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Modeling to Factor Productivity of the United Kingdom Food Chain: Using a New Lifetime-Generated Family of Distributions

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Abstract: This article proposes a new lifetime-generated family of distributions called the sine-exponentiated Weibull-H (SEW-H) family, which is derived from two well-established families of distributions of entirely different nature: the sine-G (S-G) and the exponentiated Weibull-H (EW-H) families. Three new special models of this family include the sine-exponentiated Weibull exponential (SEWE$_x$), the sine-exponentiated Weibull Rayleigh (SEWR) and sine-exponentiated Weibull Burr X (SEWBX) distributions. The useful expansions of the probability density function (pdf) and cumulative distribution function (cdf) are derived. Statistical properties are obtained, including quantiles (Q_U), moments (M_O), incomplete M_O (IM_O), and order statistics (O_S) are computed. Six numerous methods of estimation are produced to estimate the parameters: maximum likelihood (M_L), least-square (L_S), a maximum product of spacing (MP_RS_P), weighted L_S (WL_S), Cramér–von Mises (C_RVM), and Anderson–Darling (A_D). The performance of the estimation approaches is investigated using Monte Carlo simulations. The total factor productivity (TFP) of the United Kingdom food chain is an indication of the efficiency and competitiveness of the food sector in the United Kingdom. TFP growth suggests that the industry is becoming more efficient. If TFP of the food chain in the United Kingdom grows more rapidly than in other nations, it suggests that the sector is becoming more competitive. TFP, also known as multi-factor productivity in economic theory, estimates the fraction of output that cannot be explained by traditionally measured inputs of labor and capital employed in production. In this paper, we use five real datasets to show the relevance and flexibility of the suggested family. The first dataset represents the United Kingdom food chain from 2000 to 2019, whereas the second dataset represents the food and drink wholesaling in the United Kingdom from 2000 to 2019 as one factor of FTP; the third dataset contains the tensile strength of single carbon fibers (in GPa); the fourth dataset is often called the breaking stress of carbon fiber dataset; the fifth dataset represents the TFP growth of agricultural production for thirty-seven African countries from 2001–2010. The new suggested distribution is very flexible and it outperforms many known distributions.

Keywords: total factor productivity; food chain; time series; forecast; maximum likelihood estimation; maximum product spacing; sine class of distributions; exponentiated Weibull class of distributions

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1. Introduction

In the last few years, various techniques of adding a parameter to distributions have been proposed and discussed. These extended distributions give the flexibility in particular applications, such as economics, engineering, biomedical, biological studies, engineering, physics, food, environmental sciences, COVID-19, and many more. Several famous families are the Marshall–Olkin-G given in [1], odd Fréchet-G [2], beta-G [3], logarithmic-X family of distributions [4], the extended cosine-G [5], the arcsine-exponentiated-X family [6], truncated Cauchy power Weibull-G [7], the odd-exponentiated half logistic-G [8], generalized transmuted-G [9], the generalized odd log-logistic-G [10], transmuted odd Fréchet-G [11], the logistic-X [12], beyond the Sin-G family [13], Cos-G class of distributions [14], odd Perks-G [15], U family of distributions [16], the extended odd Fréchet-G [17], exponentiated M-G [18], transmuted geometric-G [19], half logistic Burr X-G [20], a new sine-G in [21], exponentiated-truncated inverse Weibull-G [22], Burr X-G [23], sec-G [24], odd Nadarajah-Haghighi-G [25], Topp-Leone-G [26], sine Topp-Leone-G family of distributions by [27], a new power Topp-Leone-G by [28], truncated inverted Kumaraswamy generated-G by [29], among others.

The authors of [30] proposed the EW-H family; this class extended the Weibull-H family of distribution introduced by [31]. The cdf of the EW-H family is provided via

$$G_{EW-H}(x;\lambda,\theta,\beta,\delta)=\left[1-e^{-\lambda\left(\frac{H(x;\delta)}{\overline{H}(x;\delta)}\right)^{\theta}}\right]^{\beta},\lambda,\theta,\beta>0,x\in R,\delta\in R, \tag{1}$$

and the pdf reduces to

$$g_{EW-H}(x;\lambda,\theta,\beta,\delta)=\lambda\theta\beta h(x;\delta)\frac{H(x;\delta)^{\theta-1}}{\overline{H}(x;\delta)^{\theta+1}}e^{-\lambda\left(\frac{H(x;\delta)}{\overline{H}(x;\delta)}\right)^{\theta}}\left[1-e^{-\lambda\left(\frac{H(x;\delta)}{\overline{H}(x;\delta)}\right)^{\theta}}\right]^{\beta-1}. \tag{2}$$

where $H(x;\delta)$, $h(x;\delta)$, and $\overline{H}(x;\delta)$ signify the cdf, pdf, and survival function (sf) of a baseline model considering a vector of parameters δ.

The creation of trigonometric classes of distributions has recently garnered considerable attention. These families have the benefit of maintaining a balance between their definitions' relative simplicity, which makes it possible to fully understand their mathematical features, and their broad application for modeling numerous sorts of real-world datasets. These two conclusions result from the proper use of adaptable trigonometric functions. The authors of [32] presented another idea of generating a new life distributions by modification of trigonometric functions to give new statistical distributions. They transformed the sine function into a new statistical distribution called the S-G family where the cdf and pdf are provided via

$$F(x)=\sin(\frac{\pi}{2}G(x)),\quad x\in R, \tag{3}$$

and

$$f(x)=\frac{\pi}{2}g(x)\cos(\frac{\pi}{2}G(x)),\quad x\in R. \tag{4}$$

The associated hazard rate function (hrf) is provided via

$$\xi(x)=\frac{\pi}{2}g(x)\tan(\frac{\pi}{4}(1+G(x))). \tag{5}$$

There are several further trigonometric families of distributions. For illustration, consider the beta trigonometric distribution by [33], sine square distribution by [34], a cosine approximation to the normal distribution by [35], odd hyperbolic cosine exponential-distribution by [36], new trigonometric classes of probabilistic distributions by [37], odd hyperbolic cosine family of lifetime distributions by [38], transmuted arcsine distribution by [39], among others.

In the article under consideration, our primary focus lies in introducing a new family of sine-generated distributions by considering the exponentiated Weibull-H family as the baseline distribution in the sine family. This new family is referred to as the SEW-H family of distributions. The following arguments give enough motivation to study the proposed model. We specify it as follows: (i) the new suggested family of distributions is very flexible and contains many generated family of distributions (see Table 1); (ii) the shapes of the probability density function (pdf) for the new models can be decreasing, right skewness, left skewness, unimodal, and heavy-tailed; (iii) the new suggested model has a closed form for quantile function and this makes the calculation of some properties such as skewness and kurtosis very easy; also to generate random numbers from the new suggested family becomes easy; (iv) some statistical and mathematical properties of the new suggested family such as Q_U, M_O, IM_O and O_S are explored; (v) six different methods of estimation, including M_L, L_S, MP_RS_P, WL_S, C_RVM, and A_D, are produced to estimate the parameters. We hope that the proposed model can be implemented to fit data in diverse scientific entities. This ability of the model is explored using five real life datasets proving the practical utility of the model being featured:

The first dataset: It represents the food chain in the United Kingdom from 2000 to 2019. The food sector plays a significant part in our economy, accounting for about 9 per cent of the Gross Value Added of the UK non-financial business economy. Four sectors make up the food chain: manufacture, wholesale, retail and non-residential catering. Both alcoholic and non-alcoholic drinks are included in food. Total factor productivity is a measure of the efficiency with which inputs are converted into outputs. For example, TFP increases if the volume of outputs increases while the volume of inputs stays the same. Similarly, TFP increases if the volume of inputs decreases while the volume of outputs stays the same. Although there is a practical limit on how much food people want to buy, the volume of output can increase due to increases in quality of products and by increases in exports.

The second dataset: It represents the food and drink wholesaling in the United Kingdom from 2000 to 2019 as one factor of FTP.

The third dataset: It is called the Single carbon fiber data and it is contains the tensile strength of single carbon fibers (in GPa).

The fourth dataset: It is often called the breaking stress of carbon fibers dataset.

The fifth dataset: It represents the TFP growth agricultural production for thirty-seven African countries from 2001–2010 as reported in Figure 1. Increasing the efficiency of agricultural production—getting more output from the same amount of resources—is critical for improving food security. To measure the efficiency of agricultural systems, we use TFP. TFP is an indicator of how efficiently agricultural land, labor, capital, and materials (agricultural inputs) are used to produce a country's crops and livestock (agricultural output)—it is calculated as the ratio of total agricultural output to total production inputs. When more output is produced from a constant amount of resources, meaning that resources are being used more efficiently, TFP increases. Measures of land and labor productivity—partial factor productivity (PFP) measures—are calculated as the ratio of total output to total agricultural area (land productivity) and to the number of economically active persons in agriculture (labor productivity). Because PFP measures are easy to estimate, they are often used to measure agricultural production performance. These measures normally show higher rates of growth than TFP because growth in land and labor productivity can result not only from increases in TFP but also from a more intensive use of other inputs (such as fertilizer or machinery). Indicators of both TFP and PFP contribute to the understanding of agricultural systems needed for policy and investment decisions by enabling comparisons across time and across countries and regions. These TFP and PFP estimates were generated using the most recent data from Economic Research Service of the United States Department of Agriculture (ERS-USDA), the FAOSTAT database of the Food and Agriculture Organization of the United Nations (FAO), and national statistical sources.

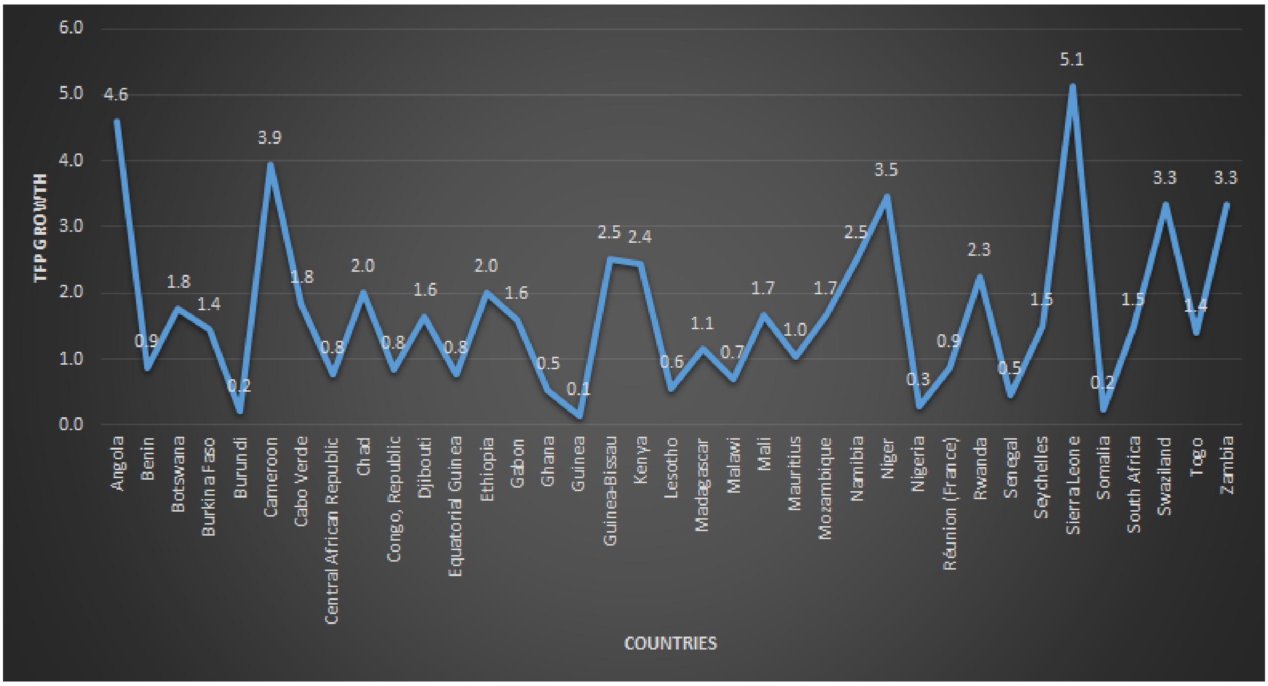

Figure 1. TFP growth for African countries.

This paper is organized as follows. In Section 2, we present a new extended generator of the exponentiated Weibull-H family and its submodels. In Section 3, we demonstrate that the SEW-H density is given by a linear combination of exponentiated-H (exp-H) densities. Three new special models of this family include SEWEx, the SEWR, and SEWBX distributions. They are are introduced in Section 4. Some statistical features of the SEW-H family including the Q_U function, M_Os, IM_Os, and O_S are provided in Section 5. Six numerous methods of estimation, including M_L, L_S, MP_RS_P, WL_S, C_RVM, and, A_D, are produced to estimate the parameters in Section 6. In Section 7, simulation results to assess the performance of the different estimate procedures are discussed. In Section 8, we provide application to five real datasets to illustrate the importance and flexibility of the new family. Finally, some concluding remarks are presented in Section 9.

2. The Sine-Exponentiated Weibull-H Family

Here, in this section, we construct a new flexible family of distributions called the SEW-H family of distributions. By inserting Equation (1) into Equation (3), we obtain the cdf as follows

$$F_{SEW-H}(x;\lambda,\theta,\beta,\delta) = \sin\left\{\frac{\pi}{2}\left[1-e^{-\lambda\left(\frac{H(x;\delta)}{\overline{H}(x;\delta)}\right)^{\theta}}\right]^{\beta}\right\}, x \in R, \tag{6}$$

as well as the associated pdf is provided via

$$f_{SEW-H}(x;\lambda,\theta,\beta,\delta) = \frac{\pi}{2}\lambda\theta\beta h(x;\delta)\frac{H(x;\delta)^{\theta-1}}{\overline{H}(x;\delta)^{\theta+1}}e^{-\lambda\left(\frac{H(x;\delta)}{\overline{H}(x;\delta)}\right)^{\theta}}\left[1-e^{-\lambda\left(\frac{H(x;\delta)}{\overline{H}(x;\delta)}\right)^{\theta}}\right]^{\beta-1} \cos\left\{\frac{\pi}{2}\left[1-e^{-\lambda\left(\frac{H(x;\delta)}{\overline{H}(x;\delta)}\right)^{\theta}}\right]^{\beta}\right\}. \tag{7}$$

For, a random variable (R_V) X that has the pdf given in Equation (7) is indicated with $X \backsim SEW\ (\lambda,\theta,\beta,\delta)$.

The reliability functions, the sf, hrf, and reversed with for the SEW-H family are respectively provided via

$$\overline{F}_{SEW-H}(x;\lambda,\theta,\beta,\delta) = 1 - \sin\left\{ \frac{\pi}{2}\left[1 - e^{-\lambda\left(\frac{H(x;\delta)}{\overline{H}(x;\delta)}\right)^{\theta}}\right]^{\beta}\right\},$$

$$\begin{aligned}\zeta_{SEW-H}(x) =& \frac{\pi}{2}\lambda\theta\beta h(x;\delta)\frac{H(x;\delta)^{\theta-1}}{\overline{H}(x;\delta)^{\theta+1}}e^{-\lambda\left(\frac{H(x;\delta)}{\overline{H}(x;\delta)}\right)^{\theta}}\left[1 - e^{-\lambda\left(\frac{H(x;\delta)}{\overline{H}(x;\delta)}\right)^{\theta}}\right]^{\beta-1} \\ &\times \frac{\cos\left\{ \frac{\pi}{2}\left[1 - e^{-\lambda\left(\frac{H(x;\delta)}{\overline{H}(x;\delta)}\right)^{\theta}}\right]^{\beta}\right\}}{1 - \sin\left\{ \frac{\pi}{2}\left[1 - e^{-\lambda\left(\frac{H(x;\delta)}{\overline{H}(x;\delta)}\right)^{\theta}}\right]^{\beta}\right\}},\end{aligned}$$

and

$$\begin{aligned}\tau_{SEW-H}(x) =& \frac{\pi}{2}\lambda\theta\beta h(x;\delta)\frac{H(x;\delta)^{\theta-1}}{\overline{H}(x;\delta)^{\theta+1}}e^{-\lambda\left(\frac{H(x;\delta)}{\overline{H}(x;\delta)}\right)^{\theta}}\left[1 - e^{-\lambda\left(\frac{H(x;\delta)}{\overline{H}(x;\delta)}\right)^{\theta}}\right]^{\beta-1} \\ &\times \cot\left\{ \frac{\pi}{2}\left[1 - e^{-\lambda\left(\frac{H(x;\delta)}{\overline{H}(x;\delta)}\right)^{\theta}}\right]^{\beta}\right\}.\end{aligned}$$

Several functions could be used in a variety of mathematical techniques within the family. Table 1 lists certain special models of the SEW-H family.

Table 1. Some Special models of the SEW-H.

Model	λ	θ	β
SW-H family	-	-	1
SBX-H family	1	2	-
SEE-H family	-	1	-
SE-H family	-	1	1

3. Linear Representations

The accompanying result demonstrates the growth of the SEW-H family's pdf and cdf via parent function modifications. We presume that integration and differentiation term by term under the infinite sum are technically conceivable. By using the Taylor expansion of the cosine function,

$$\cos\left[\frac{\pi}{2}G(x)\right] = \sum_{i=0}^{\infty}\frac{(-1)^i}{(2i)!}\left(\frac{\pi}{2}G(x)\right)^{2i},$$

we have

$$\cos\left\{ \frac{\pi}{2}\left[1 - e^{-\lambda\left(\frac{H(x;\delta)}{\overline{H}(x;\delta)}\right)^{\theta}}\right]^{\beta}\right\} = \sum_{i=0}^{\infty}\frac{(-1)^i}{(2i)!}\left(\frac{\pi}{2}\right)^{2i}\left[1 - e^{-\lambda\left(\frac{H(x;\delta)}{\overline{H}(x;\delta)}\right)^{\theta}}\right]^{2\beta i}. \quad (8)$$

Inserting Equation (8) in Equation (7), the SEW-H density function reduces to

$$f_{SEW-H}(x;\lambda,\theta,\beta,\delta) = \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i)!}\left(\frac{\pi}{2}\right)^{2i+1} \lambda\theta\beta h(x;\delta) \frac{H(x;\delta)^{\theta-1}}{\overline{H}(x;\delta)^{\theta+1}} e^{-\lambda\left(\frac{H(x;\delta)}{\overline{H}(x;\delta)}\right)^{\theta}} \left[1-e^{-\lambda\left(\frac{H(x;\delta)}{\overline{H}(x;\delta)}\right)^{\theta}}\right]^{\beta(2i+1)-1}. \tag{9}$$

Let $\mid z \mid < 1$ and $a > 0$ is a real non-integer, the generalized binomial series expansion holds

$$(1-z)^{a-1} = \sum_{j=0}^{\infty} (-1)^j \binom{a-1}{j} z^j, \tag{10}$$

and we can write

$$\left[1-e^{-\lambda\left(\frac{H(x;\delta)}{\overline{H}(x;\delta)}\right)^{\theta}}\right]^{\beta(2i+1)-1} = \sum_{j=0}^{\infty} (-1)^j \binom{\beta(2i+1)-1}{j} e^{-\lambda j\left(\frac{H(x;\delta)}{\overline{H}(x;\delta)}\right)^{\theta}}. \tag{11}$$

Inserting the above expression in Equation (11), the SEW-H density reduces to

$$f_{SEW-H}(x;\lambda,\theta,\beta,\delta) = \sum_{i,j=0}^{\infty} \frac{(-1)^{i+j}}{(2i)!} \binom{\beta(2i+1)-1}{j} \left(\frac{\pi}{2}\right)^{2i+1} \lambda\theta\beta h(x;\delta) \frac{H(x;\delta)^{\theta-1}}{\overline{H}(x;\delta)^{\theta+1}} e^{-\lambda(j+1)\left(\frac{H(x;\delta)}{\overline{H}(x;\delta)}\right)^{\theta}}. \tag{12}$$

By expanding $e^{-\lambda(j+1)\left(\frac{H(x;\delta)}{\overline{H}(x;\delta)}\right)^{\theta}}$ in a power series, we get

$$e^{-\lambda(j+1)\left(\frac{H(x;\delta)}{\overline{H}(x;\delta)}\right)^{\theta}} = \sum_{m=0}^{\infty} \frac{(-1)^m [\lambda(j+1)]^m}{m!} \frac{H(x;\delta)^{\theta m}}{\overline{H}(x;\delta)^{\theta m}}. \tag{13}$$

By applying Equation (13) to the last term in Equation (12) gives

$$f_{SEW-H}(x;\lambda,\theta,\beta,\delta) = \sum_{i,j=0}^{\infty} \frac{(-1)^{i+j}}{(2i)!} \binom{\beta(2i+1)-1}{j} \left(\frac{\pi}{2}\right)^{2i+1} \lambda\theta\beta h(x;\delta) \times \sum_{m=0}^{\infty} \frac{(-1)^m [\lambda(j+1)]^m}{m!} \frac{H(x;\delta)^{\theta(m+1)-1}}{\overline{H}(x;\delta)^{\theta(m+1)+1}}. \tag{14}$$

The generalized binomial expansion is used to obtain $(1-H(x;\delta))^{-[\theta(m+1)+1]}$. We are able to write

$$(1-H(x;\delta))^{-[\theta(m+1)+1]} = \sum_{k=0}^{\infty} \frac{\Gamma(\theta(m+1)+k+1)}{k!\Gamma(\theta(m+1)+1)} H(x;\delta)^k. \tag{15}$$

Inserting Equation (15) in Equation (14), the SEW-H pdf may be written as an infinite linear combination of exponentiated-H pdfs

$$f_{SEW-H}(x) = \sum_{m,k=0}^{\infty} \varphi_{m,k}\, \Omega_{\theta(m+1)+k}(x) \tag{16}$$

where

$$\varphi_{m,k} = \sum_{i,j=0}^{\infty} \frac{(-1)^{i+j+m}\lambda\theta\beta[\lambda(j+1)]^m\Gamma(\theta(m+1)+k+1)}{m!k!(2i)!\Gamma(\theta(m+1)+1)[\theta(m+1)+k]} \binom{\beta(2i+1)-1}{j}\left(\frac{\pi}{2}\right)^{2i+1}.$$

$\Omega_{\varkappa}(x) = \rho h(x;\delta)H^{\rho-1}(x;\delta)$ is the exponentiated-G pdf with the power parameter ρ. As a result, the SEW-H pdf may be communicated as a finite combination of exponentiated-H pdfs with parameter $(\theta(m+1)+k)$. Similarly, the cdf of the SEW-H family may also be communicated as a mixture of exponentiated-H cdfs with

$$F_{SEW-H}(x) = \sum_{m,k=0}^{\infty} \varphi_{m,k}\Omega_{\theta(m+1)+k}(x).$$

4. Some Special Models of the SEW-H Family

Naturally, the features of any special distribution of the SEW-H family depend on those of the parent distribution. In this spirit, we focus our attention on the three new SEW-H family special distributions represented by the accompanying pliant mother distributions: the exponential, Rayleigh, and Burr X distributions.

The first special distribution: Sine-exponentiated Weibull exponential (SEWE$_X$) distribution with cdf and pdf as well as

$$F_{SEWE_X}(x;\lambda,\theta,\beta,\rho) = \sin\left\{\frac{\pi}{2}\left[1-e^{-\lambda(e^{\rho x}-1)^{\theta}}\right]^{\beta}\right\},$$

and

$$f_{SEWE_X}(x;\lambda,\theta,\beta,\rho) = \frac{\pi\lambda\theta\beta\rho}{2}\frac{(1-e^{-\rho x})^{\theta-1}}{e^{-\rho\theta x}}e^{-\lambda(e^{\rho x}-1)^{\theta}} \left[1-e^{-\lambda(e^{\rho x}-1)^{\theta}}\right]^{\beta-1}\cos\left\{\frac{\pi}{2}\left[1-e^{-\lambda(e^{\rho x}-1)^{\theta}}\right]^{\beta}\right\}.$$

Different pdf forms of the SEWE$_x$ distribution are shown in Figure 2.

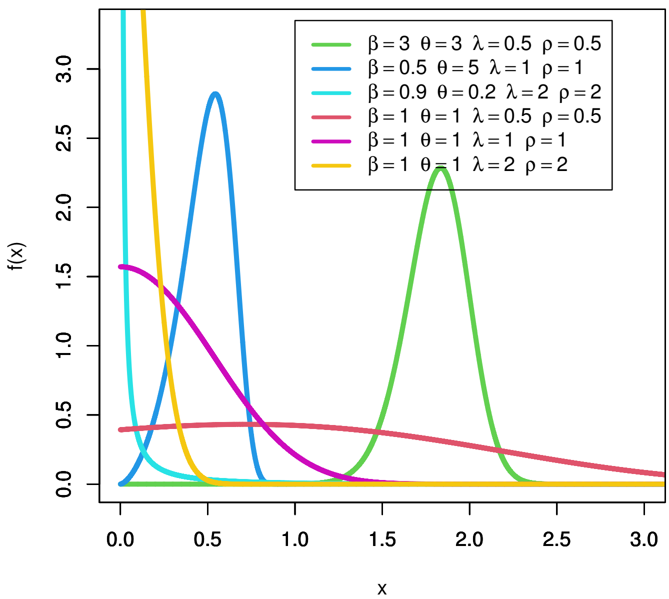

Figure 2. Density for the SEWE$_x$ distribution.

The second special distribution: Sine-exponentiated Weibull Rayleigh (SEWR) distribution with cdf and pdf as follows

$$F_{SEWR}(x;\lambda,\theta,\beta,\rho)=\sin\left\{\frac{\pi}{2}\left[1-e^{-\lambda\left(e^{\frac{\rho}{2}x^2}-1\right)^{\theta}}\right]^{\beta}\right\},$$

and

$$\begin{aligned} f_{SEWR}(x;\lambda,\theta,\beta,\rho) &= \frac{\pi}{2}\lambda\theta\beta\rho x\frac{(1-e^{-\frac{\rho}{2}x^2})^{\theta-1}}{e^{-\theta\frac{\rho}{2}x^2}}e^{-\lambda\left(e^{\frac{\rho}{2}x^2}-1\right)^{\theta}} \\ &\quad \left[1-e^{-\lambda\left(e^{\frac{\rho}{2}x^2}-1\right)^{\theta}}\right]^{\beta-1}\cos\left\{\frac{\pi}{2}\left[1-e^{-\lambda\left(e^{\frac{\rho}{2}x^2}-1\right)^{\theta}}\right]^{\beta}\right\}. \end{aligned}$$

Different pdf forms of the SEW-H Rayleigh distribution are shown in Figure 3.

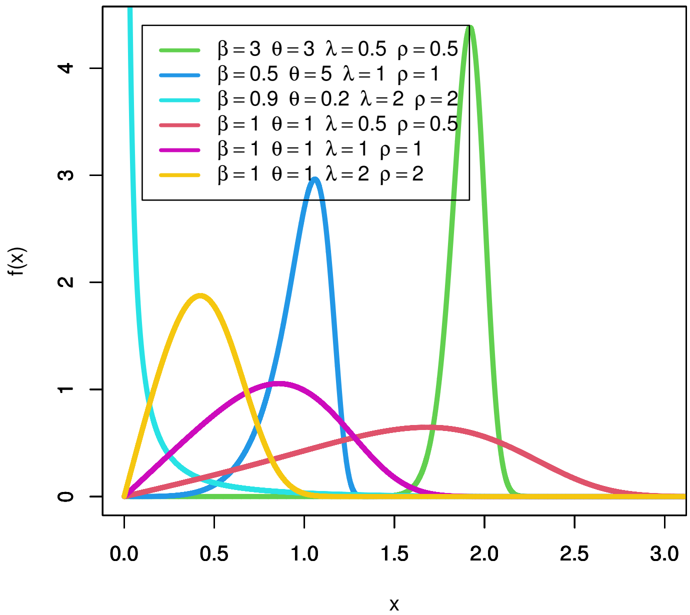

Figure 3. Density function for the SEWR distribution.

The third special distribution: Sine-exponentiated Weibull Burr X (SEWBX) distribution with cdf and pdf provided via

$$F_{SEWBX}(x;\lambda,\theta,\beta,\eta)=\sin\left\{\frac{\pi}{2}\left[1-e^{-\lambda\left((1-e^{-x^2})^{-\eta}-1\right)^{-\theta}}\right]^{\beta}\right\},$$

and

$$\begin{aligned} f_{SEWBX}(x;\lambda,\theta,\beta,\eta) &= \pi\lambda\theta\beta\eta x e^{-x^2}\frac{(1-e^{-x^2})^{\eta\theta-1}}{(1-(1-e^{-x^2})^{\eta})^{\theta+1}}e^{-\lambda\left((1-e^{-x^2})^{-\eta}-1\right)^{-\theta}} \\ &\quad \left[1-e^{-\lambda\left((1-e^{-x^2})^{-\eta}-1\right)^{-\theta}}\right]^{\beta-1}\cos\left\{\frac{\pi}{2}\left[1-e^{-\lambda\left((1-e^{-x^2})^{-\eta}-1\right)^{-\theta}}\right]^{\beta}\right\}. \end{aligned}$$

Different pdf forms of the SEW-H Burr distribution are shown in Figure 4.

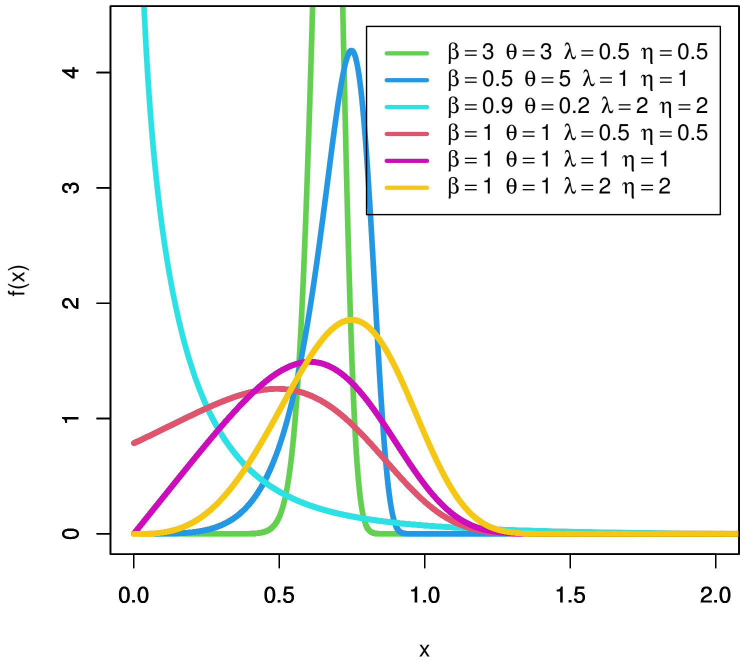

Figure 4. Density function for the SEWBX distribution.

5. Statistical Properties

We looked at the statistical features of the SEW-H family of distributions in this part, specifically the Q_U function, M_Os, IM_Os, and O_S.

5.1. Quantile Function

Theoretical considerations, statistical applications, and Monte Carlo techniques all make use of Q_U functions. Q_U functions are used in Monte Carlo simulations to generate simulated R_Vs for classical and novel continuous distributions. By inverting Equation (6), we may derive the SEW-H Q_U function, $x = Q\,(u)$.

$$F^{-1}(u) = Q_H(u) = H^{-1}\left[\frac{\left\{-\frac{1}{\lambda}\log\left[1-\left(\frac{2}{\pi}\arcsin(u)\right)^{\frac{1}{\beta}}\right]\right\}^{\frac{1}{\theta}}}{1+\left\{-\frac{1}{\lambda}\log\left[1-\left(\frac{2}{\pi}\arcsin(u)\right)^{\frac{1}{\beta}}\right]\right\}^{\frac{1}{\theta}}}\right]. \tag{17}$$

Here, $Q_{H(u)}$ signifies the Q_U function corresponding to the baseline distribution. Let us consider the nonlinear equation $F(Q\,(u)) = Q\,(F(u)) = u$, $u \in (0,1)$ distinguishes $Q\,(u)$. The median is computed by putting $u = 0.5$,

$$Median = H^{-1}\left[\frac{\left\{-\frac{1}{\lambda}\log\left[1-\left(\frac{2}{\pi}\arcsin(0.5)\right)^{\frac{1}{\beta}}\right]\right\}^{\frac{1}{\theta}}}{1+\left\{-\frac{1}{\lambda}\log\left[1-\left(\frac{2}{\pi}\arcsin(0.5)\right)^{\frac{1}{\beta}}\right]\right\}^{\frac{1}{\theta}}}\right].$$

5.2. Various Types of Moments

In this part, we obtain the expressions for the ordinary and moment generating functions of the SEW-H family of distributions. The M_Os of different orders will aid in calculating the predicted lifetime of a device, as well as the dispersion, skewness, and kurtosis in a given collection of observations occurring in dependability applications.

Let $W_{(\theta(m+1)+k)}$ be a R_V having the exponentiated-H pdf $\Omega_{\theta(m+1)+k}$ with power parameter $\theta(m+1)+k$. The r_{th} M_O of the SEW-H family of distributions can be computed from Equation (16)

$$\mu_r^{/} = E(X^r) = \sum_{m,k=0}^{\infty} \varphi_{m,k} E(W^r_{(\theta(m+1)+k)}), \tag{18}$$

where $W_{(\theta(m+1)+k)}$ signifies the exponentiated-H distribution with power parameter $(\theta(m+1)+k)$. Another formula for the r_{th} M_O follows from Equation (16) as

$$\mu_r^{/} = E(X^r) = \sum_{m,k=0}^{\infty} \varphi_{m,k} E(W^r_{(\theta(m+1)+k)}),$$

where

$$E(W^r_\kappa) = \kappa \int_{-\infty}^{\infty} x^r h(x) H(x)^{\kappa-1}, \nu > 0$$

could be computed mathematically in relation to the baseline Q_U function, i.e., $Q_H(u) = H^{-1}(u)$ as

$$E(W^r_\kappa) = \kappa \int_0^1 u^{\kappa-1} Q_H(u)^r du.$$

Some numerical values of the first four moments $\mu_1^{'}$, $\mu_2^{'}$, $\mu_3^{'}$, $\mu_4^{'}$, variance (V), skewness (CS), kurtosis (CK), and coefficient of variation (CV) for the SEWE$_X$ and SEWR models are mentioned in Tables 2 and 3.

In this step, we present two formulas for the M_O generating function ($M_O G_F$). The first formula may be determined using Equation (16), as shown below

$$M_X(t) = E(e^{tX}) = \sum_{m,k=0}^{\infty} \varphi_{m,k} M_{k+1}(t), \tag{19}$$

where $M_{(\theta(m+1)+k)}(t)$ is the $M_O G_F$ of $W_{(m(m+1)+k)}$. Consequently, we can easily compute $M_X(t)$ from the exp-G generating function. The second formula for the $M_X(t)$ follows from Equation (16) as

$$M_X(t) = E(e^{tX}) = \sum_{m,k=0}^{\infty} \varphi_{m,k}\, M_{(\theta(m+1)+k)}(t),$$

where $M_v(t)$ is the $M_O G_F$ of R_V W_v provided via

$$\begin{aligned} M_v(t) &= \int_{-\infty}^{\infty} e^{tX} h(x) h(x)^{v-1}, \varkappa > 0 \\ &= v \int_0^1 u^{v-1} e^{tQ_H(u)} du, \end{aligned} \tag{20}$$

which could be investigated numerically from the baseline R_V function, i.e., $Q_H(u) = G^{-1}(u)$.

The s_{th} IM_O of X defined by $\eta_s(t)$ for any real $s > 0$ can be computed from Equation (16) as

$$\eta_s(t) = \int_{-\infty}^{t} x^s f(x) dx = \sum_{m,k=0}^{\infty} \varphi_{m,k} \int_{-\infty}^{t} x^s \eta_{s,(\theta(m+1)+k)}(t) dx, \tag{21}$$

where

$$\eta_{s,\omega}(t) = \int_0^{H(t)} u^{\omega-1} Q_H(u)^s du,$$

and $\eta_{s,\omega}(t)$ can be investigated numerically.

Table 2. Results of $\mu_1^{'}$, $\mu_2^{'}$, $\mu_3^{'}$, $\mu_4^{'}$, V, CS, CK, and CV for the SEWE$_X$ model at λ = 1.5 and ρ = 0.5.

$\beta\downarrow$	$\theta\downarrow$	$\mu_1^{'}\downarrow$	$\mu_2^{'}\downarrow$	$\mu_3^{'}\downarrow$	$\mu_4^{'}\downarrow$	$V\downarrow$	$CS\uparrow$	$CK\uparrow$	$CV\uparrow$
0.7	1.2	0.287	0.234	0.273	0.402	0.151	2.02	8.056	1.352
	1.5	0.333	0.299	0.370	0.567	0.189	1.773	6.64	1.305
	1.8	0.368	0.360	0.468	0.742	0.225	1.607	5.75	1.289
	2	0.386	0.397	0.533	0.862	0.248	1.527	5.327	1.289
	2.3	0.409	0.448	0.627	1.044	0.281	1.437	4.853	1.297
	2.6	0.426	0.495	0.718	1.227	0.313	1.375	4.513	1.312
	3	0.444	0.550	0.834	1.469	0.353	1.322	4.194	1.339
0.9	1.2	0.285	0.223	0.233	0.296	0.142	1.656	5.9	1.322
	1.5	0.315	0.273	0.305	0.404	0.174	1.496	5.051	1.326
	1.8	0.335	0.316	0.372	0.513	0.204	1.405	4.541	1.348
	2	0.345	0.34	0.415	0.585	0.223	1.368	4.311	1.369
	2.3	0.355	0.375	0.475	0.691	0.249	1.336	4.072	1.403
	2.6	0.362	0.404	0.532	0.795	0.272	1.324	3.918	1.44
	3	0.368	0.436	0.600	0.928	0.301	1.326	3.798	1.49
1.2	1.2	0.255	0.200	0.196	0.222	0.135	1.512	4.795	1.443
	1.5	0.266	0.232	0.242	0.288	0.161	1.467	4.414	1.505
	1.8	0.271	0.256	0.283	0.351	0.182	1.468	4.245	1.573
	2	0.273	0.27	0.307	0.391	0.195	1.482	4.2	1.619
	2.3	0.273	0.286	0.340	0.447	0.211	1.514	4.194	1.685
	2.6	0.271	0.299	0.369	0.500	0.225	1.554	4.237	1.749
	3	0.268	0.312	0.403	0.564	0.240	1.612	4.34	1.828
1.5	1.2	0.211	0.170	0.163	0.176	0.126	1.654	4.929	1.677
	1.5	0.211	0.188	0.192	0.218	0.143	1.701	4.901	1.792
	1.8	0.208	0.200	0.216	0.257	0.157	1.77	5.017	1.902
	2	0.205	0.21	0.230	0.280	0.164	1.821	5.136	1.971
	2.3	0.201	0.213	0.248	0.311	0.172	1.899	5.352	2.069
	2.6	0.196	0.217	0.263	0.340	0.179	1.976	5.592	2.159
	3	0.19	0.222	0.279	0.373	0.186	2.074	5.93	2.268
1.9	1.2	0.153	0.128	0.122	0.128	0.105	2.09	6.415	2.117
	1.5	0.145	0.134	0.137	0.152	0.113	2.239	6.919	2.308
	1.8	0.138	0.137	0.148	0.171	0.118	2.386	7.504	2.481
	2	0.134	0.14	0.154	0.182	0.120	2.479	7.91	2.586
	2.3	0.128	0.139	0.161	0.197	0.122	2.611	8.524	2.731
	2.6	0.123	0.139	0.166	0.209	0.124	2.734	9.13	2.862
	3	0.117	0.138	0.172	0.223	0.124	2.882	9.916	3.018
2.1	1.2	0.127	0.108	0.104	0.109	0.092	2.384	7.729	2.39
	1.5	0.119	0.111	0.114	0.126	0.097	2.583	8.588	2.626
	1.8	0.111	0.112	0.121	0.139	0.099	2.769	9.498	2.836
	2	0.107	0.11	0.125	0.147	0.100	2.885	10.107	2.962
	2.3	0.101	0.111	0.129	0.157	0.101	3.046	11.006	3.135
	2.6	0.096	0.110	0.132	0.165	0.100	3.193	11.878	3.291
	3	0.091	0.108	0.134	0.174	0.100	3.37	12.991	3.476

Table 3. Results of μ_1', μ_2', μ_3', μ_4', V, CS, CK, and CV for the SEWR model at λ = 2.2 and ρ = 0.1.

$\beta \downarrow$	$\theta \downarrow$	$\mu_1' \downarrow$	$\mu_2' \downarrow$	$\mu_3' \downarrow$	$\mu_4' \downarrow$	$V \downarrow$	$CS \uparrow$	$CK \uparrow$	$CV \uparrow$
0.4	1.2	1.015	1.965	5.278	17.352	0.935	1.532	5.589	0.953
	1.5	1.25	2.648	7.461	25.237	1.085	1.273	4.616	0.833
	1.8	1.462	3.331	9.782	33.946	1.195	1.088	4.036	0.748
	2	1.591	3.782	11.382	40.119	1.251	0.99	3.77	0.703
	2.3	1.769	4.446	13.833	49.817	1.315	0.87	3.48	0.648
	2.6	1.932	5.092	16.321	59.937	1.361	0.772	3.277	0.604
	3	2.127	5.923	19.659	73.922	1.4	0.667	3.091	0.556
0.6	1.2	1.456	3.008	7.622	22.215	0.887	0.788	3.293	0.647
	1.5	1.703	3.827	10.225	30.886	0.926	0.621	3.016	0.565
	1.8	1.911	4.592	12.819	39.928	0.938	0.502	2.873	0.507
	2	2.033	5.072	14.525	46.072	0.937	0.44	2.816	0.476
	2.3	2.196	5.751	17.033	55.365	0.928	0.365	2.764	0.439
	2.6	2.339	6.384	19.469	64.674	0.913	0.306	2.737	0.408
	3	2.505	7.163	22.598	77.008	0.888	0.245	2.723	0.376
0.8	1.2	1.795	4.008	10.28	29.141	0.785	0.384	2.694	0.493
	1.5	2.034	4.898	13.235	38.962	0.762	0.259	2.648	0.429
	1.8	2.227	5.691	16.036	48.721	0.73	0.174	2.65	0.384
	2	2.338	6.172	17.815	55.126	0.708	0.131	2.663	0.36
	2.3	2.482	6.833	20.351	64.526	0.674	0.082	2.688	0.331
	2.6	2.606	7.432	22.739	73.647	0.643	0.044	2.715	0.308
	3	2.747	8.15	25.711	85.347	0.605	0.008	2.75	0.283
1	1.2	2.056	4.901	12.914	36.675	0.675	0.119	2.565	0.4
	1.5	2.279	5.817	16.103	47.486	0.622	0.023	2.624	0.346
	1.8	2.456	6.603	19.013	57.817	0.573	−0.039	2.687	0.308
	2	2.554	7.069	20.812	64.414	0.544	−0.069	2.725	0.289
	2.3	2.682	7.698	23.321	73.871	0.506	−0.101	2.776	0.265
	2.6	2.79	8.256	25.629	82.823	0.472	−0.123	2.818	0.246
	3	2.912	8.912	28.437	94.03	0.434	−0.142	2.864	0.226
1.1	1.2	2.163	5.303	14.177	40.471	0.623	0.017	2.574	0.365
	1.5	2.379	6.221	17.445	51.683	0.563	−0.068	2.662	0.315
	1.8	2.547	6.997	20.378	62.221	0.511	−0.12	2.741	0.281
	2	2.64	7.453	22.172	68.871	0.481	−0.144	2.785	0.263
	2.3	2.761	8.063	24.652	78.313	0.443	−0.169	2.839	0.241
	2.6	2.862	8.6	26.911	87.16	0.41	−0.185	2.882	0.224
	3	2.975	9.227	29.636	98.125	0.374	−0.197	2.926	0.206
1.4	1.2	2.421	6.35	17.672	51.513	0.49	−0.214	2.722	0.289
	1.5	2.612	7.245	21.057	63.569	0.422	−0.271	2.853	0.249
	1.8	2.758	7.978	23.981	74.443	0.37	−0.301	2.945	0.221
	2	2.838	8.399	25.723	81.114	0.343	−0.311	2.99	0.206
	2.3	2.94	8.952	28.08	90.364	0.309	−0.318	3.038	0.189
	2.6	3.025	9.431	30.182	98.823	0.281	−0.32	3.072	0.175
	3	3.119	9.981	32.664	109.062	0.252	−0.316	3.1	0.161

5.3. Order Statistics

Order statistics (O_S) is a very important statistical dimension that deals with the order in data. It is defined as follows. If X_1, X_2,..., X_n are the independent R_Vs following a SEW-H family of distributions of size n and if we arrange these variables in ascending order as $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$, then the variables $X_{(1)} \leq X_{(2)} \leq \ldots, \leq X_{(n)}$ are O_Ss of R_Vs. O_S have many applications in survival, reliability, failure analysis, and it is a natural way to perform a reliability analysis of a system. The cdf of ith O_S can provided via

$$\begin{aligned} F_{i;n}(x) =& \frac{1}{B(i,n-i+1)}\sum_{j=0}^{n-i}\frac{(-1)^j}{i+j}\binom{n-i}{j}F^{i+j}(x) \\ =& \frac{1}{B(i,n-i+1)(i+j)}\sum_{j=0}^{n-i}(-1)^j\binom{n-i}{j}\left[\sin\left\{\frac{\pi}{2}\left[1-e^{-\lambda\left(\frac{H(x;\delta)}{\overline{H}(x;\delta)}\right)^{\theta}}\right]^{\beta}\right\}\right]^{i+j}. \end{aligned} \tag{22}$$

The corresponding pdf is provided via

$$\begin{aligned} f_{i;n}(x) =& \frac{f\,(x)}{B(i,n-i+1)}\sum_{j=0}^{n-i}(-1)^j\binom{n-i}{j}F^{i+j-1}(x) \\ =& \sum_{j=0}^{n-i}\frac{\frac{\pi\lambda\theta}{2}(-1)^j\binom{n-i}{j}}{B(i,n-i+1)}\frac{\pi}{2}\lambda\theta\beta h(x;\delta)\frac{H(x;\delta)^{\theta-1}}{\overline{H}(x;\delta)^{\theta+1}}e^{-\lambda\left(\frac{H(x;\delta)}{\overline{H}(x;\delta)}\right)^{\theta}}\left[1-e^{-\lambda\left(\frac{H(x;\delta)}{\overline{H}(x;\delta)}\right)^{\theta}}\right]^{\beta-1} \\ & \cos\left\{\frac{\pi}{2}\left[1-e^{-\lambda\left(\frac{H(x;\delta)}{\overline{H}(x;\delta)}\right)^{\theta}}\right]^{\beta}\right\}\left[\sin\left\{\frac{\pi}{2}\left[1-e^{-\lambda\left(\frac{H(x;\delta)}{\overline{H}(x;\delta)}\right)^{\theta}}\right]^{\beta}\right\}\right]^{i+j}. \end{aligned} \tag{23}$$

The r_{th} M_O of of the ith v is provided via

$$\begin{aligned} \mu_r =& E(X_{i:r}^r) = \int_{-\infty}^{\infty}x^r f_{i;n}(x)dx = \\ &= \frac{1}{B(i,n-i+1)}\sum_{j=0}^{n-i}(-1)^j\binom{n-i}{j}\int_{-\infty}^{\infty}x^r f\,(x)F^{i+j-1}(x)dx \\ &= \frac{1}{B(i,n-i+1)}\sum_{j=0}^{n-i}(-1)^j\binom{n-i}{j}\mu'_{r,i+j-1} \end{aligned}$$

where the integral can be investigated numerically.

6. Estimation Methods

This section employs six estimation techniques to assess the estimation problem of the SEW-H family parameters: M_L, L_S, MP_RS_P, WL_S, C_RVM, and A_D. For more details, see [40–43].

6.1. Maximum Likelihood Estimation

In this section, we examine the estimation of the SEW-H family's λ, θ, β, and δ parameters using the M_L method while ensuring the M_L estimates (M_LEs) have nice convergence features. M_LEs have useful qualities, and they can be applied to test statistics as well as the construction of confidence intervals and regions. The following is a presentation of the method's key components as they relate to the SEW-H family: assume $x_1,\ldots,x_n$ be a random sample of size n from the SEW-H family given by (7). Then, the total log-likelihood function for the vector $\Omega = (\lambda,\theta,\beta,\delta)$ is provided via

$$\begin{aligned} L_n(\Omega) =& n\log(\frac{\pi}{2}) + n\log(\lambda) + n\log(\theta) + n\log(\beta) + \sum_{i=1}^{n}\log h(x_i;\delta) + (\theta-1)\sum_{i=1}^{n}\log H(x_i;\delta) \\ & -(\theta+1)\sum_{i=1}^{n}\log(\overline{H}(x_i;\delta)) - \lambda\sum_{i=1}^{n}d_i^{\theta} + (\beta-1)\sum_{i=1}^{n}\log\left[1-e^{-\lambda d_i^{\theta}}\right] \\ & + \sum_{i=1}^{n}\log\left\{\cos\left\{\frac{\pi}{2}\left[1-e^{-\lambda d_i^{\theta}}\right]^{\beta}\right\}\right\}, \end{aligned} \tag{24}$$

where $d_i = \frac{H(x;\delta)}{\overline{H}(x;\delta)}$. The corresponding score vector components, say $U_n(\Omega) = (\frac{\partial L_n(\Omega)}{\partial\lambda}, \frac{\partial L_n(\Omega)}{\partial\theta}, \frac{\partial L_n(\Omega)}{\partial\beta}, \frac{\partial L_n(\Omega)}{\partial\delta})^T$ are given by

$$\begin{aligned}\frac{\partial L_n(\Omega)}{\partial\lambda} =& \frac{n}{\lambda} + \sum_{i=1}^{n} d_i^{\theta} + (\beta-1)\sum_{i=1}^{n} \frac{d_i^{\theta} e^{-\lambda d_i^{\theta}}}{1-e^{-\lambda d_i^{\theta}}} \\ & - \frac{\pi}{2}\sum_{i=1}^{n} \frac{\beta d_i^{\theta} e^{-\lambda d_i^{\theta}} \sin\left\{\frac{\pi}{2}\left[1-e^{-\lambda d_i^{\theta}}\right]^{\beta}\right\}\left[1-e^{-\lambda d_i^{\theta}}\right]^{\beta-1}}{\cos\left\{\frac{\pi}{2}\left[1-e^{-\lambda d_i^{\theta}}\right]^{\beta}\right\}},\end{aligned} \tag{25}$$

$$\begin{aligned}\frac{\partial L_n(\Omega)}{\partial\theta} =& \frac{n}{\theta} + \sum_{i=1}^{n} \log H(x_i;\delta) - \sum_{i=1}^{n} \log(\overline{H}(x_i;\delta)) \\ & - \lambda\sum_{i=1}^{n} d_i^{\theta}\log d_i + (\beta-1)\sum_{i=1}^{n} \frac{d_i^{\theta} e^{-\lambda d_i^{\theta}}\log d_i}{1-e^{-\lambda d_i^{\theta}}} \\ & - \frac{\pi}{2}\sum_{i=1}^{n} \frac{\beta d_i^{\theta} e^{-\lambda d_i^{\theta}} \sin\left\{\frac{\pi}{2}\left[1-e^{-\lambda d_i^{\theta}}\right]^{\beta}\right\}\left[1-e^{-\lambda d_i^{\theta}}\right]^{\beta-1}\log d_i}{\cos\left\{\frac{\pi}{2}\left[1-e^{-\lambda d_i^{\theta}}\right]^{\beta}\right\}},\end{aligned} \tag{26}$$

$$\begin{aligned}\frac{\partial L_n(\Omega)}{\partial\beta} =& \frac{n}{\beta} + \sum_{i=1}^{n} \log\left[1-e^{-\lambda d_i^{\theta}}\right] \\ & - \frac{\pi}{2}\sum_{i=1}^{n} \frac{\left[1-e^{-\lambda d_i^{\theta}}\right]^{\beta} \sin\left\{\frac{\pi}{2}\left[1-e^{-\lambda d_i^{\theta}}\right]^{\beta}\right\}\log\left[1-e^{-\lambda d_i^{\theta}}\right]}{\cos\left\{\frac{\pi}{2}\left[1-e^{-\lambda d_i^{\theta}}\right]^{\beta}\right\}},\end{aligned} \tag{27}$$

and

$$\begin{aligned}\frac{\partial L_n(\Omega)}{\partial\delta_k} =& \sum_{i=1}^{n} \frac{h'(x_i;\delta)}{h(x_i;\delta)} + (\theta-1)\sum_{i=1}^{n} \frac{H'(x_i;\delta)}{H(x_i;\delta)} - (\theta+1)\sum_{i=1}^{n} \frac{\overline{H}'(x_i;\delta)}{\overline{H}(x_i;\delta)} \\ & - \lambda\theta\sum_{i=1}^{n} d_i^{\theta-1}\left(\frac{\partial d_i}{\partial\delta_k}\right) + (\beta-1)\sum_{i=1}^{n} \frac{\lambda d_i^{\theta-1} e^{-\lambda d_i^{\theta}}\left(\frac{\partial d_i}{\partial\delta_k}\right)}{1-e^{-\lambda d_i^{\theta}}} \\ & - \frac{\pi}{2}\sum_{i=1}^{n} \frac{\beta\lambda d_i^{\theta-1} e^{-\lambda d_i^{\theta}} \sin\left\{\frac{\pi}{2}\left[1-e^{-\lambda d_i^{\theta}}\right]^{\beta}\right\}\left[1-e^{-\lambda d_i^{\theta}}\right]^{\beta-1}\left(\frac{\partial d_i}{\partial\delta_k}\right)}{\cos\left\{\frac{\pi}{2}\left[1-e^{-\lambda d_i^{\theta}}\right]^{\beta}\right\}},\end{aligned} \tag{28}$$

where $h'(x_i;\delta) = \frac{\partial h(x_i;\delta)}{\partial\delta_k}, H'(x_i;\delta) = \frac{\partial H(x_i;\delta)}{\partial\delta_k}, \overline{H}'(x_i;\delta) = \frac{\partial\overline{H}(x_i;\delta)}{\partial\delta_k}$. Setting the nonlinear system of equations $\frac{\partial L_n(\Omega)}{\partial\lambda} = \frac{\partial L_n(\Omega)}{\partial\theta} = \frac{\partial L_n(\Omega)}{\partial\beta} = \frac{\partial L_n(\Omega)}{\partial\delta_k} = 0$ and solving these equations simultaneously, we can obtain the $M_LE(\widehat{\Omega})$. These equations can be numerically solved using iterative techniques using statistical software since analytical solutions are not possible.

6.2. Weighted and Ordinary Least Square

To determine the parameters of different distributions by L_S, the WL_S, and ordinary L_S (OL_S) approaches are utilized. If $\Theta = (\lambda, \beta, \theta)^T$ parameters from the SEW-H family class have parameters, then let $x_{i:n}, \ldots, x_{n:n}$ be a random sample. By minimizing the following estimators of OL_S (OL_SE) and WL_S (WL_SE) of the $\Omega = (\Theta, \delta)^T$, distribution parameters of SEW-H family could be derived.

$$V(\Omega)=\sum_{i=1}^{n}H_i\left[\sin\left\{\frac{\pi}{2}\left[1-e^{-\lambda\left(\frac{H(x;\delta)}{\overline{H}(x;\delta)}\right)^{\theta}}\right]^{\beta}\right\}-\frac{j}{n+1}\right]^2, \tag{29}$$

H_i is equal to one for OL_SE and H_i is $\frac{(n+1)^2(n+2)}{[i(n-I+1)]}$ with respect to Ω for WL_SE. Furthermore, the OL_SE and WL_SE with regard to Ω are obtained by solving the nonlinear equations.

6.3. Product Spacing's Method

In the case of a random sample of size n, $x_{1:n}<\cdots<x_{n:n}$, the uniform spacing of the SEW-H family can be described as:

$$PS_i(\Omega)=F(x_{i:n},\Omega)-F(x_{i-1:n},\Omega);\ i=1,...,n+1. \tag{30}$$

In this case, $PS_i(\Omega)$ stands for the uniform spacings, $F(x_{0:n},\Omega)=0$, $F(x_{n+1:n},\Omega)=1$, and $\sum_{i=1}^{n+1}PS_i(\Omega)=1$. By MP_RS_P of the SEW-H family parameters,

$$G(\Omega)=\frac{1}{n+1}\sum_{i=1}^{n+1}\ln\left\{\sin\left\{\frac{\pi}{2}\left[1-e^{-\lambda\left(\frac{H(x_{i:n};\delta)}{\overline{H}(x_{i:n};\delta)}\right)^{\theta}}\right]^{\beta}\right\}-\sin\left\{\frac{\pi}{2}\left[1-e^{-\lambda\left(\frac{H(x_{(i-1)};\delta)}{\overline{H}(x_{(i-1)};\delta)}\right)^{\theta}}\right]^{\beta}\right\}\right\}, \tag{31}$$

with regard to Ω. Further, the MP_RS_P estimates (MP_RS_PE) of the SEW-H family can also be computed by solving the nonlinear Equation (31) of derivatives of $G(\Omega)$ with respect to Ω.

6.4. Cramér–von-Mises

By minimizing the following function with respect to Ω, the C_RVM estimators (C_RVME) of the SEW-H family with vector parameters Ω are derived.

$$C(\Omega)=\frac{1}{12}+\sum_{i=1}^{n}\left(\sin\left\{\frac{\pi}{2}\left[1-e^{-\lambda\left(\frac{H(x_{i:n};\delta)}{\overline{H}(x_{i:n};\delta)}\right)^{\theta}}\right]^{\beta}\right\}-\frac{2i-1}{2n}\right)^2. \tag{32}$$

In addition, one can solve the nonlinear equations of derivatives of $C(\Omega)$ with respect to Ω.

6.5. Anderson–Darling Method

Different kinds of minimal distance estimators in A_D are the A_D estimators (A_DE). By minimizing, the A_DE of the SEW-H family's parameters is obtained.

$$AD(\Omega)=-n-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\left(\ln\left(\sin\left\{\frac{\pi}{2}\left[1-e^{-\lambda\left(\frac{H(x_{i:n};\delta)}{\overline{H}(x_{i:n};\delta)}\right)^{\theta}}\right]^{\beta}\right\}\right)-\ln\left(1-\sin\left\{\frac{\pi}{2}\left[1-e^{-\lambda\left(\frac{H(x_{(n+1-i)};\delta)}{\overline{H}(x_{(n+1-i)};\delta)}\right)^{\theta}}\right]^{\beta}\right\}\right)\right)^2, \tag{33}$$

The nonlinear equations of the derivatives of $A_D(\Omega)$ with respect to each parameter of the vector Ω may also be solved to yield the A_DE.

7. Simulation

To assess the consistency and accuracy of the six estimating techniques used by the new class, Monte Carlo simulations are performed in this section. For illustration purposes, the simulations are carried out using estimators for the parameters of the SEW-H Exp distribution. Using the inverse transformation, samples of sizes $n=40$, $n=80$, and $n=160$ are produced for the simulated replication with 1000 iterations.

$$x_i=\frac{1}{\lambda}\left[-\log\left(1-\frac{1}{1+[-\frac{1}{\beta}\log(\frac{U}{2-U})]^{\frac{1}{\theta}}}\right)\right]^{\frac{1}{\rho}}, i=1,2,\ldots,n, \tag{34}$$

where a uniform distribution on $(0, 1)$ is represented by U. The mean square error values (MSEV) and estimated relative bias values (RBV) are used to analyze the numerical results. The estimated RBV and the MSEV for the parameter estimators are shown in Tables A1–A3. We establish four arbitrary true values for $(\beta, \theta, \lambda, \rho)$, such as:

In Table A1, set I: $(3, 0.75, 0.75, 0.5)$ and set II: $(3, 0.75, 0.75, 3)$;
In Table A2, set III: $(3, 0.75, 3, 0.5)$, and set IV: $(3, 0.75, 3, 3)$;
In Table A3, set III: $(3, 3, 3, 0.5)$, and set IV: $(3, 3, 3, 3)$.

Numerous calculations were made using the R statistical programming language, with the 'stats' package, which used the Conjugate-gradient maximization algorithm being the most helpful statistical package. We can draw the following conclusions from Tables A1–A3. The results showed that, as the sample size increases, RBV and MSEV decrease, which is consistent with expectations. The proposed estimates of β, θ, λ and δ perform better in terms of their RBV and MSEV as n increases. These results unequivocally show the reliability and consistency of the estimating techniques. In order to estimate the parameters of the SEW-H Exp distribution, the six estimation approaches perform effectively.

8. Applications

8.1. Food Chain Data

The first dataset represents the food chain in the United Kingdom from 2000 to 2019, see https://www.gov.uk/government/statistics/food-chain-productivity, accessed on 30 June 2022. The data are as follows: 100, 99.9, 98.5, 100.1, 101.9, 101.4, 103.1, 103.2, 104.2, 102.9, 104.1, 104.8, 104.7, 105.8, 103.4, 104.1, 105.5, 107.2, 108.6, 109. For the first dataset, the numerical values of $\hat{\beta}$, $\hat{\theta}$, $\hat{\lambda}$, $\hat{\mu}$, and $\hat{\rho}$ are provided in Table 4. From the numerical comparison of the competing distributions in Table 4, we observe that the proposed SEWE$_x$ model is the best choice to implement for fitting of the food chain data. For the SEWE$_x$ distribution, the values of the analytical measures are AIC = 105.5160, BIC = 109.4989, CVMV = 0.0316, ADV = 0.2317, and KSD = 0.0973, with PVKS = 0.9915.

To support the best fitting power of the SEWE$_x$ model, a visual illustration is provided in Figure 5. From the visual illustration in Figure 5, we can see that the SEWE$_x$ distribution follows the fitted pdf, cdf, PP and QQ plot very closely. To support the results of Table 4, a visual illustration is provided in Figures 5, 6 and 7.

Table 4. M_LE with SEs and different measures for food chain data.

	β	θ	λ	ρ	μ	AIC	BIC	CVMV	ADV	KSD	PVKS
SEWE$_x$	25.4578	5.8544	0.0969	0.0100		105.5160	109.4989	0.0316	0.2317	0.0973	0.9915
	2.5656	0.6520	0.0416	0.0002							
EGWGP	12.9987	0.0028	0.2820	0.1226	0.9072	119.7390	124.7177	0.0325	0.2321	0.1969	0.4202
	6.9675	0.0003	1.1229	0.0319	0.1267						
KEBXII	200.4707	104.7107	1151.0364	44.4507	0.0386	137.3374	142.3161	0.0329	0.2400	0.3543	0.0132
	91.5733	47.5651	823.2539	7.5603	0.0005						
WL	39.6383	94.6265	0.2092	4.3605		108.0184	112.0013	0.0679	0.4812	0.1416	0.8177
	500.3255	5.4938	0.1859	1.3285							
MOAPW	8.6847	13.4815	14.5558	94.1638		108.9627	112.9457	0.0486	0.3695	0.1314	0.8800
	12.1874	3.3482	14.2031	3.8432							
KW	1.0714	0.0250	9.9170	0.5157		255.0566	259.0395	0.1066	0.6499	0.5813	0.0000
	0.0069	0.0056	0.0022	0.0014							
ESW	19.3720	1.2484	0.0054			174.1453	177.1325	0.0331	0.2310	0.4250	0.0015
	7.4081	0.0472	0.0011								

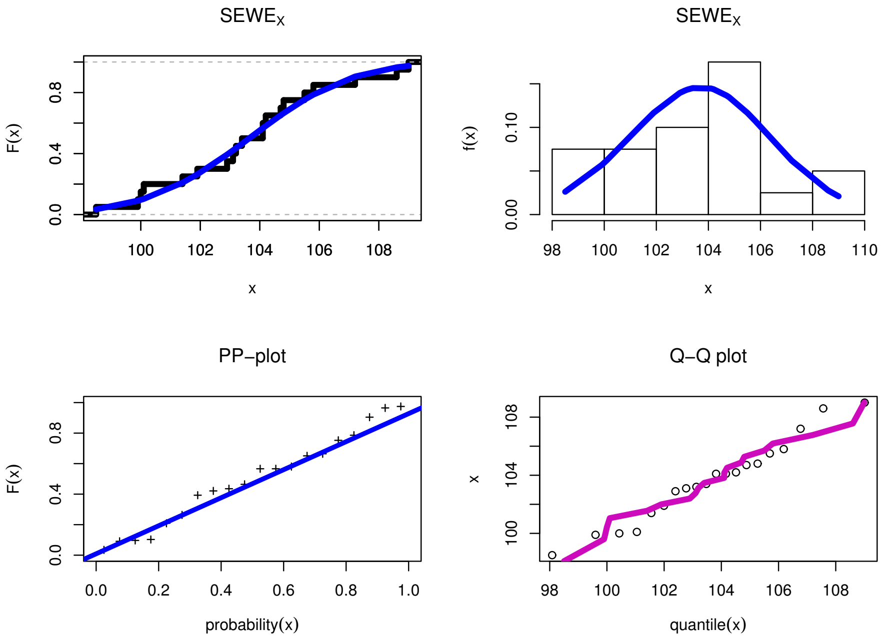

Figure 5. Plots of empirical cdf with fitted cdf, histogram with fitted pdf, PP and Q-Q plot of the SEWE$_X$ model for the food chain data.

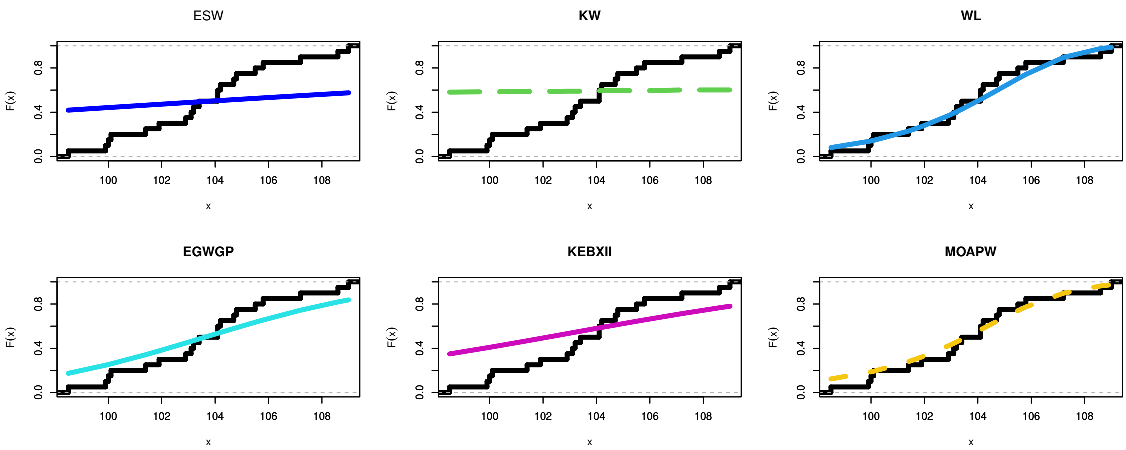

Figure 6. Plots of empirical cdf with fitted cdf of the models for the food chain data.

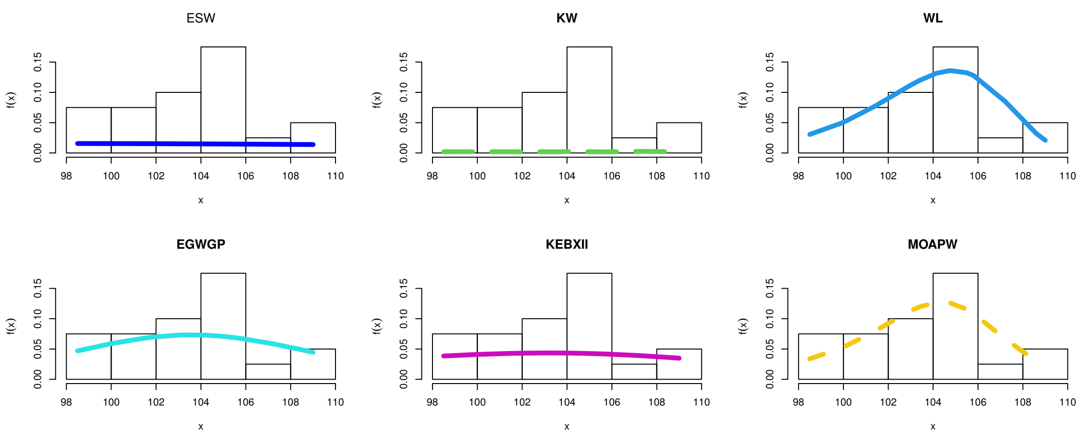

Figure 7. Plots of histogram with fitted pdf of the models for the food chain data.

8.2. Wholesale Data

The second dataset represents the food and drink wholesaling in the United Kingdom from 2000 to 2019 as one factor of FTP, see https://www.gov.uk/government/statistics/food-chain-productivity, accessed on 30 June 2022. The data are as follows: 100,101.7,99.6, 101, 102.7, 101.1, 104.2, 104.6, 106.3, 104.8, 105.6, 107.1, 107.5, 108.6, 107.5, 106.6, 109.1, 112, 114.4, 112.5.

For second dataset, the numerical values of $\hat{\beta}$, $\hat{\theta}$, $\hat{\lambda}$, $\hat{\mu}$, and $\hat{\rho}$ are provided in Table 5. From the numerical comparison of the competing distributions in Table 5, we observe that the proposed SEWE$_x$ model is the best choice to implement for fitting the Wholesale data. For the SEWE$_x$ distribution, the values of the analytical measures are AIC = 121.2337, BIC = 125.2169, CVMV = 0.0292, ADV = 0.2512, and KSD = 0.0937, with PVKS = 0.9947.

To support the best fitting power of the SEWE$_x$ model, a visual illustration is provided in Figure 8. From the visual illustration in Figure 8, we can see that the SEWE$_x$ distribution follows the fitted pdf, cdf, PP and QQ plot very closely. To support the results of Table 4, a visual illustration is provided in Figures 9 and 10.

Table 5. M_LE with SEs and different measures for Wholesale data.

	β	θ	λ	ρ	μ	AIC	BIC	CVMV	ADV	KSD	PVKS
SEWE$_x$	27.5666	2.6193	0.0172	0.0201		121.2337	125.2167	0.0292	0.2512	0.0937	0.9947
	2.5646	1.5169	0.0073	0.0166							
EGWGP	39.1871	0.0038	8.5649	0.6128	0.9686	162.5654	167.5440	0.1286	0.7957	0.4030	0.0030
	16.1161	0.0004	0.0057	0.0018	0.0012						
KEBXII	537.5850	649.1720	375.1731	4.2158	0.5565	135.8559	140.8345	0.0318	0.2694	0.2525	0.1561
	5.6995	21.6522	194.6549	0.3778	0.0499						
WL	0.0025	45.0467	0.3496	13.7505		124.2758	128.2587	0.0720	0.5231	0.1491	0.7653
	0.0007	3.7874	0.1985	1.3843							
MOAPW	378.1695	5.1839	449.6787	71.0195		123.1665	127.1494	0.0369	0.3183	0.1064	0.9774
	1288.6846	1.5231	948.4074	11.2737							
KW	1.0993	0.0279	10.0478	0.5114		256.5495	260.5324	0.0632	0.4927	0.6143	0.0000
	0.0032	0.0062	0.0034	0.0013							
ESW	20.8703	1.3592	0.0032			171.1671	174.1543	0.0268	0.2254	0.4003	0.0033
	8.1773	0.0295	0.0003								

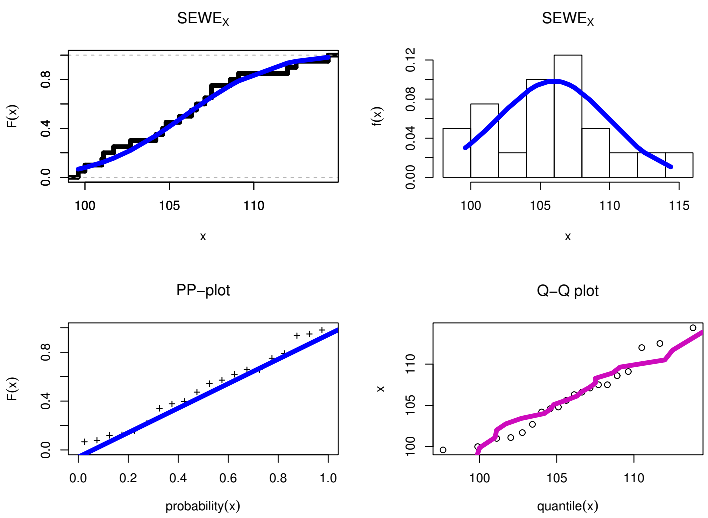

Figure 8. Plots of empirical cdf with fitted cdf, histogram with fitted pdf, PP and Q-Q plot of the SEWE$_X$ model for the Wholesale data.

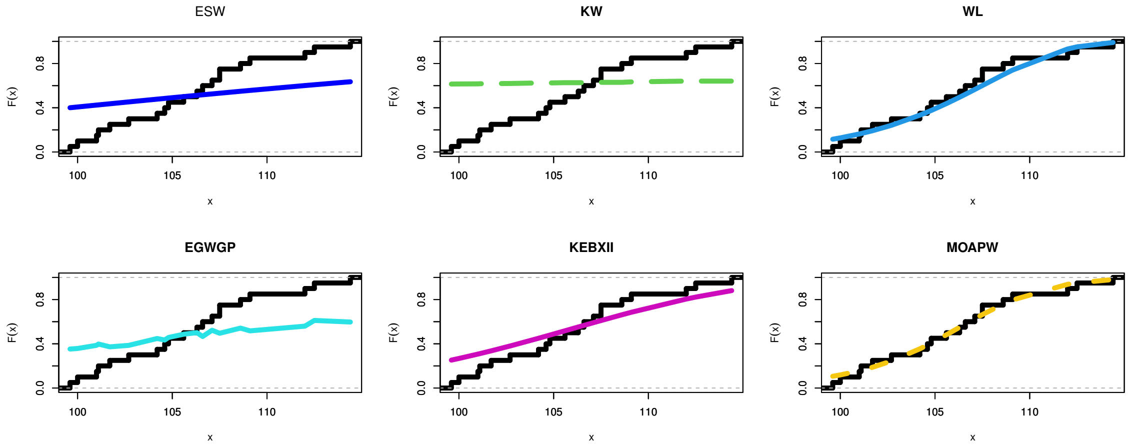

Figure 9. Plots of empirical cdf with fitted cdf of the models for the Wholesale data.

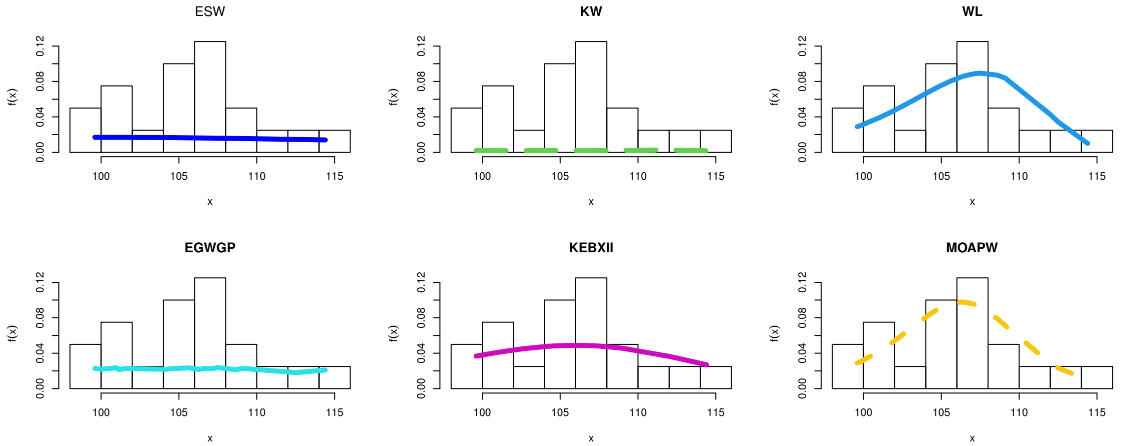

Figure 10. Plots of histogram with fitted pdf of the models for the Wholesale data.

8.3. Single Carbon Fiber Data

The source of this dataset is given in [44]. It contains the tensile strength of single carbon fibers (in GPa). This information is provided by 0.312, 0.314, 0.479, 0.552, 0.700, 0.803, 0.861, 0.865, 0.944, 0.958, 0.966, 0.997, 1.006, 1.021, 1.027, 1.055, 1.063, 1.098, 1.140, 1.179, 1.224, 1.240, 1.253, 1.270, 1.272, 1.274, 1.301, 1.301, 1.359, 1.382, 1.382, 1.426, 1.434, 1.435, 1.478, 1.490, 1.511, 1.514, 1.535, 1.554, 1.566, 1.570, 1.586, 1.629, 1.633, 1.642, 1.648, 1.684, 1.697, 1.726, 1.770, 1.773, 1.800, 1.809, 1.818, 1.821, 1.848, 1.880, 1.954, 2.012, 2.067, 2.084, 2.090, 2.096, 2.128, 2.233, 2.433, 2.585, 2.585. For the third dataset, the numerical values of $\hat{\beta}$, $\hat{\theta}$, $\hat{\lambda}$, $\hat{\mu}$, and $\hat{\rho}$ are provided in Table 6. From the numerical comparison of the competing distributions in Table 6, we observe that the proposed SEWE$_x$ model is the best choice to implement for fitting the food chain data. For the SEWE$_x$ distribution, the values of the analytical measures are AIC = 105.2991, BIC = 114.2356, CVMV = 0.0172, ADV = 0.1557, and KSD = 0.0403, with PVKS = 0.9999. To support the best fitting power of the SEWE$_x$ model, a visual illustration is provided in Figures 11–13. From the visual illustration in Figures 11–13, we can see that the SEWE$_x$ distribution follows the fitted pdf, cdf, PP and QQ plot very closely.

Table 6. M_LE with SEs and different measures for single carbon fiber data.

		β	θ	λ	ρ	AIC	BIC	KSD	PVKS	CVMV	ADV
$SEWE_x$	estimates	14.2450	0.2028	1.1062	2.9008	105.2991	114.2356	0.0403	0.9999	0.0172	0.1557
	SE	7.1414	0.1436	1.0753	1.4375						
OLLMW	estimates	28.7708	0.0560	0.6093	0.0125	106.6756	115.6121	0.0468	0.9982	0.0222	0.1625
	SE	39.8918	0.0814	0.1193	0.0310						
KW	estimates	0.7560	0.1473	1.0575	3.4344	105.5197	114.4561	0.0487	0.9967	0.0220	0.1947
	SE	0.0286	0.0202	0.0209	0.0151						
GMW	estimates	0.4879	4.4450	0.8570	0.3466	105.4390	114.3755	0.0422	0.9997	0.0185	0.1654
	SE	0.7596	8.9394	0.2829	1.0786						
EOWL	estimates	2.6361	0.1152	8.7047	19.2595	105.5875	114.5240	0.0435	0.9995	0.0211	0.1878
	SE	0.4997	0.2885	43.9097	100.6442						
EOWINH	estimates	3.2607	0.0821	0.5014	2.9358	109.0365	117.9729	0.0424	0.9997	0.0405	0.3270
	SE	1.2895	0.3173	0.2179	1.9671						

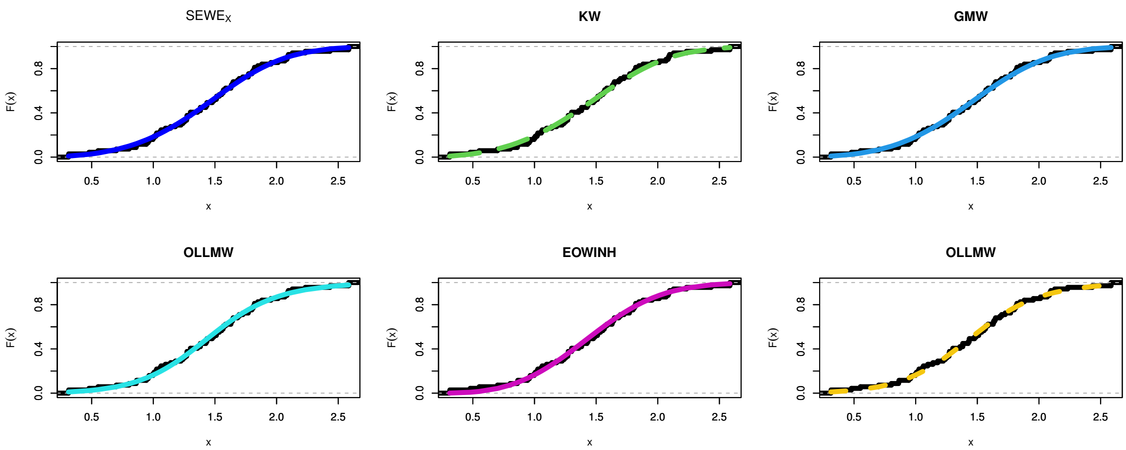

Figure 11. Plots of empirical cdf with fitted cdf of the models for the third dataset.

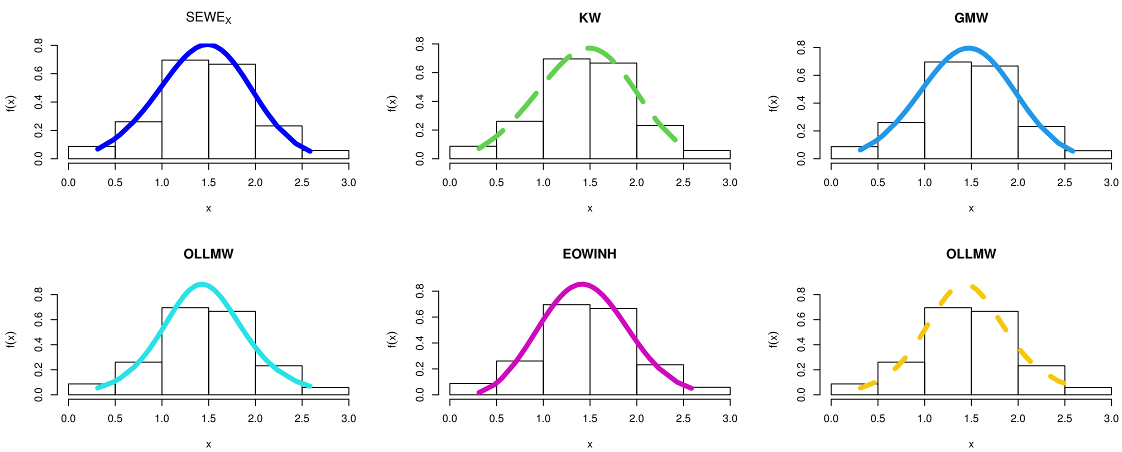

Figure 12. Plots of histogram with fitted pdf of the models for the third dataset.

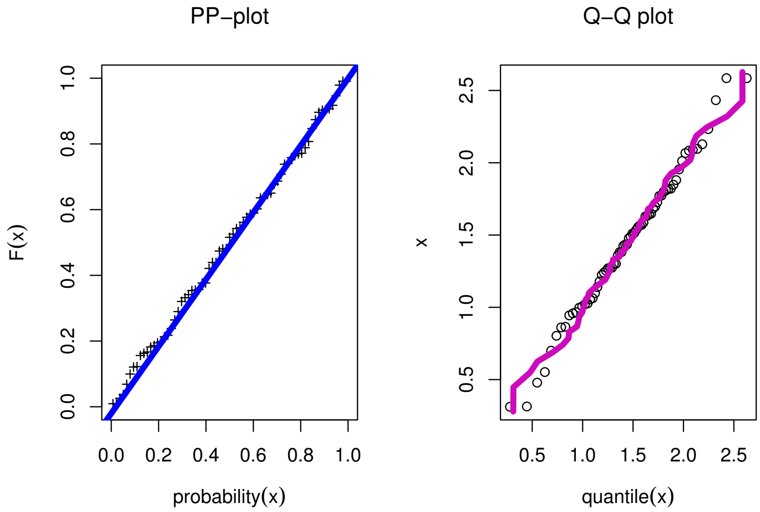

Figure 13. Plots of PP and Q-Q plot of the SEWE$_X$ model for the third dataset.

8.4. Breaking Stress Dataset

The fourth dataset, often called breaking stress of carbon fiber dataset, was used by [45]. This dataset is given by: "3.70, 2.74, 2.73, 2.50, 3.60, 3.11, 3.27, 2.87, 1.47, 3.11, 3.56, 4.42, 2.41, 3.19, 3.22, 1.69, 3.28, 3.09, 1.87, 3.15, 4.90, 1.57, 2.67, 2.93, 3.22, 3.39, 2.81, 4.20, 3.33, 2.55, 3.31, 3.31, 2.85, 1.25, 4.38, 1.84, 0.39, 3.68, 2.48, 0.85, 1.61, 2.79, 4.70, 2.03, 1.89, 2.88, 2.82, 2.05, 3.65, 3.75, 2.43, 2.95, 2.97, 3.39, 2.96, 2.35, 2.55, 2.59, 2.03, 1.61, 2.12, 3.15, 1.08, 2.56, 1.80, 2.53". For the fourth dataset, the numerical values of $\hat{\beta}$, $\hat{\theta}$, $\hat{\lambda}$, $\hat{\mu}$, and $\hat{\rho}$ are provided in Table 7. From the numerical comparison of the competing distributions in Table 7, we observe that the proposed SEWE$_x$ model is the best choice to implement for fitting the food chain data. For the SEWE$_x$ distribution, the values of the analytical measures are AIC = 178.4290, BIC = 187.1876, CVMV = 0.0631, ADV = v, and KSD = 0.0707, with PVKS = 0.8967.

To support the best fitting power of the SEWE$_x$ model, a visual illustration is provided in Figures 14–16. From the visual illustration in Figures 14–16, we can see that the SEWE$_x$ distribution follows the fitted pdf, cdf, PP and QQ plot very closely.

Table 7. M_LE with SEs and different measures for carbon fiber dataset.

		β	θ	λ	ρ	AIC	BIC	KSD	PVKS	CVMV	ADV
SEWE$_x$	estimates	24.8869	0.0945	1.4235	3.0217	178.4290	187.1876	0.0707	0.8967	0.0631	0.3704
	SE	75.6618	0.1957	2.3635	4.4106						
OLLMW	estimates	2.761189	0.140337	0.054904	1.73303	182.6582	191.4168	0.0835	0.7467	0.1416	0.7459
	SE	25.9085	0.0357	0.0977	0.0260						
KW	estimates	0.7246	0.1677	0.5057	3.8408	179.2803	188.0390	0.0841	0.7392	0.0730	0.4549
	SE	0.0144	0.0244	0.0102	0.0170						
GMW	estimates	0.4364	5.4989	0.5161	0.1485	178.7462	187.5048	0.0761	0.8398	0.0654	0.3940
	SE	0.6527	8.0561	0.1731	0.5409						
EOWINH	estimates	4.5935	0.0028	0.2527	21.5314	180.3045	189.0631	0.0832	0.7515	0.0946	0.5382
	SE	1.9504	0.2126	0.1221	24.5623						

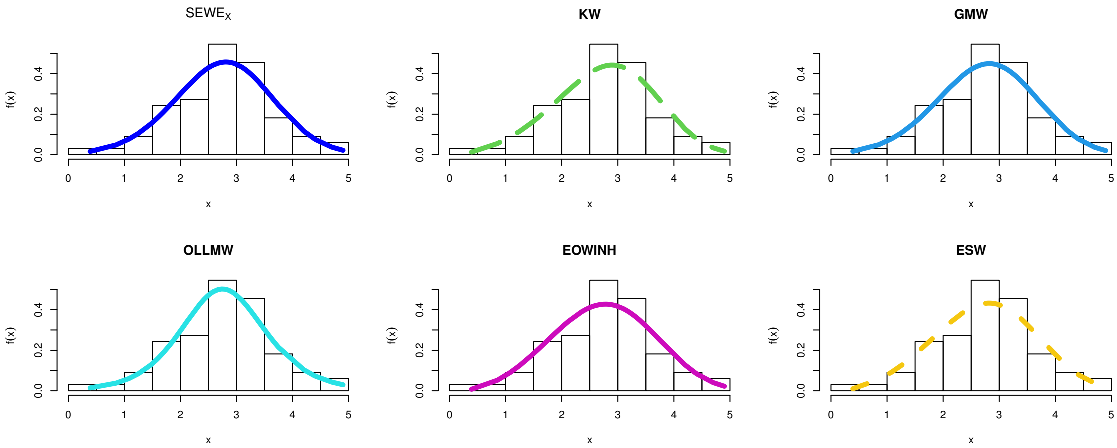

Figure 14. Plots of histogram with fitted pdf of the models for the carbon fiber dataset.

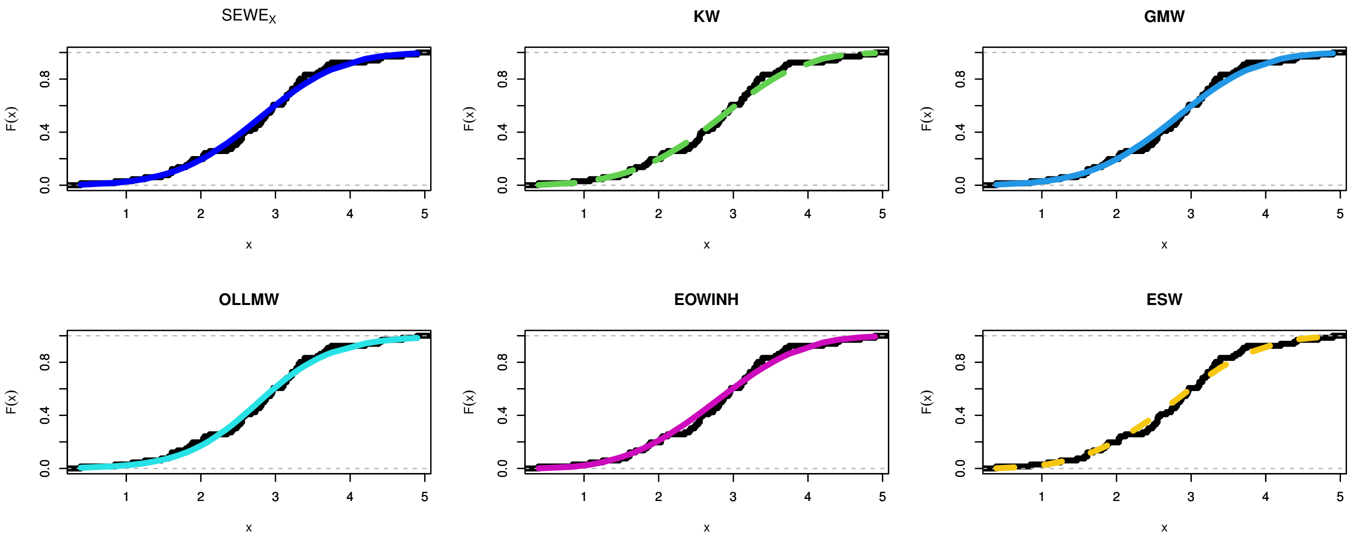

Figure 15. Plots of empirical cdf with fitted cdf of the models for the carbon fiber dataset.

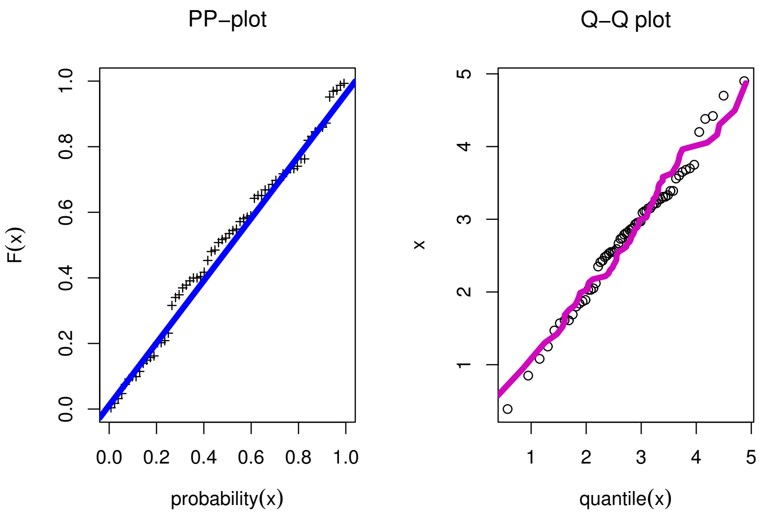

Figure 16. Plots of PP and Q-Q plot of the $SEWE_X$ model for the carbon fiber dataset.

8.5. TFP Growth Dataset

The fifth dataset represents the TFP growth agricultural production for thirty-seven African countries from 2001–2010, see https://dataverse.harvard.edu/dataset.xhtml?persistent\Id=doi:10.7910/DVN/9IOAKR, accessed on 30 June 2022. The dataset is given as 4.6, 0.9, 1.8, 1.4, 0.2, 3.9, 1.8, 0.8, 2.0, 0.8, 1.6, 0.8, 2.0, 1.6, 0.5, 0.1, 2.5, 2.4, 0.6, 1.1, 0.7, 1.7, 1.0, 1.7, 2.5, 3.5, 0.3, 0.9, 2.3, 0.5, 1.5, 5.1, 0.2, 1.5, 3.3, 1.4, 3.3.

For the fifth dataset, the numerical values of $\hat{\beta}$, $\hat{\theta}$, $\hat{\lambda}$, $\hat{\mu}$, and $\hat{\rho}$ are provided in Table 8. From the numerical comparison of the competing distributions in Table 8, we observe that the proposed SEWE$_x$ model is the best choice to implement for fitting the TFP growth data. For the SEWE$_x$ distribution, the values of the analytical measures are AIC = 114.7737, BIC = 116.0237, CVMV = 0.0329, ADV = 0.1988, and KSD = 0.0826, with PVKS = 0.9622.

To support the best fitting power of the SEWE$_x$ model, a visual illustration is provided in Figure 14. From the visual illustration in Figures 14–16, we can see that the SEWE$_x$ distribution follows the fitted pdf, cdf, PP and QQ plot very closely. To support results of Table 8, a visual illustration is provided in Figures 14 and 15.

Table 8. M_LE with SEs and different measures for TFP growth data.

		β	θ	λ	ρ	μ	AIC	BIC	CVMV	ADV	KSD	PVKS
SEWE$_x$	estimates	18.9511	0.2121	2.6809	0.6078		114.7737	116.0237	0.0329	0.1988	0.0826	0.9622
	SE	686.6481	3.1465	27.8125	5.5706							
EGWGP	estimates	0.9855	0.6312	1.3171	0.4959	0.0860	116.8016	118.7371	0.0332	0.1994	0.0830	0.9618
	SE	4.0934	4.3180	0.4487	0.2928	0.9245						
KEBXII	estimates	39.6448	1.1351	569.6257	0.1379	2.7644	117.6760	119.6115	0.0385	0.2376	0.1061	0.7991
	SE	81.7057	2.3102	1917.6428	0.1862	4.2189						
WL	estimates	2.6661	1.3872	1.1264	4.3821		114.9927	116.2427	0.0330	0.1989	0.0828	0.9622
	SE	73.0566	0.5367	4.0920	102.3363							
MOAPW	estimates	0.9331	1.4697	0.8656	2.0064		114.9835	116.2335	0.0340	0.1989	0.0843	0.9550
	SE	9.1323	0.4757	4.3685	1.1887							
KW	estimates	3.1522	0.1048	5.1531	1.0782		116.6211	117.8711	0.0566	0.3540	0.1279	0.5805
	SE	0.0914	0.0174	0.0097	0.0095							
		1.0655	1.2939	0.2672			113.0408	113.7681	0.0289	0.1795	0.0793	0.9741
		0.7932	0.5831	0.2602								

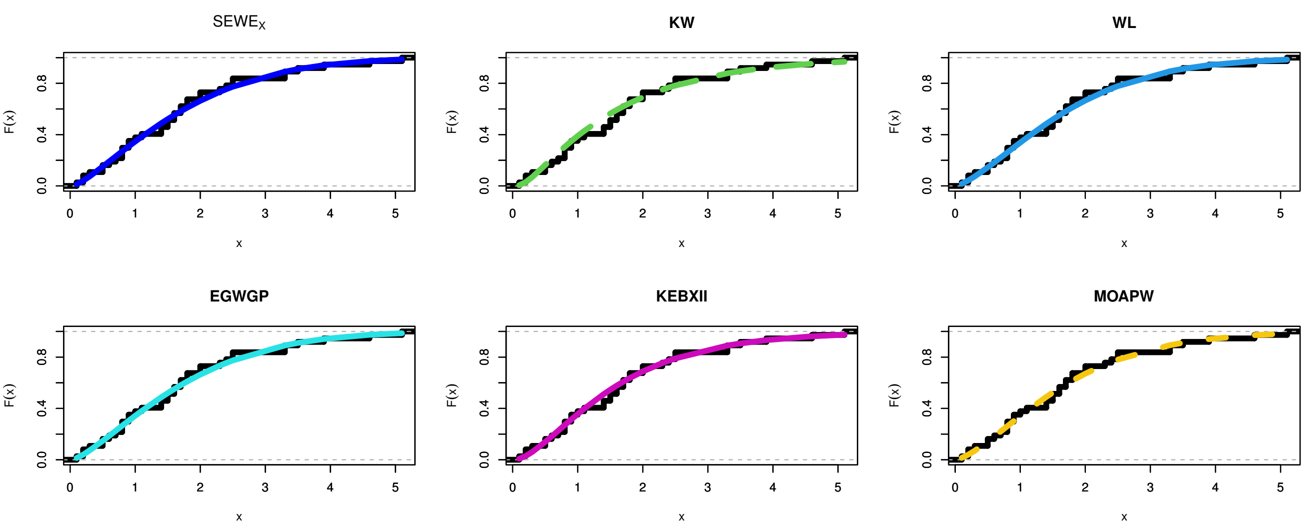

Figure 17. Plots of empirical cdf with fitted cdf of the models for the TFP growth dataset.

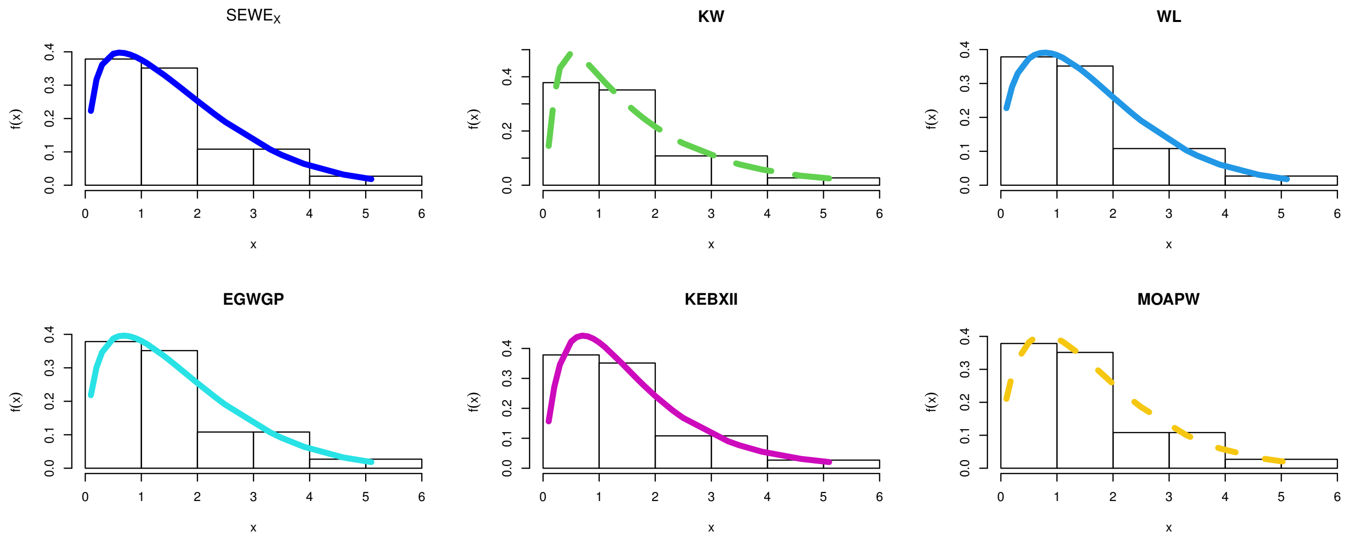

Figure 18. Plots of histogram with fitted pdf of the models for the TFP growth dataset.

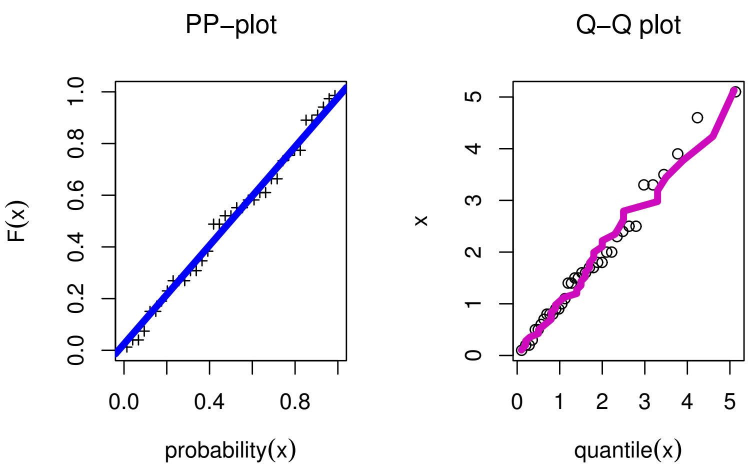

Figure 19. Plots of PP and Q-Q plot of the SEWE$_X$ model for the TFP growth dataset.

9. Conclusions and Summary

In this article, a new lifetime-generated family of distributions called the sine-exponentiated Weibull-H family is proposed; this family is obtained from two well-established families of distributions of completely different nature: the sine-G and the exponentiated Weibull-H families. Three new sub-models were proposed and discussed, including the sine-exponentiated Weibull Rayleigh (SEWR), sine-exponentiated Weibull Burr X (SEWBX), and Sine-exponentiated Weibull exponential (SEWE$_x$) distributions. Some important statistical features of the new family of distributions are investigated, such as quantiles, moments, incomplete moments, and order statistics. Six methods of estimation, namely M_L, L_S, MP_RS_P, WL_S, C_RVM, and A_D, are produced to estimate the parameters. The performance of the estimation approaches is investigated using Monte Carlo simulation. In this article, we use five real datasets to show the relevance and flexibility of the suggested family. The first dataset represents the United Kingdom food chain from 2000 to 2019, whereas the second dataset represents the food and drink wholesaling in the United Kingdom from 2000 to 2019 as one factor of FTP; the third dataset contains the tensile strength of single carbon fibers (in GPa); the fourth dataset is often called the breaking stress of carbon fiber dataset; the fifth dataset represent the TFP growth agricultural production for thirty-seven African countries from 2001–2010. The SEWE$_x$ model as example of the suggested family gives the best fit for all datasets against all competitive models. In the future, we hope to introduce many new statistical models from the suggested family of distributions and study their statistical properties. We also hope that these models have many applications in different fields, including agricultural sciences, environmental sciences, biomedical sciences, engineering sciences, economics, and lifetime data.

Author Contributions: Data curation, S.A.A.; Formal analysis, E.M.A.; Investigation, N.A.; Methodology, I.E.; Resources, N.A.; Software, E.M.A.; Supervision, I.E.; Writing—original draft, M.E.; Writing—review & editing, M.E. All authors have read and agreed to the published version of the manuscript.

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Data Availability Statement: Datasets are available in the application section.

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Conflicts of Interest: The authors declare no conflicts of interest.

Appendix A

Table A1. RBv and MSEV for different estimation techniques of the parameters of $SEWE_X$ distribution: set-I and set-II.

			M_LE		L_SE		WL_SE		MP_RS_PE		C_RVME		A_DE	
ρ	n		RBV	MSEV	RBV	MSEV	RBV	MSEV	RBV	MSEV	RBV	MSEV	RBV	MSEV
0.5	40	β	−0.0051	0.0065	0.0005	0.0008	−0.0014	0.0013	−0.0051	0.0048	0.0023	0.0008	0.0005	0.0009
		θ	−0.0786	0.0226	0.0029	0.0118	−0.0030	0.0135	−0.0786	0.0263	0.0350	0.0130	0.0059	0.0120
		λ	0.0405	0.0963	−0.0073	0.0114	−0.0102	0.0228	0.0407	0.1002	−0.0185	0.0116	−0.0167	0.0193
		ρ	0.2161	0.0725	0.0180	0.0042	0.0540	0.0155	0.2154	0.0931	0.0136	0.0035	0.0446	0.0104
	80	β	−0.0032	0.0038	−0.0005	0.0003	0.0009	0.0005	−0.0032	0.0027	0.0003	0.0004	−0.0003	0.0005
		θ	−0.0553	0.0139	−0.0021	0.0054	−0.0028	0.0059	−0.0552	0.0156	0.0046	0.0054	−0.0057	0.0058
		λ	0.0412	0.0768	0.0028	0.0065	0.0140	0.0137	0.0413	0.0672	−0.0008	0.0070	0.0039	0.0133
		ρ	0.1263	0.0441	0.0135	0.0024	0.0141	0.0055	0.1259	0.0496	0.0108	0.0023	0.0253	0.0068
	160	β	−0.0008	0.0016	−0.0001	0.0003	0.0002	0.0003	−0.0008	0.0016	0.0002	0.0003	−0.0001	0.0005
		θ	−0.0298	0.0087	−0.0021	0.0048	−0.0001	0.0035	−0.0300	0.0091	0.0048	0.0050	−0.0035	0.0054
		λ	0.0567	0.0494	0.0027	0.0049	0.0108	0.0073	0.0406	0.0490	0.0006	0.0069	0.0035	0.0127
		ρ	0.0483	0.0223	0.0129	0.0024	0.0046	0.0025	0.0486	0.0230	0.0131	0.0022	0.0203	0.0067
3	40	β	−0.0053	0.3367	−0.0011	0.0141	0.0090	0.0331	−0.0051	0.2493	0.0010	0.0168	0.0039	0.0312
		θ	−0.0041	0.0103	0.0082	0.0098	0.0144	0.0084	−0.0043	0.0069	0.0392	0.0114	0.0201	0.0080
		λ	0.1231	0.2721	−0.0038	0.0126	0.0280	0.0278	0.1231	0.1563	−0.0162	0.0125	0.0086	0.0187
		ρ	−0.0199	0.4947	0.0017	0.0267	−0.0128	0.0609	−0.0200	0.4374	0.0012	0.0278	−0.0055	0.0541
	80	β	−0.0009	0.1493	0.0011	0.0132	−0.0026	0.0268	−0.0049	0.1569	0.0010	0.0138	0.0036	0.0161
		θ	−0.0089	0.0038	−0.0061	0.0043	0.0009	0.0036	−0.0042	0.0032	0.0083	0.0044	0.0025	0.0034
		λ	0.0936	0.0744	0.0032	0.0093	0.0079	0.0136	0.0936	0.0890	0.0069	0.0081	0.0072	0.0126
		ρ	−0.0195	0.2347	−0.0013	0.0243	0.0019	0.0483	−0.0196	0.2829	−0.0015	0.0237	−0.0049	0.0316
	160	β	0.0120	0.1345	−0.0001	0.0026	0.0021	0.0128	0.0012	0.0895	0.0004	0.0038	0.0014	0.0106
		θ	−0.0100	0.0025	−0.0021	0.0031	0.0006	0.0024	−0.0010	0.0021	0.0080	0.0032	0.0022	0.0024
		λ	0.0923	0.0733	0.0031	0.0032	0.0071	0.0061	0.0919	0.0516	0.0017	0.0034	0.0106	0.0068
		ρ	−0.0297	0.2404	0.0001	0.0044	−0.0014	0.0221	−0.0130	0.1752	0.0000	0.0063	−0.0025	0.0193

Table A2. RBv and MSEV for different estimation techniques of the parameters of $SEWE_X$ distribution: set-III and set-IV.

			M_LE		L_SE		WL_SE		MP_RS_PE		C_RVME		A_DE	
ρ	n		**RBV**	**MSEV**	**RBV**	**MSEV**	**RBV**	**MSEV**	**RBV**	**MSEV**	**RBV**	**MSEV**	**RBV**	**MSEV**
0.5	40	β	−0.0436	0.4199	−0.0074	0.0834	−0.0027	0.0220	−0.0435	0.3166	−0.0032	0.0935	−0.0041	0.1598
		θ	0.0194	0.0322	0.0152	0.0093	0.0148	0.0075	0.0191	0.0176	0.0421	0.0113	0.0283	0.0100
		λ	−0.0099	0.2653	−0.0052	0.0554	−0.0017	0.0123	−0.0198	0.1630	−0.0033	0.0580	0.0031	0.0936
		ρ	−0.0382	0.0089	−0.0062	0.0042	0.0013	0.0033	−0.0387	0.0048	0.0258	0.0045	0.0019	0.0035
	80	β	−0.0234	0.2154	−0.0013	0.0643	−0.0017	0.0076	−0.0233	0.1976	−0.0016	0.0841	0.0040	0.0916
		θ	0.0023	0.0124	−0.0059	0.0036	0.0084	0.0033	0.0182	0.0091	0.0077	0.0038	0.0031	0.0032
		λ	0.0014	0.1411	−0.0013	0.0105	−0.0008	0.0037	0.0164	0.1059	−0.0016	0.0163	0.0020	0.0198
		ρ	−0.0313	0.0054	−0.0027	0.0019	−0.0005	0.0016	−0.0314	0.0030	0.0075	0.0020	−0.0009	0.0016
	160	β	−0.0026	0.1627	−0.0008	0.0144	−0.0015	0.0068	−0.0027	0.1034	−0.0015	0.0167	0.0006	0.0339
		θ	−0.0118	0.0049	−0.0013	0.0028	0.0015	0.0022	−0.0117	0.0032	0.0068	0.0029	0.0022	0.0025
		λ	0.0145	0.0984	−0.0011	0.0099	−0.0006	0.0036	0.0145	0.0691	−0.0016	0.0108	0.0017	0.0123
		ρ	−0.0332	0.0025	−0.0004	0.0014	0.0002	0.0011	−0.0334	0.0021	0.0065	0.0015	0.0007	0.0013
3	40	β	−0.0687	1.5287	−0.0094	0.0275	0.0055	0.2252	−0.0684	0.8406	−0.0585	0.0237	−0.0073	0.0911
		θ	0.1085	0.2062	0.0097	0.0100	0.0241	0.0148	0.1080	0.0950	0.0424	0.0120	0.0259	0.0097
		λ	−0.0277	0.4156	−0.0091	0.0491	0.0047	0.0940	−0.0278	0.2324	0.0096	0.0491	0.0012	0.0586
		ρ	−0.0358	0.1394	−0.0036	0.0345	−0.0037	0.0410	−0.0360	0.0744	0.0143	0.0393	0.0033	0.0300
	80	β	−0.0467	0.9585	−0.0019	0.0275	−0.0073	0.1440	−0.0465	0.4781	−0.0078	0.0153	−0.0041	0.0916
		θ	0.0440	0.0741	−0.0043	0.0050	0.0162	0.0149	0.0438	0.0361	0.0142	0.0057	0.0066	0.0062
		λ	−0.0158	0.2451	−0.0046	0.0249	−0.0048	0.0621	−0.0159	0.1192	−0.0063	0.0341	0.0012	0.0396
		ρ	−0.0253	0.0718	−0.0032	0.0178	0.0009	0.0227	−0.0253	0.0423	0.0051	0.0189	−0.0013	0.0205
	160	β	−0.0153	0.4215	−0.0015	0.0175	−0.0063	0.0682	−0.0155	0.2706	−0.0045	0.0108	−0.0035	0.0893
		θ	0.0060	0.0168	0.0037	0.0039	0.0085	0.0051	0.0061	0.0097	0.0132	0.0042	0.0062	0.0060
		λ	−0.0016	0.1069	−0.0033	0.0140	−0.0009	0.0243	−0.0017	0.0825	0.0018	0.0318	−0.0010	0.0259
		ρ	−0.0237	0.0315	0.0012	0.0136	0.0004	0.0124	−0.0239	0.0336	0.0045	0.0150	0.0012	0.0194

Table A3. RBv and MSEV for different estimation techniques of the parameters of $SEWE_X$ distribution: set-V and set-VI.

			M_LE		L_SE		WL_SE		MP_RS_PE		C_RVME		A_DE	
0.5	40	β	−0.01121	0.03921	−0.00557	0.00859	−0.00381	0.01597	−0.01126	0.01443	0.00544	0.01353	0.00411	0.01517
		θ	−0.02074	0.05917	0.00910	0.02844	−0.00946	0.02595	−0.02086	0.04968	0.01512	0.03550	0.00706	0.04065
		λ	0.00108	0.00050	−0.00027	0.00012	−0.00019	0.00025	0.00109	0.00021	−0.00125	0.00022	−0.00041	0.00021
		ρ	−0.01047	0.00018	−0.00231	0.00016	−0.00391	0.00014	−0.01057	0.00020	0.00220	0.00015	−0.00031	0.00015
	80	β	−0.00782	0.03262	−0.00246	0.00646	0.00328	0.00878	−0.00784	0.00683	0.00330	0.00886	−0.00136	0.01373
		θ	−0.01516	0.02808	−0.00515	0.01619	0.00809	0.01861	−0.01519	0.02192	0.00417	0.01970	−0.00504	0.02411
		λ	0.00085	0.00050	0.00017	0.00007	−0.00077	0.00011	0.00085	0.00009	−0.00038	0.00011	0.00037	0.00016
		ρ	−0.00590	0.00009	−0.00177	0.00009	0.00223	0.00008	−0.00592	0.00009	0.00150	0.00010	−0.00028	0.00009
	160	β	−0.00507	0.00652	−0.00085	0.00175	−0.00123	0.00323	−0.00509	0.00388	0.00081	0.00173	−0.00064	0.00189
		θ	−0.01107	0.01619	−0.00123	0.00619	−0.00147	0.00856	−0.01109	0.01347	0.00156	0.00652	−0.00167	0.00679
		λ	0.00067	0.00009	0.00001	0.00002	−0.00010	0.00005	0.00067	0.00005	−0.00013	0.00002	0.00011	0.00002
		ρ	−0.00413	0.00006	−0.00018	0.00005	−0.00021	0.00005	−0.00417	0.00006	0.00106	0.00005	−0.00013	0.00005
	40	β	−0.02534	0.77945	−0.00540	0.04216	−0.02703	0.20585	−0.02518	0.69258	0.01059	0.04814	0.00493	0.06843
		θ	0.02476	0.32320	0.00587	0.09164	0.02061	0.13155	0.02441	0.33022	0.02969	0.10620	0.01548	0.08509
		λ	−0.00364	0.01487	−0.00160	0.00058	−0.00235	0.00319	−0.00362	0.01415	−0.00134	0.00062	−0.00121	0.00103
		ρ	−0.01226	0.00699	−0.00307	0.00931	−0.00151	0.00793	−0.01235	0.00856	0.00415	0.00925	0.00144	0.00731

3

Table A3. *Cont.*

		M_LE		L_SE		WL_SE		MP_RS_PE		C_RVME		A_DE	
80	β	−0.01514	0.61948	−0.00496	0.02314	0.01265	0.09885	−0.01502	0.46116	−0.00125	0.03397	−0.00456	0.05829
	θ	0.00867	0.31488	−0.00473	0.05100	−0.00403	0.05293	0.00850	0.20105	0.00856	0.05780	0.00614	0.06378
	λ	−0.00136	0.01230	−0.00070	0.00042	0.00123	0.00165	−0.00135	0.00891	−0.00082	0.00061	−0.00098	0.00100
	ρ	−0.00795	0.00375	−0.00301	0.00445	−0.00074	0.00383	−0.00797	0.00396	0.00160	0.00439	−0.00129	0.00362
160	β	0.00461	0.21543	0.00208	0.02135	0.00069	0.02479	0.00433	0.29192	0.00105	0.03412	0.00396	0.04742
	θ	−0.00775	0.09234	−0.00280	0.03551	0.00059	0.02697	−0.00766	0.11092	0.00549	0.03685	−0.00070	0.04183
	λ	0.00126	0.00390	0.00010	0.00036	−0.00011	0.00036	0.00121	0.00540	0.00008	0.00057	0.00091	0.00096
	ρ	−0.00553	0.00234	−0.00098	0.00325	−0.00012	0.00262	−0.00558	0.00278	0.00139	0.00326	0.00021	0.00257

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