j} = e^{\frac{-\xi q_{\xi}}{\tau}}$ (Table 2);
γ_j	The average fraction (cumulative average up to ξ) of returned items that can be remanufactured for their ith time in cycle j, where $\gamma_j = \frac{\sum_{i=1}^{\xi} \gamma_{ij}}{\xi}$ (Table 4);
c_{pm}	The unit purchasing cost for new items;
c_{prj}	The unit purchasing price for retuned items in cycle j, where $c_{prj} = c_{pm} e^{\frac{-1}{q_j}}$;
c_{inv}	The remanufacturing investment cost in the design process of an item in order to make it remanufactured τ number of times;
c_{invj}	The remanufacturing investment cost in cycle j in the design process of an item in order to make it remanufactured ξ number of times, where $c_{invj} = c_{inv}\left(1 - e^{\frac{-\xi}{q_j}}\right)$;
c_m	The unit manufacturing cost;
c_r	The unit remanufacturing cost;
c_s	The unit screening cost;
h_z	The holding cost per unit per unit time;

c_w	The unit cost for disposing deteriorated and defective waste items;
w_m	The switching cost from remanufacturing phase to manufacturing phase;
w_r	The switching cost from manufacturing phase to remanufacturing phase;
S_z	The setup/order cost per cycle.

Table 3. The average fraction (cumulative average up to ζ) of the quality level of items that recover for their *i*th time when $\tau = 1,\ 2, \ldots 8$.

1	**2**	**3**	**4**	**5**	**6**	**7**	**8**
0.368	0.607	0.717	0.779	0.819	0.846	0.867	0.882
	0.487	0.615	0.693	0.745	0.782	0.809	0.831
		0.533	0.619	0.679	0.723	0.757	0.783
			0.556	0.622	0.671	0.709	0.739
				0.571	0.624	0.665	0.698
					0.581	0.625	0.660
						0.588	0.626
							0.593

Table 4. The average fraction (cumulative average up to ζ) of returned items that can be remanufactured for their *i*th time when $\tau = 1,\ 2, \ldots 8$.

1	**2**	**3**	**4**	**5**	**6**	**7**	**8**
0.692	0.738	0.788	0.823	0.849	0.868	0.884	0.896
	0.715	0.749	0.781	0.807	0.828	0.845	0.859
		0.730	0.754	0.778	0.798	0.816	0.830
			0.739	0.758	0.776	0.793	0.807
				0.745	0.760	0.775	0.789
					0.749	0.762	0.775
						0.752	0.763
							0.754

Below is a list of all assumptions used in the paper:

1. The collection of returns occurs throughout the time interval at a rate $c_j(t)$.
2. Only a proportion $\gamma_j(0 \leq \gamma_j \leq 1)$ of the returned items can be remanufactured, and the amount $\left(1-\gamma_j\right)c_j(t)$ is disposed as waste outside the system.
3. New items are manufactured at a rate $P_{mj}(t)$ and the accepted returned items are remanufactured at a rate $P_{rj}(t)$ as good as new.
4. Items deteriorate at a rate $\delta_{zj}(t)$ while they are effectively in stock.
5. The demand rate $D_j(t)$ is satisfied from produced and remanufactured items.
6. The return rate is a varying demand-dependent rate, which is a decision variable.
7. The demand, product deterioration, manufacturing, and remanufacturing rates are arbitrary functions of time.
8. The values of all functions and input parameters can be adjusted for subsequent cycles.
9. Shortages are not allowed; this implies that $P_{mj}(t) > D_j(t)$, $P_{rj}(t) > \gamma_j c_j(t)$ and $P_{rj}(t) > D_j(t)\ \forall t \geq 0$.
10. There is no repair or replacement of deteriorated items.

4.2. The General Model

Figure 3 depicts a general framework of production and remanufacturing unified system, and Figure 4 depicts the behavior of such a unified system. In our model, demand in the first cycle is satisfied from production only (see Figure 4), as the inventory of returns at the beginning of the first cycle is zero (there are no returned items to be remanufactured). The process is repeated until inventory of product returns can be technologically attainable. Then, at the beginning of each cycle j, the system starts the production prosses until time

T_{1j}, by which point in time Q_{mj} units have been produced and stored in the serviceable stock. The inventory level of new items declines continuously and becomes zero by time T_{2j}. The deteriorated quantity is d_{gmj} units, which refers to the difference between Q_{mj} units that have been manufactured in cycle j and the satisfied demand during production cycle. The remanufacturing process starts at time T_{2j} until time T_{3j}, by which point in time the remanufactured quantity Q_{rj} units have been accumulated and stored in the serviceable stock. The collection of returns occurs throughout the time interval at a rate $c_j(t)$, in which a fraction $\gamma_j c_j(t)$ has been remanufactured and the remaining quantity $(1-\gamma_j)c_j(t)$ is disposed as waste outside the system. The remaining quantity $(1-\gamma_j)c_j(t)$ refers to returned items that have been remanufactured ξ number of times or items that do not meet the minimum acceptance quality level, q_{min}. The inventory level of remanufacturing items becomes zero by time T_{4j}. The deteriorated quantity is d_{grj} units, which refers to the difference between Q_{rj} units that have been remanufactured in cycle j and the satisfied demand during remanufacturing cycle. At time T_{4j}(the end of cycle j), Δ_j units have been accumulated and stored in the repairable stock, which constitutes the initial inventory of returned items for the next cycle. In our model, the term Δ_{j-1} governs the flow of products returned, which in turn affects the inventory of all quantities for each cycle and at the beginning of the first remanufacturing cycle, $\Delta_{j-1} = \Delta_0 = 0$. That is, the initial inventory of returns in the first remanufacturing cycle is set equal to zero. The deteriorated quantity in the repairable stock is d_{rj}, which denotes the difference between the returned quantity that have been accepted to be remanufactured and Q_{rj} units that have been remanufactured in cycle j. The process is repeated.

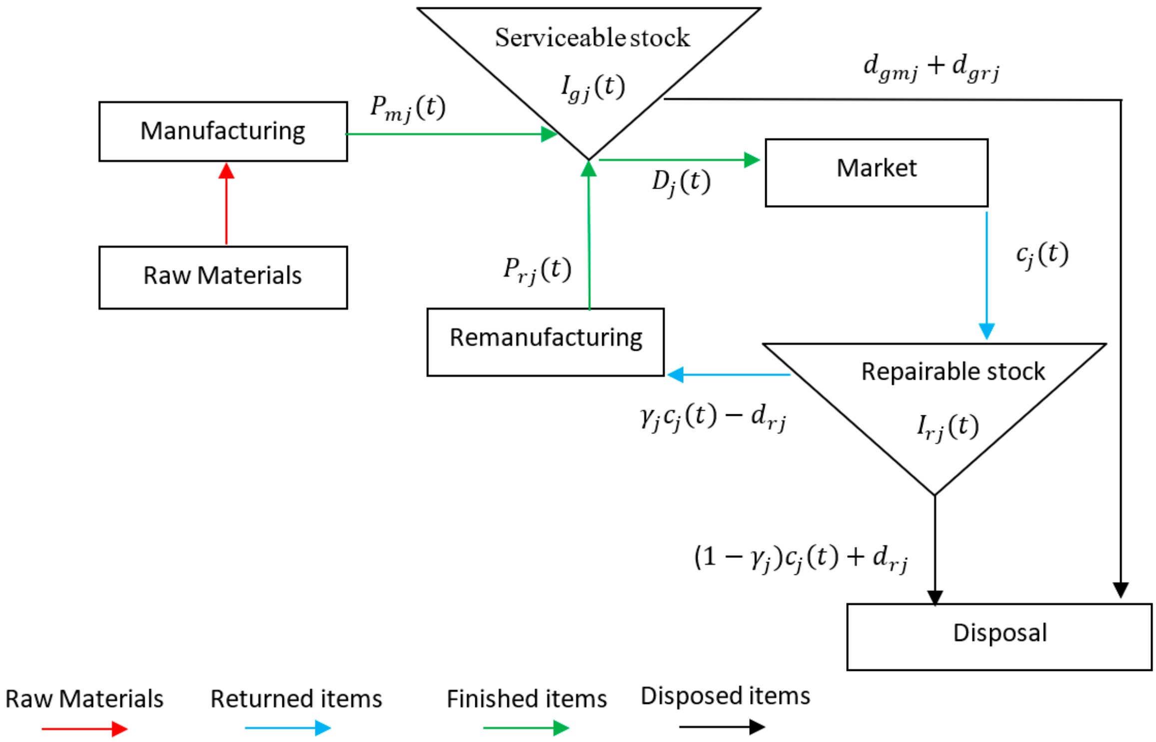

Figure 3. Product flow for production and remanufacturing system in one cycle.

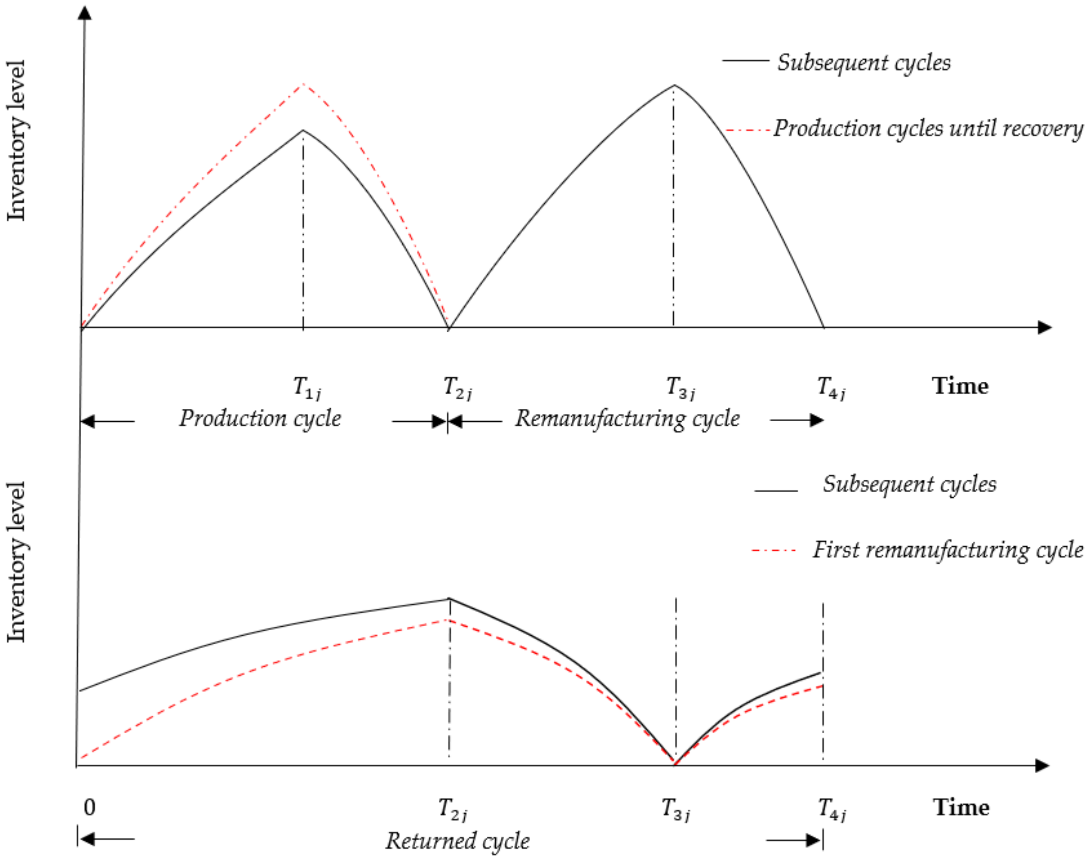

Figure 4. Inventory variation of manufactured, remanufactured, and returned items for one cycle.

The variations in the inventory levels, as shown in Figure 4, can be represented by the following differential equations:

$$\frac{dI_{gmj}(t)}{dt} = P_{mj}(t) - D_j(t) - \delta_{gmj} I_{gmj}(t), \qquad 0 \le t < T_{1j} \tag{1}$$

$$\frac{dI_{gmj}(t)}{dt} = -D_j(t) - \delta_{gmj} I_{gmj}(t), \qquad T_{1j} \le t \le T_{2j} \tag{2}$$

$$\frac{dI_{grj}(t)}{dt} = P_{rj}(t) - D_j(t) - \delta_{grj} I_{grj}(t), \qquad T_{2j} \le t < T_{3j} \tag{3}$$

$$\frac{dI_{grj}(t)}{dt} = -D_j(t) - \delta_{grj} I_{grj}(t), \qquad T_{3j} \le t \le T_{4j} \tag{4}$$

$$\frac{dI_{rj}(t)}{dt} = \gamma_j c_j(t) - \delta_{rj} I_{rj}(t), \qquad 0 \le t < T_{2j} \tag{5}$$

$$\frac{dI_{rj}(t)}{dt} = \gamma_j c_j(t) - P_{rj}(t) - \delta_{rj} I_{rj}(t), \qquad T_{2j} \le t < T_{3j} \tag{6}$$

$$\frac{dI_{rj}(t)}{dt} = \gamma_j c_j(t) - \delta_{rj} I_{rj}(t), \qquad T_{3j} \le t \le T_{4j} \tag{7}$$

with the boundary conditions

$$I_{gmj}(0) = 0,\ I_{gmj}(T_{2j}) = 0,\ I_{grj}(T_{2j}) = 0,\ I_{grj}(T_{4j}) = 0,\ I_{rj}(0) = \Delta_{j-1} \text{ and } I_{rj}(T_{3j}) = 0$$

Taking into account the above boundary conditions, the solutions of Equations (1)–(7) are given by

$$I_{gmj}(t) = e^{-h_{gmj}(t)} \int_0^t \left[P_{mj}(u) - D_j(u)\right] e^{h_{gmj}(u)} du, \qquad 0 \le t < T_{1j} \tag{8}$$

$$I_{gmj}(t) = e^{-h_{gmj}(t)} \int_t^{T_{2j}} D_j(u) e^{h_{gmj}(u)} du, \qquad T_{1j} \le t \le T_{2j} \tag{9}$$

$$I_{grj}(t) = e^{-h_{grj}(t)} \int_{T_{2j}}^t \left[P_{rj}(u) - D_j(u)\right] e^{h_{grj}(u)} du, \qquad T_{2j} \le t < T_{3j} \tag{10}$$

$$I_{grj}(t) = e^{-h_{grj}(t)} \int_t^{T_{4j}} D_j(u) e^{h_{grj}(u)} du, \qquad T_{3j} \le t \le T_{4j} \tag{11}$$

$$I_{rj}(t) = e^{-(h_{rj}(t) - h_{rj}(0))} \Delta_{j-1} + e^{-h_{rj}(t)} \int_0^t \left[\gamma_j c_j(u)\right] e^{h_{rj}(u)} du, \qquad 0 \le t < T_{2j} \tag{12}$$

$$I_{rj}(t) = e^{-h_{rj}(t)} \int_t^{T_{3j}} \left[P_{rj}(u) - \gamma_j c_j(u)\right] e^{h_{rj}(u)} du, \qquad T_{2j} \le t < T_{3j} \tag{13}$$

$$I_{rj}(t) = e^{-h_{rj}(t)} \int_{T_{3j}}^t \gamma_j c_j(u) e^{h_{rj}(u)} du, \qquad T_{3j} \le t \le T_{4j} \tag{14}$$

respectively, where

$$h_{zj}(t) = \int \delta_{zj}(t) dt. \tag{15}$$

From Equations (1)–(14), we note that each function is solely modeled and, therefore, functions may or may not be related to each other.

The total costs per cycle for the underlying inventory model are given as follows:

Purchasing price for returned items (c_{prj}) + Inspection cost (c_s) + Disposal cost for waste and deteriorated items (c_w) + Material cost for new items (c_{pm}) + Manufacturing cost (c_m) + Remanufacturing cost (c_r) + Holding costs (h_z)+ Switching cost for manufacturing (w_m) + Switching costs for remanufacturing (w_r) + Investment cost (c_{invj}) + Setup and order costs (S_z) =

$$\left(c_{pm} e^{\frac{-1}{q_j}} + c_s + c_w(1 - \gamma_j)\right) \int_0^{T_{4j}} c_j(u) du + c_w\left(d_{gmj} + d_{grj} + d_{rj}\right) + \left(c_{pm} + c_m\right) \int_0^{T_{1j}} P_{mj}(u) du$$
$$+ c_r \int_{T_{2j}}^{T_{3j}} P_{rj}(u) du + h_z + w_m + w_r + c_{inv}\left(1 - e^{\frac{-\xi}{q_j}}\right) + S_{pm} + S_{pr} + S_r.$$

Now, we denote $K = w_m + w_r + S_{pm} + S_{pr} + S_r$, and $d_j = d_{gmj} + d_{grj} + d_{rj}$.

As can be seen, the returned, manufacturing, and remanufacturing costs involve deteriorated items as well. Note that this is very well recognized in the literature because items deteriorate while they are effectively in stock (e.g., [38–41]).

By integrating Equations (8)–(14), over the proper limits, the holding costs can be obtained as follows:

Holding costs at the serviceable stock applying for both produced and remanufactured items:

$$h_{gm}\left[I_{gmj}(0, T_{1j}) + I_{gmj}(T_{1j}, T_{2j})\right] + h_{gr}\left[I_{grj}(T_{2j}, T_{3j}) + I_{grj}(T_{3j}, T_{4j})\right]$$

Holding cost for returned items at the repairable stock:

$$h_r\left[I_{rj}(0, T_{2j}) + I_{rj}(T_{2j}, T_{3j}) + I_{rj}(T_{3j}, T_{4j})\right]$$

Therefore, the per-unit time total cost function for the unified inventory system during time $\left[0, T_{4j}\right]$, as a function of T_{1j}, T_{2j}, T_{3j} and T_{4j} denoted by $L\left(T_{1j}, T_{2j}, T_{3j}, T_{4j}\right)$, is given by

$$L(T_{1j}, T_{2j}, T_{3j}, T_{4j}) = \frac{1}{T_{4j}}\Bigg\{\left(c_{pm}e^{\frac{-1}{q_j}} + c_s + c_w(1-\gamma_j)\right)\int_0^{T_{4j}} c_j(u)du + (c_{pm}+c_m)\int_0^{T_{1j}} P_{mj}(u)du + c_r\int_{T_{2j}}^{T_{3j}} P_{rj}(u)du + h_{gm}\Big[-\int_0^{T_{1j}} H_{gmj}(u)[P_{mj}(u)-D_j(u)]e^{h_{gmj}(u)}du + H_{gmj}(T_{1j})\int_0^{T_{1j}}[P_{mj}(u)- D_j(u)]\,e^{h_{gmj}(u)}du + \int_{T_{1j}}^{T_{2j}}[H_{gmj}(u) - H_{gmj}(T_{1j})]D_j(u)e^{h_{gmj}(u)}du\Big] + h_{gr}\Big[\int_{T_{2j}}^{T_{3j}}[H_{grj}(T_{3j}) - H_{grj}(u)][P_{rj}(u)- D_j(u)]\,e^{h_{grj}(u)}du + \int_{T_{3j}}^{T_{4j}}[H_{grj}(u) - H_{grj}(T_{3j})]D_j(u)e^{h_{grj}(u)}du\Big] + h_r\Big[[H_{rj}(T_{2j}) - H_{rj}(0)]e^{h_{rj}(0)}\Delta_{j-1} + H_{rj}(T_{2j})\int_0^{T_{3j}}\gamma_j c_j(u)e^{h_{rj}(u)}du + \int_{T_{2j}}^{T_{3j}}[H_{rj}(u) - H_{rj}(T_{2j})]P_{rj}(u)e^{h_{rj}(u)}du - \int_0^{T_{4j}} H_{2j}(u)\gamma_j c_j(u)e^{h_{2j}(u)}du + H_{rj}(T_{4j})\int_{T_{3j}}^{T_{4j}}\gamma_j c_j(u)e^{h_{rj}(u)}du\Big] + c_w d_j + c_{inv}\left(1 - e^{\frac{-\xi}{q_j}}\right) + K\Bigg\}, \tag{16}$$

where

$$H_{zj}(t) = \int e^{-h_{zj}(t)}dt. \tag{17}$$

Note that Equation (16) is a modified version of that of Alamri [31]. Therefore, and to avoid repetition, the existence of the solution for Equation (16), its uniqueness, and its associated global optimality can be derived by a quite similar way. Interested readers are referred to Alamri [28,31].

From Equations (8)–(14), it is clear that $T_{ij}, i(i = 1,2,3,4)$ are related to each other as follows:

$$T_{1j} < T_{2j} < T_{3j} < T_{4j}, \tag{18}$$

$$\int_0^{T_{1j}} P_{mj}(u)e^{h_{gmj}(u)}du = \int_0^{T_{2j}} D_j(u)e^{h_{gmj}(u)}du, \tag{19}$$

$$\int_{T_{2j}}^{T_{3j}} P_{rj}(u)e^{h_{rj}(u)}du = e^{h_{rj}(0)}\Delta_{j-1} + \int_0^{T_{3j}}\gamma_j c_j(u)e^{h_{rj}(u)}du, \tag{20}$$

$$\int_{T_{2j}}^{T_{3j}} P_{rj}(u)e^{h_{grj}(u)}du = \int_{T_{2j}}^{T_{4j}} D_j(u)e^{h_{grj}(u)}du, \tag{21}$$

$$R_j = \int_0^{T_{4j}} c_j(u)du, \tag{22}$$

$$\Delta_{j-1} = e^{-h_{rj-1}(T_{4j-1})}\int_{T_{3j-1}}^{T_{4j-1}}\gamma_{j-1}c_{j-1}(u)e^{h_{rj-1}(u)}du. \tag{23}$$

For example, relations 19 and 20 guarantee that the level of inventory must have equal values for the production and the remanufacturing phases for $t = T_1$ and for $t = T_3$. Note that the term Δ_{j-1} is modeled as a deterministic value, i.e., it impacts the behavior of each cycle until the system plateaus. This is a key in the mathematical formulation and, consequently, it ensures that the model remains viable and generates optimal solution for each cycle in case the values of the input parameters need to be adjusted [31].

It can be seen from Equation (19) that $P_{mj}(t) > D_j(t) \Longrightarrow$ Equation (19) $\Leftrightarrow T_{1j} < T_{2j}$. In addition, from Equation (19), $T_{1j} = 0 \Longrightarrow T_{2j} = 0 \Longrightarrow$, a pure strategy of no manufacturing option. Therefore, $P_{rj}(t) > D_j(t) \Longrightarrow$ Equation (21) $\Leftrightarrow T_{3j} < T_{4j}$. Conversely, $T_{3j} = 0 \Longrightarrow T_{4j} = 0$. Thus, from Equations (21) and (22), $T_{2j} = T_{3j} \Longrightarrow T_{3j} = T_{4j} \Longrightarrow T_{1j} < T_{2j} \Longrightarrow$, a pure strategy of no remanufacturing option. Conversely, $T_{1j} = 0 \Longrightarrow T_{2j} = 0 \Longrightarrow T_{3j} = 0 \Longrightarrow T_{4j} = 0$. Thus, $T_{1j} > 0 \Longrightarrow T_{1j} < T_{2j}$ and $T_{2j} < T_{3j} \Longrightarrow < T_{3j} < T_{4j}$. Hence, Equations (19)–(22) implies constraint (18), and, therefore, constraint (18) can be ignored. Thus, our goal is to solve the following objective function:

$$(Z) = \left\{\begin{matrix} \text{minimize } L(T_{1j}, T_{2j}, T_{3j}, T_{4j}) \text{given by Equation (16)} \\ \text{subject to} \\ \text{Equations(19–22)} \\ 0 \le \varnothing_j < 1 \end{matrix}\right\}. \tag{24}$$

Solution Procedure

As can be seen, Equations (19)–(22) can be used to obtain T_{ij} as functions of R_j, where

$$T_{ij} = f_{ij}(R_j). \tag{25}$$

Taking also into account Equations (19)–(22), the objective function (Z) is reduced to the following function, with the variable R_j (say (Z_1)) subject to $0 \leq \varnothing_j < 1$.

$$\begin{aligned} L(R_j) = \tfrac{1}{f_{4j}}\Bigg\{ & \left(c_{pm}e^{\frac{-1}{q_j}} + c_s + c_w(1-\gamma_j)\right)\int_0^{f_{4j}} c_j(u)du + (c_{pm}+c_m)\int_0^{f_{1j}} P_{mj}(u)du + c_r\int_{f_{2j}}^{f_{3j}} P_{rj}(u)du + \\ & h_{gm}\left[-\int_0^{f_{1j}} H_{gmj}(u)P_{mj}(u)e^{h_{gmj}(u)}du + \int_0^{f_{2j}} H_{gmj}(u)D_j(u)e^{h_{gmj}(u)}du\right] + \\ & h_{gr}\left[-\int_{f_{2j}}^{f_{3j}} H_{grj}(u)P_{rj}(u)e^{h_{grj}(u)}du + \int_{f_{2j}}^{f_{4j}} H_{grj}(u)D_j(u)e^{h_{grj}(u)}du\right] + h_r\Big[-H_{rj}(0)e^{h_{rj}(0)}\Delta_{j-1} + \\ & \int_{f_{2j}}^{f_{3j}} H_{rj}(u)P_{rj}(u)e^{h_{rj}(u)}du + \int_{f_{3j}}^{f_{4j}} H_{rj}(f_{4j})\gamma_j c_j(u)e^{h_{rj}(u)}du - \int_0^{f_{4j}} H_{rj}(u)\gamma_j c_j(u)e^{h_{rj}(u)}du\Big] + c_w d_j + \\ & c_{inv}\left(1 - e^{\frac{-\xi}{q_j}}\right) + K\Bigg\}, \end{aligned} \tag{26}$$

where $h_{zj}(t)$ is given by Equation (15) and $H_{zj}(t)$ is given by Equation (17).

Hence, if $L = \frac{l}{f_{4j}}$, then the necessary condition for the objective function (Z_1) to have a minimum is

$$l'_{R_j} f_{4j} = f'_{4j,R_j} l, \tag{27}$$

where l'_{R_j} and f'_{4j,R_j} represent, respectively, the derivatives of l and f_{4j} with respect to R_j.

Now, considering Equation (26), we obtain

$$\begin{aligned} l'_{R_j} = & \left(c_{pm}e^{\frac{-1}{q_j}} + c_s + c_w(1-\gamma_j)\right) + (c_{pm}+c_m)f'_{1j,R_j}P_{mj}(f_{1j}) + c_r P_{rj}\left(f'_{3j,R_j}(f_{3j}) - f'_{2j,R_j}(f_{2j})\right) + \\ & h_{gm}\left[-H_{gmj}(f_{1j})f'_{1j,R_j}P_{mj}(f_{1j})e^{h_{gmj}(f_{1j})} + H_{gmj}(f_{2j})f'_{2j,R_j}D_j(f_{2j})e^{h_{gmj}(f_{2j})}\right] + \\ & h_{gr}\Big[-H_{grj}(f_{3j})f'_{3j,R_j}P_{rj}(f_{3j})e^{h_{grj}(f_{3j})} + H_{grj}(f_{2j})f'_{2j,R_j}P_{rj}(f_{2j})e^{h_{grj}(f_{2j})} + H_{grj}(f_{4j})f'_{4j,R_j}D_j(f_{4j})e^{h_{grj}(f_{4j})} - \\ & H_{grj}(f_{2j})f'_{2j,R_j}D_j(f_{2j})e^{h_{grj}(f_{2j})}\Big] + h_r\Big[\Delta_j + f'_{3j,R_j}H_{rj}(f_{3j})P_{rj}(f_{3j})e^{h_{rj}(f_{3j})} - f'_{2j,R_j}H_{rj}(f_{2j})P_{rj}(f_{2j})e^{h_{rj}(f_{2j})} - \\ & H_{rj}(f_{4j})f'_{3j,R_j}\gamma_j c_j(f_{3j})e^{h_{rj}(f_{3j})}\Big]. \end{aligned} \tag{28}$$

From which Equation (27) $\Leftrightarrow$

$$L = \frac{l}{f_{4j}} = \frac{l'_{R_j}}{f'_{4j,R_j}}. \tag{29}$$

Equation (29) can now derive the optimal quantity of R_j and the per-unit time total cost. Then Equations (19)–(22) can be used to obtain the decision variables $T_{ij}, i(i = 1, 2, 3, 4)$.

To find the optimal ξ for a given τ, the following steps are required:

1. In the first remanufacturing cycle, start by setting $\xi = 1$, $c_{invj} = c_{inv\xi}$, $c_{prj} = c_{pr\xi}$, $\lambda_j = \lambda_\xi$ and $\Delta_{j-1} = \Delta_0 = 0$ in Equation (29) and compute L_1.
2. Repeat step 1 for Δ_{j-1}(obtained from step 1) to compute $L_{2,1}$.
3. Set $\xi = 2, c_{invj} = c_{inv\xi}, c_{prj} = c_{pr\xi}, \lambda_j = \lambda_\xi$ and Δ_{j-1}(obtained from step 1) in Equation (29) and compute L_2.
4. Repeat step 3 for Δ_{j-1}(obtained from step 3) to compute $L_{3,2}$.
5. Set $\xi = 3, c_{invj} = c_{inv\xi}, c_{prj} = c_{pr\xi}, \lambda_j = \lambda_\xi$ and Δ_{j-1}(obtained from step 3) in Equation (29) and compute L_3.
6. Repeat step 5 for Δ_{j-1}(obtained from step 5) to compute $L_{4,3}$.
7. Repeat steps 5 and 6 for $\xi = 4, 5, \ldots, \tau$ and Δ_{j-1}(obtained to find L_{j-1}) to compute $L_{j,\xi}$.

8. Set $\xi^* = \xi$ when $\left(L_{j,\xi}\right)$ at its minimum and continue to insert Δ_{j-1} in Equation (29) until the system plateaus.

Remark 1. *For a mature system, applying the above steps will generate the optimal remanufacturing policy, where Δ_{j-1} represents the current on-hand inventory of returned items.*

5. Illustrative Examples for Different Settings

In this section, we emphasize the practical application of the proposed model by presenting numerical examples and special cases that reflect different realistic situations. Products that may encounter remanufacturing include tires, motor vehicle parts, electric motors, computers, air-conditioning units, photocopiers, telecommunication equipment, aerospace devices, aircraft parts, gaming machines, medical equipment, vending machines, automotive parts, industrial equipment, televisions, etc. [42]. In real-life settings, manufacturing, remanufacturing, demand, return, and deterioration rates may vary with time or with any other factors [43–51]. Accordingly, the proposed model allows the incorporation of different forms of time-varying functions. Let us now consider the following functions for time-varying rates:

$P_{mj}(t) = \pi_{mj}t + \phi_{mj}, P_{rj}(t) = \pi_{rj}t + \phi_{rj}, D_j(t) = \alpha_j t + r_j, c_j(t) = \varnothing_j D_j(t)$ and $\delta_{zj}(t) = \frac{l_{zj}}{\vartheta_{zj} - \beta_{zj}t}$, where $\phi_{mj}, \phi_{rj}, r_j, \varnothing_j, \vartheta_{zj} > 0, \pi_{mj}, \pi_{rj}, \alpha_j, l_{zj}, \beta_{zj} \geq 0and\beta_{zj}t < \vartheta_{zj}$. As can be seen, the deteriorated function $\delta_{zj}(t)$ is an increasing function of time.

In real-life settings, all function or input parameters are subject to adjustment due to external competitiveness and/or internal challenges or due to price fluctuations. Therefore, our model is viable if all values are adjusted for subsequent cycles.

The objective function (Z_1) was coded in *MATLAB* for the input parameters that are presented in Table 5 below, and solutions were obtained using Equation (29) subject to $0 \leq \varnothing_j < 1$. Note that each of the return, manufacturing, and remanufacturing rates is solely modeled. This is so because they may or may not be considered as functions of the demand rate. Now, let us consider the following functions for varying return, manufacturing, and remanufacturing rates as functions of the demand rate:

$$c_j(t) = \varnothing_j D_j(t),\ P_{mj}(t) = \frac{D_j(t)}{0.6} \text{ and } P_{rj}(t) = \frac{D_j(t)}{0.3}$$

Table 5. Input parameters for time-varying rates.

h_{gm} 1.6 USD/unit/month	h_{gr} 1.6 USD/unit/month	h_r 1.2 USD/unit/month	r_j 1000 Unit/month	α_j 130 Unit/month	ϕ_{1j} 1666.7 Unit/month
π_{1j} 216.7 Unit/month	ϕ_{2j} 3333.3 Unit/month	π_{2j} 433.3 Unit/month	l_{gm} 1 Unit/month	l_{gr} 1 Unit/month	l_r 1 Unit/month
ϑ_{gm} 50 Unit/month	ϑ_{gr} 50 Unit/month	ϑ_r 40 Unit/month	β_{gm} 0.25 Unit/month	β_{gr} 0.25 Unit/month	β_r 0.25 Unit/month
c_{inv} 4000 USD/cycle	w_m 100 USD/cycle	w_r 100 USD/cycle	S_{pm} 2400 USD/cycle	S_{pr} 1600 USD/cycle	S_r 1200 USD/cycle
c_w 0.2 USD/unit	c_m 2 USD/unit	c_r 1.2 USD/unit	c_{pm} 5 USD/unit	c_s 0.5 USD/unit	

5.1. Example 1

In this example, we investigate how the model would behave with respect to the parameters that are listed in Table 5. In this example, we consider $\tau = 5$, i.e., the expected number of times an item can be remanufactured in its lifecycle is 5. In this case, $c_{invj} = c_{inv}(1 - e^{\frac{-\zeta}{q_j}})$ and $c_{prj} = c_{pm}e^{\frac{-1}{q_j}}$. The optimal values of ϕ_j^*, f_{4j}^*, Q_{mj}^*, Q_{rj}^*, R_j^*, Δ_j^*, d_j^*, L_j^*, and l_j^* until the system plateaus are given in Table 6. The percentage of retuned items in the first remanufacturing cycle is equal to $\gamma_1 = 0.849$ (recall Table 4), resulting in a total number of $R_1^* = 2406$ units. This retuned quantity is accumulated by time $T_{41}^* = 2.954$ months ≈ 90 days at a return rate of $\phi_1^* = 0.683$ or 68.3% of the demand rate. At time T_{4j}, $\Delta_1^* = 571$ units, which constitutes the initial inventory of returned items for the second cycle. The deteriorated quantity in the serviceable stock is $d_{g1} = 27\left(d_{gm1} = 16 \text{ and } d_{gr1} = 11\right)$ units, and $d_{r1} = 38$ units have deteriorated in the repairable stock, i.e., $d_1^* = 16 + 11 + 38 = 65$ units. This deteriorated quantity can be sold at a salvage price or (as in this example) disposed at a charge. By time $T_{11}^* = 1.178$ months ≈ 36 days, the optimal produced quantity is $Q_{m1}^* = 2113$ units, which satisfies demand until time $T_{21}^* = 1.87$ months ≈ 57 days (the time by which the remanufacturing process started). By time $T_{31}^* = 2.21$ months ≈ 67 days, the optimal remanufactured quantity is $Q_{r1}^* = 1434$ units, which satisfies demand until time $T_{41}^* = 2.954$ months ≈ 90 days. The monthly cost is $L_1^* = USD\ 11,332$ and the cost for the first remanufacturing cycle is $l_1^* = USD\ 33,475$.

Table 6. Optimal results for varying rates when $\tau = 5$ and $c_{inv} = USD\ 4000$.

j	ζ_j^*	c_{invj}	c_{prj}	λ_j^*	ϕ_j^*	f_{4j}^*	Q_{mj}^*	Q_{rj}^*	R_j^*	Δ_j^*	d_j^*	L_j^*	l_j^*
1	1	2821	1.474	0.849	0.683	2.954	2113	1434	2406	571	65	11,332	33,475
2	2	3727	1.305	0.807	0.614	2.692	1624	1562	1944	530	69	11,155	30,031
3	3	3952	1.147	0.778	0.688	2.773	1663	1634	2251	598	74	11,206	31,077
4	4	3994	1.001	0.758	0.736	2.733	1547	1697	2369	646	75	11,081	30,287
5	5	3999	0.868	0.745	0.791	2.702	1442	1760	2512	707	76	10,948	29,582
6	5	3999	0.868	0.745	0.771	2.652	1378	1755	2397	688	75	10,895	28,891
7	5	3999	0.868	0.745	0.778	2.668	1399	1757	2435	694	75	10,912	29,117
8	5	3999	0.868	0.745	0.776	2.663	1392	1756	2423	692	75	10,907	29,046
9	5	3999	0.868	0.745	0.776	2.663	1392	1756	2423	692	75	10,907	29,046

Note that in the first remanufacturing cycle, the initial inventory of returned items is zero as there are no returned items to be remanufactured. Accordingly, $Q_{rj}^*(Q_{mj}^*)$ attain their minimum (maximum) values in this cycle, resulting in a dramatic decrease in the manufactured quantity in the second cycle. Note that, ϕ_j^*, Δ_j^*, and R_j^* attain their minimum values in the second cycle because of the effect of the first cycle. Moreover, cycles $j = 2,\ 3,\ 4,$ and 5 are influenced by c_{invj}, c_{prj}, Δ_{j-1}, and λ_j; consequently, the optimal values vary from cycle to cycle, and ϕ_j^*, Δ_j^*, Q_{rj}^*, and R_j^* reach their maximum values in the fifth cycle. As a result, f_{4j}^*, L_j^*, l_j^*, and Q_{mj}^* approach their minimum values in the sixth cycle before the system plateaus in the eighth cycle (Table 6). Therefore, when the system plateaus, the buyback proportion is set equal to $\phi_8^* = 0.776$ and the use proportion (that attains its minimum value in the fifth cycle) is set equal to $\lambda_5 = 0.745$. This implies that the reusable proportion is set equal to $\phi_8^*\lambda_5 = 0.776 \times 0.745 = 0.5781$, or 57.8% of demand rate. Figure 5 depicts the effect of c_{invj}, c_{prj}, Δ_{j-1}, and λ_j on the optimal values until the fifth cycle and the sole effect of Δ_{j-1} until these values plateau. Note that in cycles $j = 1,\ 2,\ \ldots,\ 5$, all returned items have been remanufactured fewer than or equal to $j - 1$ number of times upon recovery and fewer than or equal to $\zeta_j^* - 1$ when recovered for subsequent cycles. This implies that the remanufacturing number of times for an item is tangible, definite, tractable, and modeled. Finally, in cycles $j = 1,\ \ldots\ldots,\ 5$, c_{invj}, c_{prj}, Δ_{j-1}, and λ_j vary from cycle to cycle. Unlike previous works, excluding the work of Alamri [31], this, indeed, provides evidence that the proposed model is viable for the case that the values of the input parameters are distinct for any given cycle.

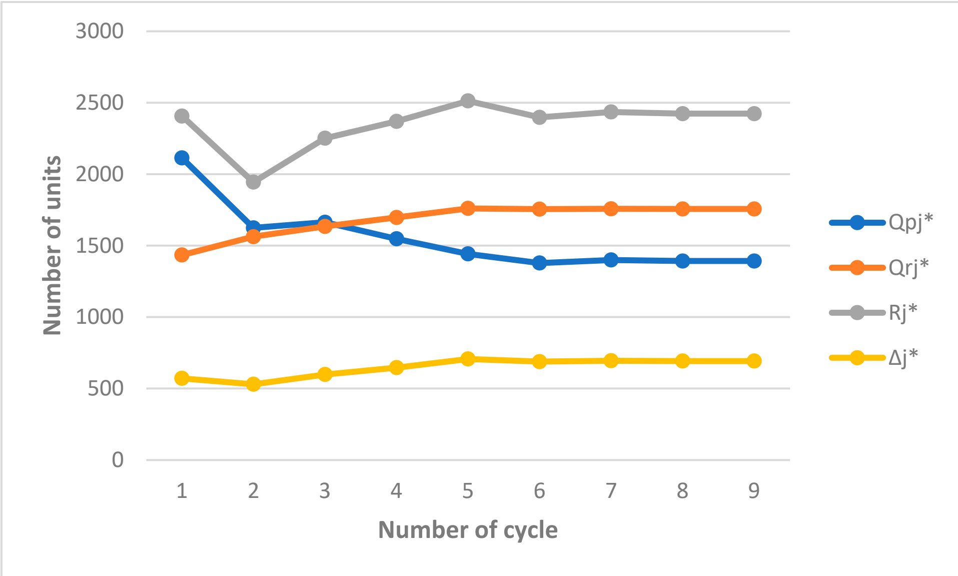

Figure 5. The effect of model parameters on the optimal values when $\tau = 5$ and $c_{inv} = USD$ 4000.

As illustrated in example 1, interested readers and practitioners can implement other forms of time-varying functions in the general model to assess the consequences of distinct strategies.

5.2. Example 2

In this example, we repeat example 1 to examine the effects on the optimal values when $\tau = 3$. As can be seen from Table 7, the optimal values behave similarly when the expected number of times an item can be remanufactured in its lifecycle decreases from five to three. The only exception is that the value of R_j^* in the third cycle experiences a slight decrease by 10 units from that accumulated in the first cycle. This can be justified by the fact that the value of Δ_j^* in the second cycle is greater than that accumulated in the first cycle (see Table 6). Note that ϕ_j^*, Δ_j^*, and Q_{rj}^* reach their maximum values in the third cycle and f_{4j}^*, L_j^*, l_j^*, and Q_{mj}^* approach their minimum values in the fourth cycle before the system plateaus in the eighth cycle (Table 7).

Table 7. Optimal results for varying rates when $\tau = 3$ and $c_{inv} = USD$ 4000.

j	ζ_j^*	c_{invj}	c_{prj}	λ_j^*	ϕ_j^*	f_{4j}^*	Q_{mj}^*	Q_{rj}^*	R_j^*	Δ_j^*	d_j^*	L_j^*	l_j^*
1	1	3009	1.238	0.788	0.770	2.981	2089	1498	2741	623	66	11,324	33,761
2	2	3845	0.983	0.749	0.736	2.684	1497	1679	2320	632	73	11,006	29,544
3	3	3986	0.765	0.730	0.855	2.716	1415	1806	2731	768	77	10,885	29,565
4	3	3986	0.765	0.730	0.808	2.604	1277	1793	2460	721	74	10,770	28,049
5	3	3986	0.765	0.730	0.825	2.645	1325	1798	2558	738	75	10,811	28,592
6	3	3986	0.765	0.730	0.819	2.630	1308	1796	2522	732	75	10,796	28,398
7	3	3986	0.765	0.730	0.821	2.636	1315	1797	2535	734	75	10,801	28,473
8	3	3986	0.765	0.730	0.820	2.634	1312	1797	2530	733	75	10,800	28,444
9	3	3986	0.765	0.730	0.820	2.634	1312	1797	2530	733	75	10,800	28,444

5.3. Example 3

In this example, we repeat example 2 by increasing the investment cost, c_{inv}, from USD 4000 to USD 6000 to investigate the behavior of the optimal values. Table 8 reveals that the optimal situation in this case is to remanufacture once ($\zeta = 1$). Note that c_{invj}, c_{prj}, and λ_j

remain static until the system plateaus and, consequently, the only factor affecting the optimal values is Δ_{j-1}. As can be seen from Table 8, the model behaves similarly with respect to ϕ_j^*, Δ_j^*, and R_j^* that reach their maximum values in the first cycle, and f_{4j}^*, L_j^*, and l_j^* attain their minimum values in the second cycle before the system plateaus in the sixth cycle (Table 8). Note that $Q_{rj}^*(Q_{mj}^*)$ reach their minimum (maximum) values in the first cycle since the inventory of returned items is zero (see also Tables 6 and 7). It is worth noting here that $L_{4,3}^* = USD\ 11,441 > L_{2,1}^* = USD\ 11,351$ (recall solution steps). However, when the system plateaus for $\zeta = 3$, the difference between the total minimum cost per month is negligible, i.e., $L^{\zeta=3} = USD\ 11,464 > L^{\zeta=1} = 11,428$.

Table 8. Optimal results for varying rates when $\tau = 3$ and $c_{inv} = USD$ 6000.

j	ζ_j^*	c_{invj}	c_{prj}	λ_j^*	ϕ_j^*	f_{4j}^*	Q_{mj}^*	Q_{rj}^*	R_j^*	Δ_j^*	d_j^*	L_j^*	l_j^*
1	1	4514	1.238	0.788	0.754	3.224	2318	1614	2940	656	79	11,809	38,073
2	1	4514	1.238	0.788	0.615	2.768	1647	1644	2010	545	77	11,351	31,421
3	1	4514	1.238	0.788	0.645	2.859	1772	1645	2187	571	78	11,446	32,730
4	1	4514	1.238	0.788	0.638	2.838	1743	1645	2145	565	78	11,424	32,425
5	1	4514	1.238	0.788	0.640	2.843	1749	1644	2154	567	78	11,429	32,489
6	1	4514	1.238	0.788	0.639	2.843	1749	1645	2153	567	78	11,428	32,487
7	1	4514	1.238	0.788	0.639	2.843	1749	1645	2153	567	78	11,428	32,487

5.4. Special Cases

5.4.1. Case 1

In this case (Case 1), we replicate Tables 6 and 7 to investigate the work of Alamri [31] for the set of input parameters as listed in Table 5. In Case 1, we let $w_r = w_m = c_{invj} = c_s = 0$, $c_w = 0.1$, $c_{prj} = 1$, $\varnothing_j = 0.231$ and $\lambda_j = 0.875$, which are identical to those of Alamri [31]. Note that $c_j(t) = \varnothing_j D_j(t)$, and an item is recovered an indefinite number of times. Now, considering the above values in Equation (29), the results are obtained as shown in Table 9. Note that Table 9 is identical to Table 3 (page 529) in Alamri [31]. This, indeed, confirms and ensures the validity and robustness of our general mathematical formulation. It is worth noting here that Case 1 regenerates the optimal values of the general model of those of [31], from which we are sure that all the examples and special cases provided and solved in Alamri [31] can also represent special cases of our model (e.g., [23,24,27]).

Table 9. Optimal results for varying rates as in Alamri [31] with $\varnothing_j = 0.231$ and $\lambda_j = 0.875$.

j	f_{4j}^*	Q_{mj}^*	Q_{rj}^*	R_j^*	Δ_j^*	d_j^*	L_j^*	l_j^*
1	2.454	2373	493	657	69	33	10,317	25,314
2	2.371	2223	533	632	75	34	10,220	24,231
3	2.364	2210	536	630	75	34	10,211	24,140
4	2.364	2210	536	630	75	34	10,211	24,140

5.4.2. Case 2

In this case (Case 2), we investigate the behavior of the model when the demand rate is adjusted within cycles. In real-life settings, all function or input parameters are subject to adjustment due to external competitiveness and/or internal challenges or due to price fluctuations. Let us now support our finding in Example 1 and show the validity of our model if the input parameters change their values for subsequent cycles. In Case 2, we illustrate how the system would behave if the decision-maker wished to increase the demand rate in the eighth cycle to evaluate the consequences of such increase. In Case 2, we assume that $D_j(t) = \alpha_j t + r_j$, where $\alpha_j = 156$ and $r_j = 1200$. Note that row one of Table 10 represents the optimal values of the eighth cycle of example 2 (Table 7). A comparison between Tables 7 and 10 reveals that in the first cycle of the adjustment, all optimal values increase except f_{4j}^*, which encounters a slight decrease. Such increase can be justified by the increase of ϕ_j^* and Δ_j^*. Note that all decision variables attain their maximum (minimum) values in the ninth (tenth) cycle, i.e., in the first (second) cycle of the adjustment of the demand rate.

Table 10. Optimal results for varying rates when $\tau = 3$, $c_{inv} = USD\ 4000$, $\alpha_j = 156$ units, and $r_j = 1200$ units.

j	ζ_j^*	c_{invj}	c_{prj}	λ_j^*	ϕ_j^*	f_{4j}^*	Q_{mj}^*	Q_{rj}^*	R_j^*	Δ_j^*	d_j^*	L_j^*	l_j^*
8 *	3	3986	0.765	0.730	0.820	2.634	1312	1797	2530	733	75	10,800	28,444
9	3	3986	0.765	0.730	0.936	2.488	1353	2139	3248	912	76	12,099	30,106
10	3	3986	0.765	0.730	0.884	2.340	1154	2104	2860	847	71	11,925	27,908
11	3	3986	0.765	0.730	0.905	2.396	1228	2118	3007	873	74	11,992	28,737
12	3	3986	0.765	0.730	0.897	2.375	1200	2113	2950	863	73	11,966	28,419
13	3	3986	0.765	0.730	0.900	2.383	1210	2115	2971	866	73	11,976	28,538
14	3	3986	0.765	0.730	0.899	2.380	1205	2115	2965	866	73	11,972	28,489
15	3	3986	0.765	0.730	0.899	2.380	1205	2115	2965	866	73	11,972	28,489

Cycle 8 *, which represents the steady state situation of Table 7 when $\alpha_j = 130$ units and $r_j = 1000$ units.

5.4.3. Case 3

In this case (Case 3), we repeat example 2 to investigate the sensitivity analysis of the optimal values in different settings. Row one of Table 11 shows the optimal values (base model) of the first cycle of example 2 (Table 7). Table 11 illustrates the sensitivity analysis of distinct model parameters for comparison purposes with the results that are derived for example 2. Note that in all cases the model behaves as expected. For example, when the holding costs are equal, i.e., $h_{gm} = h_{rm} = h_r = 1.2$, we note from Table 11 that all optimal values are higher than the values that are computed for row 1 (base model), except L_j^*, which experiences a lower cost. This can be justified by the fact that the system reduces the holding cost at the serviceable stock. Note that similar behavior is also observed for $S_{gm} = S_{gr} = S_r = 2000$ except ϕ_j^*, which is associated with a slight decrease. This can be attributed to the increase of the order cost for returned items. Similarly, when $c_{pm} = 6$, which also affects c_{pr}, all optimal values are higher than the values that are computed for row 1 (base model), except f_{4j}^* and Q_{mj}^*, which are associated with lower values. This can be attributed to the fact that ϕ_j^* increased by 7.8% $\left(\frac{0.835-0.770}{0.835} = 0.0778\right)$. For $c_w = 0.3$, we note that f_{4j}^*, L_j^*, l_j^*, and Q_{mj}^* are associated with greater values than those of the base model, and ϕ_j^*, Δ_j^*, Q_{rj}^*, and R_j^* are associated with lower values. This can be justified by the fact that the system reaps the benefit of not disposing of more items. Finally, when the deterioration rates are equal, i.e., $\vartheta_{gm} = \vartheta_{gr} = \vartheta_r = 30$, all optimal values are less than the values that are computed for row 1 (base model), except L_j^* and ϕ_j^*, which are associated with greater values. As expected, more items (97 units) are deteriorated and disposed of outside the system due to the increase of the deterioration rates.

Table 11. Sensitivity analysis of the optimal results for varying rates when $\tau = 3$ and $c_{inv} = USD\ 4000$.

Parameters	j	ζ_j^*	c_{invj}	c_{prj}	λ_j^*	ϕ_j^*	f_{4j}^*	Q_{mj}^*	Q_{rj}^*	R_j^*	Δ_j^*	d_j^*	L_j^*	l_j^*
Base model *	1	1	3009	1.238	0.788	0.770	2.981	2089	1498	2741	623	66	11,324	33,761
$h_z = 1.2$	1	1	3009	1.238	0.788	0.772	3.090	2178	1564	2866	652	72	11,139	34,420
$S_z = 2000$	1	1	3009	1.238	0.788	0.761	3.113	2212	1561	2849	641	73	11,586	36,072
$c_{pm} = 6$	1	1	3009	1.486	0.788	0.835	2.888	1926	1530	2866	690	65	12,244	35,364
$c_w = 0.3$	1	1	3009	1.238	0.788	0.760	2.982	2104	1484	2705	609	66	11,345	33,832
$\vartheta_z = 30$	1	1	3009	1.238	0.788	0.778	2.950	2074	1486	2737	618	97	11,377	33,564

* Row one of Table 7.

5.4.4. Case 4

In this case (Case 4), we replicate example 2 with respect to constant rates without deterioration. As can be seen from Table 12, the model behaves in a similar way to that observed in Table 7, in particular, the behavior of R_j^* in the third cycle and Δ_j^* in the second cycle (recall the justification in example 2). Note that ϕ_j^* attains its maximum value when the system plateaus, i.e., it differs from that observed in Table 7. A comparison between Tables 7 and 12 shows that for each cycle j, all optimal values are higher than those of example 2 (Table 7), except L_j^* and ϕ_j^*, which are associated with lower values.

Table 12. Optimal results for constant rates without deterioration when $\tau = 3$ and $c_{inv} = USD\ 4000$.

j	ζ_j^*	c_{invj}	c_{prj}	λ_j^*	ϕ_j^*	f_{4j}^*	Q_{mj}^*	Q_{rj}^*	R_j^*	Δ_j^*	L_j^*	l_j^*
1	1	3009	1.238	0.788	0.635	4.808	3207	1781	3051	623	9479	45,577
2	2	3845	0.983	0.749	0.736	4.257	2277	1980	2685	655	9362	39,851
3	3	3986	0.765	0.730	0.717	4.225	2128	2097	3028	768	9264	39,138
4	3	3986	0.765	0.730	0.695	4.071	1980	2091	2829	743	9218	37,532
5	3	3986	0.765	0.730	0.700	4.107	2014	2093	2875	749	9229	37,906
6	3	3986	0.765	0.730	0.699	4.099	2006	2092	2864	747	9227	37,816
7	3	3986	0.765	0.730	0.699	4.101	2008	2093	2867	748	9227	37,837
8	3	3986	0.765	0.730	0.820	4.100	2008	2093	2866	748	9227	37,832
9	3	3986	0.765	0.730	0.820	4.100	2008	2093	2866	748	9227	37,832

5.4.5. Case 5

In this case (Case 5), we replicate example 3 with respect to constant rates without deterioration. As Table 13 shows, the model behaves similarly with respect to constant rates without deterioration (see Table 8). As can be seen from Table 13, ϕ_j^*, Δ_j^*, and R_j^* reach their maximum values in the first cycle and f_{4j}^*, L_j^*, and l_j^* attain their minimum values in the second cycle before the system plateaus in the fifth cycle (Table 13). Similarly, Q_{rj}^* (Q_{mj}^*) reach their minimum (maximum) values in the first cycle since the inventory of returned items is zero. A comparison between Tables 8 and 13 shows that for each cycle j, all optimal values are higher than those of example 3 (Table 8), except L_j^* and ϕ_j^*, which are associated with lower values. Note that this finding is also observed in Case 4. In addition, $L_{4,3}^* = USD\ 9662 > L_{2,1}^* = USD\ 9603$ (recall solution steps). However, when the system plateaus for $\zeta = 3$, the difference between the total minimum cost per month is negligible, i.e., $L^{\zeta=3} = USD\ 9667 > L^{\zeta=1} = 9625$ (see also example 3).

Table 13. Optimal results for constant rates without deterioration when $\tau = 3$ and $c_{inv} = USD\ 6000$.

j	ζ_j^*	c_{invj}	c_{prj}	λ_j^*	ϕ_j^*	f_{4j}^*	Q_{mj}^*	Q_{rj}^*	R_j^*	Δ_j^*	L_j^*	l_j^*
1	1	4514	1.238	0.788	0.614	5.243	3348	1895	3219	642	9779	51,268
2	1	4514	1.238	0.788	0.534	4.476	2526	1950	2390	574	9603	42,983
3	1	4514	1.238	0.788	0.544	4.568	2620	1949	2486	585	9628	43,984
4	1	4514	1.238	0.788	0.543	4.554	2605	1949	2471	583	9624	43,828
5	1	4514	1.238	0.788	0.543	4.556	2607	1949	2474	584	9625	43,852
6	1	4514	1.238	0.788	0.543	4.556	2607	1949	2474	584	9625	43,852

6. Implications and Managerial Insights

- Considering that returned items may arrive with different number of remanufacturing times reduces the total system cost as well as ensures reducing the disposal of unnecessary amount.
- The optimal policy is either to remanufacture once or remanufacture up to the expected number of times an item can be remanufactured in its lifecycle.
- Modeling the return rate as a decision variable not only increases the remanufactured quantity, but also decreases the consumption of the produced quantity.

- When the return rate is a decision variable, it increases the reusable proportion, and subsequently impacts the economic opportunities, which in turn influence both environmental and social interests.
- All functions may or may not be related to each other and, therefore, each is solely modeled.
- The remanufacturing number of times for an item is tangible, definite, tractable, and modeled.
- The purchasing price of recovery items, remanufacturing investment cost, return rate, and the percentage of returns vary until the number of cycles reaches the expected number of times an item can be remanufactured in its lifecycle. Such variation implies further reduction in the total cost and ensures a positive environmental impact.
- The return rate is a varying demand-dependent rate, which is a decision variable. This consideration reduces the total cost and solid waste disposal and, consequently, the system emphasizes sustainability because it reflects the influence of economic, social, and environmental interests.
- The initial inventory of returns in the first remanufacturing cycle is zero and it differs for subsequent cycles, which in turn affects the optimal values that vary until the system plateaus. This consideration is key in that it allows for the adjustment of all functions and input parameters for subsequent cycles.
- Incorporating the initial inventory of returns in the mathematical formulation enables the system to reap further cost reduction until all optimal values plateau.
- The proposed model is a viable solution for different forms of time-varying functions as well as for systems encountering periodic review applications.
- The solution quality of the special cases is identical to that of published sources, which implies that the robustness, viability, and validity of the general mathematical formulation are ascertained.

7. Summary and Conclusions

This paper is concerned with the number of times an item can be remanufactured. The mathematical modeling of reverse logistics inventory systems assumes that all returned items have been remanufactured an equal number of times. Nevertheless, this assumption ignores the fact that returned items may arrive out of sequence. The present paper developed a new mathematical expression of the percentage of returns that can be remanufactured a finite number of times. The proposed expression was modeled as a function of the expected number of times an item can be remanufactured in its lifecycle and the number of times an item can be technologically (or optimally) remanufactured based on its quality upon recovery. The mathematical expression was incorporated in a general joint model for production and remanufacturing options.

In the proposed model, demand, product deterioration, production, and remanufacturing rates are arbitrary functions of time so as to reflect a diverse range of time-varying forms. The return rate is a varying demand-dependent rate, which is a decision variable. The model considers the initial inventory of returned items in the mathematical formulation, which enables decision-makers to adjust all functions and input parameters for subsequent cycles.

We evaluated the impact of varying rates on the optimal quantities subject to the expected number of times an item can be remanufactured in its lifecycle. We found that the effect of varying purchasing price of recovery items, remanufacturing investment cost, return rate, the percentage of returns, and the initial inventory of returned items significantly impact the behavior of the model. Consequently, the optimal policy is either to remanufacture once or remanufacture up to the expected number of times an item can be remanufactured in its lifecycle. We tested and observed the behavior of the optimal values in different realistic situations and discussed some important managerial insights for policymakers and practitioners and highlighted some implications that may interest researchers. The versatile nature of the proposed model was emphasized by presenting sensitivity analysis and special cases, where the solution quality was identical with that of published sources. This implies that the robustness, viability, and validity of the proposed model are ascertained.

The results indicate that modeling the return rate as a decision variable not only decreases the per-unit time cost but also decreases the consumption of produced quantity. Such modeling also reduces solid waste disposal, which in turn has a direct impact on environmental sustainability. In parallel, when the return rate is a decision variable, it increases the reusable proportion and subsequently impacts the economic opportunities, which in turn influence both environmental and social interests. The results also indicate that incorporating the initial inventory of returns in the mathematical formulation enables the system to reap further cost reduction until all optimal values plateau.

Further research may include extensions such as allowing for shortages and incorporating learning and forgetting curves in the manufacturing and remanufacturing rates. In addition, the formulation of greenhouse gas (GHG) emissions for inventory models considering manufacturing, remanufacturing, and transportation options can also be addressed. Another research option is the consideration of energy consumption during manufacturing and remanufacturing processes. In parallel, it seems plausible to extend the general model to multiple manufacturing and remanufacturing cycles while accounting for different emission trading schemes.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The author would like to express his great thanks to the anonymous reviewers for their valuable remarks and suggestions that improved the presentation of the paper.

Conflicts of Interest: The author declares no conflict of interest.

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