Supplementary 3 - Theory

Here, we characterize the best reply functions of agents having *extreme* objective functions and look for possible game equilibria: in particular we focus on agents who are *pure profit maximizers* or *pure impact maximizers.*

S.1 Market shares

In our experiment the market shares are defined as follows. Letting p_i, q_i, the price and the quality chosen by subject $i = L, H$, we have:
if $q_L < q_H$

$$s_L(p_L, p_H, q_L, q_H) = \begin{cases} 1 & if \quad p_L < p_H - \frac{3}{2}(q_H - q_L) \\ \frac{2}{3}\frac{p_H - p_L}{q_H - q_L} & if \quad \frac{2}{3}\frac{p_H - p_L}{q_H - q_L} \in [0, 1] \\ 0 & if \quad p_L > p_H \end{cases} \tag{1}$$

while if $q_L = q_H$

$$s_L(p_L, p_H, q_L, q_H) = \begin{cases} 1 & if \quad p_L < p_H \\ \frac{1}{2} & if \quad p_L = p_H \\ 0 & if \quad p_L > p_H \end{cases} \tag{2}$$

and in both cases

$$s_H(p_L, p_H, q_L, q_H) = 1 - s_L(p_L, p_H, q_L, q_H) \tag{3}$$

S.2 A methodological remark

Given the discontinuous nature of this optimization problem, in what follows it will be often necessary to parcel it out in sub-problems defined over non-compact sets where, technically, the optimal choice does not exist but where, practically, it is optimal *"to stay as close to the border as possible"*. Given that the objective functions involved are bounded, for the sake of simplicity, we will approximate the value of $\lim_{x \to x_0} f(x)$ with $f(x_0 \pm \varepsilon)$, with no consequence for the results.[1]

S.3 Case 1: Best reply function for an individual-impact-maximizing agent

Let's consider the case of an agent i willing to maximize his social impact under the non-negative profit constraint (implying $p_i \geq q_i$). The social impact of player i is given by $q_i s_i$. Then, the problem for i is

$$\arg\max_{p_i, q_i} I(p_i, q_i; p_j, q_j) = \arg\max_{p_i, q_i} q_i s_i(p_i, q_i; p_j, q_j) \qquad \text{subject to } 0 \leq q_i \leq p_i \leq 400$$

Observe that, in this case, p_i only plays a role in shaping s_i which is a non increasing function of it. Hence it is a weakly dominant strategy that of chosing $p_i = q_i$ and the problem can be simplified in

$$\arg\max_{q_i} q_i s_i(p_i = q_i; p_j, q_j) \qquad \text{subject to } 0 \leq q_i = p_i \leq 400$$

The following proposition holds.

Proposition 1 *The best reply function for an impact maximizing agent is*

$$q_i^* = p_i^* = \begin{cases} 400 & if \quad q_j < \frac{400}{3} \text{ and } p_j > q_j \\ 400 & if \quad q_j \leq 200 \text{ and } p_j = q_j \\ q_j - \varepsilon & if \quad p_j = q_j \geq 200 \\ p_j & if \quad p_j > q_j \geq \frac{400}{3} \end{cases}$$

[1] This choice, could also be explained as the result of optimization with respect to a non continuous variable, which indeed was the case in the experiment. Unfortunately, this argument would be at odds with the usual analytical approach followed throughout this Appendix.

and the corresponding generated impact is

$$I^* = \begin{cases} 400\left(1 - \frac{2}{3}\frac{400-p_j}{400-q_j}\right) & \text{if} \quad q_j < \frac{400}{3} \text{ and } p_j > q_j \\ \frac{400}{3} & \text{if} \quad q_j \le 200 \text{ and } p_j = q_j \\ \frac{2}{3}\left(q_j - \varepsilon\right) & \text{if} \quad p_j = q_j \ge 200 \\ p_j & \text{if} \quad p_j > q_j \ge \frac{400}{3} \end{cases}$$

Proof.

1. Consider first the case $p_j = q_j$.
   Player i's market share will be equal to 2/3, 1/2 or 1/3 (refer to equations 1 - 3) if q_i is smaller, equal or greater than q_j respectively. Hence, it is

$$I_i = \begin{cases} \frac{2}{3}q_i & \text{if} \quad q_i < q_j \\ \frac{q_i}{2} & \text{if} \quad q_i = q_j \\ \frac{q_i}{3} & \text{if} \quad q_i > q_j \end{cases} \Rightarrow q_i^* = \begin{cases} q_j - \varepsilon & \text{if} \quad q_i < q_j \\ q_j & \text{if} \quad q_i = q_j \\ 400 & \text{if} \quad q_i > q_j \end{cases}$$

   and the optimal solution is

$$q_i^* = \begin{cases} 400 & \text{if} \quad q_j < 200 \\ q_j - \varepsilon & \text{if} \quad q_j \ge 200 \end{cases} \text{ with } I_i^* = \begin{cases} \frac{400}{3} & \text{if} \quad q_j < 200 \\ \frac{2}{3}\left(q_j - \varepsilon\right) & \text{if} \quad q_j \ge 200 \end{cases} \tag{4}$$

2. Let's consider now the case $p_j > q_j$.
   Observe that $q_j < q_i = p_i = p_j$ implies $s_i = 1$ (eqn. 1 - 3), so there is no reason to choose $q_i < p_j$ as this would necessarily yield a smaller impact. At the same time, for $q_j < p_j < q_i = p_i$ the impact function is convex given that

$$\frac{\partial_2\left(I_i\right)}{\partial q_i^2} = \frac{\partial_2\left(q_i\left(1 - \frac{2}{3}\frac{q_i - p_j}{q_i - q_j}\right)\right)}{\partial q_i^2} = \frac{4}{3}q_j\frac{p_j - q_j}{\left(q_i - q_j\right)^3} > 0$$

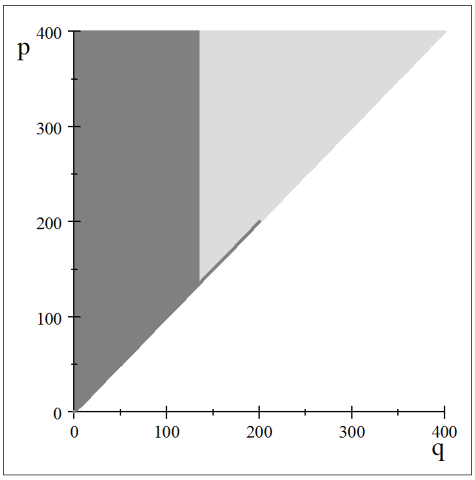
   As a consequence, the solution to the problem has to be either $q_i = p_j$ or $q_i = 400$. Evaluating player i's impact we get

$$I_i\left(q_i = p_i = p_j, q_j < p_j\right) = p_j$$
$$I_i\left(q_i = p_i = 400, q_j < p_j\right) = 400\left(1 - \frac{2}{3}\frac{400 - p_j}{400 - q_j}\right)$$

   and $p_j \ge 400\left(1 - \frac{2}{3}\frac{400-p_j}{400-q_j}\right)$ if and only if $q_j \ge \frac{400}{3}$, so the solution (given $p_j > q_j$) is

$$q_i^* = \begin{cases} 400 & \text{if} \quad q_j < \frac{400}{3} \\ p_j & \text{if} \quad q_j \ge \frac{400}{3} \end{cases} \text{ with } I_i^* = \begin{cases} 400\left(1 - \frac{2}{3}\frac{400-p_j}{400-q_j}\right) & \text{if} \quad q_j < \frac{400}{3} \\ p_j & \text{if} \quad q_j \ge \frac{400}{3} \end{cases} \tag{5}$$

Putting equations (4) and (5) together we get the desired result. ■
The following figure illustrates the findings.



Best reply of agent i when (q_j, p_j) belongs to the dark grey region is $q_i^* = p_i^* = 400$ while for (q_j, p_j) in the light grey region it is $q_i^* = p_i^* = p_j$.

S.3.1 Best reply function for an aggregate-impact-maximizing agent

What if the agent also includes the impact generated by the competitor in his own objective function? In this case the solution simplifies as follows. The problem for i is now

$$\arg\max_{q_i} q_i s_i + q_j s_j \qquad \text{subject to } 0 \le q_i = p_i \le 400$$

The following proposition holds.

Proposition 2 *The best reply function for an aggregate-impact-maximizing agent is*

$$q_i^* = 400$$

and the corresponding generated impact is

$$I^* = \frac{400 + 2p_j}{3}$$

Proof. When $p_j = q_j$ it is

$$I_i = \begin{cases} \frac{2q_i+q_j}{3} & \text{if} \quad q_i < q_j \\ \frac{q_i+q_j}{2} & \text{if} \quad q_i = q_j \\ \frac{q_i+2q_j}{3} & \text{if} \quad q_i > q_j \end{cases} \Rightarrow q_i^* = \begin{cases} q_j - \varepsilon & \text{if} \quad q_i < q_j \\ q_j & \text{if} \quad q_i = q_j \\ 400 & \text{if} \quad q_i > q_j \end{cases}$$

and the optimal solution is

$$q_i^* = 400 \text{ with } I_i^* = \frac{400 + 2q_j}{3} \tag{6}$$

Instead, when $p_j > q_j$, following the same reasoning as in the proof of Proposition 1 we get

$$\begin{aligned} I_i &= \begin{cases} q_i \frac{2}{3}\frac{p_j-q_i}{q_j-q_i} + q_j\left(1 - \frac{2}{3}\frac{p_j-q_i}{q_j-q_i}\right) = \frac{2}{3}q_i - \frac{2}{3}p_j + q_j & \text{if} \quad q_i < \min\{q_j; 3q_j - 2p_j\} \\ q_i & \text{if} \quad \min\{q_j; 3q_j - 2p_j\} \le q_i \le p_j \\ q_i\left(1 - \frac{2}{3}\frac{q_i-p_j}{q_i-q_j}\right) + q_j \frac{2}{3}\frac{q_i-p_j}{q_i-q_j} = \frac{2}{3}p_j + \frac{1}{3}q_i & \text{if} \quad q_i > p_j \end{cases} \Rightarrow \\ \Rightarrow \quad q_i^* &= \begin{cases} \min\{q_j; 3q_j - 2p_j\} & \text{if} \quad q_i < \min\{q_j; 3q_j - 2p_j\} \\ p_j & \text{if} \quad \min\{q_j; 3q_j - 2p_j\} < q_i \le p_j \\ 400 & \text{if} \quad q_i > p_j \end{cases} \end{aligned}$$

and the optimal solution is

$$q_i^* = 400 \text{ with } I_i^* = \frac{400 + 2p_j}{3} \tag{7}$$

■

S.4 Case 2: Best reply function for a profit-maximizing agent

Let's consider now the case of an agent i willing to maximize his own payoff. Differently from the previous case, there is no reason to restrict to $p_i = q_i$ (in fact, only the case $p_i > q_i$ deserves now some attention given that profits would be null otherwise). The problem for i is now

$$\arg\max_{p_i, q_i} \Pi\left(p_i, q_i; p_j, q_j\right) = \arg\max_{p_i, q_i} \left(p_i - q_i\right) \cdot s_i\left(p_i, q_i; p_j, q_j\right) \qquad \text{subject to } 0 \le q_i \le p_i \le 400$$

The following proposition holds.

Proposition 3 *The best reply function for a profit maximizing agent is*

$$\left(q_i^*, p_i^*\right) = \begin{cases} \left(q_j + \varepsilon, p_j\right) \ \textit{or} \ \left(q_j - \varepsilon, p_j - \frac{3}{2}\varepsilon\right) & \textit{if} \quad (q_j, p_j) \in I \\ \left(q_j + \frac{\sqrt{6(400-p_j)(400-q_j)}}{3}, 400\right) & \textit{if} \quad (q_j, p_j) \in II \\ \left(0, \frac{p_j}{2}\right) & \textit{if} \quad (q_j, p_j) \in III \end{cases}$$

and the corresponding generated profit is

$$\Pi^* = \begin{cases} p_j - q_j - \varepsilon & \text{if} \quad (q_j, p_j) \in I \\ \frac{\left(1200 - 3q_j - \sqrt{6(400-p_j)(400-q_j)}\right)^2}{3600 - 9q_j} & \text{if} \quad (q_j, p_j) \in II \\ \frac{p_j^2}{6q_j} & \text{if} \quad (q_j, p_j) \in III \end{cases}$$

where, given the constraints $0 \leq q_j \leq p_j \leq 400$*, it is*

$$\begin{aligned} I &= \left\{ (q_j, p_j) : p_j \geq \max \left\{ \left(3 - \sqrt{3}\right) q_j, 16 + \frac{24}{25} q_j \right\} \right\} \\ II &= \left\{ (q_j, p_j) : p_j \leq 16 + \frac{24}{25} q_j \wedge \left(3(400 - q_j) + 2(400 - p_j) - \frac{p_j^2}{2q_j} \right)^2 \geq 24(400 - p_j)(400 - q_j) \right\} \\ III &= \left\{ (q_j, p_j) : p_j \leq \left(3 - \sqrt{3}\right) q_j \wedge \left(3(400 - q_j) + 2(400 - p_j) - \frac{p_j^2}{2q_j} \right)^2 \leq 24(400 - p_j)(400 - q_j) \right\} \end{aligned}$$

Proof.

1. As before, let's consider first the case $p_j = q_j$. We can distinguish three alternatives depending on whether q_i is equal, smaller or greater than q_j.

   (a) For $i = L$, that is $q_i < q_j$, the problem is

   $$\arg\max_{p_i, q_i} \Pi = \arg\max_{p_i, q_i} \begin{cases} (p_i - q_i) & \text{if} \quad \frac{2}{3}\frac{q_j - p_i}{q_j - q_i} > 1 \\ (p_i - q_i) \cdot \frac{2}{3}\frac{q_j - p_i}{q_j - q_i} & \text{if} \quad \frac{2}{3}\frac{q_j - p_i}{q_j - q_i} \in [0, 1] \\ 0 & \text{if} \quad \frac{2}{3}\frac{q_j - p_i}{q_j - q_i} < 0 \end{cases}$$

   If $\frac{2}{3}\frac{q_j - p_i}{q_j - q_i} \in [0, 1]$ holds, first order conditions are

   $$FOC \rightarrow \begin{cases} \frac{\partial \Pi}{\partial p_i} = \frac{2}{3}\left(\frac{q_j - p_i}{q_j - q_i} - (p_i - q_i)\frac{1}{q_j - q_i}\right) = \frac{2}{3}\frac{q_j + q_i - 2p_i}{q_j - q_i} = 0 \rightarrow p_i = \frac{q_j + q_i}{2} \\ \frac{\partial \Pi}{\partial q_i} = -\frac{2}{3}\frac{q_j - p_i}{q_j - q_i} + (p_i - q_i)\frac{2}{3}\frac{q_j - p_i}{(q_j - q_i)^2} = -\frac{2}{3}\frac{(p_i - q_j)^2}{(q_i - q_j)^2} < 0 \end{cases}$$

   and $\frac{\partial_2 \Pi}{\partial p_i^2} = -\frac{4}{3(q_j - q_i)} < 0$. Hence $q_i^* = 0$, $p_i^* = \frac{q_j}{2}$ and $\Pi_{p_j = q_j > q_i} = \frac{q_j}{6}$. Observe that for these values, condition $\frac{2}{3}\frac{q_j - p_i}{q_j - q_i} \in [0, 1]$ becomes $q_j \geq 0$ which always holds.

   (b) For $i = H$, that is $q_i > q_j$, the problem is

   $$\arg\max_{p_i, q_i} \Pi = \arg\max_{p_i, q_i} \begin{cases} (p_i - q_i) & \text{if} \quad 1 - \frac{2}{3}\frac{p_i - q_j}{q_i - q_j} > 1 \\ (p_i - q_i) \cdot \left(1 - \frac{2}{3}\frac{p_i - q_j}{q_i - q_j}\right) & \text{if} \quad 1 - \frac{2}{3}\frac{p_i - q_j}{q_i - q_j} \in [0, 1] \\ 0 & \text{if} \quad 1 - \frac{2}{3}\frac{p_i - q_j}{q_i - q_j} < 0 \end{cases}$$

   If $1 - \frac{2}{3}\frac{p_i - q_j}{q_i - q_j} \in [0, 1]$ holds, first order conditions are

   $$FOC \rightarrow \begin{cases} \frac{\partial \Pi}{\partial p_i} = \left(1 - \frac{2}{3}\frac{p_i - q_j}{q_i - q_j}\right) - \frac{2}{3}(p_i - q_i)\frac{1}{q_i - q_j} = \frac{5q_i - 4p_i - q_j}{3(q_i - q_j)} = 0 \\ \frac{\partial \Pi}{\partial q_i} = -\left(1 - \frac{2}{3}\frac{p_i - q_j}{q_i - q_j}\right) + (p_i - q_i)\left(-\frac{2}{3}\right)\frac{q_j - p_i}{(q_i - q_j)^2} = \frac{2p_i^2 - 4p_i q_j - g3q_i^2 + 6q_i q_j - q_j^2}{3(q_i - q_j)^2} = 0 \end{cases}$$

   Solving the second equation and considering second order conditions (as well as condition $q_i > q_j$) we see that there is a candidate solution for $q_i^* = q_j + \frac{\sqrt{6}}{3}(p_i - q_j)$. By substitution in $\frac{\partial \Pi}{\partial p_i}$ we get

   $$\left.\frac{\partial \Pi}{\partial p_i}\right|_{q_i = q_i^*} = \frac{5 - 2\sqrt{6}}{3} > 0$$

   Hence it is $p_i^* = 400$, $q_i^* = \frac{3 - \sqrt{6}}{3} q_j + \frac{400\sqrt{6}}{3}$ and $\Pi_{q_i > q_j = p_j} = \frac{5 - 2\sqrt{6}}{3}(400 - q_j)$. It's easy to see that $q_i^* \in (q_j, 400)$ and that condition $1 - \frac{2}{3}\frac{p_i - q_j}{q_i - q_j} \in [0, 1]$ becomes $q_j \leq 400$ which always holds.

(c) The case $q_i = q_j = p_j$ is uninteresting due to either i's market share or i's mark-up (and hence profits) going down to zero.

A comparison between profits obtained in *(a)*, *(b)* and *(c)* shows that

$$\Pi_{p_j=q_j>q_i} = \frac{q_j}{6} \geq \frac{5-2\sqrt{6}}{3}(400-q_j) = \Pi_{q_i>q_j=p_j} \Leftrightarrow q_j \geq 224-64\sqrt{6}$$

meaning that i's optimal choice when $p_j = q_j$ is

$$q_i^* = \begin{cases} \frac{3-\sqrt{6}}{3}q_j + \frac{400\sqrt{6}}{3} & \text{and} \quad p_i^* = 400 \quad \text{if} \quad q_j \leq 224-64\sqrt{6} \\ 0 & \text{and} \quad p_i^* = \frac{q_j}{2} \quad \text{if} \quad q_j \geq 224-64\sqrt{6} \end{cases}$$

resulting in

$$\Pi^* = \begin{cases} \frac{5-2\sqrt{6}}{3}(400-q_j) & \text{if} \quad q_j \leq 224-64\sqrt{6} \quad \text{with} \quad q_i^* = \frac{3-\sqrt{6}}{3}q_j + \frac{400\sqrt{6}}{3}, p_i^* = 400 \\ \frac{q_j}{6} & \text{if} \quad q_j \geq 224-64\sqrt{6} \quad \text{with} \quad q_i^* = 0, p_i^* = \frac{q_j}{2} \end{cases} \quad (8)$$

2. Let's now consider the case $p_j > q_j$. Again, we separately analyze $q_i \lesseqgtr q_j$.

(a) For $i = L$, that is $q_i < q_j$, the problem is

$$\arg\max_{p_i,q_i} \Pi = \arg\max_{p_i,q_i} \begin{cases} (p_i - q_i) & \text{if} \quad \frac{2}{3}\frac{p_j-p_i}{q_j-q_i} > 1 \\ (p_i - q_i)\cdot\frac{2}{3}\frac{p_j-p_i}{q_j-q_i} & \text{if} \quad \frac{2}{3}\frac{p_j-p_i}{q_j-q_i} \in [0,1] \\ 0 & \text{if} \quad \frac{2}{3}\frac{p_j-p_i}{q_j-q_i} < 0 \end{cases}$$

If $\frac{2}{3}\frac{p_j-p_i}{q_j-q_i} \in [0,1]$ holds, the Hessian determinant, $\det H_\Pi = -\frac{4}{9}\frac{(p_j-q_j)^2}{(q_j-q_i)^4}$, is always negative. This implies either a border solution with $q_i = 0$ or no solution at all if $q_i = q_j - \varepsilon$ proves to be a better choice when $\varepsilon \to 0$ (whereas $p_i = q_i$ or $p_i = 400$ can be excluded as they would imply zero profits). Given that

$$\frac{\partial \Pi}{\partial p_i} = \frac{2}{3}\left(\frac{p_j-p_i}{q_j-q_i} - (p_i-q_i)\frac{1}{q_j-q_i}\right) = \frac{2}{3}\frac{p_j+q_i-2p_i}{q_j-q_i} = 0 \Rightarrow p_i = \frac{p_j+q_i}{2}$$

using $q_i = 0$, and checking for condition $\frac{2}{3}\frac{p_j-p_i}{q_j-q_i} \in [0,1]$ we find that a candidate solution is $(q_i,p_i) = \begin{cases} \left(0,\frac{p_j}{2}\right) & \text{if} \quad p_j < 3q_j \\ \left(0, p_j - \frac{3}{2}q_j\right) & \text{if} \quad p_j \geq 3q_j \end{cases}$. Instead, when $q_i = q_j - \varepsilon$ (and using $p_j > q_j$), we have that $p_i = \frac{p_j+q_i}{2}$ always implies $\frac{2}{3}\frac{p_j-p_i}{q_j-q_i} > 1$. Hence, by forcing $\frac{2}{3}\frac{p_j-p_i}{q_j-q_i} = 1$ we get $\left(q_j-\varepsilon, p_j - \frac{3}{2}\varepsilon\right)$. Substituting in the profit function we get

$$\Pi\left(q_i = 0, p_i = \frac{p_j}{2}\right) = \frac{p_j^2}{6q_j} \quad \text{if} \quad p_j < 3q_j$$
$$\Pi\left(q_i = 0, p_i = p_j - \frac{3}{2}q_j\right) = p_j - \frac{3}{2}q_j \quad \text{if} \quad p_j \geq 3q_j$$
$$\Pi\left(q_i = q_j - \varepsilon, p_i = p_j - \frac{3}{2}\varepsilon\right) = p_j - q_j - \frac{\varepsilon}{2}$$

and comparing the profits we see that the best reply (q_i^*, p_i^*) is $\left(q_j - \varepsilon, p_j - \frac{3}{2}\varepsilon\right)$ if $p_j \geq q_j\left(3-\sqrt{3}\right)$, and $\left(0,\frac{p_j}{2}\right)$ otherwise. Hence, under the constraint $q_i < q_j < p_j$, it is

$$\Pi_{p_j>q_j>q_i} = \begin{cases} p_j - q_j - \frac{\varepsilon}{2} & \text{if} \quad p_j \geq q_j\left(3-\sqrt{3}\right) \quad \text{with} \quad q_i = q_j - \varepsilon, p_i = p_j - \frac{3}{2}\varepsilon \\ \frac{p_j^2}{6q_j} & \text{if} \quad p_j < q_j\left(3-\sqrt{3}\right) \quad \text{with} \quad q_i = 0, p_i = \frac{p_j}{2} \end{cases}$$

(b) For $i = H$, that is $q_i > q_j$, the problem is

$$\arg\max_{p_i,q_i} \Pi = \arg\max_{p_i,q_i} \begin{cases} (p_i - q_i) & \text{if } \; 1 - \frac{2}{3}\frac{p_i-p_j}{q_i-q_j} > 1 \\ (p_i - q_i) \cdot \left(1 - \frac{2}{3}\frac{p_i-p_j}{q_i-q_j}\right) & \text{if } \; 1 - \frac{2}{3}\frac{p_i-p_j}{q_i-q_j} \in [0,1] \\ 0 & \text{if } \; 1 - \frac{2}{3}\frac{p_i-p_j}{q_i-q_j} < 0 \end{cases}$$

If $1 - \frac{2}{3}\frac{p_i-p_j}{q_i-q_j} \in [0,1]$ holds then the Hessian determinant, $\det H_\Pi = -\frac{4}{9}\frac{(p_j-q_j)^2}{(q_i-q_j)^4}$, is the same as in the previous case and is always negative. Again, this implies either a border solution with $p_i = 400$ or no solution at all if $q_i = q_j + \varepsilon$ proves to be a better choice when $\varepsilon \to 0$ (whereas $p_i = q_i$ or $q_i = 400$ can be excluded as they would imply no profits). From

$$\frac{\partial \Pi}{\partial q_i} = \frac{2p_i^2 - 2p_iq_j - 2p_jp_i - 3q_i^2 + 6q_iq_j - 3q_j^2 + 2p_jq_j}{3\left(q_i - q_j\right)^2} = 0$$

considering the constraint $q_i > q_j$, and substituting for $p_i = 400$ we get the solution[2]

$$q_i = q_j + \frac{\sqrt{6\left(400 - p_j\right)\left(400 - q_j\right)}}{3}$$

We shall now check whether conditions $q_i \in (q_j, 400)$ and $1 - \frac{2}{3}\frac{p_i-p_j}{q_i-q_j} \in [0,1]$ are satisfied.
The inequality $q_j + \frac{\sqrt{6(400-p_j)(400-q_j)}}{3} > q_j$ is always satisfied (exception made for the uninteresting case $p_j = 400$ or $q_j = 400$) while $q_j + \frac{\sqrt{6(400-p_j)(400-q_j)}}{3} < 400$ is satisfied if and only if

$$p_j > \frac{3}{2}q_j - 200$$

which is always met for $p_j > q_j$ and $q_j < 400$. As for $1 - \frac{2}{3}\frac{p_i-p_j}{q_i-q_j} \in [0,1]$, by substitution we obtain

$$1 - \frac{2}{3}\frac{p_i - p_j}{q_i - q_j}\bigg|_{q_i = q_j + \frac{\sqrt{6(400-p_j)(400-q_j)}}{3},\, p_i = 400} = \frac{1200 - 3q_j - \sqrt{6\left(400 - p_j\right)\left(400 - q_j\right)}}{1200 - 3q_j}$$

which is trivially smaller than 1 in the relevant region. Furthermore

$$\begin{aligned} & 1200 - 3q_j - \sqrt{6\left(400 - p_j\right)\left(400 - q_j\right)} \geq 0 \Leftrightarrow \left(1200 - 3q_j\right)^2 \geq 6\left(400 - p_j\right)\left(400 - q_j\right) \Leftrightarrow \\ & 480000 + 9q_j^2 - 4800q_j + 6p_j\left(400 - q_j\right) \geq 0 \Leftrightarrow 6p_j\left(400 - q_j\right) \geq -480000 - 9q_j^2 + 4800q_j \Leftrightarrow \\ & p_j \geq \frac{-480000 - 9q_j^2 + 4800q_j}{6\left(400 - q_j\right)} \Leftrightarrow p_j \geq \frac{3}{2}q_j - 200 \end{aligned}$$

so $1 - \frac{2}{3}\frac{p_i-p_j}{q_i-q_j} \geq 0$ is always true when $q_j, p_j \in [0, 400]$ and $p_j > q_j$.
Let's consider now the case $q_i = q_j + \varepsilon$. Using the first order condition

$$\frac{\partial \Pi}{\partial p_i} = \frac{2p_j - 3q_j - 4p_i + 5q_i}{3\left(q_i - q_j\right)} = 0 \Leftrightarrow p_i = \frac{2p_j - 3q_j + 5q_i}{4}$$

we have

$$1 - \frac{2}{3}\frac{p_i - p_j}{q_i - q_j}\bigg|_{p_i = \frac{2p_j - 3q_j + 5q_i}{4}}\bigg|_{q_i = q_j + \varepsilon} = \frac{2p_j - 2q_j + \varepsilon}{6\varepsilon}$$

[2] Observe that

$$\frac{\partial_2 \Pi}{\partial q_i^2} = -\frac{4}{3}\left(p_i - p_j\right)\frac{p_i - q_j}{\left(q_i - q_j\right)^3}$$

which is always negative with $p_i = 400$ and $q_i > q_j$.

which, given $p_j > q_j$, implies $1 - \frac{2}{3}\frac{p_j - p_i}{q_j - q_i} > 1$ for ε small enough. Hence, by forcing $1 - \frac{2}{3}\frac{p_j - p_i}{q_j - q_i} = 1$ we get $(q_i, p_i) = (q_j - \varepsilon, p_j)$.[3]
Substituting in the profit function we obtain

$$\Pi\left(q_i = q_j + \frac{\sqrt{6\,(400 - p_j)\,(400 - q_j)}}{3}, p_i = 400\right) = \frac{\left(1200 - 3q_j - \sqrt{6\,(400 - p_j)\,(400 - q_j)}\right)^2}{3600 - 9q_j}$$
$$\Pi\,(q_i = q_j + \varepsilon, p_i = p_j) = p_j - q_j - \varepsilon$$

and comparing the profits we get

$$\begin{aligned}
&p_j - q_j > \frac{\left(1200 - 3q_j - \sqrt{6\,(400 - p_j)\,(400 - q_j)}\right)^2}{3600 - 9q_j} \Leftrightarrow \\
&(p_j - q_j)\,(3600 - 9q_j) > \left(1200 - 3q_j - \sqrt{6\,(400 - p_j)\,(400 - q_j)}\right)^2 \Leftrightarrow \\
&6\,(400 - q_j)\,\sqrt{6\,(400 - p_j)\,(400 - q_j)} > 15\,(400 - p_j)\,(400 - q_j) \Leftrightarrow \\
&36\,(400 - q_j)^2\,6\,(400 - p_j)\,(400 - q_j) > 225\,(400 - p_j)^2\,(400 - q_j)^2 \Leftrightarrow \\
&24\,(400 - q_j) > 25\,(400 - p_j) \Leftrightarrow p_j > 16 + \frac{24}{25}q_j
\end{aligned}$$

showing that the best reply (q_i^*, p_i^*) is $(q_j + \varepsilon, p_j)$ if $p_j > 16 + \frac{24}{25}q_j$, and $\left(q_j + \frac{\sqrt{6(400-p_j)(400-q_j)}}{3}, 400\right)$ otherwise. Hence, under the constraint $p_j > q_j \wedge q_i > q_j$, it is

$$\Pi_{p_j > q_j \wedge q_i > q_j} =$$
$$= \begin{cases} p_j - q_j - \varepsilon & \text{if} \quad p_j > 16 + \frac{24}{25}q_j \quad \text{with} \quad q_i = q_j + \varepsilon, p_i = p_j \\ \frac{\left(1200 - 3q_j - \sqrt{6(400-p_j)(400-q_j)}\right)^2}{3600 - 9q_j} & \text{if} \quad p_j \leq 16 + \frac{24}{25}q_j \quad \text{with} \quad q_i = q_j + \frac{\sqrt{6(400-p_j)(400-q_j)}}{3}, p_i = 400 \end{cases}$$

(c) Finally, for $q_i = q_j$ it will also be optimal to choose $p_i = p_j - \varepsilon$. In this case profits are $\Pi_{p_j > q_j = q_i} = p_j - q_j - \varepsilon$.

Summarizing, if $p_j > q_j$ the best we can get is

$$\Pi_{p_j > q_j > q_i} = \begin{cases} p_j - q_j - \frac{\varepsilon}{2} & \text{if} \quad p_j \geq q_j\left(3 - \sqrt{3}\right) \quad \text{with} \quad q_i = q_j - \varepsilon, p_i = p_j - \frac{3}{2}\varepsilon \\ \frac{p_j^2}{6q_j} & \text{if} \quad p_j < q_j\left(3 - \sqrt{3}\right) \quad \text{with} \quad q_i = 0, p_i = \frac{p_j}{2} \end{cases}$$

with $q_i < q_j$ while with $q_i > q_j$ it is

$$\Pi_{p_j > q_j \wedge q_i > q_j} = \begin{cases} p_j - q_j - \varepsilon & \text{if} \quad p_j > 16 + \frac{24}{25}q_j \quad \text{with} \quad q_i = q_j + \varepsilon, p_i = p_j \\ \frac{\left(1200 - 3q_j - \sqrt{6(400-p_j)(400-q_j)}\right)^2}{3600 - 9q_j} & \text{if} \quad p_j \leq 16 + \frac{24}{25}q_j \quad \text{with} \quad q_i = q_j + \frac{\sqrt{6(400-p_j)(400-q_j)}}{3}, p_i = 400 \end{cases}$$

and with $q_i = q_j$
$$\Pi_{p_j > q_j = q_i} = \quad p_j - q_j - \varepsilon \quad \text{with} \quad q_i = q_j, p_i = p_j - \varepsilon$$

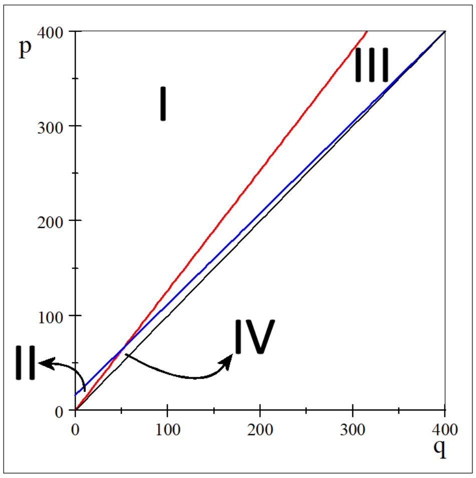
The alternative is between *staying close* to the competitor's quality and price or to move away and differentiate the product by *staying low* ($q_i = 0$, in the first case) or *staying high* ($p_i = 400$, in the second). We can distinguish

[3] Another argument is the following.
The partial derivative $\frac{\partial \Pi}{\partial p_i}$ evaluated in $q_i = q_j + \varepsilon$ is

$$\frac{5\varepsilon - 4p_i + 2p_j + 2q_j}{3\varepsilon} < \frac{5\varepsilon - 4p_i + 4p_j}{3\varepsilon} = \frac{5\varepsilon - 4\,(p_i - p_j)}{3\varepsilon}$$

which is negative for $p_i > p_j$ and $\varepsilon \to 0$. Hence a candidate optimal solution is $\left(q_i^*, p_i^*\right) = (q_j + \varepsilon, p_j)$.

four different regions depending on the previous results (see figure below).



It's easy to see that the best choice is *staying close* in region **I**, *staying high* in region **II**, and *staying low* in region **III**. As for region **IV** we must compare the *high/low* alternatives. It is

$$\begin{aligned}
&\frac{\left(1200-3q_j-\sqrt{6\,(400-p_j)\,(400-q_j)}\right)^2}{3600-9q_j} > \frac{p_j^2}{6q_j} \Leftrightarrow \\
&\left(1200-3q_j-\sqrt{6\,(400-p_j)\,(400-q_j)}\right)^2 > \frac{3p_j^2\,(400-q_j)}{2q_j} \Leftrightarrow \\
&9\,(400-q_j)^2+6\,(400-p_j)\,(400-q_j)-6\,(400-q_j)\sqrt{6\,(400-p_j)\,(400-q_j)} > \frac{3p_j^2\,(400-q_j)}{2q_j} \Leftrightarrow \\
&3\,(400-q_j)+2\,(400-p_j)-\frac{p_j^2}{2q_j} > 2\sqrt{6\,(400-p_j)\,(400-q_j)}
\end{aligned}$$

To solve the inequality first observe that

$$3\,(400-q_j)+2\,(400-q_j)-\frac{p_j^2}{2q_j} < 3\,(400-q_j)+2\,(400-q_j)-\frac{q_j}{2}$$

that

$$2\sqrt{6\,(400-p_j)\,(400-q_j)} > 2\sqrt{6}\,(400-q_j)$$

and that

$$3\,(400-q_j)+2\,(400-q_j)-\frac{q_j}{2} < 2\sqrt{6}\,(400-q_j)$$

is always true for all $q_j > 224-64\sqrt{6} \simeq 67.233$. Hence the inequality can be studied, without loss of generality, under the constraint $q_j < 224-64\sqrt{6}$. Now observe that we shall compare the inequality subject to $p_j < \left(3-\sqrt{3}\right)q_j$. Because

$$3\,(400-q_j)+2\,(400-q_j)-\frac{p_j^2}{2q_j} > 3\,(400-q_j)+2\,(400-q_j)-\frac{\left(\left(3-\sqrt{3}\right)q_j\right)^2}{2q_j} > 0$$

for all $q_j < 224-64\sqrt{6}$, we can write

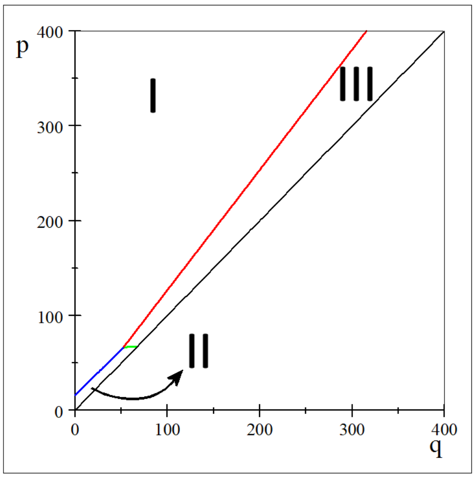
$$\begin{aligned}
3\,(400-q_j)+2\,(400-p_j)-\frac{p_j^2}{2q_j} &> 2\sqrt{6\,(400-p_j)\,(400-q_j)} \Leftrightarrow \\
\left(3\,(400-q_j)+2\,(400-p_j)-\frac{p_j^2}{2q_j}\right)^2 &> 24\,(400-p_j)\,(400-q_j)
\end{aligned}$$

to obtain a fourth degree inequality in p_j. To understand some qualitative properties of the solution, consider *(i)* the intersection[4] between the lines $p_j = \left(3-\sqrt{3}\right) q_j$ and $p_j = 16+\frac{24}{25}q_j$, $(q_j, p_j) = \left(\frac{10200+5000\sqrt{3}}{363}, \frac{5200+1600\sqrt{3}}{121}\right)$, and *(ii)* the point discriminating between the *low/high* optimal strategies when $q_j = p_j$, $(q_j, p_j) = \left(224-64\sqrt{6}, 224-64\sqrt{6}\right)$. Also observe that, consistently with continuity, it is

$$\left(3\,(400-q_j)+2\,(400-p_j)-\frac{p_j^2}{2q_j}\right)^2 - 24\,(400-p_j)\,(400-q_j)\Bigg|_{q_j=\frac{10200+5000\sqrt{3}}{363},\,p_j=\frac{5200+1600\sqrt{3}}{121}} = 0$$

$$\left(3\,(400-q_j)+2\,(400-p_j)-\frac{p_j^2}{2q_j}\right)^2 - 24\,(400-p_j)\,(400-q_j)\Bigg|_{q_j=p_j=224-64\sqrt{6}} = 0$$

The complete curve is numerically plotted and the three relevant regions are depicted in the figure below



where best reply and profits are

$$\Pi^* = \begin{cases} p_j - q_j - \varepsilon & \text{if} \quad (q_j, p_j) \in I \quad \text{with} \quad q_i^* = q_j + \varepsilon, p_i^* = p_j \text{ or } q_i^* = q_j - \varepsilon, p_i^* = p_j - \frac{3}{2}\varepsilon \\ \frac{\left(1200-3q_j-\sqrt{6(400-p_j)(400-q_j)}\right)^2}{3600-9q_j} & \text{if} \quad (q_j, p_j) \in II \quad \text{with} \quad q_i^* = q_j + \frac{\sqrt{6(400-p_j)(400-q_j)}}{3}, p_i^* = 400 \\ \frac{p_j^2}{6q_j} & \text{if} \quad (q_j, p_j) \in III \quad \text{with} \quad q_i^* = 0, p_i^* = \frac{p_j}{2} \end{cases} \tag{9}$$

Finally, remark that these results extends to the line $p_j = q_j$, corresponding to those in eqn. (8). ■

S.5 Market equilibrium

We now want to check whether a dynamic equilibrium is possible in a market populated with 2 agents.

S.5.1 Both agents are individual-impact maximizers

In this case both players will play $p_i = q_i$, $i = 1, 2$, so both optimal strategies are described by equation (4). It's easy to understand that no equilibrium exists under this setting. If player 1 offers a quality $q_1 < 200$ then the best reply is $q_2 = 400$. Then, a sequence of quality choices such that $q_i = q_j - \varepsilon$ will emerge until one of the two competitors' quality will fall under 200, starting again an analogous path. The dynamics for any other possible initial condition is described by the same argument.

[4] It is

$$\left(3-\sqrt{3}\right) q_j = 16 + \frac{24}{25} q_j \Leftrightarrow q_j = \frac{16}{\frac{51}{25}-\sqrt{3}} = \frac{10200+5000\sqrt{3}}{363} \simeq 51.957$$

and $p_j = 16 + \frac{24}{25}\frac{10200+5000\sqrt{3}}{363} = \frac{5200+1600\sqrt{3}}{121}$.

S.5.2 Both agents are profit maximizers

In this case both players will play $p_i > q_i$, $i = 1, 2$, so both optimal strategies are described by equation (9). Again, no equilibrium exists under this setting. This is a very well known result. To understand why this is true, first observe that no equilibrium is possible having both competitors inside region **I**. Indeed, if player 1 plays a strategy in region **I** then the best reply for player 2 is to *stay close* generating a non-stationary sequence which could (after a possibly long time) exit the region. On the other hand, if player 1 plays a strategy in region **II** then the best reply for player 2 is to *stay high* (hence in region **III**) as well as the best reply to a strategy in region **III** is to *stay low* (hence in region **II**): so no equilibrium having both players in the same region is possible. What if player 1 is in region **II** and player 2 is in region **III**? From equations (9) we have that the candidate equilibrium shall assume the form

$$\begin{cases} q_1^* = 0, p_1^* = 200 \\ q_2^* = \frac{400\sqrt{3}}{3}, p_2^* = 400 \end{cases}$$

which is not compatible with the constraints $p_1^* < 16$ and $q_2^* > \frac{600+200\sqrt{3}}{3}$.

S.5.3 One agent is individual-impact maximizer and the other is profit maximizer

Now let's consider the mixed case. The strategies of the impact maximizing player 1 are described by equation (5) while for the profit maximizing player 2 the strategies are described by equation (8). It is:

$$\begin{aligned} I_1^* &= \begin{cases} 400\left(1 - \frac{2}{3}\frac{400-p_2}{400-q_2}\right) & \text{if} \quad q_2 < \frac{400}{3} \quad \text{with} \quad q_1^* = 400 \\ p_2 & \text{if} \quad q_2 \geq \frac{400}{3} \quad \text{with} \quad q_1^* = p_2 \end{cases} \\ \Pi_2^* &= \begin{cases} \frac{5-2\sqrt{6}}{3}(400 - q_1) & \text{if} \quad q_1 \leq 224 - 64\sqrt{6} \quad \text{with} \quad q_2^* = \frac{3-\sqrt{6}}{3}q_1 + \frac{400\sqrt{6}}{3}, p_2^* = 400 \\ \frac{q_1}{6} & \text{if} \quad q_1 \geq 224 - 64\sqrt{6} \quad \text{with} \quad q_2^* = 0, p_2^* = \frac{q_1}{2} \end{cases} \end{aligned}$$

In this case an equilibrium exists. If we assume $q_2 < \frac{400}{3}$ then player 1 best reply is $q_1^* = p_1^* = 400$ implying player 2 best reply $q_2^* = 0, p_2^* = 200$ which is compatible with the initial assumption. Hence

$$\begin{aligned} (q_1^*, p_1^*) &= (400, 400) \\ (q_2^*, p_2^*) &= (0, 200) \end{aligned}$$

is a Nash Equilibrium. No other equilibria exist. Indeed, assume by contradiction that $q_2 > \frac{400}{3}$, then player 1 best reply is $q_1^* = p_1^* = p_2 > q_2 > \frac{400}{3}$. In turn this implies that player 2 best reply is $q_2^* = 0, p_2^* = \frac{p_1}{2}$ which is at odds with the initial assumption.

S.5.4 One agent is aggregate-impact maximizer

To conclude, let's consider the case with at least one aggregate-impact-maximizing agent. By Proposition 2, this kind of agent has an optimal strategy irrespective of the competitor's choice. This implies that there will always be a unique Nash equilibrium.

Indeed, when the competitor is another impact-maximizing agent (individual or aggregate) the Nash equilibrium implies that both agents will choose $q^* = p^* = 400$ (see Proposition 1).

Instead, when the competitor is a profit-maximizing agent, the Nash equilibrium implies $q_1^* = p_1^* = 400$ for the aggregate-impact maximizer and (from Proposition 3) $q_2^* = 0$, $p_2^* = 200$ for the profit maximizer.