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Government Reserve of Rare Earths under Total Quota Management: An Interactive Game between Government and Rare-Earth Firms

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Abstract: The total quota control and reserve of rare earths are important means for the sustainable development and utilization of rare-earth resources. Focusing on the government reserve of rare-earth products under stochastic demand, this paper analyses the interactive decisions of the government and the rare-earth firms from a game-theoretic perspective. The government determines the total quantity, reserve quantity and reserve–release quantity of the rare-earth products to maximize social warfare, while the firm decides the price of rare-earth products to maximize its own profit. The results show that the production cost and the expected net present value (NPV) of the reserve are important factors affecting the government's decisions. When the expected NPV of the reserve is below a threshold, the government adopts the *no-reserve strategy*: it determines only a total quota index that maximizes the current-period social welfare but keeps no reserve. When the expected NPV of a reserve is higher than the above threshold but lower than the production cost, the government adopts the *low-reserve strategy*: it determines a total quota index and a low reserve that are both in increasing in the expected NPV of reserve, and will release the reserve as many as possible if there is a supply shortage. When the expected NPV of a reserve is higher than both the above threshold and the production cost, the government adopts the *high-reserve strategy*: it sets a total quota index which is sufficiently large to cover the entire market demand, reserves a large amount, and releases part of the reserve to completely fill the demand gap (if any).

Keywords: rare-earth products; government reserve; total quota management; reserve release; rare earth price

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1. Introduction

Rare earth is a non-renewable strategic mineral resource and a material treasure chest in the new era. Known as "industrial vitamin" and "industrial monosodium glutamate", rare earth is an indispensable element in modern industry [1]. With good characteristics in photo-electro-magnetism, superconductivity, activity, catalysis and other aspects, rare earth plays an important role in improving product performance and productivity, and is widely used in areas such as aerospace, new energy, environmental protection, new materials, electronic information, etc. [2]. As a non-renewable resource experiencing increasing demand, the sustainable utilization of rare earth is very important. However, there still exist various problems in the development of the rare-earth industry, including illegal mining, the rapid expansion of smelting and separation capacities, the serious waste of resources, ecological and environmental damage, backward research and the development of high-end applications, and a chaotic export order [3]. These problems seriously threaten the sustainable development and utilization of rare-earth resources.

Total quota control and strategic reserves of rare earth are effective means to achieve rare-earth sustainability. On the one hand, rare-earth total quota control and reserves are beneficial to balance the intergenerational allocation of rare-earth resources and their benefits. On the other hand, rare-earth total quota control and reserve can promote the progress of utilization technologies and the development of alternative products, reduce predatory exploitation under the condition of the current immature technological level, reduce the waste of resources, protect the ecological environment, realize the orderly development of rare-earth mines, and promote the sustainable development of national economies and society.

In order to strengthen the sustainable utilization and effective protection of rare-earth resources, the Chinese government has introduced a series of policies on the total quota control and strategic reserve of rare earth. On 10 May 2011, China proposed a strategic reserve of rare-earth products (including rare-earth mineral products and rare-earth smelting separation products) for the first time, including a government reserve, enterprise (commercial) reserve, resource (land) reserve and physical reserve [4]. On 18 October 2016, the Ministry of Industry and Information Technology issued the Rare Earth Industry Development Plan (2016–2020), which clearly stipulates that, by the end of 2020, the six major rare-earth groups in China will complete the integration of all rare-earth enterprises (including rare-earth mining enterprises, rare-earth smelting and separation enterprises, rare-earth comprehensive utilization enterprises and rare-earth metal-smelting enterprises). The "13th Five-Year Plan" for rare earth states that the mining quantity of rare earth in 2020 should be strictly controlled to within 140,000 tons [5]. On 15 January 2021, the Ministry of Industry and Information Technology issued Rare Earth Management Regulations (Draft for Comments), in which Article 16 pointed out that China should implement the strategic reserve of rare-earth resources and rare-earth products [6]. On 30 September 2021, the Ministry of Industry and Information Technology and the Ministry of Natural Resources issued the 2021 annual total quota control indexes of rare-earth mining (168,000 tons) and smelting and separation (162,000 tons), with both increased by 20% compared with the corresponding indexes in 2020 [7].

Although the total quota management and strategic reserve of rare-earth products are being increasingly scrutinized, the related theoretical research is very scarce. Previous studies mainly discussed, at the conceptual level, the effects of rare-earth reserves on rare-earth price stabilization [8–10], national defense security [11,12], and ecological environment protection [13–15], etc. Some articles study the rare-earth industry or resources from the perspective of game theory. For example, Han et al. (2015) attempt to analyze adjustments in China's rare-earth regulation policies and the effects on the rare-earth market supply by using a static game-theoretic model and a dynamic game-theoretic model with complete information in the contextof China strengthening protection of rare-earth resources and the environment [16]. Brown and Eggert (2018) use a Stackelberg model to explore the effects of Chinese rare-earth reserves, environmental taxation, and improvements in recovery rates on rare-earth markets [17]. Lee et al. (2018) investigate how competition for limited resources influences firms' adoption of environmentally sustainable strategies [18]. None of the above studies consider the interactions between total quota control and rare-earth reserve or pricing, which is the very focus of this study.

We also employ the game-theoretic framework to analyze the decision-making interactions of the rare-earth industry chain. However, in contrast to the above studies, our focus falls on the interactions between the government's total quota management and strategic reserve of rare-earth products and rare-earth firm's pricing decisions, which is not examined by the above studies. We also examine the influence of key parameters such as production cost, expected net present value (NPV) of reserve, market potential and price sensitivity coefficient on the above decision results. Although the literature on government reserves of rare-earth products under total quota management is small, the literature on government reserves of other mineral resources (e.g., petroleum, iron ore and coal) and emergency supplies (e.g., protective suits and goggles) is relatively rich. Using a dynamic programming model, Wu et al. (2012) examine the optimal stockpiling and drawdown strategies for China's strategic petroleum reserve under various scenarios, with the goal of minimizing the total cost of reserves [19]. Cui and Wei (2017) study the phenomenon of thermal-coal price distortion through economic theoretical modeling and empirical cointegration analysis from the perspective of market forces [20]. Rademeyer et al. (2021) find that a profit-maximizing trader will seek to sell more volumes to domestic consumers of higher grade coal to compensate for earnings lost due to lower export volumes [21]. With the goal of minimizing the expected cost of the government (buyer) and maximizing the profit of the enterprise (supplier), Hu et al. (2019) examine the optimal reserve of emergency materials and investigate the coordination of an emergency supply chain [22]. Hu and Dong (2019) design a two-stage stochastic programming model to optimize firms' material storage management in a humanitarian supply chain [23]. Liang et al. (2012) design an option-contract pricing model for a relief-material supply chain [24]. These studies provide important references for our study of reserves of rare-earth products. However, rare-earth products have an important difference from the aforementioned minerals or materials, that is, the government implements total quota management for rare-earth products. The total quota control of rare-earth products will affect the reserve decision, so the two should be considered together. Then, under the total quota management of rare-earth products, how will the government make decisions on the reserve and release of rare-earth products, taking into account the potential response of enterprises and market demand? How will rare-earth companies set prices? In addition, how should the government optimize the total quota management plan? This paper aims to address these issues.

To sum up, this study employs a game-theoretic framework to investigate the interactions between the government and rare-earth enterprises under the total quota management policy, examining issues of rare-earth products reserve, pricing, and reserve release. This study contributes to the literature by enriching the theoretical results on the optimization of operation decisions related to the quantity and price of rare-earth products, and by providing managerial insights for the government and rare-earth enterprises on the sustainable exploitation and utilization of rare-earth resources.

2. Model Description

Consider a single-period, multi-stage decision-making problem for a rare-earth product supply chain composed of the government, a rare-earth firm (hereafter referred to as the firm), and consumers (or downstream firms) (see Figure 1). At time T1, the beginning of the period, the government determines the total quota control index $\overline{q}$ of rare-earth products in this period, and issues production plans to the rare-earth firm according to the relevant requirements of the *Interim Measures for the Management of Rare Earth Mandatory Production Plans* (which came into effect on 13 June 2012) [25]. Let the firm's unit production cost in the current period be c, and assume that the firm always arranges the maximum production allowed by the total quota index $\overline{q}$. It is worth mentioning that the total quota index $\overline{q}$ includes the quantities from mining, smelting and separation (which are collectively called production in this paper), but not imports. Although rare-earth imports and exports may have an impact on the model results, the current study does not explicitly model them due to the following reasons. First, rare-earth imports are small compared with the total quota control index. For example, in 2021, China's total quota control index of rare earth (converted to REO) was 330,000 tons (https://bit.ly/3ThW519, accessed on 28 September 2022), while the import quantity (converted to REO) was at the level of 20,000~30,000 tons (http://www.customs.gov.cn/, accessed on 28 September 2022). Second, rare-earth exports do not need to be characterized separately, because they are already included in the market demand. Third, the current model analysis is already complicated, and considering the influences of imports and exports will further increase analytical difficulties.

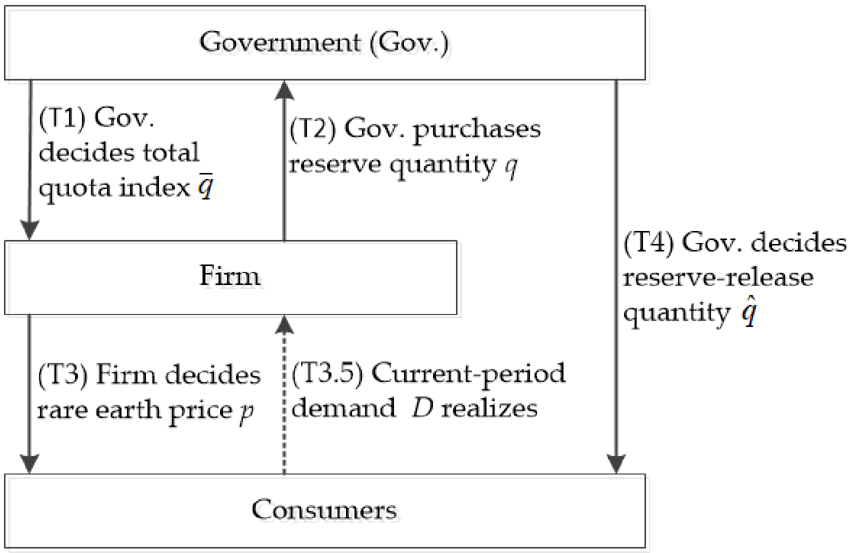

Figure 1. Model structure and sequence of events.

At T2, the government decides the amount $q \in [0, \overline{q}]$ of rare-earth products to be reserved in the period. In general, the government buys rare-earth products from rare-earth firms at a price (set as w) below the market level, and the rare-earth products bought are reserved and managed by the China State Reserve Bureau. The unit reserve cost of the government is denoted as c_G. At T3, the firm decides the market price p of rare-earth products in the current period. Influenced by price p and other factors (e.g., supply, demand, policy, etc.), the demand D of rare-earth products in this period is realized. At time T4, according to the market supply and demand situation, the government can choose to release part of the reserve $\hat{q} \in [0, q]$ to alleviate the market demand when necessary. The rare-earth products put into the market but not sold in the current period will be disposed of with a certain salvage value; the unit salvage value of residual rare-earth products processed by the government and enterprises are v_G and v_F, respectively. If the rare-earth products put on the market in the current period fail to meet all the demand, the seller (possibly the government or the firm) shall bear the loss of shortage. Assume that the unit loss of shortage of the government and the firm are t_G and t_F, respectively. Unreleased reserve will enter the next period and generate some value in the future. Assume the expected net present value (NPV) of the unreleased government reserve to be r_G per unit. In order to make the model conform to reality, it is assumed that the expected NPV of (unreleased) reserve is higher than the salvage value in the current period, i.e., $r_G > v_G$.

In order to formulate the consumer surplus, this paper refers to the modeling of market demand by Xue et al. (2014) [26] and Li Yao et al. (2006) [27], and assumes that consumers' purchase willingness ξ for rare-earth products follows the cumulative distribution function (cdf) $\Phi(\xi) = b\xi/a, \xi \in [0, a/b]$ (the probability density function (pdf) is denoted as $\phi(\xi)$). Thus, given the price p of rare-earth products, the probability that consumers are willing to buy is $\Pr(\xi \geq p) = 1 - \Phi(p) = (a - bp)/a$, which represents the proportion of consumers who can form demand when the market price is p. Denote this ratio as $y(p) = (a - bp)/a$ and refer to it as demand ratio, where $a > 0$ is the market potential and $b > 0$ is the price sensitivity coefficient. Assuming the size of potential consumers as a random variable ε, the demand at price p (assuming that each consumer can only buy one product at most) can be expressed as $D = y(p)\varepsilon$. Similar multiplicative form of price-dependent stochastic demand is very common in the operation management literature (e.g., [28,29]). Assume that the cdf, pdf and mean of ε are $F(\cdot)$, $f(\cdot)$ and μ, respectively, and write $\overline{F}(\cdot) = 1 - F(\cdot)$. Suppose that ε satisfies the property of increasing generalized failure rate (IGFR), i.e., $xf(x)/\overline{F}(x)$ increases with x. Many common distributions satisfy this assumption, such as uniform distribution, normal distribution, and exponential distribution, etc. (e.g., [30–32]). Based on the above model assumptions, it can be known that, given the market price $p \in [0, a/b]$, the expected surplus of a single consumer is $\int_p^{a/b} (\xi - p)\phi(\xi)d\xi = ay(p)^2/(2b)$. After taking into account the government's reserve (q) and release $(\hat{q})$ decisions, the final product supply is $\overline{q} - q + \hat{q}$, and the volume of consumers that can be satisfied is $\min\{\varepsilon, (\overline{q} - q + \hat{q})/y(p)\}$. Thus, the total expected consumer surplus is:

$$\pi_C(\overline{q}, q, p, \hat{q}) = \min\left\{\varepsilon, \frac{\overline{q} - q + \hat{q}}{y(p)}\right\} \int_p^{\frac{a}{b}} (\xi - p)\phi(\xi)d\xi = ay(p)\frac{\min\{D, \overline{q} - q + \hat{q}\}}{2b} \tag{1}$$

For the firm, it faces a demand D and offers a supply quantity of $\overline{q} - q$. Thus, the profit of the firm is:

$$\begin{aligned} \pi_F(\overline{q}, q, p) &= -c\overline{q} + wq + p\min\{D, \overline{q} - q\} - t_F(D - \overline{q} + q)^+ + v_F(\overline{q} - q - D)^+ \\ &= (v_F - c)\overline{q} + (w - v_F)q - t_F D + (p + t_F - v_F)\min\{D, \overline{q} - q\} \end{aligned} \tag{2}$$

where $(D - \overline{q} + q)^+ = D - \min\{D, \overline{q} - q\}$ and $(\overline{q} - q - D)^+ = \overline{q} - q - \min\{D, \overline{q} - q\}$.

In the first equation of Equation (2), the first to the fifth terms are the production cost, reserve-offer revenue, sales revenue, backorder loss, and salvage value, respectively. Moreover, since the firm's supply is assumed to be sold before the government's supply, the firm's profit is independent of the government's release of reserve $\hat{q}$.

For the government, it faces a demand of $(D - \overline{q} + q)^+$ (assuming that the firm's supply is sold before the government's in the case of market supply shortage), offers a supply of $\hat{q}$, and keeps an unreleased reserve of $q - \hat{q}$. Thus, the government's profit is:

$$\begin{aligned} \pi_G(\overline{q}, q, p, \hat{q}) \quad &= \quad -wq - c_G q + p\min\left\{(D - \overline{q} + q)^+, \hat{q}\right\} - t_G[(D - \overline{q} + q)^+ - \hat{q}]^+ \\ &\quad +v_G[\hat{q} - (D - \overline{q} + q)^+]^+ + r_G(q - \hat{q}) \\ &= \quad (r_G - c_G - w)q + (v_G - r_G)\hat{q} + (p + t_G - v_G)\min\{D, \overline{q} - q + \hat{q}\} \\ &\quad -(p - v_G)\min\{D, \overline{q} - q\} - t_G D \end{aligned} \tag{3}$$

where

$\min\left\{(D - \overline{q} + q)^+, \hat{q}\right\} = \min\{D, \overline{q} - q + \hat{q}\} - \min\{D, \overline{q} - q\}$,

$[(D - \overline{q} + q)^+ - \hat{q}]^+ = D - \min\{D, \overline{q} - q + \hat{q}\}$, and

$[\hat{q} - (D - \overline{q} + q)^+]^+ = \hat{q} - \min\{D, \overline{q} - q + \hat{q}\} + \min\{D, \overline{q} - q\}$.

In the first equation of Equation (3), the first to the sixth terms are the government's procurement cost, reserve cost, sales revenue, backorder loss, salvage value, and total expected NPV of unreleased reserve, respectively.

In line with previous studies examining the interactions between rare-earth firms and the government (e.g., [33]), the firm in our model aims to maximize its own profit π_F, while the government aims to maximize the social welfare, defined as the sum of producer surplus and consumer surplus ([33–35]). Since the government purchasing and selling reserve plays part of producer's role, the social welfare in our model (denoted as π_{SW}) is the sum of the firm's profit, the consumer's surplus, and the government's profit, i.e.,$\pi_{SW} = \pi_G + \pi_F + \pi_C$. From the perspective of sustainable supply-chain management, social-welfare maximization reflects the government's emphasis on both economic sustainability (i.e., the firm's and government's profit) and social sustainability (i.e., the consumer surplus) [36]. The third dimension of sustainability is environmental sustainability (e.g., [37]), which is beyond the scope of the current study and may be worth investigating in future research.

Table 1 summarizes the notations used in this paper.

Table 1. Summary of notations.

Notation	Description
$\overline{q}$	Government's total quota index;
q	Government's reserve quantity, $q \in [0, \overline{q}]$;
$\hat{q}$	Government's reserve-release quantity, $\hat{q} \in [0, q]$;
p	Market price of rare-earth products;
w	Government's purchasing price for reserve;
c	Firm's unit production cost;
c_G	Government's unit reserve cost;
v_G/v_F	Government's/firm's unit salvage value;
t_G/t_F	Government's/firm's unit loss of shortage;
r_G	Expected net present value (NPV) of the unreleased reserve;
ε	Size of potential consumers, with cdf $F(\cdot)$, pdf $f(\cdot)$ and mean μ;
$y(p)$	Demand ratio at price p, $y(p) = (a - bp)/a$;
D	Demand at price p, $D = y(p)\varepsilon$;
π_G/Π_G	Government's profit/expected profit, $\Pi_G = \mathbb{E}_\varepsilon[\pi_G]$;
π_F/Π_F	Firm's profit/expected profit, $\Pi_F = \mathbb{E}_\varepsilon[\pi_F]$;
π_C/Π_C	Consumer's surplus/expected surplus, $\Pi_C = \mathbb{E}_\varepsilon[\pi_C]$;
π_{SW}/Π_{SW}	Social welfare/expected social welfare, $\pi_{SW} = \pi_G + \pi_F + \pi_C$, $\Pi_{SW} = \mathbb{E}_\varepsilon[\pi_{SW}]$.

3. Model Analysis

Since the model involves a multi-stage game, by backward induction, we successively analyzed the government's optimal release strategy, the firm's optimal pricing strategy, and the government's optimal reserve and total quota control strategy.

3.1. The Government's Optimal Release Decision

At time T4, given the total quota index $\overline{q}$, the reserve quantity q, the price p, and the current-period demand D, the government's problem is to choose the optimal release quantity $\hat{q} \in [0, q]$ to maximize the social welfare π_{SW}:

$$\begin{aligned} \pi_{SW}(\overline{q}, q, p, \hat{q}) \quad & \equiv \pi_G(\overline{q}, q, p, \hat{q}) + \pi_F(\overline{q}, q, p) + \pi_C(\overline{q}, q, p, \hat{q}) \\ & = (v_F - c)\overline{q} + (r_G - c_G - v_F)q + (v_G - r_G)\hat{q} \\ & -(t_G + t_F)D + (v_G + t_F - v_F)\min\{D, \overline{q} - q\} \\ & + \begin{cases} [(a+bp)/(2b) + t_G - v_G]D & \hat{q} > D - \overline{q} + q \\ [(a+bp)/(2b) + t_G - v_G](\overline{q} - q + \hat{q}) & \hat{q} \le D - \overline{q} + q \end{cases} \end{aligned} \tag{4}$$

One can verify that $\pi_{SW}(\overline{q}, q, p, \hat{q})$ is a piecewise linear function of the government's decision variable $\hat{q}$. Therefore, the government's optimal decision on $\hat{q}$ can be easily obtained by examining the first-order condition.

Proposition 1. *The government's optimal release quantity $\hat{q}^*$ at time T4 is given by*

$$\hat{q}^* = \begin{cases} 0 & (a+bp)/(2b) + t_G \le r_G \\ \min\{D, \overline{q}\} - \min\{D, \overline{q} - q\} & (a+bp)/(2b) + t_G > r_G \end{cases} \tag{5}$$

and the corresponding social welfare is given by

$$\begin{aligned} \pi_{SW}(\overline{q}, q, p) \quad & \equiv \pi_{SW}(\overline{q}, q, p, \hat{q}^*) \\ & = (v_F - c)\overline{q} + (r_G - c_G - v_F)q - (t_G + t_F)D + (r_G + t_F - v_F)\min\{D, \overline{q} - q\} \\ & + \begin{cases} [(a+bp)/(2b) + t_G - r_G]\min\{D, \overline{q} - q\} & (a+bp)/(2b) + t_G \le r_G \\ [(a+bp)/(2b) + t_G - r_G]\min\{D, \overline{q}\} & (a+bp)/(2b) + t_G > r_G \end{cases} \end{aligned} \tag{6}$$

All proofs are provided in Appendix A. Proposition 1 shows that the government's reserve release strategy at T4 is either not to release at all ($\hat{q}^* = 0$), or to release according to the demand gap within its maximum reserve ($\hat{q}^* = \min\{q, D - \overline{q} + q\}$). There are two factors that determine the government's release strategy. One is whether the current market supply can meet the demand. If the market is adequately supplied ($D \in [0, \overline{q} - q]$), there is certainly no reason for the government to release the reserve. If the market supply is insufficient ($D \in [\overline{q} - q, \infty)$), the government needs to weigh the expected NPV of reserve r_G against the social welfare loss $(a + bp)/(2b) + t_G$ caused by the current-period supply shortage. If and only if the latter exceeds the former, will the government release reserves to meet the demand gap as much as possible.

3.2. The Firm's Optimal Pricing Decision

At time T3, given the total quota index $\overline{q}$ and the reserve quantity q, and expecting the subsequent demand D, the firm's objective is to determine the optimal price that maximizes its own expected profit:

$$\Pi_F(\overline{q}, q, p) = (v_F - c)\overline{q} + (w - v_F)q - \mu t_F y(p) + (p + t_F - v_F)S(z_1)y(p) \tag{7}$$

where $S(z) = \mathbb{E}_\varepsilon[\min\{\varepsilon, z\}]$ and $z_1 = (\overline{q} - q)/y(p)$. In order to simplify the subsequent analysis, we define $\eta(z) = z\overline{F}(z)/S(z)$ and $g(z) = zf(z)/\overline{F}(z)$ as in [26], and then the following lemma can be obtained by following [26] and [38]:

Lemma 1. *For any $z > 0$, (a) $S(z) = \int_0^z \overline{F}(\xi)d\xi \le \mu$; (b) $\eta(z)$ is monotonically non-increasing with respect to z; and*(c) *$\eta(z) < 1$.*

Based on Lemma 1, the optimal pricing decision of the firm can be solved, as characterized in the following proposition:

Proposition 2. *The firm's optimal price p^* at time T3 is the unique solution of the equation $T(p^*) = 0$ and satisfies $p^* \in (v_F, a/b)$, where:*

$$T(p) \equiv (a/b - t_F + v_F - 2p)S(z_1) + (p + t_F - v_F)z_1\overline{F}(z_1) + \mu t_F \tag{8}$$

By investigating the influence of relevant parameters on the firm's optimal price, the following proposition is obtained:

Proposition 3. *The firm's optimal price p^* is:*

(a) *decreasing in the rare-earth supply $(\overline{q} - q)$, i.e., $\partial p^*/\partial(\overline{q} - q) < 0$;*

(b) *increasing in the firm's unit shortage loss t_F, i.e., $\partial p^*/\partial t_F > 0$;*

(c) *increasing in the firm's unit salvage value v_F, i.e., $\partial p^*/\partial v_F > 0$;*

(d) *increasing in the market potential a, i.e., $\partial p^*/\partial a > 0$;*

(e) *decreasing in the price sensitivity coefficient b, i.e., $\partial p^*/\partial b < 0$.*

Proposition 3 has important implications for the market pricing of rare-earth firms. First, the larger the market supply $\overline{q} - q$, the more rare-earth firms should lower the price p^* to reduce the loss of shortage or salvage. If the government wants to boost the price of rare earth, it should cut the total quota index $\overline{q}$ or increase the reserve q to reduce the amount of rare earths in circulation in the market $\overline{q} - q$. Second, when the unit loss of shortage t_F is larger or the salvage value v_F of rare-earth products is higher, the firm should increase the price p^* to reduce the market demand, because in this case, the backorder will cause great losses to the firm, but the value loss due to over-supply is small. Third, when the market potential a is larger, the firm has more pricing power due to the favourable demand situation. Finally, when the price sensitivity of rare-earth products is higher, the firm should reduce the price, because a small price reduction can bring about a large increase in demand.

3.3. The Government's Optimal Reserve Decision

By substituting the firm's optimal price p^* in Proposition 2 into Equation (6) and calculating the expectation of potential consumer size ε, the expected social welfare at time T2 can be obtained as follows:

$$\begin{aligned}\Pi_{SW}(\overline{q},q) &= \mathbb{E}_\varepsilon[\pi_{SW}(\overline{q},q,p^*)] \\ &= (v_F-c)\overline{q} + (r_G-c_G-v_F)q + \left[(r_G+t_F-v_F)S(z_1^*) - (t_F+t_G)\mu\right]y(p^*) \\ &+ \begin{cases}[(a+bp^*)/(2b)+t_G-r_G]S(z_1^*)y(p^*) & (a+bp^*)/(2b)+t_G \le r_G \\ [(a+bp^*)/(2b)+t_G-r_G]S(z_2^*)y(p^*) & (a+bp^*)/(2b)+t_G > r_G\end{cases}\end{aligned} \tag{9}$$

where $z_1^* = (\overline{q}-q)/y(p^*)$ and $z_2^* = \overline{q}/y(p^*)$. The government's objective at time T2 is to select the optimal reserve quantity $q \in [0,\overline{q}]$ to maximize $\Pi_{SW}(\overline{q},q)$.

Under the general demand distribution $F(\cdot)$, the firm's optimal price p^* is given by the implicit function (Equation (8)), which makes it very difficult to solve the government's reserve decision. In order to obtain an analytical solution, the following assumptions are made hereafter to simplify the analysis:

Assumption 1. *Reserve cost, salvage value and loss of shortage are not considered, i.e.,* $c_G = v_G = v_F = t_G = t_F = 0$.

Assumption 2. *The size of potential consumers* ε *is drawn from the uniform distribution on* $[0, 2\mu]$, *which means that:*

$$F(\varepsilon) = \begin{cases}\varepsilon/(2\mu) & \varepsilon \in [0,2\mu] \\ 1 & \varepsilon \in (2\mu,\infty)\end{cases} \text{ and } S(z) = \begin{cases} z - z^2/(4\mu) & z \in [0,2\mu] \\ \mu & z \in (2\mu,\infty)\end{cases}$$

Assumption 3. *The market supply* $\overline{q}-q$ *is no greater than the upper bound of demand* $2\mu y(p^*)$, *i.e.,* $\overline{q}-q \le 2\mu y(p^*)$, *which is equivalent to* $\overline{q}-q \le \mu$ *after transformation. To ensure that this condition holds for any reserve quantity* $q \in [0,\overline{q}]$, *assume* $\overline{q} \le \mu$.

Under the above assumptions, the expected social welfare (Equation (9)) can be reformulated as a function of Q:

$$\Pi_{SW}(Q) = \begin{cases}\Pi_{SW1} = -\mu t Q[4(1-X)Q^2 + Q - 4X]/8 + \mu Y^2(t - tX - c) & Q \in [0, A_1] \\ \Pi_{SW2} = -\mu t[2(1-X)Q^4 + Y^2Q^2 + 2Y^4X]/(4Q) + \mu Y^2\left(tY^2/8 + t - c\right) & Q \in (A_1, A_2] \\ \Pi_{SW3} = \mu t Q^2[Q^2 - 6Q + 8X]/8 + \mu Y^2(t - tX - c) & Q \in (A_2, Y]\end{cases} \tag{10}$$

where $t = a/b$, $X = 1 - r_G/t \in (0,1)$, $Q = \sqrt{(\overline{q}-q)/\mu}$, $Y = \sqrt{\overline{q}/\mu} \le 1$, $A_1 = \min\{4X, Y^2\}$ and $A_2 = \min\{4X, Y\}$. Due to the one-to-one mapping between Q and q, optimizing Π_{SW} with respect to q is equivalent to optimizing Π_{SW} with respect to Q.

Proposition 4. *Under Assumptions 1–3, the government's optimal reserve quantity* q^* *at time T2 is given by:*

$$q^* = \begin{cases} q_1^* = \overline{q} - \mu\kappa^2 & r_G \in [t/8, t) \text{ and } \overline{q} \ge \mu\kappa \\ q_2^* = \overline{q} - \overline{q}\kappa & r_G \in [t/8, t) \text{ and } \overline{q} < \mu\kappa \\ q_3^* = 0 & r_G \in (0, t/8)\end{cases} \tag{11}$$

where $\kappa \equiv [-1 + \sqrt{1 + 48X(1-X)}]/[12(1-X)]$ *(increasing in X),* $X_1 \equiv (6Y^2+1)Y^2/[2(3Y^4+1)]$*(increasing in Y). The corresponding expected social welfare is:*

$$\Pi_{SW}^{*}=\begin{cases}\Pi_{SW1}^{*}=\mu t[8(1-X)\kappa+1]\kappa^{2}/8+\mu Y^{2}[t(1-X)-c] & r_G\in[t/8,t)\ and\ \overline{q}\geq\mu\kappa\\ \Pi_{SW2}^{*}=-\mu tY^{3}[4(1-X)\kappa+1]\sqrt{\kappa}/2+\mu Y^{2}(8t-8c+tY^{2})/8 & r_G\in[t/8,t)\ and\ \overline{q}<\mu\kappa\\ \Pi_{SW3}^{*}=\mu Y^{2}(8t-8c+tY^{2}-6tY)/8 & r_G\in(0,t/8)\end{cases} \tag{12}$$

Proposition 4 shows that, depending on the expected NPV of its unreleased reserve (r_G), the government may adopt three different reserve strategies. When r_G is low ($r_G \in (0, t/8)$), there is little value in reserving the rare-earth products. Therefore, the government's strategy is not to reserve ($q_3^* = 0$), and all rare-earth products under the total quota index $\overline{q}$ enter the market circulation. When r_G is not too low ($r_G \geq t/8$), rare-earth products have reserve value, but the amount of the government reserve will vary according to the size of the total quota index $\overline{q}$: if $\overline{q}$ is low ($\overline{q} < \mu\kappa$), the government will keep a low reserve level ($q^* = q_2^*$), and will release as much as possible to fill the demand gap in the case of market supply shortage. If the total quota index $\overline{q}$ is high ($\overline{q} > \mu\kappa$), the government will keep a high reserve level ($q^* = q_1^*$), which will cause the firm to set a high price. In this case, even if there is a market supply shortage, the government can fully fill the demand gap by only releasing a part of the reserve. Figure 2 illustrates the conditions where each of the above three strategies is applicable.

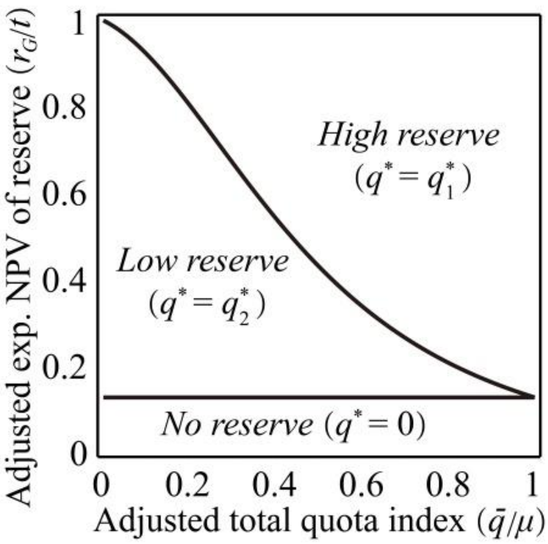

Figure 2. The government's reserve strategies.

By investigating the influence of parameters on the government's optimal reserve, we obtain the following proposition:

Proposition 5. *The government's optimal reserve quantity q^* is:*

(a) *Increasing in the expected NPV r_G of unreleased reserve, i.e., $\partial q^*/\partial r_G \geq 0$;*

(b) *Decreasing in the market potential a, i.e., $\partial q^*/\partial a \leq 0$;*

(c) *Increasing in the price sensitivity factor b, i.e., $\partial q^*/\partial b \geq 0$;*

(d) *Increasing in the total quota index $\overline{q}$, i.e., $\partial q^*/\partial q_H \geq 0$;*

(e) *Decreasing in the average size μ of potential consumers, i.e., $\partial q^*/\partial \mu \leq 0$.*

The implications of Proposition 5 are as follows. With a higher expected NPV r_G of its unreleased reserve, the government should naturally stockpile to meet future demand. When the market potential a is larger, the market demand of the current period is larger, and the government should reduce the reserve quantity to better meet the current market demand. When the rare-earth product is more sensitive to the price (b is larger), the government should reduce the current market supply to maintain a high market price, so it should increase the reserve quantity. A higher total quota index $\overline{q}$ means there may be more leftover if demand remains unchanged, so the government should increase the reserve quantity. When the expected size μ of potential consumers is larger, the total demand in the current period is larger, so the government should reduce the reserve quantity to better meet the current market demand.

3.4. The Government's Total Quota Index Decision

At time T1, the government needs to make a decision on the total quota index $\overline{q}$ by maximizing the expected social welfare Π_{SW}^{*} (Equation (12)). Due to the one-to-one correspondence between $Y = \sqrt{\overline{q}/\mu}$ and $\overline{q}$, the optimization of Π_{SW}^{*} with respect to $\overline{q}$ is equivalent to the optimization of Π_{SW}^{*} with respect to Y.

Proposition 6. *Under Assumptions 1–3, the government's optimal decision of total quota index at time T1 is $\overline{q}^{*} = \mu Y^{*2}$, where:*

$$Y^{*} = \begin{cases} Y_1^{*} = 1 & r_G \in [t/8, t) \text{ and } r_G \geq c \\ Y_2^{*} = 3\sqrt{\kappa}[2(1-X)\kappa + 1/2] - \sqrt{9\kappa[2(1-X)\kappa + 1/2]^2 - 4(1-c/t)} & r_G \in [t/8, t) \text{ and } r_G < c = \\ Y_3^{*} = 9/4 - \sqrt{64ct + 17t^2}/(4t) & r_G \in (0, t/8) \end{cases} \quad (13)$$

The corresponding expected social welfare is:

$$\Pi_{SW}^{**} = \begin{cases} \Pi_{SW1}^{**} = \mu t[8(1-X)\kappa + 1]\kappa^2/8 + \mu[t(1-X) - c] & r_G \in [t/8, t) \text{ and } r_G \geq c \\ \Pi_{SW2}^{**} = (t-c)\mu Y_2^{*2}/2 - \mu Y_2^{*2} t\sqrt{\kappa}[4(1-X)\kappa + 1]Y_2^{*}/8 & r_G \in [t/8, t) \text{ and } r_G < c \\ \Pi_{SW3}^{**} = \mu t(3 - Y_3^{*})Y_3^{*3}/8 & r_G \in (0, t/8) \end{cases} \quad (14)$$

By Proposition 6, the government will formulate the total quota index according to the relationship between the expected NPV r_G of the unreleased reserve and the unit production cost c. When r_G is very low ($r_G < t/8$), the government only needs to consider meeting the market demand of the current period, but will not consider the need of reserving, so its optimal decision of total quota index does not include r_G. When r_G exceeds a certain threshold ($r_G > t/8$) but is not enough to cover the production cost ($r_G < c$), the government will appropriately increase the total quota index to meet the current market demand as much as possible. At the same time, if the current supply is sufficient, the government will reserve part of it. When r_G exceeds the aforementioned threshold ($r_G > t/8$) and is higher than the production cost ($r_G < c$), the government will set the total quota index to the maximum extent and supply the current period according to the maximum possible value of market demand, with the remaining part being reserved.

4. Numerical Analysis

In this section, numerical simulation is used to verify the model results and observe the relevant equilibrium properties. Firstly, we investigate the decision results regarding the rare-earth quantities (i.e., the total quota index $\overline{q}^{*}$, the reserve quantity q^{*}, and expected unreleased reserve $q^{*} - \mathbb{E}[\hat{q}^{*}]$) and their influencing factors. Assume that the mean value of potential consumer size is $\mu = 1$ and the relative market potential is $t = a/b = 1$. Figure 3 depicts how the total quota index $\overline{q}^{*}$, the reserve quantity q^{*}, and the expected unreleased reserve $q^{*} - \mathbb{E}[\hat{q}^{*}]$ change with the expected NPV r_G of the unreleased reserve, under production costs (a) $c = 0.3$, (b) $c = 0.5$ and (c) $c = 0.7$, respectively.

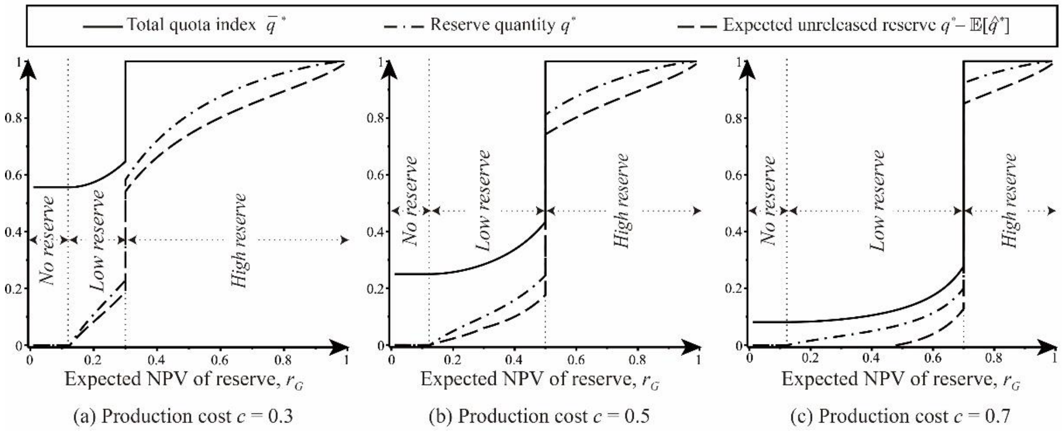

Figure 3. The total quota index $\bar{q}^*$, reserve quantity q^* and expected unreleased reserve $q^* - \mathbb{E}[\hat{q}^*]$ change with parameters.

As can be seen from any sub-figure in Figure 3, when r_G changes from small to large, the government's reserve strategy will go through three different stages: no reserve, low reserve and high reserve (Proposition 4). Among them, the total quota index $\bar{q}^*$ in the no-reserve stage is low, and is independent of r_G. Both $\bar{q}^*$ and q^* increase in r_G, and the government will release part of the reserve in the case of insufficient market supply (the release amount is equal to the difference between the dot-dashed line and the dashed line). In this case, the supply may still be insufficient after the release of the reserve (part $r_G \in (0.125, 0.5)$ in Figure 3c), but the expected unreleased reserve increases in r_G. When r_G crosses the threshold of production cost c, the government will enter the stage of high reserve, and both the total quota and reserve quantity will see a sharp increase. In this case, even if there is a shortage of supply, the government only needs to release a part of the reserve to fully satisfy the market, and the expected quantity of unreleased reserve still increases in r_G. Note that Figure 3 shows that the expected release quantity $\mathbb{E}[\hat{q}^*]$ (the difference between the dotted line and the dashed line) first increases and then decrease in r_G. A horizontal comparison of the three sub-figures in Figure 3 shows that, given that other conditions remain unchanged, when the production cost c increases: (i) the possibilities of no reserve, low reserve, and high reserve are unchanged, increased, and decreased, respectively; and (ii) the government's total quota index decreases under the strategy of no reserve and low reserve, and the reserve quantity decreases under the low-reserve strategy.

Next, the influence of parameters on the equilibrium rare-earth price p^* is investigated. Figure 4 plots p^* as a function of r_G under (a) relative market potential $t = a/b = 1$ and rare-earth production cost $c = 0.5$ (solid line), (b) $t = 1$ and $c = 0.7$ (dashed line), and (c) $t = 1.1$ and $c = 0.5$ (dot-dashed line).

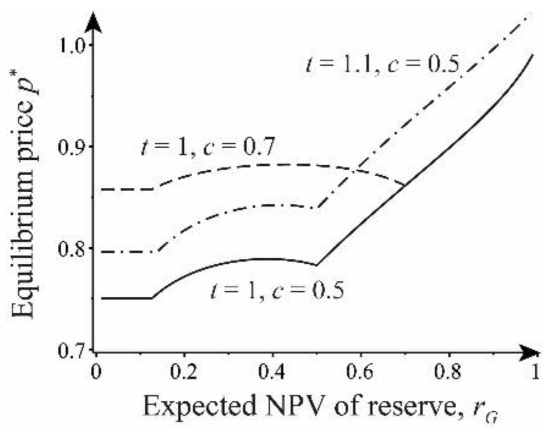

Figure 4. Equilibrium price p^* changes with parameters.

As can be seen from any curve in Figure 4, with the increase in r_G, p shows three different changes: When r_G is small, the government adopts the no-reserve strategy, and the equilibrium price p is independent of r_G. When r_G is at a medium level, the government adopts the low-reserve strategy. In this case, p increases first and then decreases in r_G, because the market supply $\overline{q}^* - q^*$ decreases first and then increases in r_G. When r_G is high, the government adopts the high-reserve strategy, and p increases with r_G. By comparing the solid line with the dashed line in Figure 4, it can be seen that when the production cost c increases, the equilibrium price will increase both in the case of no reserve and low reserve, while the price will remain unchanged in the case of high reserve. The former is because the firm's rare-earth supply $\overline{q}^* - q^*$ decreases with the production cost in the case of no reserve or low reserve, so the firm should adjust the price in the opposite direction to balance the supply and demand. The latter is because, in the case of high-reserve, the government will always control the firm's supply quantity according to $\overline{q}^* - q^* = \mu\kappa^2$ (Equation (11)), which is not affected by the firm's production cost. Comparing the solid line with the dot-dashed line in Figure 4, it is found that an increase in the relative market potential $t = a/b$ leads to an increase in rare-earth price. This is because, when the relative market potential t increases, either the potential market demand faced by the firm increases (i.e., the market potential a increases) or the consumer becomes less sensitive to the rare-earth price (i.e., the price sensitivity factor b decreases), both of which will prompt the firm to set a higher price.

Finally, the influence of parameters on the expected social welfare and the firm's expected profit is investigated. Figure 5a shows how the expected social welfare Π_{SW}^{**} varies with r_G under different parameter conditions. For the whole system, the expected NPV r_G of unreleased reserve, the mean size of potential consumers μ and the relative market potential t can be regarded as efficiency parameters, while the firm's production cost c is a cost parameter. Therefore, Figure 5a shows that the expected social welfare increases with r_G, μ, and t and decreases with c. Figure 5b shows how the firm's expected profit is affected by parameters when the government acquires rare-earth reserves at the price equaling the firm's production cost. It can be seen that the firm's expected profit increases with the mean size of potential consumers μ and the relative market potential t, and decreases with the production cost c, for the same reasons as before. However, Figure 5b also shows that the impact of r_G on the firm's expected profit is not monotone: when r_G is small, the government does not reserve, and the firm's expected profit is independent of r_G; when r_G is not very small, the government adopts the low-reserve or high-reserve strategy, and an increase in r_G will impact both the firm's price and sales volume, so the firm's expected profit may either increase or decrease. In addition, the firm's expected profit is also related to the government's reserve price w. Figure 5c simulates the relationship between the firm's expected profit and r_G when the government purchases reserves at a price that is $\alpha \in \{1, 1.1, 1.2\}$ times of the firm's production cost.

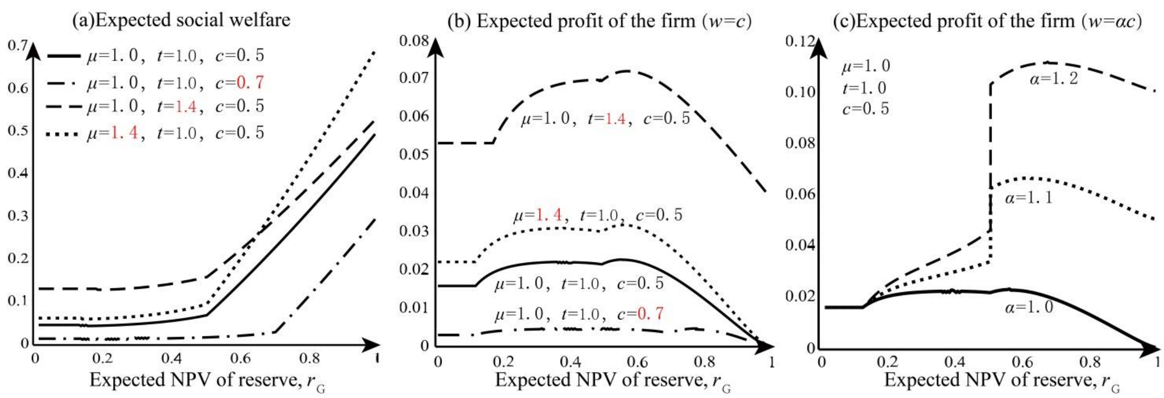

Figure 5. Impacts of parameters on the expected social welfare and the firm's expected profit. In the left and middle plots, the solid curve is the benchmark curve, and all other curves involve a parameter (marked in red font) different from the benchmark curve.

5. Conclusions

In the new wave of scientific and technological revolution and industrial transformation, the application value of rare earth in social development and national economies has been further enhanced, and the importance of the total quota management and strategic reserve of rare-earth products has become increasingly prominent. This paper examines a multi-stage decision model of a rare-earth supply chain in a game-theoretic framework, and examines the government's reserve decision and the rare-earth firm's pricing decision under total quota management.

Several meaningful implications are generated for both the government and rare-earth firms. For the government, critical conditions of whether and at what level the government should keep rare-earth reserves are identified. The production cost and the expected net present value (NPV) of unreleased reserves are found to be key factors affecting the above decision. When the expected NPV of reserves is below a threshold, the government should not keep any reserve. Otherwise, it should keep a low or high level of reserve, depending on the relationship between the expected NPV of the reserve and the production cost: a low-reserve (high-reserve) strategy should be adopted if the former is lower (higher). Under situations where the low-reserve strategy is appropriate, the total quota index should be set as an increasing function of the expected NPV of the reserve, and the government should also release as much of the reserve as possible to meet the demand gap (if any). Under situations where the high-reserve strategy is appropriate, the total quota index should be set to be sufficiently large to cover the entire market demand, and the government can fill the demand gap (if any) by releasing only part of the reserve.

For the rare-earth firm, implications are derived from how the rare-earth price should be determined under different reserve strategies of the government. Under situations where the government adopts the no-reserve strategy, the firm should set a relatively low price that is independent of the expected NPV of reserve. Under situations where a low-reserve strategy is adopted, the firm should determine a price that first increases and then decreases with the expected NPV of reserves. Under situations where a high-reserve strategy is adopted, the firm should set a price that is increasing with the expected NPV of reserves. Moreover, a somewhat counterintuitive finding is observed: Under situations where the government adopts a high-reserve strategy, the equilibrium price does not change with the production cost. This is because the government, in this case, will always control the firm's supply quantity to a certain level that is not affected by the production cost.

It should be noted that in order to obtain analytical solutions to the government's optimization problem, we made some simplifications to the model, such as assuming a uniform demand distribution and ignoring the reserve costs of the government and the firm. In addition, China implements the policy of combining government reserves and commercial reserves. The preliminary study of the government reserve of rare earth in this paper will provide useful reference for future research on commercial and hybrid reserves of rare-earth products.

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Appendix A

Proof of Proposition 1. The first-order derivative of $\pi_{SW}(\overline{q}, q, p, \hat{q})$ with respect to $\hat{q}$ is:

$$\frac{\partial \pi_{SW}}{\partial \hat{q}} = \begin{cases} v_G - r_G < 0 & \hat{q} > D - \overline{q} + q \\ (a+bp)/(2b) + t_G - r_G & \hat{q} \le D - \overline{q} + q \end{cases}$$

(i) when $(a+bp)/(2b) + t_G - r_G \le 0$, obviously, $\partial \pi_{SW}/\partial \hat{q} \le 0$ always holds, and the optimal solution is $\hat{q} = 0$. (ii) When $(a+bp)/(2b) + t_G - r_G > 0$, $\pi_{SW}(\overline{q}, q, p, \hat{q})$ increases in $\hat{q}$ first and then decreases, and reaches the maximum value at $\hat{q} = D - \overline{q} + q$. Combined with the range of $\hat{q}$, i.e., $[0, q]$, the optimal solution can be obtained as follows:

$$\begin{aligned} \hat{q} &= \begin{cases} q & D \in [\overline{q}, \infty) \\ D - \overline{q} + q & D \in [\overline{q} - q, \overline{q}) \\ 0 & D \in [0, \overline{q} - q) \end{cases} \\ &= \min\{D, \overline{q}\} - \min\{D, \overline{q} - q\} \end{aligned}$$

Summarizing the above two cases, it can be seen that the government's optimal release quantity is Equation (5).

Substituting Equation (5) back into the government's objective function (Equation (4)), the social welfare under the optimal release quantity can be obtained after necessary transformations, as given by Equation (6). □

Proof of Lemma 1. For part (a), $S(z) = \int_0^z \overline{F}(\xi) d\xi$ is easily obtained by applying integration by parts according to the definition of expectation. In addition, $S(z) \le \mathbb{E}_\varepsilon[\varepsilon] = \mu$ since $\min\{\varepsilon, z\} \le \varepsilon$. Part (b) is directly obtained from [38]. Part (c) is because $S(z) = \int_0^z \overline{F}(\xi) d\xi > \int_0^z \overline{F}(z) d\xi = z\overline{F}(z)$. □

Proof of Proposition 2. The first-order derivative of Π_F with respect to p is $\partial \Pi_F / \partial p = bT(p)/a$. Since both a and b are positive, the sign of $\partial \Pi_F / \partial p$ depends entirely on $T(p)$. Easy to verify is that $T(p)$ is decreasing in p:

$$T'(p) = -2S(z_1)[1 - \eta(z_1)] - bz_1^2(p + t_F - v_F) f(z_1)/(a - bp) < 0$$

when $p = v_F (< a/b)$, we have: $T(v_F) = (a/b - v_F)S(z_1) + t_F z_1 \overline{F}(z_1) + t_F[\mu - S(z_1)] > 0$ (where $\mu - S(z_1) > 0$ is guaranteed by Lemma 1a); when $p \to a/b$, we have: $z_1 \to \infty, S(z_1) \to \mu$ and $z_1 \overline{F}(z_1) \to 0$, so $\lim_{p \to a/b} T(p) = -(a/b - v_F)\mu < 0$ holds. Therefore, there exists a unique $p^* \in (v_F, a/b)$ such that $T(p^*) = 0$, and p^* is the firm's optimal price. □

Proof of Proposition 3. Proposition 2 and its proof have shown that p^* is the unique solution to $T(p^*) = 0$, and $T'(p) < 0$ holds for all $p \in [0, a/b]$ and $z > 0$. Since $\eta'(z) = \overline{F}(z)[1 - g(z) - \eta(z)]/S(z)$ (by definition) and $\eta'(z) \le 0$ (by Lemma 1b), it follows that $1 - g(z) - \eta(z) \le 0$, i.e., $g(z) \ge 1 - \eta(z)$, there is:

$$\begin{aligned} \partial T(p)/\partial z_1 &= \overline{F}(z_1)[a/b - p - (p + t_F - v_F) g(z_1)] \\ &\le \overline{F}(z_1)\{a/b - p - (p + t_F - v_F)[1 - \eta(z_1)]\} \\ &= \overline{F}(z_1)[T(p) - \mu t_F]/S(z_1) \end{aligned}$$

Thus, at $p = p^*$, $\partial T(p)/\partial z_1 \le -\mu t_F \overline{F}(z_1)/S(z_1) < 0$ holds. Below, we prove the conclusions in Proposition 3 one by one.

(a) Take $(\overline{q} - q)$ as a whole, and take the derivatives with respect to $(\overline{q} - q)$ from both sides of Equation $T(p^*) = 0$, there is:

$$T'(p^*)\frac{\partial p^*}{\partial(\overline{q} - q)} + \frac{\partial T(p)}{\partial z_1}\frac{1}{y(p)}\bigg|_{p=p^*} = 0 \tag{A1}$$

As $T'(p^*) < 0, [\partial T(p)/\partial z_1]\big|_{p=p^*} < 0$ and $y(p^*)^{-1} > 0$, there must be $\partial p^*/\partial(\overline{q} - q) < 0$ to make the equation hold.

(b) Take the derivatives with respect to t_F on both sides of the equation $T(p^*) = 0$, and one obtains

$$T'(p^*)\frac{\partial p^*}{\partial t_F} - S(z_1^*) + z_1^*\overline{F}(z_1^*) + \mu = 0 \tag{A2}$$

As $T'(p^*) < 0$ and $-S(z_1^*) + z_1^*\overline{F}(z_1^*) + \mu > 0$, there must be $\partial p^*/\partial t_F > 0$ for the equation to hold.

(c) Take the derivatives with respect to v_F on both sides of Equation $T(p^*) = 0$, and one obtains:

$$T'(p^*)\frac{\partial p^*}{\partial v_F} + S(z_1^*) - z_1^*\overline{F}(z_1^*) = 0 \tag{A3}$$

As $S(z_1^*) - z_1^*\overline{F}(z_1^*) = S(z_1^*)[1 - \eta(z_1^*)] > 0$ (Lemma 1c) and $T'(p^*) < 0$, there must be $\partial p^*/\partial v_F > 0$ for the equation to hold.

(d) Take the derivatives with respect to a on both sides of Equation $T(p^*) = 0$, and we obtain

$$T'(p^*)\frac{\partial p^*}{\partial a} + \frac{\partial T(p)}{\partial z_1}\frac{\partial z_1^*}{\partial a}\bigg|_{p=p^*} + \frac{S(z_1^*)}{b} = 0 \tag{A4}$$

As $T'(p^*) < 0$, $[\partial T(p)/\partial z_1]\big|_{p=p^*} < 0, \partial z_1^*/\partial a = -bp^*z_1^*/(a^2 y(p^*)) < 0$, and $S(z_1^*)/b > 0$, there must be $\partial p^*/\partial a > 0$ to make the equation hold.

(e) Take the derivatives with respect to b on both sides of the equation $T(p^*) = 0$, and we obtain

$$T'(p^*)\frac{\partial p^*}{\partial b} + \frac{\partial T(p)}{\partial z_1}\frac{\partial z_1^*}{\partial b}\bigg|_{p=p^*} - \frac{aS(z_1^*)}{b^2} = 0 \tag{A5}$$

As $T'(p^*) < 0$, $[\partial T(p)/\partial z_1]\big|_{p=p^*} < 0, -aS(z_1^*)/b^2 < 0$, and $\partial z_1^*/\partial b = p^*z_1^*/(a - bp^*) > 0$, there must be $\partial p^*/\partial b < 0$ for the equation to hold. $\square$

Proof of Proposition 4. Noting that $\Pi_{SW}(Q)$ is a piecewise function of Q, we will solve the local optimal solution for each piece, and then determine the global optimal solution by comparing the local optimal values of the three pieces.

(1) When $Q \in [0, A_1]$, the second-order derivative of $\Pi_{SW1}(Q)$ with respect to Q is $\Pi''_{SW1}(Q) = -\mu t[12Q(1 - X) + 1]/4 < 0$, implying that $\Pi_{SW1}(Q)$ is concave in Q. Therefore, its local optimal solution can be uniquely determined by the following Kuhn–Tucker (KT) conditions:

$$\begin{gathered} \Pi'_{SW1}(Q) + \lambda_1 - \lambda_2 = 0, \\ \lambda_1 Q = 0, \lambda_2(A_1 - Q) = 0, \\ 0 \le Q \le A_1, \lambda_1 \ge 0, \lambda_2 \ge 0 \end{gathered}$$

where $\Pi'_{SW1}(Q) = \mu t(6XQ^2 - 6Q^2 - Q + 2X)/4$. Solving the above KT conditions yields the local optimal solution on the interval $[0, A_1]$:

$$Q_1 = \begin{cases} Q_1^* \equiv \kappa & if\ 6A_1^2(1 - X) + A_1 - 2X \ge 0 \\ A_1 & if\ 6A_1^2(1 - X) + A_1 - 2X < 0 \end{cases} \tag{A6}$$

(2) When $Q \in [A_1, A_2]$, the second-order derivative of $\Pi_{SW2}(Q)$ with respect to Q is $\Pi''_{SW2}(Q) = -\mu t[3Q^4(1 - X) + XY^4]/Q^3 < 0$, implying that $\Pi_{SW2}(Q)$ is concave in

Q. Therefore, its local optimal solution can be uniquely determined by the following KT conditions:

$$\begin{aligned} &\Pi'_{SW2}(Q) + \lambda_1 - \lambda_2 = 0, \\ &\lambda_1(Q - A_1) = 0, \lambda_2(A_2 - Q) = 0, \\ &A_1 \le Q \le A_2, \lambda_1 \ge 0, \lambda_2 \ge 0 \end{aligned}$$

where $\Pi'_{SW2}(Q) = \mu t(6XQ^4 + 2XY^4 - 6Q^4 - Q^2Y^2)/(4Q^2)$. Solving the above KT conditions yields the local optimal solution on the interval $[A_1, A_2]$:

$$Q_2 = \begin{cases} A_1 & if\ 6A_1^4(1-X) + A_1^2Y^2 - 2XY^4 > 0 \\ Q_2^* = Y\sqrt{\kappa} & if \begin{cases} 6A_1^4(1-X) + A_1^2Y^2 - 2XY^4 \le 0 \\ 6A_2^4(1-X) + A_2^2Y^2 - 2XY^4 \ge 0 \end{cases} \\ A_2 & if\ 6A_2^4(1-X) + A_2^2Y^2 - 2XY^4 < 0 \end{cases} \tag{A7}$$

(3) When $Q \in [A_2, Y]$, the first-order derivative of $\Pi_{SW3}(Q)$ with respect to Q is $\Pi'_{SW3}(Q) = \mu tQ(2Q^2 - 9Q + 8X)/4$, whose sign is determined by the term $2Q^2 - 9Q + 8X$. On the interval $[A_2, Y]$, $2Q^2 - 9Q + 8X$ is decreasing in Q, and $Q'_3 = [9 - \sqrt{81 - 64X}]/4$ is the root of $2Q^2 - 9Q + 8X = 0$ (the other larger root is rounded off). Therefore, the local optimal solution of $\Pi_{SW3}(Q)$ on $[A_2, Y]$ is

$$Q_3 = \begin{cases} A_2 & if\ 2A_2^2 - 9A_2 + 8X < 0 \\ Q'_3 & if \begin{cases} 2A_2^2 - 9A_2 + 8X \ge 0 \\ 2Y^2 - 9Y + 8X \le 0 \end{cases} \\ Y & if\ 2Y^2 - 9Y + 8X > 0 \end{cases} \tag{A8}$$

Below we determine the global optimal solution by comparing the above three local optimal solutions.

(1) when $4X < Y^2$, we have $A_1 = A_2 = 4X$, and $6A_1^2(1 - X) + A_1 - 2 = 48X(1 - X) + 1 > 0$ in Equation (A6) is always true. Therefore, the optimal solution of the first piece (Equation (A6)) is simplified to $Q_1 = Q_1^*$. Similarly, the optimal solutions of the second and third pieces (Equations (A7) and (A8)) can be simplified to $Q_2 = A_1 = 4X$ and $Q_2 = A_2 = 4X$, respectively. Since $Q = 4X$ is the feasible but non-optimal solution of the first piece, the local optimal solution $Q_1 = Q_1^*$ of the first piece is the global optimal solution, namely

$$Q^* = Q_1^*, X < Y^2/4 \tag{A9}$$

(2) When $Y^2 \le 4X < Y$, we have $A_1 = Y^2$ and $A_2 = 4X$. In this case, $6A_1^2(1 - X) + A_1 - 2X \ge 0$ in Equation (A6) is equivalent to $X \le X_1$, where $X_1 \equiv (6Y^2 + 1)Y^2/[2(3Y^4 + 1)]$. It is easy to verify that $X_1 > Y^2/4$, but X_1 can be either larger or smaller than $Y/4$, so the optimal solution (Equation (A20)) of the first piece is

$$Q_1 = \begin{cases} Q_1^* & X \in (Y^2/4, \min\{Y/4, X_1\}] \\ Y^2 & X \in (\min\{Y/4, X_1\}, Y/4] \end{cases} \tag{A10}$$

Similarly, $6A_1^4(1 - X) + A_1^2Y^2 - 2XY^4 > 0$ in Equation (A7) is equivalent to $X < X_1$, and $6A_2^4(1 - X) + A_2^2Y^2 - 2XY^4 > 0$ is always true, so Equation (A7) is converted to

$$Q_2 = \begin{cases} Y^2 & X \in (Y^2/4, \min\{Y/4, X_1\}] \\ Q_2^* & X \in (\min\{Y/4, X_1\}, Y/4] \end{cases} \tag{A11}$$

$2A_2^2 - 9A_2 + 8X < 0$ in Equation (A8) is equivalent to $X < 7/8$, which obviously holds, and $2Y^2 - 9Y + 8X \le 0$ also always holds. Therefore, Equation (A8) can be simplified to $Q_3 = 4X$. Since $4X$ is a feasible but non-optimal solution of the second piece, Equation (A8) cannot be a global optimal solution. Comparing Equations (A10) (A11), we can see that the global optimal solution for $Y^2 \le 4X < Y$ is

$$Q^* = \begin{cases} Q_1^* & X \in (Y^2/4, \min\{Y/4, X_1\}] \\ Q_2^* & X \in (\min\{Y/4, X_1\}, Y/4] \end{cases} \tag{A12}$$

(3) When $Y \le 4X < 4$, we have $A_1 = Y^2$ and $A_2 = Y$. In this case, $6A_1^2(1-X) + A_1 - 2X \ge 0$ in Equation (A6) is equivalent to $X \le X_1$, while X_1 can be either larger or smaller than $Y/4$; therefore, Equation (A6) becomes

$$Q_1 = \begin{cases} Q_1^* & X \in (Y/4, \max\{Y/4, X_1\}] \\ Y^2 & X \in (\max\{Y/4, X_1\}, 1) \end{cases} \tag{A13}$$

In Equation (A7), $6A_1^4(1-X) + A_1^2Y^2 - 2XY^4 > 0$ is equivalent to $X < X_1$, $6A_2^4(1-X) + A_2^2Y^2 - 2XY^4 > 0$ is equivalent to $X < 7/8$, and $X_1 < 7/8$; therefore, Equation (A7) becomes

$$Q_2 = \begin{cases} Y^2 & X \in (Y/4, \max\{Y/4, X_1\}] \\ Q_2^* & X \in (\max\{Y/4, X_1\}, 7/8] \\ Y & X \in (7/8, 1) \end{cases} \tag{A14}$$

In Equation (A8), letting $A_2 = Y$ yields $Q_3 = Y$. Since Y is a feasible but non-optimal solution to the second piece, Equation (A8) cannot be a globally optimal solution. By comparing Equations (A13) and (A14), the global optimal solution for $Y \le 4X < 4$ can be obtained as follows:

$$Q^* = \begin{cases} Q_1^* & X \in (Y/4, \max\{Y/4, X_1\}] \\ Q_2^* & X \in (\max\{Y/4, X_1\}, 7/8] \\ Y & X \in (7/8, 1) \end{cases} \tag{A15}$$

By combining Equations (A9), (A12) and (A15), the complete global optimal solution can be obtained as follows:

$$Q^* = \begin{cases} Q_1^* = \kappa & X \in (0, X_1] \\ Q_2^* = Y\sqrt{\kappa} & X \in (X_1, 7/8] \\ Q_3^* = Y & X \in (7/8, 1) \end{cases} \tag{A16}$$

Substituting Equation (A30) into $q^* = \overline{q} - \mu(Q^*)^2$, the optimal reserve quantity is obtained as shown in Equation (11). Then, substituting Equation (A16) back to Equation (10) obtains the maximized social welfare as shown in Equation (12). □

Proof of Proposition 5. By Equation (11), q^* is increasing in $\overline{q}$ and decreasing in μ, so parts (d)–(e) are immediately proved. As (i) q^* is decreasing in κ, (ii) κ is increasing in X, and (iii) $X = 1 - br_G/a$ is decreasing in b and r_G and increasing in a, respectively, one can verify that q^* is increasing in b and r_G and decreasing in a; hence, parts (a)–(c) are proved. □

Proof of Proposition 6. Since $\prod_{SW}^*$ (Equation (12)) is a piecewise function of Y, below we first find the local optimal solution in each piece, and then determine the global optimal solution through comparison.

(1) Case $r_G \in [t/8, 1)$ and $\overline{q} \ge \mu\kappa$ (i.e., $Y^2 \ge \kappa$). Since $\partial \Pi_{SW1}^* / \partial Y = 2\mu(r_G - c)Y$, Π_{SW1}^* is increasing in Y when $r_G \ge c$. Thus, the local optimal solution is $Y = 1$, and the corresponding expected social welfare is Π_{SW1}^{**}, as given by Equation (14); when $r_G < c$, Π_{SW1}^* is decreasing in Y. Thus, the local optimal solution is $Y = \sqrt{\kappa}$, and the corresponding expected social welfare is $\Pi_{SWB}^{**} = \Pi_{SW1}^{**} - \mu(1-\kappa)(r_G - c)$.

(2) Case $r_G \in [t/8, 1)$ and $\bar{q} \le \mu\kappa$ (i.e., $Y^2 \le \kappa$). The following KT conditions are necessary for finding the local optimal solution of Y:

$$\partial\Pi^*_{SW2}/\partial Y - \lambda = 0,\ \lambda(\sqrt{\kappa} - Y) = 0,\ \sqrt{\kappa} \ge Y,\ \lambda \ge 0$$

Solving the above K-T conditions yields: if $r_G < c$, then $Y = Y^*_2$, and the corresponding expected social welfare is Π^{**}_{SW2}, as given in Equation (14); if $r_G \ge c$, then $Y = \sqrt{\kappa}$, and the corresponding expected social welfare is Π^{**}_{SWB}.

(3) Case $r_G \in (0, t/8)$. Solving $\partial\Pi^*_{SW3}/\partial Y = \mu Y(2tY^2 - 9tY - 8c + 8t)/4 = 0$ yields the optimal solution $Y = Y^*_3$, and the corresponding expected social welfare is Π^{**}_{SW3}.

Comparing the above three cases, one can verify that the global optimal solution is as shown in Equation (13), and the corresponding expected social welfare is as shown in Equation (14). □

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