Article

Maximizing the Influence of Innovative Green Product Propagation

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Abstract: This article considers the competition between the propagation of traditional product information and innovative green product information, and it proposes a hybrid model with advertisement and promotion strategies. On this basis, an innovation green product information propagation model is developed through the optimization of the advertisement strategies of the adopters of innovative green product information and the promotion strategies of the adopters of traditional product information, according to Pontryagin's maximum principle to seek the optimal strategies for maximizing the influence of innovative green product information, and using numerical calculations to simulate the propagation state of product information. The results show that advertisement strategies play a decisive role in the propagation of innovative green product information in the market. If the promotion strategies are also considered, the propagation effect of innovative green product information will be more effective.

Citation: Huo, L.; Wang, Q.; Lin, T.; He, H. Maximizing the Influence of Innovative Green Product Propagation. *Sustainability* **2021**, *13*, 4110. <https://doi.org/10.3390/su13084110>

Academic Editors: Antonino Galati and Tracy Bhamra

Received: 12 March 2021

Accepted: 31 March 2021

Published: 7 April 2021

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Keywords: advertisement strategies; promotion strategies; innovative green product information; traditional product information; optimal control

1. Introduction

With the advancement of technology and urbanization, environmental problems are becoming more and more serious. People's awareness of environmental protection has increased, the concept of green consumption has gained popularity, and innovative green products are favored [1–3]. Production of innovative green products is considered one of the key factors in achieving growth and environmental sustainability, and in improving quality of life [4]. Green product innovation requires the attention of scholars and important continuous research to support managers and enterprises interested in green product marketing [5]. Green product marketing plays a positive role in promoting the popularization of green consumption [6,7], and, in particular, the emergence of online social networks (OSNs) has changed the existing marketing methods and competition rules for product propagation [8]. Individuals always search their favorite product information through the OSNs without having to go to the site [9–11]. As a result, the competition of green products has gradually evolved into the competition of green product information propagation.

Information propagation can be traced back to the classic rumor propagation model (DK model) proposed by Daley and Kendall in 1964. The DK model is the basis for exploring information propagation [12]. Then, Maki and Thomson proposed the MT model based on the modification of the DK model [13]. With the emergence of complex networks, a new viewpoint and tool to study information propagation was provided. Watts and Barabasi discussed information propagation from the perspective of complex networks [14,15]. Based on the process of information propagation and various influencing factors, summarizing the basic law of innovation propagation in social systems, Rogers proposed the S curve theory of innovation propagation, and the theory has continually improved from different

perspectives [16]. After Rogers, Bass proposed an innovative purchase model called the Bass model [17]. It provided an important market analysis and forecasting tool, and then subsequent research on market propagation was proposed [18–20]. At the same time, the rationality of the innovation propagation model has been effectively explained from various angles, and the innovation propagation theories have been further developed. Robinson involved price into the Bass model and studied the price function to improve the predictive ability of the model [21]. Kalish introduced market potential into the Bass model and used the functional relationship between market potential and price to study the propagation rate of products [22]. Easingword relaxed the assumption that the model coefficients were fixed and constructed a more flexible and non-uniform innovation propagation model [23].

With the advent of the green era, consumers' purchasing demands have changed, and the propagation of innovative green product information has become a research hotspot for scholars. From the perspective of enterprise competitiveness, green innovation products play an important role in improving the competitiveness of enterprises. The importance and potential of green innovation in the hotel industry in Malaysia in promoting the sustainable performance of the hotel industry have been highlighted [6]. Li's research showed that enterprise green technology innovation ability has a significant positive impact on enterprise competitiveness [24]. From the perspective of consumer psychology, psychologists discovered many years ago that building good information could change human behavior and attitudes [25–27]. Heidenreich indicated that the promotion of green product information can enhance consumers' purchasing intention for green products to a certain extent [28]. Leonidou et al.'s research on the content of green advertising found that if the audience is differentiated between enterprise-oriented and consumer-oriented, green advertising focuses more on consumers [29]. Mishra studied the difference in the effect of increasing the quantity without a price increase and the sales of green food, and the results showed that price discounts are effective for non-green foods, and the promotion of green food without a price increase is more effective [30]. Regarding the study of competitive propagation, the Norton model was proposed based on the case study of the propagation of four generations of computer random access memory products [31]. Chitra studied the impact of consumers' environmental awareness on their purchase of green products and showed that the more environmentally conscious consumers were, the more likely they were to buy green products [32]. Lee incorporated environmental concerns into new product development and proposed a way to enhance business competitiveness through green new product development activities [33]. In the context of powerful practical applications, an increasing number of scholars have studied and discussed the issue of competitive propagation from different perspectives. Li proposed a competitive propagation model from the perspective of predictive propagation and pointed out that when competitive propagation exists, word of mouth was mainly determined by the information of the product itself and the brand [34]. Schramm believed that the factors that affect the proliferation of competition include prices and promotion strategies [35]. Tashiro proposed a new method of estimating the number of advertisements with fitted parameters and used this model to predict iPod and iPhone sales data [36]. However, there are few studies on the optimization of multi-angle marketing strategies for innovative green products. Especially in the fierce market competition, the propagation of innovative green products requires not only advertisement strategies for potential customers but also promotion strategies for the adopters of traditional product information. Traditional products in this paper refer to traditional non-green products, which may harm the environment in their whole life cycle. In the context of sustainable development, we define innovative green products as products that enterprises produce new categories of, and that identify and respond to new green markets. We believe that the fundamental difference between traditional products and innovative green products lies in their function of improving the environment and the quality of social life. In the propagation process, consumers choose between innovative green products and traditional products. These two types of products are homogeneous in function and heterogeneous in environmental attributes.

This article considers the real market situation and combines the advertisement and promotion strategies in market competition propagation to expand the influence of propagation of innovative green product information. Since the advertisement strategies and promotion strategies are closely related to the propagation process of innovation green product information and traditional product information, we establish an innovation green product information propagation model with advertisement–promotion strategies.

This paper is organized as follows: Section 2 models the advertisement–promotion strategies as a hybrid model, and Section 3 presents the optimal system to seek the strategies for maximizing the influence of innovative green product information based on Pontryagin’s maximum principle. Section 4 experimentally examines the validity of the advertisement–promotion strategies. Section 5 concludes this work.

2. The Modeling of the Advertisement–Promotion Strategies

Discussing and researching the information propagation strategies of innovative green products make enterprises more market-competitive. Well-known examples such as the propagation of Philips’s lighting LED lamp and fluorescent energy-saving lamp series products quickly gained market shares by the research of the high efficiency and low energy consumption of green lighting. We consider the existence of both innovative green product information and traditional product information, which have homogeneity in function and heterogeneity in environmental attributes, and there exists three types of consumer groups: (1) the potential consumers who never receive any information; (2) the adopters of innovative green product information who receive information about innovative green products and choose to purchase them; (3) the adopters of traditional product information who receive information about traditional products and choose to purchase them. On the one hand, advertisement strategies will affect consumers’ acceptance of innovative green products and purchase behavior. A portion of potential consumers in social networks are influenced by external factors such as advertisement and may turn into the adopters of innovative green product information. On the other hand, the implementation of reasonable promotion strategies can also effectively promote the propagation of innovative green product information. Under the influence of promotion strategies implemented for the adopters of traditional product information, the adopters of traditional product information may turn into the adopters of innovative green product information. At the same time, word-of-mouth marketing can satisfy consumers’ mentality of acquiring information, and strong word of mouth can improve the conversion rate of the adopters of innovative green product information. Considering the word of mouth effect, the adopters of innovative green product information may turn into the adopters of traditional product information after interacting with the adopters of traditional product information; similarly, the adopters of traditional product information may turn into the adopters of innovative green product information after interacting with the adopters of innovative green product information. Enterprises hope to expand the market of innovative green products through the above three ways, so that innovative green products can quickly occupy the market. Based on the optimization of the marketing strategies of innovative green products, this article is dedicated to formulating advertisement strategies and promotion strategies to maximize the influence of innovative green product information.

2.1. Preliminary Terms and Notations

In the fierce market competition, enterprises expand the influence of innovative green product information through three ways of advertisement and promotion strategies, plus word-of-mouth marketing of customers. As advertisement strategies and promotion strategies are the most direct methods for companies to implement, this article mainly starts from these two strategies. Consider a population of persons labeled 1 through N who are interconnected by OSNs. a_{ij} refers to whether person i can send a message to person j . Let $A = (a_{ij})_{N \times N}$ denote the adjacency matrix of the underlying communication network

of the population, i.e., $a_{ij} = 0$ or 1 according to whether the person can send messages directly to person i or not.

We assume that traditional product information and innovative green product information will compete in the consumer market. Suppose an advertisement of an innovative green product implemented for potential consumers and promotion strategies implemented for the adopters of traditional product information are propagating in the population in the finite time $[0, t_f]$. Assume that at any time in the time horizon, every person in the population is in one of three possible states: potential consumers (P), the adopters of innovative green product information (I), or the adopters of traditional product information (T). Let $X_i(t)$ denote the set of all the people at time t . Let $\mathbf{X}(t) = (x_1(t), x_2(t), \dots, x_N(t))$. This is a random vector we refer to as the state of the population at time t .

Let $P_i(t)$, $I_i(t)$, and $T_i(t)$ denote the probabilities that person i is a potential consumer, an adopter of innovative green product information, or an adopter of traditional product information at time t , respectively. The vector

$$\mathbf{E}(t) = (P_1(t), \dots, P_N(t), I_1(t), \dots, I_N(t), T_1(t), \dots, T_N(t))$$

stands for the expected state of the population at time t . For brevity, let

$$\mathbf{P}(t) = (P_1(t), \dots, P_N(t)), \mathbf{I}(t) = (I_1(t), \dots, I_N(t)), \mathbf{T}(t) = (T_1(t), \dots, T_N(t))$$

Then, $\mathbf{E}(t) = (\mathbf{P}(t), \mathbf{I}(t), \mathbf{T}(t))$.

We assume that under the advertisement–promotion strategies, the changing state rules are as follows:

- (1) The change process of P nodes: The potential consumer i is influenced by advertisement strategies and turns into the adopter of innovative green product information with the probability of $\theta_i(t)$; or influenced by an advertisement of a traditional product and turns into the adopter of traditional product information with the probability of $\gamma_i(t)$. According to the theory of continuous-time Markov chains [37], the P_i turns into the adopter of innovative green product information at time t with the probability of $\sum_{j=1}^N \beta_{ij}^P I_j(t)$ after contact with the adopter of the innovative green product information I_j , and it turns into the adopter of traditional product information at time t with the probability of $\sum_{j=1}^N \alpha_{ij}^P T_j(t)$ after contact with adopters of traditional product information T_j .
- (2) The dynamic change process of I nodes: due to promotion of traditional products, the adopter of innovative green product information P_i turns into the adopter of traditional product information T_i with the probability of $\delta_i(t)$; considering the word of mouth effect, the adopter of the innovative green product information turns I_i into the adopter of the traditional product information with the probability of $\sum_{j=1}^N \alpha_{ij}^I T_j(t)$ after the adopter of the innovative green product information I_i contacts the adopter of the traditional product information T_j .
- (3) The dynamic change process of T nodes: due to promotion strategies implemented for the adopters of traditional product information, the adopter of traditional product information T_i turns into the adopter of innovative green product information I_i with the probability of $\eta_i(t)$; considering the word of mouth effect, the adopter of the traditional product information T_i turns into the adopter of the innovative green product information with the probability of $\sum_{j=1}^N \beta_{ij}^T I_j(t)$ after the adopter of the traditional product T_i contacts the adopter of the innovative green product information I_j .

Figure 1 shows this collection of assumptions schematically.

Figure 1. Diagram of innovative green product propagation.

For our purpose, let us introduce the following notation.

$\theta_i(t)$: due to the influence of advertisements, the average probability of a potential consumer i turning into an adopter of innovative green product information, i.e., the probability of advertisement propagation. The number of potential consumers i turning into adopters of innovative green product information per unit time t is $\theta_i(t)P_i(t)$ due to the influence of advertisements.

$\gamma_i(t)$: due to the influence of advertisement, the average probability of a potential consumer i turning into an adopter of traditional product information. The number of potential consumers i turning to adopters of traditional product information per unit time t is $\gamma_i(t)P_i(t)$ due to the influence of advertisements.

β_{ij}^P : due to the influence of the adopter of innovative green product information j , the average probability of which a potential consumer i turns into an adopter of innovative green product information. The number of potential consumers i turning into adopters of innovative green product information I_j per unit time t is $P_i(t) \sum_{j=1}^N \beta_{ij}^P I_j(t)$ due to the influence of the adopter of innovative green product information I_j .

α_{ij}^P : due to the influence of the adopter of traditional product information j , the average probability of which a potential consumer i turns into an adopter of traditional product information. The number of potential consumers i turning into adopters of traditional product information T_j per unit time t is $P_i(t) \sum_{j=1}^N \alpha_{ij}^P T_j(t)$ due to the influence of the adopter of traditional product information T_j .

$\eta_i(t)$: due to the influence of promotion for innovative green products, the average probability of an adopter of traditional product information i turning into an adopter of innovative green product information, i.e., the probability of promotion propagation. The number of adopters of traditional product information T_i turning into adopters of innovative green product information per unit time t is $\eta_i(t)T_i(t)$ due to influence of promotion for innovative green products.

$\delta_i(t)$: due to the influence of promotion for traditional products, the average probability of an adopter of innovative green product information i turning into an adopter of traditional product information. The number of adopters of innovative green product information I_i turning into adopters of traditional product information per unit time t is $\delta_i(t)I_i(t)$ due to the influence of promotion for traditional products.

α_{ij}^I : due to the influence of word of mouth, the average probability of a potential consumer i turning into an adopter of innovative green product information. The number of adopters of innovative green product information I_i turning into adopters of traditional product information T_j per unit time t is $I_i(t) \sum_{j=1}^N \alpha_{ij}^I T_j(t)$ due to the influence of word of mouth.

α_{ij}^I : due to the influence of word of mouth, the average probability of an adopter of innovative green product information i turning into an adopter of traditional product information. The number of adopters of traditional product information T_i turning into adopters of innovative green product information I_j per unit time t is $T_i(t) \sum_{j=1}^N \beta_{ij}^T I_j(t)$ due to the influence of word of mouth.

By the total probability formula, we obtain the following system:

$$\begin{cases} \frac{dP_i(t)}{dt} = -P_i(t) \sum_{j=1}^N \beta_{ij}^P I_j(t) - P_i(t) \sum_{j=1}^N \alpha_{ij}^I T_j(t) - \theta_i(t) P_i(t) - \gamma P_i(t) \\ \frac{dI_i(t)}{dt} = P_i(t) \sum_{j=1}^N \beta_{ij}^P I_j(t) + \theta_i(t) P_i(t) + T_i(t) \sum_{j=1}^N \beta_{ij}^T I_j(t) - I_i(t) \sum_{j=1}^N \alpha_{ij}^I T_j + \eta_i(t) T_i(t) - \delta I_i(t) \\ \frac{dT_i(t)}{dt} = P_i(t) \sum_{j=1}^N \alpha_{ij}^P T_j(t) + \gamma P_i(t) + I_i(t) \sum_{j=1}^N \alpha_{ij}^I T_j - T_i(t) \sum_{j=1}^N \beta_{ij}^T I_j(t) - \eta_i(t) T_i(t) + \delta I_i(t) \end{cases} \quad (1)$$

Remark 1. The above parameters can be estimated by collecting and analyzing the message-spreading probabilities in OSNs.

Remark 2. Obviously, $\beta_{ij}^P = \beta_{ij}^T = \alpha_{ij}^P = \alpha_{ij}^I = 0$ if $a_{ij} = 0$ (person i cannot send messages directly to person j).

$$\mathbf{B}^P = (\beta_{ij}^P)_{N \times N}, \mathbf{B}^T = (\beta_{ij}^T)_{N \times N}, \mathbf{C}^P = (\alpha_{ij}^P)_{N \times N}, \mathbf{C}^I = (\alpha_{ij}^I)_{N \times N}.$$

2.2. Formulating AP Strategies

This article focuses on the marketing strategies of innovative green products, and we assume that $\gamma_i(t) = \gamma$, $\delta_i(t) = \delta$. Let $\theta_i(t)$ denote the probability of advertisement propagation, and let $\eta_i(t)$ denote the probability of promotion propagation. We refer to the N -dimensional vector-valued function $\mathbf{v}$, defined by $\mathbf{v}(t) = (\theta_1(t), \dots, \theta_N(t))$, $0 \leq t \leq t_f$, as advertisement strategies, the N -dimensional vector-valued function $\mathbf{w}$, defined by $\mathbf{w}(t) = (\eta_1(t), \dots, \eta_N(t))$, $0 \leq t \leq t_f$, as promotion strategies, and the $2N$ -dimensional vector-valued function $\mathbf{w}$, defined by $\mathbf{w}(t) = (\theta_1(t), \dots, \theta_N(t), \eta_1(t), \dots, \eta_N(t))$, $0 \leq t \leq t_f$, as advertisement–promotion (AP) strategies. For our purpose, we assume that the admissible set of the AP strategies is

$$\omega = \left\{ \mathbf{w} \in L[0, T]^{2N} \mid \theta_i(t) \leq \bar{\theta}_i, \eta_i(t) \leq \bar{\eta}_i, 0 \leq t \leq t_f, 1 \leq i \leq N \right\}$$

where $L[0, t_f]$ stands for the set of all the Lebesgue integrable functions defined on $[0, t_f]$ [38].

Remark 3. For brevity, let $\bar{\mathbf{u}} = (\bar{\theta}_1, \dots, \bar{\theta}_N)$, $\bar{\mathbf{v}} = (\bar{\eta}_1, \dots, \bar{\eta}_N)$.

We refer to system (1) as the innovative green product propagation model. This model is schematically shown in Figure 1. The model may be written in matrix notation as

$$\begin{cases} \frac{d\mathbf{E}(t)}{dt} = \mathbf{f}(\mathbf{E}(t), \mathbf{w}(t)), & 0 \leq t \leq t_f, \\ \mathbf{E}(0) = \mathbf{E}_0. \end{cases} \quad (2)$$

2.3. The Optimal Propagation Strategies

The influence of innovative green product information consists of three parts: the number of adopters of innovative green product information increased by the implementation of advertisement strategies and promotion strategies, the cost for implementing the advertisement strategies, and the cost for carrying out the promotion strategies.

Firstly, we assume that the number of adopters of innovative green product information per unit time increased by the implementation of advertisement strategies and promotion strategies is a_i units (USD, say), which is a constant. Then, the expected income in the horizon $[0, t_f]$ increased by the advertisement and promotion is

$$N(\mathbf{w}) = \int_0^{t_f} \sum_{i=1}^N a_i I_i(t) dt \quad (3)$$

We use Equation (3) to measure the number of adopters of innovative green product information increased by the implementation of advertisement strategies and promotion strategies.

Secondly, assume the cost per unit time for advertisement to a potential customer i at the possibility of $\theta_i(t)$ is b_i , which is a constant. Then, the expected cost for implementing advertisement strategies $\mathbf{u}$ is

$$C_A(\mathbf{w}) = \int_0^{t_f} \sum_{i=1}^N b_i \theta_i(t) P_i(t) dt \quad (4)$$

We use Equation (4) to measure the expected cost for implementing advertisement strategies $\mathbf{u}$.

Lastly, assume the cost per unit time for promoting the adopter of traditional product information i at the possibility of $\eta_i(t)$ is c_i , which is a constant. Then, the expected cost for implementing promotion strategies $\mathbf{v}$ is

$$C_P(\mathbf{w}) = \int_0^{t_f} \sum_{i=1}^N c_i \eta_i(t) T_i(t) dt \quad (5)$$

We use Equation (5) to measure the cost for implementing promotion strategies $\mathbf{v}$.

Remark 4. a_i is determined by estimating the potential influence of the innovative green product information and the social influence of the person i , b_i is assessed by estimating the advertisement costs of a potential consumer i who has transformed into an adopter of innovative green product information, and c_i is figured out by estimating the promotion cost of the transition from the adopter of traditional product information i to the adopter of innovative green product information.

Combining the above discussions, the expected influence of the innovative green product is

$$J(\mathbf{w}) = N(\mathbf{w}) - C_A(\mathbf{w}) - C_P(\mathbf{w}) = \int_0^{t_f} \sum_{i=1}^N a_i I_i(t) dt - \int_0^{t_f} \sum_{i=1}^N b_i \theta_i(t) P_i(t) dt - \int_0^{t_f} \sum_{i=1}^N c_i \eta_i(t) T_i(t) dt \quad (6)$$

We use quantity (6) to measure the influence of the innovative green product information. For brevity, let $\mathbf{a} = (a_1, \dots, a_N)$, $\mathbf{b} = (b_1, \dots, b_N)$, $\mathbf{c} = (c_1, \dots, c_N)$.

2.4. Modeling of the AP Strategies

Based on the previous discussions, the AP strategies are modeled as the following optimal control problem.

$$\begin{aligned} \text{Max } J(\mathbf{w}) &= \int_0^{t_f} F(\mathbf{E}(t), \mathbf{w}(t)) dt \\ \text{subject to } \begin{cases} \frac{d\mathbf{E}(t)}{dt} = \int_0^{t_f} F(\mathbf{E}(t), \mathbf{w}(t)) dt, & 0 \leq t \leq t_f, \\ \mathbf{E}(0) = \mathbf{x}_0. \end{cases} \end{aligned} \quad (7)$$

Here, $F(\mathbf{E}(t), \mathbf{w}(t)) = \sum_{i=1}^N a_i I_i(t) - \sum_{i=1}^N b_i \theta_i(t) P_i(t) - \sum_{i=1}^N c_i \eta_i(t) T_i(t)$. We refer to the optimal control problem as the AP strategies. In this model, the control stands for AP

strategies, the objective functional stands for the influence of the innovative green product information, and the optimal control stands for optimal AP strategies.

The AP strategies (7) can be characterized as the following 10-tuple:

$$\Gamma = (t_f, \bar{\mathbf{u}}, \bar{\mathbf{v}}, \mathbf{B}^P, \mathbf{B}^T, \mathbf{C}^P, \mathbf{C}^I, \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{E}_0) \quad (8)$$

3. A Method for Solving the AP Strategies

In this section, we determine the sufficient conditions for the existence of the proposed optimal control problem by using the calculus of variations [39]. Now, we obtain the optimal conditions for the control problem by the classical Pontryagin maximum principle [40].

3.1. The Existence of an Optimal Control

According to market changes, innovative green products will have different dissemination intensities in propagation. With the change in the communication intensity, we implement advertisement and promotion strategies to control them and continuously optimize the influence. Optimal control is the advertisement and promotion strategies with the lowest economic cost and the largest number of adopters of innovative green products. To show the AP strategies (8) admit an optimal control, we need to introduce the following lemma [39].

Lemma 1. $(c_1)W$ is bounded and convex.

(c_2) There is $\mathbf{w} \in W$ such that the corresponding state evolution model is solvable.

$(c_3)F(\mathbf{E}, \mathbf{w})$ is bounded by a linear function in $\mathbf{E}$.

$(c_4)F(\mathbf{E}, \mathbf{w})$ is convex on W .

$(c_5)F(\mathbf{E}, \mathbf{w}) \leq d_1 \|\mathbf{w}\|^\rho + d_2$ for some $\rho > 1, d_1 > 0$ and d_2 .

We are ready to show the following result.

Theorem 1. The AP strategies admit an optimal control.

Proof. The proof of this lemma is given in Appendix A. $\square$

3.2. A Necessary Condition for the Optimal Control

The Hamiltonian of the AP strategies is

$$\begin{aligned} H(\mathbf{x}(t), \mathbf{w}(t), \mathbf{z}(t)) = & \sum_{i=1}^N a_i I_i(t) - \sum_{i=1}^N b_i \theta_i(t) P_i(t) - \sum_{i=1}^N c_i \eta_i(t) T_i(t) \\ & + \sum_{i=1}^N \lambda_i(t) [-P_i(t) \sum_{j=1}^N \beta_{ij}^P I_j(t) - P_i(t) \sum_{j=1}^N \alpha_{ij}^P T_j(t) - \theta_i(t) P_i(t) - \gamma P_i(t)] \\ & + \sum_{i=1}^N \mu_i(t) [P_i(t) \sum_{j=1}^N \beta_{ij}^P I_j(t) + \theta_i(t) P_i(t) + T_i(t) \sum_{j=1}^N \beta_{ij}^T I_j(t) - I_i(t) \sum_{j=1}^N \alpha_{ij}^I T_j - \delta I_i(t) + \eta_i(t) T_i(t)] \\ & + \sum_{i=1}^N v_i(t) [P_i(t) \sum_{j=1}^N \alpha_{ij}^P T_j(t) + \gamma P_i(t) + I_i(t) \sum_{j=1}^N \alpha_{ij}^I T_j - T_i(t) \sum_{j=1}^N \beta_{ij}^T I_j(t) + \delta I_i(t) - \eta_i(t) T_i(t)] \end{aligned} \quad (9)$$

where $\mathbf{z}(t) = (\lambda_1(t), \dots, \lambda_N(t), \mu_1(t), \dots, \mu_N(t), v_1(t), \dots, v_N(t))$ is the adjoint.

We give a necessary condition for the optimal control of the AP strategies as follows.

Theorem 2. Suppose $\mathbf{w}$ is an optimal control of AP strategies (8), and $\mathbf{E}$ is the solution to the corresponding model [41]. Then, there exists $\mathbf{z}$ such that

$$\left\{ \begin{array}{l} \frac{d\lambda_i(t)}{dt} = b_i\theta_i(t) - \lambda_i(t)\theta_i(t) + \lambda_i(t)\gamma - \mu_i(t)\theta_i(t) + v_i(t)\gamma \\ \quad + [\lambda_i(t) - \mu_i(t)] \sum_{j=1}^N \beta_{ij}^P I_j(t) + \sum_{j=1}^N \alpha_{ij}^P [\lambda_i(t) - v_i(t)] T_j(t), \\ \frac{d\mu_i(t)}{dt} = -a_i + v_i(t)\delta - \mu_i(t)\delta + [\mu_i(t) - v_i(t)] \sum_{i=1}^N \alpha_{ij}^I T_j \\ \quad + \sum_{j=1}^N \beta_{ij}^T [v_i(t) - \mu_i(t)] T_j(t) + \sum_{j=1}^N \beta_{ij}^P [\lambda_i(t) - \mu_i(t)] P_j(t), \\ \frac{dv_i(t)}{dt} = -c_i\eta_i(t) + v_i(t)\eta_i(t) - \mu_i(t)\eta_i(t) + \sum_{j=1}^N \alpha_{ij}^I [\mu_i(t) - v_i(t)] I_j \\ \quad + \sum_{j=1}^N \alpha_{ij}^P [\lambda_i(t) - v_i(t)] P_j(t) + [v_i(t) - \mu_i(t)] \sum_{j=1}^N \beta_{ij}^T I_j(t). \end{array} \right. \quad (10)$$

with $\mathbf{z}(T) = 0$. Moreover,

$$\theta_i(t) = \begin{cases} \theta_i & \text{if } \mu_i < \lambda_i(t) - b_i, P_i(t) > 0 \\ \bar{\theta}_i & \text{if } \mu_i > \lambda_i(t) - b_i, P_i(t) > 0 \end{cases} \quad (11)$$

$$\eta_i(t) = \begin{cases} \eta_i & \text{if } \mu_i(t) > v_i - c_i, T_i(t) > 0 \\ \bar{\eta}_i & \text{if } \mu_i(t) < v_i - c_i, T_i(t) < 0 \end{cases} \quad (12)$$

Proof. The proof of this theorem is given in Appendix B. $\square$

3.3. The Optimal System for the AP Strategies

In the maximization case, when there are no state space constraints, it showed that the Pontryagin necessary conditions plus concavity of the Hamiltonian function with respect to the state and control variables were sufficient and optimal [42,43].

Theorem 3. Consider the problem

Max $J(\mathbf{w})$ when $f(\mathbf{w}) \geq 0$ and $\mathbf{w} \in W$.

Let $W = \{\mathbf{w} : f(\mathbf{w}) \geq 0\}$. Assume that

(d₁) J has a subgradient Q at $\bar{\mathbf{w}}$ w.r.t. the set W .

(d₂) f has a subquasigradient K at $\bar{\mathbf{w}}$ w.r.t the set W .

(d₃) For some $1 \times N$ vector $\mathbf{z} \geq 0$, $(Q + \mathbf{z} \cdot K) \cdot (\mathbf{w} - \bar{\mathbf{w}}) \leq 0$ for all $\mathbf{w} \in W$.

(d₄) $\mathbf{z} \cdot f(\bar{\mathbf{w}}) = 0$.

(d₅) $f(\bar{\mathbf{w}}) \geq 0$

Then, $\bar{\mathbf{w}}$ solves problem Max $J(\mathbf{w})$ when $f(\mathbf{w}) \geq 0$ and $\mathbf{w} \in W$.

Proof. The proof of this theorem is given in Appendix C. $\square$

Based on the above discussions, Equations (2) and (10)–(12) plus $\mathbf{z}(T) = 0$ constitute the optimal system for the AP strategies (8). By using the well-known forward–backward Euler scheme in the optimal system, we get an optimal control of the model, which stands for optimal AP strategies.

Remark 5. It follows from Theorem 2 that every component of the optimal control of the AP strategies is bang-bang control. In practice, this kind of optimal control is easily implementable.

Remark 6. The optimal control obtained by solving the optimal system of AP strategies is open loop, so it lacks flexibility. In practice, a longer time interval can be flexibly divided into several

shorter time intervals, and then the proposed method can be applied to each time interval to further improve the performance of the proposed control strategies.

4. Numerical Simulation

4.1. Without Optimal Control

In this subsection, we show the number of adopters of innovative green product information without optimal control based on the dynamic system (1), where θ and η are constants.

In order to further study the changes and trends of the adopters of innovative green product information and the adopters of traditional product information in the model, simulation analysis is conducted in this section. The initial parameters are set as follows: It is assumed that there are 10,000 people in the network, and the initial number of the various types of people is set as follows. Potential customers are 8000, the adopters of innovative green product information are 300, the adopters of traditional product information are 1700. Consider the AP strategies (8) with the following parameters: $\beta_{ij}^P = 0.0000015$, $\beta_{ij}^T = 0.000001$, $\alpha_{ij}^P = 0.000001$, $\alpha_{ij}^T = 0.000001$, $\delta = 0.03$, $\eta = 0.01$ [44]. The relevant parameter values in this work are directly given constants to simulate the real situation as much as possible. Therefore, it has certain limitations. However, it can provide reference for further refining and fitting to calculate the real transformation probability function or transformation value [20].

Figure 2 shows the relationship between the number of adopters of innovative green product information and the probability of advertisement propagation of innovative green products. Under the condition that other parameters remain unchanged, the probability of advertisement propagation increases, and the number of adopters of innovative green product information increases accordingly. From the perspective of advertisement strategies, initially, potential consumers choose to purchase innovative green products, and the number of adopters of innovative green product information increases; as innovative green products gain a certain market position, the number of adopters of innovative green product information gradually stabilizes. In short, when companies increase the efficiency of the initial propaganda and the advertisement intensity of innovative green products, potential consumers are more likely to receive innovative green product information; thus, the probability of purchasing innovative green products is higher.

Figure 2. Time evolution of the number of I with θ .

Figure 3 shows the relationship between the number of adopters of innovative green product information and the probability of promotion propagation of innovative green product information. Under the condition that other parameters remain unchanged, the probability of promotion propagation increases, and the number of adopters of innovative green product information increases accordingly. From the perspective of promotion strategies, initially, the adopters of traditional product information receive innovative

green product information and choose to purchase innovative green products, and the number of adopters of innovative green product information increases, and then the number of adopters of innovative green product information tends to be stable. In short, when companies focus on improving the effectiveness of promotion strategies, the adopters of traditional product information increase their willingness to purchase innovative green products; therefore, more and more often, the adopters of traditional product information choose to purchase innovative green products.

Figure 3. Time evolution of the number of I with η .

4.2. With Optimal Control

From the previous subsection, it was found that parameters θ and η have a great influence on innovative green products. Considering the cost, it is necessary to increase their strength within a reasonable range. Therefore, it is further considered to set parameters as dynamically variable and seek the optimal market competition strategies for innovative green products under the consideration of cost.

In this subsection, we give some optimal AP strategies by solving the corresponding systems. Assume that the parameters in the social network are as follows: $\gamma = 0.01$, $\delta = 0.03$, $\beta^P = 0.0000015$, $\beta^T = 0.000001$, $\alpha^P = 0.000001$, $\alpha^I = 0.000001$. The number of direct conversion units of potential consumers who become the adopters of innovative green product information under the effect of advertisement is $4(a = 4)$, the unit cost of advertisement is $2(b = 2)$, and the unit cost of promotion is $1.5(c = 1.5)$.

Figure 4 shows a graph of the number of adopters of innovative green product information under controlled (solid line) and uncontrolled (dashed line) conditions. Obviously, when there is control (solid line), the number of adopters of innovative green product information increases faster because when companies implement AP strategies, consumers can obtain product information from many aspects and thus have a greater chance to choose to purchase innovative green products. Conversely, when there is no control (dashed line), consumers cannot effectively obtain product information, and the rate of conversion into adopters of innovative green product information is slow, and the time to reach a stable state is longer. In short, when companies implement advertisement strategies and consider promotion strategies at the same time, they can improve the level of product marketing information and thus increase the market share of innovative green products.

Figure 5 shows the dynamic changes in the control variables $\theta(t)$ and $\eta(t)$. We observe the dynamic change in advertisement and promotion over time period t and focus on 44 advertisement and promotion strategy schemes. Advertisement strategies play a leading role in the initial propagation of the influence of innovative green products, advertisement is not effective enough over time, and the probability of potential consumers turning into the adopters of innovative green product information decreases; the effect of advertisement is not good enough, and the promotion strategies gradually dominate, so the probability of promotion propagation drops first and then rises. Therefore, enterprises

should pay attention to the intensity and timing of the implementation of advertisement and promotion strategies.

Figure 4. Time evolution of the number of I with and without control.

Figure 5. Control variables $\theta(t)$ and $\eta(t)$.

The basic assumption of the genetic algorithm is to find out the maximum income (the greatest influence of innovative green products), that is, through the implementation of advertisement and promotion strategies, it is concluded that the impact of innovative green products is the largest income. In this paper, we set the fitness function as $g_w = 1/J(w)$. The genetic algorithm process is shown in Table 1.

The target value of the AP strategies involve the number of conversions and the cost. Enterprises expand the influence of innovative green product information propagation by allocating the corresponding advertisement and promotion so that advertisement and promotion strategies become the optimal problem of resource allocation. This paper carries out plan allocation, and different allocation plans will get different target values; thus, the goal of this article is to find the optimal scheme with the largest target value (the largest number of adopters of innovative green product information) in the allocation scheme through the algorithm. In this paper, multiple groups of allocation schemes are selected and the optimal allocation scheme is found by using genetic algorithms. Suppose the distribution scheme $L_1(0.01, 0.05)$, and then execute the algorithm to calculate the optimal solution J_1 of the scheme L_1 , where $L_1 \sim L_N$ corresponds to $J_1 \sim J_N$.

Table 2 shows the target values of the 44 allocation schemes that we focused on in Figure 5 and a scheme without advertisement and promotion strategies. We find that the target value under control is greater than that without control. This is consistent with real market conditions. Figure 6 indicates that different schemes assign corresponding target values. It contains 44AP strategies and allocation schemes without an AP strategy and target values. As can be seen from the figure, the target value of the scheme without an AP

strategy is the smallest, and the allocation scheme with the largest target value is (0.01,0.05), and the target value is 4,375,597.

Table 1. Genetic algorithm process.

Step 1	Initialization parameters: including advertising promotion strategy allocation scheme, the number of three groups, other parameters, and crossover probability and mutation probability;
Step 1.1	Population size: for example, (0.01, 0.05) represents one of the initial advertisement and promotion allocation schemes;
Step 1.2	The number of three groups: 10,000 people on social networks. Potential customers are 8000, the adopters of innovative green products are 300, the adopters of traditional product are 1700;
Step 1.3	Other parameters: $\alpha_{ij}^p = 0.000001$, $\alpha_{ij}^l = 0.000001$, $\delta = 0.03$, $\eta = 0.01$. Crossover probability $m = 0.95$, mutation probability: $n = 0.05$.
Step 2	Calculation of fitness value: The fitness can be used to measure the quality of the results. The goal of this model is to find the greatest influence of innovative green product information, and the smaller the fitness value, the better. The fitness function is defined as $g_w = 1/J(w)$. Therefore, when the value of g_w is smaller, the result is optimal.
Step 3	Take the roulette way to choose: after calculating the fitness function value, sort these values from good to bad.
Step 4	Repeat the following steps when conditions are not met: Select replication. Selection and replication are to preserve good individuals, remove bad individuals, and then pass these good individuals to the next generation, and replication operations are based on individual fitness calculations.
Step 5	Two pairs, cross, produce new advertisement and promotion strategies: The same location of the two strategies' exchange (also means the transformation of information) can produce new advertisement and promotion strategies. The purpose of this process is to produce new advertisement and promotion strategies. The method used is mainly two-point cross. The random two cross advertisement and promotion strategies' schemes are exchanged for data, and the strategy schemes in other locations do not change.
Step 6	Mutation operation: Based on the mutation mechanism in genetics, some individuals in the population are numerically changed to change the fitness and see whether the results are optimal. Variation probability set at 0.05.
Step 7	Repeat the above steps, once the conditions required for stopping are met, the system can output the minimum fitness value.

Figure 6. Target values for different scenarios.

Table 2. Scheme allocation and target value under advertisement–promotion strategies.

PLAN	J(W)	PLAN	J(W)	PLAN	J(W)
(0.0420, 0.0180)	345,686	(0.0580, 0.0020)	133,899	(0.0370, 0.0230)	1,766,845
(0.0350, 0.0250)	431,575	(0.0515, 0.0085)	228,767	(0.0545, 0.0055)	4,350,176
(0.0280, 0.0320)	530,370	(0.0470, 0.0130)	285,993	(0.0490, 0.0110)	260,945
(0.0210, 0.0390)	657,621	(0.0340, 0.0260)	371,660	(0.0450, 0.0150)	309,877
(0.0150, 0.0450)	809,742	(0.0260, 0.0340)	562,880	(0.0400, 0.0200)	369,626
(0.0080, 0.0520)	1,086,809	(0.0140, 0.0460)	841,074	(0.0300, 0.0300)	500,001
(0.0525, 0.0075)	215,379	(0.0110, 0.0490)	949,573	(0.0130, 0.0470)	874,641
(0.0505, 0.0095)	241,838	(0.0050, 0.0550)	1,266,275	(0.0065, 0.0535)	1,152,032
(0.0460, 0.0140)	297,838	(0.0550, 0.0050)	180,225	(0.0320, 0.0280)	471,550
(0.0390, 0.0210)	381,710	(0.0535, 0.0065)	201,631	(0.0230, 0.0370)	617,069
(0.0240, 0.0360)	598,203	(0.0600, 0.0100)	248,269	(0.0200, 0.0400)	679,497
(0.0170, 0.0430)	770,811	(0.0430, 0.0170)	333,768	(0.0180, 0.0420)	721,992
(0.0075, 0.0525)	1,113,317	(0.0360, 0.0240)	442,682	(0.0085, 0.0515)	1,061,453
(0.0020, 0.0580)	1,511,058	(0.0250, 0.0350)	644,081	(0.0100, 0.0500)	4,375,597
(0.0055, 0.0545)	99,895	(0.0095, 0.0505)	1,013,911	without AP strategy	46,271

5. Conclusions

Based on the innovative green product propagation model, this article has discussed how to establish *AP* strategies to maximize the influence of innovative green product information. In terms of theoretical contributions, firstly, this article has considered the competition between traditional product information and innovative green product information, has proposed an innovative green product propagation model, and has introduced advertisement and promotion strategies to expand the influence of innovative green product information propagation. Secondly, in order to expand the influence of the propagation of innovative green product information, by introducing two control strategies of advertisement and promotion, we have proposed an optimal control problem by maximizing the number of adopters of innovative green products. Finally, the realization of different strategies has been studied through algorithms to obtain the optimal target value. The analysis results of the model show that in the process of product competition, advertisement strategies have promoted the information propagation of innovative green products, and promotion strategies have further increased the market share of innovative green products. In terms of implications for strategy managers, this study provides directions to a growing number of firms that are embracing environmental sustainability as part of their strategies and have started or are planning to implement advertisement and promotion strategies. On the one hand, when the company promoted innovative green products, the early intervention of advertisement, the improvement in the network coverage, and the promotion of multiple channels were conducive to the increase in user traffic and the propagation of information; on the other hand, while increasing the propaganda of innovative green products, we also needed to pay attention to the importance of word-of-mouth marketing to further increase the conversion rate. In the fierce market competition environment, this research can provide decision support for green product marketing strategies. In terms of implications for policy makers, this study suggests that public policy can improve environmental quality. For example, it can arouse people's awareness of environmental protection. In addition, public policies can work through joint country-to-country cooperation projects to address climate change and greenhouse gas reduction.

We duly acknowledge potential limitations of this study. We use fixed parameters to simulate the propagation process of innovative green products. Collecting real data and refining and fitting to calculate the true transformation probability function or transformation value are the next research directions. The design method of the dynamic optimal control law is given by using the Pontryagin maximum principle. There are some gaps and limitations in applying the research results to practice. Towards this direction, there are some topics of research that merit additional study. First, empirical analysis to increase data credibility should be carried out. Second, some factors such as the impact of time-lapse risks, psychology, and marketing on the model can be incorporated in this study. Furthermore, it is worthwhile to extend this work to situations in which sensitivity

analysis must be considered to deal with the impact of uncertain environmental changes on the sensitivity of the decision-making process. We hope that our work stimulates further theoretical refinement and empirical investigation in this important area of research.

Author Contributions: All authors developed and tested the presented models and methodologies. L.H.: conceptualization, methodology, supervision. Q.W.: writing—original draft, software. T.L.: visualization, investigation. H.H.: visualization, validation. All authors have read and agreed to the published version of the manuscript.

Funding: This work was partially supported by The Program for Professor of Special Appointment (Eastern Scholar) at Shanghai Institutions of Higher Learning, and the National Natural Science Foundation of China (71774111, 61702331, 71804047).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: No new data were created or analyzed in this study. Data sharing is not applicable to this article.

Conflicts of Interest: The authors declare no conflict of interest.

Appendix A. Proof of Theorem 1

Proof. Let $\mathbf{w} = (\theta_1, \dots, \theta_N, \eta_1, \dots, \eta_N)$ be a limit point of ω . Then, there exists a sequence of points of $\mathbf{w}$, $\mathbf{w}^{(n)} = (\theta_1^{(n)}, \dots, \theta_N^{(n)}, \eta_1^{(n)}, \dots, \eta_N^{(n)})$, $n = 1, 2, \dots$, that approaches ω . As $L[0, t_f]^{2N}$ is complete, we have $\mathbf{w} \in L[0, t_f]^{2N}$. Therefore, $\mathbf{w} \in \omega$ follows from the observation that $\theta_i = \lim_{n \rightarrow \infty} \theta_i^{(n)}(t) \leq \bar{\theta}_i$, $\eta_i = \lim_{n \rightarrow \infty} \eta_i^{(n)}(t) \leq \bar{\eta}_i$, $0 \leq t \leq t_f$, $1 \leq i \leq N$.

Hence, ω is closed. Let $\mathbf{w}^{(1)}, \mathbf{w}^{(2)} \in \omega$, $0 < \sigma < 1$. As $L[0, t_f]^{2N}$ is a real vector space, we have $(1 - \sigma)\mathbf{w}^{(1)} + \sigma\mathbf{w}^{(2)} \in L[0, t_f]^{2N}$. Therefore, $(1 - \sigma)\mathbf{w}^{(1)} + \sigma\mathbf{w}^{(2)} \in \omega$ follow from the observation that

$$(1 - \sigma)\theta_i^{(1)} + \sigma\theta_i^{(2)} \in \bar{\theta}_i, (1 - \sigma)\eta_i^{(1)} + \sigma\eta_i^{(2)} \in \bar{\eta}_i, 0 \leq t \leq t_f, 1 \leq i \leq N$$

Hence, ω is convex. The first condition in Lemma 1 is proven. Let $\bar{\mathbf{w}} = (\bar{\theta}_1, \dots, \bar{\theta}_N, \bar{\eta}_1, \dots, \bar{\eta}_N)$. Then, $\bar{\mathbf{w}} \in \omega$. As $\mathbf{f}(\mathbf{E}, \bar{\mathbf{w}})$ is continuously differentiable, it follows by the continuation theorem for differential systems that the corresponding state evolution state is solvable. The second condition is proven. The third condition follows from the boundedness of P_i, I_i, T_i and $\mathbf{w}$, and the fourth condition follows that F is linear in $\mathbf{w}$ and hence is convex. The fifth condition follows from the observation that

$$F(\mathbf{E}, \mathbf{w}) \leq \|\mathbf{w}\|_2^2 - \sum_{i=1}^N (\bar{\theta}_i^2 + \bar{\eta}_i^2).$$

By Lemma 1, the proposition holds. $\square$

Appendix B. Proof of Theorem 2

Proof. It follows from the Pontryagin principle [40] that there exists $\mathbf{z}$, where

$$\begin{cases} \frac{d\lambda_i(t)}{dt} = -\frac{\partial H(\mathbf{x}(t), \mathbf{w}(t), \mathbf{z}(t))}{\partial P_i(t)}, & 0 \leq t \leq t_f, 1 \leq i \leq N, \\ \frac{d\mu_i(t)}{dt} = -\frac{\partial H(\mathbf{x}(t), \mathbf{w}(t), \mathbf{z}(t))}{\partial I_i(t)}, & 0 \leq t \leq t_f, 1 \leq i \leq N, \\ \frac{dv_i(t)}{dt} = -\frac{\partial H(\mathbf{x}(t), \mathbf{w}(t), \mathbf{z}(t))}{\partial T_i(t)}, & 0 \leq t \leq t_f, 1 \leq i \leq N. \end{cases}$$

Equation (9) follows by direct calculations. As the terminal cost is unspecified, and the final state is free, the transversality condition $\mathbf{z}(H) = 0$ holds. Again, by the Pon-

tryagin maximum principle [40], we have $H(\mathbf{x}(t), \mathbf{w}(t), \mathbf{z}(t)) = \arg\max_{\mathbf{w} \in W} H(\mathbf{x}(t), \tilde{\mathbf{w}}(t), \mathbf{z}(t))$, $0 \leq t \leq t_f$. Equations (10) and (11) follow by direct calculations. $\square$

Appendix C. Proof of Theorem 3

Proof. Let $\mathbf{w}$ be a point of A . A $N \times N$ matrix Q is called a subgradient at $\mathbf{w}$ for the vector function F w.r.t the set A provided

$$F(\mathbf{E}, \mathbf{w}) - F(\mathbf{E}, \bar{\mathbf{w}}) \leq F \cdot (\mathbf{w} - \bar{\mathbf{w}}) \text{ for all } \mathbf{w} \in A$$

In the same setting, a $N \times N$ matrix K is called a subquasigradient at F w.r.t. the set A provided

$$F(\mathbf{E}, \mathbf{w}) \geq F(\mathbf{E}, \bar{\mathbf{w}}) \Rightarrow K \cdot (\mathbf{w} - \bar{\mathbf{w}}) \geq 0 \text{ for all } \mathbf{w} \in A$$

Let $Y = \{i : f_i(\bar{\mathbf{w}}) = 0\}$. Since $z_i \geq 0$ and $f_i(\bar{\mathbf{w}}) \geq 0$ for all i . By (d_4) , letting k_i denote the i -th row vector K , we get

$$f_i(\mathbf{w}) \geq f_i(\bar{\mathbf{w}}) \Rightarrow k_i \cdot (\mathbf{w} - \bar{\mathbf{w}}) \geq 0 \text{ for all } \mathbf{w} \in W$$

When $i \in Y$, $f_i(\bar{\mathbf{w}}) = 0$, so for all $\mathbf{w} \in W$, $k_i(\mathbf{w} - \bar{\mathbf{w}}) \geq 0$, and therefore

$$z_i k_i \cdot (\mathbf{w} - \bar{\mathbf{w}}) \geq 0 \text{ for all } i \in Y \text{ and } \mathbf{w} \in W$$

Since $z_i = 0$ for $i \notin Y$, $z_i k_i \cdot (\mathbf{w} - \bar{\mathbf{w}}) \geq 0$ actually holds for all $i = 1, \dots, N$. Summing the inequality $z_i k_i \cdot (\mathbf{w} - \bar{\mathbf{w}}) \geq 0$ from $i = 1$ to $i = N$, we obtain $z \cdot K \cdot (\mathbf{w} - \bar{\mathbf{w}}) \geq 0$ for all $\mathbf{w} \in W$.

By applying (d_3) , it follows that $Q \cdot (\mathbf{w} - \bar{\mathbf{w}}) \leq 0$ for all $\mathbf{w} \in W$. This inequality together with (d_1) gives us $f_i(\mathbf{w}) - f_i(\bar{\mathbf{w}}) \leq Q \cdot (\mathbf{w} - \bar{\mathbf{w}}) \leq 0$ for all $\mathbf{w} \in W$. Hence, $f(\mathbf{w}) \leq f(\bar{\mathbf{w}})$ for all $\mathbf{w} \in W$, so $\bar{\mathbf{w}}$ solves the problem [43]. $\square$

References

1. Dai, R.; Zhang, J. Green process innovation and differentiated pricing strategies with environmental concerns of South-North markets. *Transp. Res. Part E Logist. Transp. Rev.* **2017**, *98*, 132–150. [CrossRef]
2. Gao, J.; Xiao, Z.; Cao, B.; Chai, Q. Green supply chain planning considering consumer's transportation process. *Transp. Res. Part E Logist. Transp. Rev.* **2018**, *109*, 311–330. [CrossRef]
3. Dong, C.; Liu, Q.; Shen, B. To be or not to be green? Strategic investment for green product development in a supply chain. *Transp. Res. Part E Logist. Transp. Rev.* **2019**, *131*, 193–227. [CrossRef]
4. Pujari, D.D. Mainstreaming Green Product Innovation: Why and How Companies Integrate Environmental Sustainability. *J. Bus. Ethics* **2010**, *95*, 471–480.
5. Dangelico, R.M.; Pujari, D.; Pontrandolfo, P. Green Product Innovation in Manufacturing Firms: A Sustainability—Oriented Dynamic Capability Perspective. *Bus. Strategy Environ.* **2017**, *26*, 490–506. [CrossRef]
6. Asadi, S.; Pourhashemi, S.O.; Nilashi, M.; Abdullah, R.; Samad, S.; Yadegaridehkordi, E.; Aljojo, N.; Razali, N.S. Investigating influence of green innovation on sustainability performance: A case on Malaysian hotel industry. *J. Clean. Prod.* **2020**, *258*, 1–15. [CrossRef]
7. Awan, U.; Sroufe, R.; Kraslawski, A. Creativity enables sustainable development: Supplier engagement as a boundary condition for the positive effect on green innovation. *J. Clean. Prod.* **2019**, *226*, 172–185. [CrossRef]
8. Figueroa-Perez, J.F.; Leyva-Lopez, J.C.; Santillan, L.C.; Contreras, E.O.P.; Sánchez, P.J. The Use of Marketing Decision Support Systems for New Product Design: A Review. *Atlantis Press SARL* **2019**, *12*, 761–774. [CrossRef]
9. Dawood, A.; Shahrokh, V. Diversified viral marketing: The power of sharing over multiple online social networks. *Knowl. Based Syst.* **2019**, *193*, 105430.
10. De Souza, R.C.; Figueiredo, D.R.; Rocha, A.D.A.; Ziviani, A. Efficient network seeding under variable node cost and limited budget for social networks. *Inf. Sci.* **2020**, *514*, 369–384. [CrossRef]
11. Robles, J.F.; Chica, M.; Cordon, O. Evolutionary multi-objective optimization to target social network influentials in viral marketing. *Expert Syst. Appl.* **2020**, *147*, 113183. [CrossRef]
12. Daley, D.J.; Kendall, D.G. Epidemics and rumors. *Nature* **1964**, *204*, 1118. [CrossRef]
13. Maki, D.P.; Thompson, M. *Mathematical Models and Applications: With Emphasis on the Social, Life, and Management Sciences*; Prentice-Hall: Englewood Cliffs, NJ, USA, 1973.
14. Watts, D.J.; Strogatz, S.H. Collective dynamics of 'small-world' networks. *Nature* **1998**, *393*, 440–442. [CrossRef] [PubMed]

15. Barabasi, A.L.; Albert, R. Emergence of scaling in random networks. *Science* **1999**, *286*, 509–512. [[CrossRef](#)]
16. Rogers, E.M. *Diffusion of Innovations*; Free Press: New York, NY, USA, 1983.
17. Bass, F.M. Comments on a new product growth for model consumer durables. *Manag. Sci.* **2004**, *50*, 1833–1841. [[CrossRef](#)]
18. Lancaster, G.A.; Wright, G. Forecasting the future of video using a diffusion model. *Eur. J. Mark.* **1983**, *17*, 70–79. [[CrossRef](#)]
19. Lilien, G.L.; Rangaswamy, A.; Van den Bulte, C. Diffusion models: Managerial application and software. *New Prod. Diffus. Models* **2000**, *3004*, 295–336.
20. Peres, R.; Muller, E.; Mahajan, V. Innovation diffusion and new product growth models: A critical review and research directions. *Int. J. Res. Mark.* **2010**, *27*, 91–106. [[CrossRef](#)]
21. Robinson, B.; Lakhani, C. Dynamic Price Models for New-Product Planning. *Manag. Sci.* **1975**, *21*, 1113–1122. [[CrossRef](#)]
22. Kalish, S. A new product adoption model with price, advertisement and uncertainty. *Manag. Sci.* **1985**, *31*, 1569–1585. [[CrossRef](#)]
23. Easingwood, C.; Mahajan, V.; Muller, E. A nonuniform influence innovation diffusion model of new product acceptance. *Mark. Sci.* **1983**, *2*, 273–295. [[CrossRef](#)]
24. Li, G.; Wang, X.; Su, S.; Su, Y. How green technological innovation ability influences enterprise competitiveness. *Technol. Soc.* **2019**, *59*, 101136. [[CrossRef](#)]
25. Benson, V.; Ezingard, J.N.; Hand, C. An empirical study of purchase behaviour on social platforms: The role of risk, beliefs and characteristics. *Inf. Technol. People* **2019**, *32*, 876–896. [[CrossRef](#)]
26. Briffett, P. The psychology of the social shopper. *J. Digit. Soc. Media Mark.* **2014**, *2*, 64–69.
27. Ahmed, W.; Zhang, Q. Green purchase intention: Effects of electronic service quality and customer green psychology. *J. Clean. Prod.* **2020**, *267*, 122053. [[CrossRef](#)]
28. Heidenreich, S.; Spieth, P.; Petschnig, M. Ready, Steady, Green: Examining the Effectiveness of External Policies to Enhance the Adoption of Eco-Friendly Innovations. *J. Prod. Innov. Manag.* **2017**, *34*, 343–359. [[CrossRef](#)]
29. Leonidou, L.C.; Hultman, M. Evaluating the green advertising practices of international firms: A trend analysis. *Int. Mark. Rev.* **2011**, *28*, 6–33. [[CrossRef](#)]
30. Mishra, A.; Mishra, H. The Influence of Price Discount Versus Bonus Pack on the Preference for virtue and vice foods. *J. Mark. Res.* **2011**, *48*, 196–206. [[CrossRef](#)]
31. Notron, J.A.; Bass, F.M. Evolution of Technological Generations: The Law of Capture. *Sloan Manag. Rev.* **1992**, *33*, 66–77.
32. Chitra, K. In search of the green consumers: A perceptual study. *J. Serv. Res.* **2007**, *7*, 173–191.
33. Ki-Hoon, L.; Ji-Whan, K. Green new product development and supplier involvement: Strategic partnership for green innovation. *Int. J. Innov. Sustain. Dev.* **2012**, *6*, 290–304.
34. Libai, B.; Muller, E.; Peres, R. The role of within-brand and cross-brand communications in competitive growth. *J. Mark.* **2009**, *73*, 19–34. [[CrossRef](#)]
35. Schramm, M.E.; Trainor, K.J.; Shanker, M.; Hu, M.Y. An agent-based diffusion model with consumer and brand agents. *Decis. Support Syst.* **2010**, *50*, 234–242. [[CrossRef](#)]
36. Tashiro, T. Hierarchical Bass model: A product diffusions model considering a diversity of sensitivity to fashion. *Phys. A Stat. Mech. Its Appl.* **2016**, *461*, 824–832. [[CrossRef](#)]
37. Stewart, W.J. *Probability, Markov Chains, Queues, and Simulation: The Mathematical Basis of Performance Modeling*; Princeton University Press: Princeton, NJ, USA, 2009.
38. Stein, E.M.; Shakarchi, R. *Real Analysis: Measure Theory, Integration and Hilbert Spaces*; Princeton University Press: Princeton, NJ, USA, 2005.
39. Lenhart, S.; Workman, J.T. *Optimal Control Applied to Biological Models*; Chapman & Hall/CRC: London, UK, 2007.
40. Potryagin, L.S. *The Mathematical Theory of Optimal Processes*; CRC Press: Boca Raton, FL, USA; Taylor & Francis Group: Abingdon, UK, 1987.
41. Liberzon, D. *Calculus of Variations and Optimal Control Theory: A Concise Introduction*, 1st ed.; Princeton University Press: Princeton, NJ, USA, 2012.
42. Mangasarian, O.L. Sufficient Conditions for the Optimal Control of Nonlinear Systems. *Siam J. Control* **1966**, *4*, 83–93. [[CrossRef](#)]
43. Seierstad, A.; Sydsaeter, K. Sufficient Conditions in Optimal Control Theory. *Int. Econ. Rev.* **1977**, *18*, 367–391. [[CrossRef](#)]
44. Jiang, Z.; Deceased, F.B.; Bass, P. The Virtual Bass Model and the Left-Hand Data-Truncation Bias in Diffusion of Innovation Studies. *Soc. Sci. Electron. Publ.* **2006**, *23*, 93–106. [[CrossRef](#)]