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Concreteness Fading Strategy: A Promising and Sustainable Instructional Model in Mathematics Classrooms

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Abstract: Conceptual understanding has been emphasized in the national curriculum and principles and standards across nations as it is the key in mathematical learning. However, mathematics instruction in classrooms often relies on rote memorization of mathematical rules and formulae without conceptual connections. This study considers the *concreteness fading instruction strategy*—starting with physical activities with manipulatives and gradually fading concreteness to access abstract concepts and representations—as a promising and sustainable instructional model for supporting students in accessing conceptual understanding in mathematics classrooms. The results from the case study support the validity of the concreteness fading framework in providing specific instructional strategies in each phase of concept development. This study implies the development of sustainable teacher education and professional development by providing specific instructional strategies for conceptual understanding.

Keywords: Bruner; conceptual understanding; concreteness fading strategy; high school mathematics

1. Introduction

While a reform of how mathematics is taught in school is a worldwide issue, the implementation of such a reform in U.S. schools has been advocated by the Principles and Standards for School Mathematics [1], and more recently by the Common Core State Standards for Mathematics (CCSS-M) [2]. These standards emphasize the importance of mathematics by learning aspects such as conceptual understanding, communication, and productive disposition in teaching and learning mathematics. While national curricular in some countries and mathematics instruction have influenced CCSS-M in the U.S., the U.S. reform in school mathematics has also influenced mathematics education in other countries. This educational reform movement in such countries has made a change in mathematics textbooks and has recommended changing teaching practices in mathematics classrooms. However, mathematics teachers often face challenges in employing these principles and standards in their classrooms as they are too generally written or advocated, and they should be supported by explicit instructional guidance in how to do so.

In order to support teachers and students, mathematics textbooks have been revised and a variety of contextually rich problems have been introduced. Features of the new approach include story-telling mathematics (i.e., contextualizing the mathematical content and connecting it to history, visualization, music, design, and additional real-life situations) and problem solving. In this context, the significance of *mathematizing* from Realistic Mathematics Education (RME) [3] has been more emphasized. RME is a mathematical instructional theory that argues that students' mathematical knowledge gradually becomes more abstract and less context specific by doing mathematics in rich *realistic* situations [4]. In his method of didactical phenomenology, Freudenthal considers mathematics as a human activity, so students should engage in active *mathematizing* processes rather than be passive receivers of ready-made mathematics, which he views as an *anti-didactic inversion.*

Similarly, Bruner [5] proposes three forms for children's conceptual development: concrete, pictorial, and symbolic forms. While RME proposes six core principles for teaching mathematics (i.e., the activity principle, reality principle, level principle, intertwinement principle, interactivity principle, and guidance principle) [6], it is still hard for K–12 mathematics teachers to apply these principles in their instructional practices due to the lack of specificity in terms of application to K–12 situations. However, the concreteness fading technique recommended by Bruner [5] provides implications on specific instructional strategies that help students engage in mathematizing. Concreteness fading is a technique that is used to support students in gradually fading from manipulating concrete materials to representing abstract concepts in mathematical learning.

In this article, I argue that the concreteness fading strategy in the context of RME provides fruitful insights on teaching for mathematical thinking with active mathematizing processes. The case of a high school mathematics lesson was used to illustrate how the hybrid framework provides benefits both in theory and practice.

2. Theoretical Backgrounds

2.1. Concreteness Fading

Concreteness fading was originally proposed by Bruner [5]. According to Bruner, children's cognitive development is processed in three forms: (1) an enactive or concrete form, in which students develop mathematical concepts by manipulating concrete objectives physically; (2) an iconic form, in which they learn to represent a mathematical concept in a graphic or pictorial form; and (3) a symbolic form, in which they learn to represent a concept with an abstract model or symbols. For example, students can count three blocks or three apples physically (concrete) to represent the quantity "three", and the quantity three can then be represented by three dots in a graphic form or as graphic bars on a number line, which is still concrete but more abstract than the physical forms. Finally, the quantity "three" can be represented by a mathematical symbol, the numeral 3. In Korea, this concreteness fading is well known as Bruner's EIS (Enactive-Iconic-Symbolic) theory [6,7]. In early childhood education or special education in the U.S., it is also known as Bruner's concrete-representational-abstract (CRA) [8,9]. In Singapore, it is known as the concrete-pictorial-abstract (CPA) sequence, and it is nationally recommended as a key instructional strategy for the development of mathematical concepts [10]. To emphasize the techniques of gradual progression in a continuum from concreteness to abstractness (see Figure 1), I adopt the term, *concreteness fading* [5,11–13].

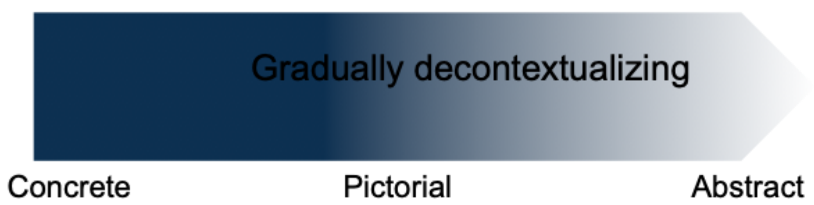


Figure 1. Continuum of concreteness fading.

The concreteness fading strategy helps not only in providing students with various types of representations of an abstract mathematical concept, but also in supporting them to make connections among those representations by gradually decontextualizing concreteness. There is a significant amount of research on the benefits for mathematical learning when generating and using multiple representations (e.g., [12–14]). However, students often fail to intellectually engage in explicit attention to key concepts by making connections between ideas, procedures, and various representations [15–17], especially when concrete manipulatives or visual representations are merely introduced without support to connect them to abstract representations. Students should be supported in making connections among the various representations toward the symbolic representations of key concepts. Thus, the concreteness fading strategy can support students in engaging in mathematizing and eventually in developing abstract concepts by generating multiple representations, comparing those representations, and analyzing representations to extract key mathematical concepts.

2.2. Theoretical Framework

The theoretical framework in this study was developed based on previous literature on concreteness fading [5,11]. Previous studies on RME [3,5,18,19] also contributed to our understanding of the importance of mathematizing and its process, and studies on concreteness fading suggest key techniques for learning mathematics using concrete materials, iconic, and symbolic representations. However, the core principles for teaching mathematizing did not provide specific strategies for teachers to apply and implement it in their practice. Studies on CRA or CPA such as Fyfe, McNeil, and Borjas [20], or Flores [21] provide evidence of how concrete and pictorial representations help students to develop mathematical concepts better; however, clear suggestions on how to provide the connections between phases (e.g., a connection between the concrete phase and the pictorial phase, or a connection between the pictorial phase and the symbolic phase) had not been provided until very recently. Thus, in this section, I use a theoretical framework that helps teachers to implement the ideas of mathematizing and concreteness fading in mathematical teaching and learning. The framework is shown in Figure 2 and the critical features of connections are additionally elaborated in the last part of this section.

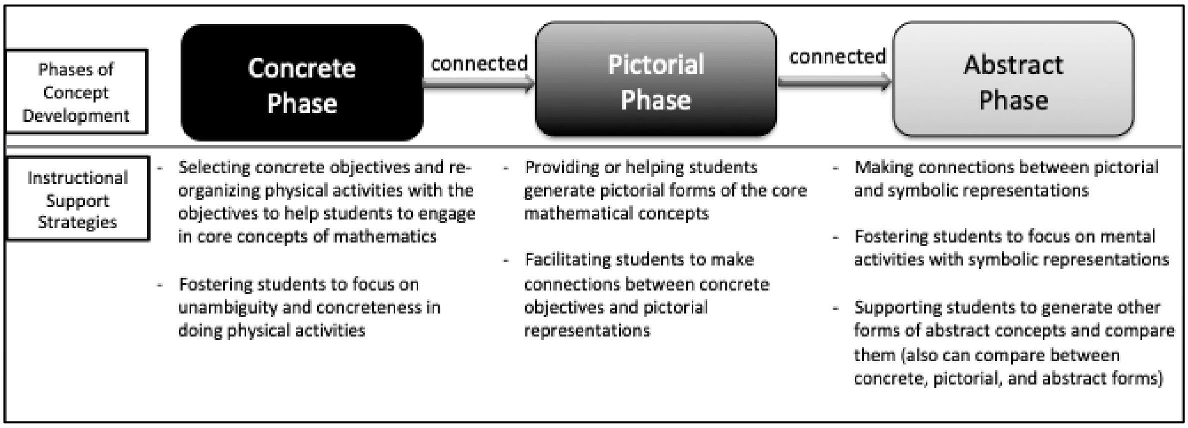


Figure 2. Concreteness fading strategies in the phases of concept development.

Concreteness fading strategies support students in developing mathematical concepts and in experiencing mathematizing, starting with concrete models and physical activities with concrete manipulatives in the enactive/concrete phase. In the enactive/concrete phase, teachers select and provide concrete objectives to students and support students in engaging with unambiguous and familiar concepts. These concrete activities with familiar objectives help students organize informal concepts that will then be connected to the core mathematical concepts later on.

In the iconic/pictorial phase, teachers can either encourage students to generate pictorial forms or provide visual representations, depending on the mathematical concepts. This pictorial phase is still concrete when compared with abstract forms; however, superficial features from the concrete phase are deleted so that it can be seen as a scaffold or a ladder for connection toward the abstract phase.

The last phase is the symbolic or abstract phase, which is an alternate goal for the development of mathematical concepts. In this phase, students finally have a schema of the concept so that they can manipulate these schematized concepts mentally.

The critical features of the framework of concrete fading are the suggestions of instructional strategies in connecting the stages between phases (e.g., a connection between the enactive and the iconic phase, or a connection between the iconic and the abstract phase). The instructional strategies for the connecting stage between the enactive and iconic phases are as follows:

(1) Teachers support students in interpreting their activities with manipulatives in mathematically meaningful ways (e.g., generating/providing graphical representations for concrete objectives); and
(2) Teachers delete superficial features of physical activities whenever they occur in the concrete phase.

By deleting unrelated and unimportant features of physical activities, students can gradually step into making connections between these physical activities and pictorial forms.

When students enter the iconic/pictorial phase, teachers support them by focusing on the core mathematical concepts with visual representations. In the connection stage between the iconic/pictorial phase and the symbolic/abstract phase, teachers can

(1) Introduce symbolic forms of mathematical concepts;
(2) Support students in interpreting their activities with visual representations in mathematically meaningful ways; and
(3) Delete superficial and unrelated features from activities with visual representations by highlighting the core structures of concepts.

By implementing these strategies, teachers can help students focus on the core structures of mathematical concepts, which is a stepping-stone to the symbolic phase of concept development.

Teachers help students make connections between pictorial and symbolic representations, and even connections with concrete activities when students finally enter into the symbolic/abstract phase. This is indeed helpful for students for understanding abstract concepts through interpreting unambiguous objects [22,23]. In this phase, it is also important to foster students in engaging in mental activities with symbolic representations by generating alternative symbolic forms and comparing these representations and forms [24].

3. Methods: The Empirical Case to Support Concreteness Fading Strategies

Empirical studies for Bruner's EIS or CRA model have increased, but they are mostly studies in early childhood mathematics or in elementary school level mathematics. Furthermore, the empirical supports have focused more on each phase of EIS or CRA in tutoring settings, rather than in classroom settings where teachers face more complicated real-life situations. In this article, I provide the case of a high school lesson with the topic of the *summation of finite sequences* to support the theoretical framework of the concreteness fading strategy. The summation of finite sequences is often considered to be one of the most difficult topics in high school because the derivation of formulae for the sums of different sequences is not easily understandable. Furthermore, the symbol $\sum$ is new to high school students following the Korean national mathematics curriculum.

3.1. Participants and Data Sources for the Case

The participant of this study was a high school mathematics teacher in Korea, Ms. K, with four years of experience in teaching middle school mathematics and three years of experience in teaching high school mathematics at the time of data collection. The videotaping and observation of her classroom was conducted in 2015. The high school was an all-female high school and was located in an urban school district in a metropolitan city in South Korea. The classroom consisted of 35 female students in second grade, 16–17 years old, at the same high school. Three cameras were used to collect classroom video data: one focused on the front of the classroom and was located at the back; another focused on the students as a whole group and was located at the front; and the third one followed the teacher whenever she led a group discussion, and focused on a small group of students while students worked in small groups. These classroom video data are the main data used in this study.

As supplementary data, I collected teacher interview video data asking questions about her lesson goals and reflections of her own teaching; video data for reflections and discussions among the observing teachers including myself, Ms. K, and other teachers who observed Ms. K's class; and the lesson plans that Ms. K had crafted.

3.2. Context of the Case

The teacher, Ms. K, taught the topic of "finding the generalized sum of finite sequences" and the class, particularly focused on the concepts of $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$ and $\sum_{k=1}^{n} k^3 = \left\{\frac{n(n+1)}{2}\right\}^2$, by using blocks, visual representations, and algebraic representation.

The textbook that Ms. K's school adopted, "High School Mathematics I" [25], mainly presents the concept by deriving the sum of the squares of consecutive natural numbers, $1^2 + 2^2 + 3^2 + \ldots + n^2 = \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$, using conventional methods at the beginning of the lesson. To motivate the students to engage in the abstraction of mathematical concepts using conventional methods, Korean textbooks usually present each mathematical concept starting with the *Problems of Inquiry* (or Opening Problems) before introducing conventional methods. This lesson unit in the textbook also presents the *Problems of Inquiry* first, which initiates students' thinking on the topic more broadly, and motivates students toward abstract mathematical thinking through numerical representations listed in the textbook (see Figure 3).

We build a pyramid with cubes. Answer the following questions.
How many cubes are in Stage 4?
Represent the total number of cubes in Stages 1 to n using $\sum$.

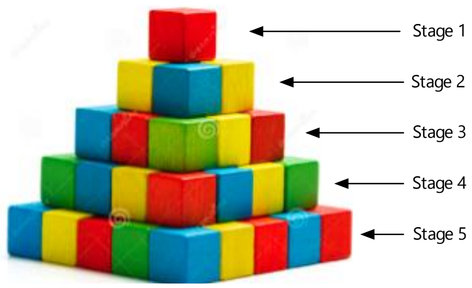


Figure 3. Problems of inquiry in a Korean textbook (opening problems).

Next, the textbook guides the learners toward an understanding of how to derive the formula for the sum of the squares of consecutive natural numbers using conventional methods (see Figure 4). Figure 4 shows (1) a reminder of the previous lesson's concept $1 + 2 + 3 + \ldots + n = \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$, and (2) a derivation of the formula $1^2 + 2^2 + 3^2 + \ldots + n^2 = \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$ using numerical representation.

For $\sum_{k=1}^{n} k^3 = \left\{\frac{n(n+1)}{2}\right\}^2$, the textbook recommends that students use the identity, $(x-1)^4 - x^4 = 4x^3 + 6x^2 + 4x + 1$, similar to $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$. Next, the textbook provides some exercises related to these two formulae, and at the end of the lesson, there are some visual representations of $\sum_{k=1}^{n} k^3 = \left\{\frac{n(n+1)}{2}\right\}^2$ (Figure 5) and $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$ (Figure 6).

The sequence of natural numbers
1, 2, 3, 4, …, n, …
is the arithmetic sequence with 1 as the first term, and 1 as the common difference between the terms. By using the formula of the sum of the arithmetic sequence, we now know that

$$\sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

However, the sequence of the squares of natural numbers

$$1^2,\ 2^2, 3^2, 4^2, \ldots, n^2, \ldots$$

is neither the arithmetic sequence nor the geometric sequence.
Next, let us find the sum of the sequence from the 1-st to the n-th term, i.e.,

$$\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + 4^2 + \cdots + n^2$$

When you substitute each of $1^2,\ 2^2, 3^2, 4^2, \ldots, n^2$ into the x of the identity, $(x+1)^3 - x^3 = 3x^2 + 3x + 1$, you obtain

$$x = 1; \quad 2^3 - 1^3 = 3 \times 1^2 + 3 \times 1 + 1$$
$$x = 2; \quad 3^3 - 2^3 = 3 \times 2^2 + 3 \times 2 + 1$$
$$x = 3; \quad 4^3 - 3^3 = 3 \times 3^2 + 3 \times 3 + 1$$

…

$$x = n; \quad (n+1)^3 - n^3 = 3n^2 + 3n + 1$$

Further, when you add all n equations on each side, you obtain

$$(n+1)^3 - 1^3 = 3\sum_{k=1}^{n} k^2 + 3\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1$$

Here, $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$, $\sum_{k=1}^{n} 1 = n$; therefore,

$$(n+1)^3 - 1^3 = 3\sum_{k=1}^{n} k^2 + 3\frac{n(n+1)}{2} + n$$

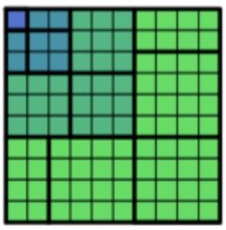
$$3\sum_{k=1}^{n} k^2 = (n+1)^3 - 1^3 - 3\frac{n(n+1)}{2} - n = \frac{n(n+1)(2n+1)}{2}$$

Therefore,

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

Figure 4. Algebraic representation of the $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$ in the textbook.

Using the diagram, show the equation, $\sum_{k=1}^{n} k^3 = \left\{\frac{n(n+1)}{2}\right\}^2$.



Let the length of the small square in the diagram be 1.
Then, the area of the blue square is 1^2, and there is only one blue square.
There are 2 squares with area 2^2.
There are 3 squares with area 3^2.
There are 4 squares with area 4^2.
The area of the big square is $(1+2+3+4)^2$.
Hence, $1 \times 1^2 + 2 \times 2^2 + 3 \times 3^2 + 4 \times 4^2 = 1^3 + 2^3 + 3^3 + 4^3 = (1+2+3+4)^2 = \left\{\frac{4(4+1)}{2}\right\}^2$
Therefore, $\sum_{k=1}^{4} k^3 = (1+2+3+4)^2 = \left\{\frac{4(4+1)}{2}\right\}^2$
When we generalize this equation, we obtain $\sum_{k=1}^{n} k^3 = (1+2+3+\cdots+n)^2 = \left\{\frac{n(n+1)}{2}\right\}^2$

Figure 5. Visual representation of $\sum_{k=1}^{n} k^3 = \left\{\frac{n(n+1)}{2}\right\}^2$.

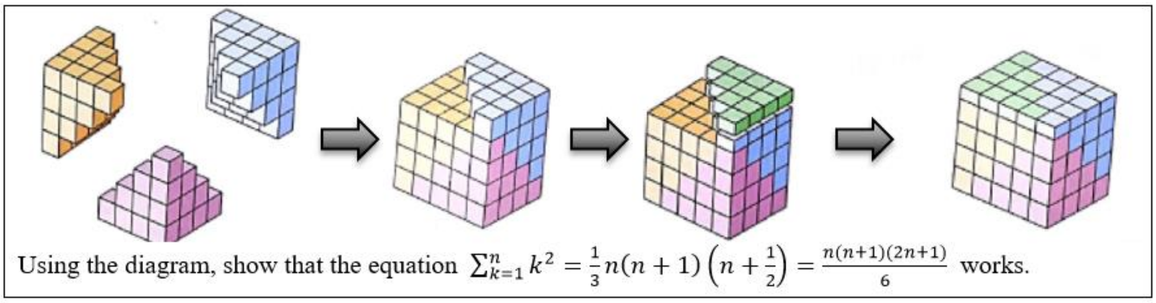


Figure 6. Visual representations of $\sum\limits_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$.

3.3. Data Analysis

Video observation data were analyzed to illustrate how Ms. K used concreteness fading strategies to support the students' conceptual understanding. Video data were transcribed and summarized in video logs with time stamps so that the instructional ways and students' works were summarized in order of time. The instructional supports related to concreteness fading strategies were identified as coding. After the video logs with transcripts were coded, the video data were reviewed to verify the coding. Two trained research assistants coded the video logs separately, and three researchers, the two assistants and the principal investigator, discussed the identified concreteness fading strategies to assess inter-rater reliability. As the phases and strategies were clearly identifiable, the agreement was 100%. The textbook and students' works were also analyzed with regard to the presentation of such instructional strategies.

4. Results

4.1. Textbook: Abstract Instruction

The ways in which the textbook provides concrete, visual, and abstract representations on the topic of "finding the generalized sum of finite sequences" is not based on the concreteness fading strategy. The concreteness fading strategy differs from merely providing concrete materials and visual representations. The textbook provides abstract representations first (Figure 4), and then the visual representations are provided at the end of the lesson unit (Figures 5 and 6). This is not an implementation of the concreteness fading strategy but abstract instruction [14]. This is typically used in school algebra classes in the U.S. [26], or traditional mathematics classes in Korea. However, Ms. K, did not use the textbook in a written way per se, but taught the lesson using concreteness fading strategy by creating her own student learning activity sheets, which differed from the textbook.

4.2. Activity Sheet: Concreteness Fading Strategies

Ms. K created an activity sheet with different ways of explaining mathematical concepts and representations compared with the textbook at her school-based lesson study group with mathematics teachers of the same grade. The activity sheet is attached in Appendix A as well as another activity sheet that she prepared that includes visual representation hints, and is attached in Appendix B.

The activity sheet that Ms. K created and provided to her students included concreteness fading strategies. While the textbook provides abstract representation first, the activity sheet guides students to use concrete materials such as magnetic blocks to represent each concept. For example, the first activity on the activity sheet guides students to represent 1 + 2 + 3 using magnetic blocks, first with the goal of representing $\sum\limits_{k=1}^{n} k = \frac{n(n+1)}{2}$. The second question aims to represent 1 + 2 + 3 + 4 with a fading of the concreteness, (i.e., students can use magnetic blocks to represent 1 + 2 + 3 + 4 but are starting to broaden their thinking of how to expand the concept (see Figure 7)). Therefore, the third question asks how to represent 1 + 2 + 3+ ... + 10,000 using multiplication (i.e., symbolic notion). The last question is on how to represent $\sum_{k=1}^{n} k$ using multiplication without $\sum$ (i.e., $1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$).

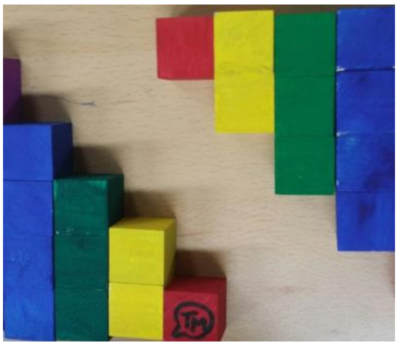


Figure 7. Concrete representations of 1 + 2 + 3 + 4 and the possibility of broadening the concept to deriving the multiplication formula $\frac{4(4+1)}{2}$.

The activities Ms. K provided in her lesson activity sheet have already been shown to be concreteness fading strategies. The concreteness fading instructional strategy, using the instructional support by Ms. K during the lesson, is described in the next section.

4.3. Concreteness Fading Instructional Strategies

Ms. K's concreteness fading instructional strategies are summarized in Table 1. Table 1 is summarized and described within the theoretical framework (i.e., Figure 2) that emphasizes the connecting phases.

The activity that she provided to small groups of students involved making sense of the three equations of the sums of the power of positive integers, $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$, $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$, and $\sum_{k=1}^{n} k^3 = \left\{\frac{n(n+1)}{2}\right\}^2$, by using magnetic blocks first, and guiding students in generalizing the sums step-by-step, as described in Section 4.2.

Briefly describing her instruction, in her lesson, Ms. K started the lesson with an opening inquiry question to motivate students to engage in the lesson topic and the learning objectives. She then conducted a small group activity using the activity sheet by suggesting to students the use of concrete manipulatives (i.e., the magnetic blocks such as shown in Figure 7). Students in each group were given magnetic blocks so that they could attempt to construct their own models. During the group activity, Ms. K helped the students in determining the models or rules using blocks through collaboration and communication with each other in small groups. When the students struggled to find the models, Ms. K distributed the visual representation hint sheets (Figures 6 and 8).

Table 1. Concreteness fading instructional strategies.

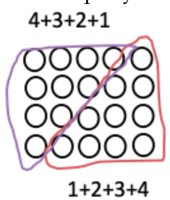
Concreteness Fading Phases	Concrete Phase	Connecting Phase	Pictorial Phase	Connecting Phase	Abstract Phase
$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$	Step-by-step approach. Representation of 1 + 2 + 3 using magnetic blocks first. 	Support in representing 1 + 2 + 3 + 4 both using a pair of blocks of 1 + 2 + 3 + 4 and implicitly expanding it into an abstract representation. 	Asking students how to represent $1 + 2 + 3 + \ldots + n$ using a diagram. One student represents the exemplary diagram: 	Supporting students in expanding this diagram to generalize it to $1 + 2 + 3 + \ldots + n$ by asking facilitating question, "How can we represent the finite sums of natural numbers using this diagram? Can you generalize how many dots are in the diagram?"	Students use two ways to represent the algebraic forms of $\sum_{k=1}^{n} k$: (a) $\mathrm{n} \times (\mathrm{n}+1) \div 2$ (b) $\frac{n}{2} \times (\mathrm{n}+1)$ Ms. K summarizes that these two algebraic forms are the same as $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$.
$\sum_{k=1}^{n} k^3 = \left\{\frac{n(n+1)}{2}\right\}^2$	Step-by-step approach. Representation of $1^3 + 2^3 + 3^3$ using magnetic blocks first. 	Representing $1^3 + 2^3 + 3^3 + 4^3$ both using blocks and implicitly expanding it into an abstract representation.  → 	Asking students how to represent $1^2 + 2^2 + 3^2 + \ldots + n^2$ using re-organizing blocks. Providing a visual hint sheet for struggling students. (Figure A2)	Supporting students in interpreting their activities with magnetic blocks or the visual hint sheet. Highlighting how the re-organized blocks can be represented in algebraic forms.	Students use an algebraic form, $\sum_{k=1}^{n} k^3 = \left\{\frac{n(n+1)}{2}\right\}^2$ for representation by making connections between blocks, visual diagrams and the formula. They first figure out that the re-organized blocks represent (1 + 2 + ... + n)x(1 + 2 + ... + n), i.e., $\left\{\frac{n(n+1)}{2}\right\}^2$. Ms. K summarizes the connection by rephrasing the students' presentation (Figure 10).

Table 1. *Cont.*

Concreteness Fading Phases	Concrete Phase	Connecting Phase	Pictorial Phase	Connecting Phase	Abstract Phase
$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$	Step-by-step approach. Representation of $1^2 + 2^2 + 3^2$ using magnetic blocks first. 	Representing $1^2 + 2^2 + 3^2 + 4^2$ both using three pairs of blocks of $1^2 + 2^2 + 3^2 + 4^2$ and implicitly expanding it into an abstract representation.  $\rightarrow$ 	Asking students how to represent $1^2 + 2^2 + 3^2 + \ldots + n^2$ using complied three sets of blocks. Providing a visual hint sheet for struggling students. (Figure A1)	Supporting students in interpreting their activities with magnetic blocks or the visual hint sheet. Selecting a different model from the small group works to present to the whole group: A student presents her group's model that is different from the hint sheet using a diagram (Figure 12)	Students use two ways of representing algebraic forms of $\sum_{k=1}^{n} k^2$: (a) $\frac{n(n+1)(2n+1)}{6}$ (b) $\mathrm{n}(\mathrm{n}+1)\frac{2n+1}{2} \div 3$ Several students demonstrate their understanding of connections among blocks, visuals and abstract forms of $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$
Instructional support strategies	Ms. K focuses on sense-making of each concept by supporting students in operating the magnetic blocks (i.e., concrete objectives and physical activities) with step-by-step questions.	Ms. K supports her students in representing simpler concepts with magnetic blocks, and also in interpreting their activities with manipulatives in mathematically meaningful ways.	Ms. K focuses on concept development and meaningful support by providing visual representations, especially when students are struggling with making connections between concrete activities and abstract representations.	Ms. K guides students in deleting superficial features from manipulatives or visual presentations. She highlights core structures of mathematical concepts.	Ms. K fosters students in using abstract representations of the concept by supporting generalizations and core abstract concepts. She also supports students in generating different algebraic forms of a concept from their concrete and visual representations.

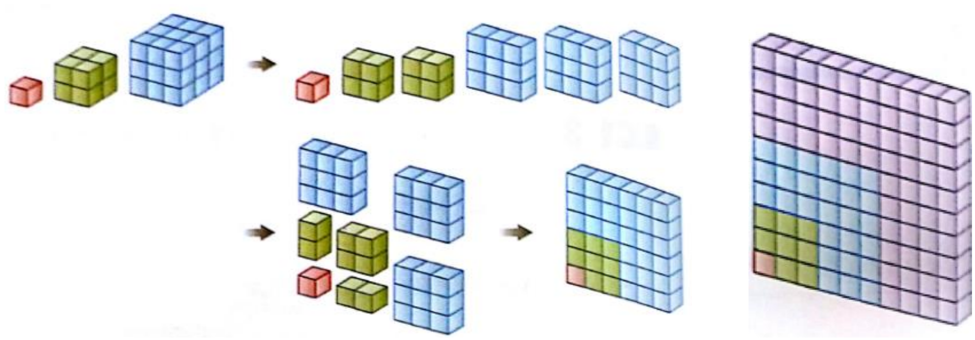


Figure 8. Visual representation hint sheet for $\sum_{k=1}^{n} k^3 = \left\{\frac{n(n+1)}{2}\right\}^2$.

After the small group activity, students presented their group work of how they made sense of the concept using magnetic blocks and abstract representations to the entire class. Students finally made connections between concrete manipulatives, visual representations, and abstract representations with the support of Ms. K.

4.3.1. Instructional Strategies in the Opening Phase

In the opening phase, Ms. K inquired whether the students were engaged with the learning objectives by asking thought-provoking questions, "How can we figure the value of $1^2 + 2^2 + 3^2 + \ldots + 1000^2$ out in easier ways?", using the opening problem provided in the textbook (Figure 3). This question would be explored throughout the whole lesson rather than receiving the answer right away, so that students could engage with the lesson objectives during the lesson.

Subsequently, she provided the learning objectives of the day's lesson: (1) students can prove the formulae of the sums of the powers of positive integers, and can solve related problems; and (2) students work collaboratively and all students engage in collaborative work. Students read the learning objectives aloud together as Korean teachers believe that this reading-aloud-together makes students more aware of what they are going to learn throughout the lesson.

4.3.2. Instructional Strategies in Concreteness Fading Phases

After reading the learning objectives together, Ms. K led small group work described on the activity sheet that contained three mathematical concepts (e.g., making sense of three equations, $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$, $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$, and $\sum_{k=1}^{n} k^3 = \left\{\frac{n(n+1)}{2}\right\}^2$) in the way of concreteness fading. The three mathematical concepts are provided in step-by-step descriptions, as described in Section 4.2. Each small group was given magnetic blocks, so that they could attempt to construct their own models (i.e., concrete representation) to represent abstract models and to make sense of the abstract model that they generalized from their concrete models.

In particular, during the small group activity, Ms. K supported students in representing simpler sums first such as 1 + 2 + 3, $1^2 + 2^2 + 3^2$, or $1^3 + 2^3 + 3^3$ using blocks. She focused on the students' understanding of each concept by fostering them in operating the blocks. For example, she asked questions while the students discussed how to calculate $1^3 + 2^3 + 3^3$ using the previous method of calculating 1 + 2 + 3.

T: Can you remember how to represent 1 + 2 + 3 + 4 using blocks? S1: Yes. T: How did you do that? S1: We first made one stair model (see Figure 7) and added another model upside down. Then, we calculated the area of all blocks using the rectangle area model. Then divided by 2. T: That's right. Can you do it in similar way for $1^2 + 2^2 + 3^2$? S2: We tried, but it is not that simple. T: Then, we can do $1^3 + 2^3 + 3^3$ first. What does 1^3 look like? What about 3^3? S2: A cube? // S1: A rectangle? Square? T: Yes, a cube. You will reorganize these blocks, but the hint is that you will do this in the plane rather than thinking of a 3-D structure.

While Ms. K fostered students in reorganizing the blocks, several groups of students were struggling to disassemble and re-assemble the blocks to model the generalization of the sums. When students struggled to build the models, Ms. K distributed the visual representation hint sheets (Figures 6 and 8). She then kept supporting students in representing $1^3 + 2^3 + 3^3 + 4^3$ using both blocks and thinking about how to disassemble and re-assemble the blocks, like that shown in the visual hint sheets (see Figure A1, Figure A2). While asking students to do so, Ms. K also supported students in interpreting their activities with blocks to generalize the sums.

T: So, are you disassembling the blocks of $1^3 + 2^3 + 3^3 + 4^3$? Then, what shape are they going to be in the plane? Students: Square! T: Yes, it looks like a square (see second diagram in Figure 8). Then, what is the length of one side of the square? S1: Oh, n? //S2: n + 1? (students had already started thinking of the generalization from the 4th model of the sum) T: That's right. A square with n-length. Then, what about the re-assembled square? What will be the length of one side of this square (see the very right diagram in Figure 8)? S2: Oh, it's the sum of n. T: Yes, it is going to be 1 + 2 + 3 + . . . + n. Then, what will be the area of the square? Students: Wow, it's square of (1 + 2 + 3 + n).

This is evidence that the instructional support strategies that Ms. K used were helpful in supporting students in making connections between the concrete, visual, and abstract representations in mathematically meaningful ways.

After the activity, Ms. K also made students present their models to the whole group. Several students demonstrated their understanding of the models for $\sum_{k=1}^{n} k^3 = \left\{\frac{n(n+1)}{2}\right\}^2$ using magnetic blocks in front of the whole class (see Figure 9) and other students also presented their models for $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$ (Figure 11).

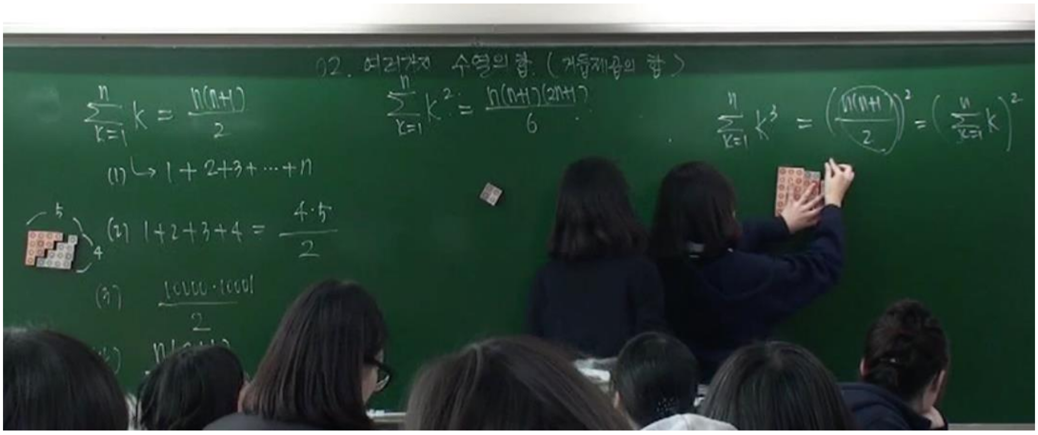


Figure 9. Students present their models for $\sum_{k=1}^{n} k^3 = \left\{\frac{n(n+1)}{2}\right\}^2$ worked in small groups using blocks.

Figure 9 shows that two students of a small group presented how they constructed their model using blocks. Students' understanding was shown in their presentation on how they made connections between concrete and abstract representations or between concrete, visual, and abstract representations. After the detailed mathematical ideas were discussed with others during the student presentation time, Ms. K rephrased the students' voices from the presentation and discussion so that other groups had a better understanding and made better connections among different representations. She also highlighted the core structures of each model with different representations. Ms. K, then summarized and made connections between the concrete materials and their corresponding symbolic representations (Figure 10).

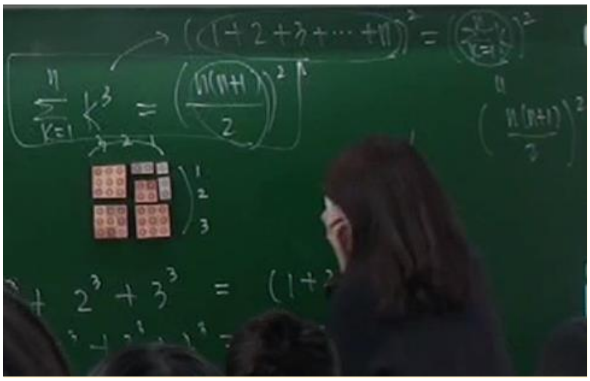


Figure 10. Ms. K's connection and generalization for $\sum_{k=1}^{n} k^3 = \left\{\frac{n(n+1)}{2}\right\}^2$.

For $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$, a student presented her group work using a diagram and block model, as shown in Figure 11 (Figure 12 shows a larger version of what the student used as a visual representation during her presentation in Figure 11). In her presentation, students in her group noticed that there would be one more layer in the cube-like model; thus, they designed another cube-like model using blocks, and the diagram on the blackboard in Figure 11 visually shows a combination of two cube-like models (this diagram is recreated in Figure 12). The upper and lower levels of the diagram each show three $\sum_{k=1}^{n} k^2$ expressions (i.e., there was a total of six $\sum_{k=1}^{n} k^2$ expressions in the diagram in Figure 12), and the volume of the cuboid was $n(n+1)(2n+1)$. Therefore, one $\sum_{k=1}^{n} k^2$ expression should be divided by six (i.e., $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$).

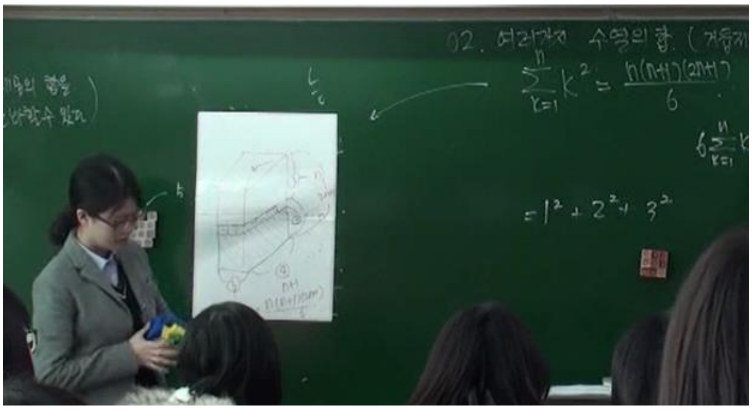


Figure 11. Students presenting their group work on connecting blocks and visual representation for $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$.

Similar to the instructional support strategies that she used in the lesson episodes of the other two sums, Ms. K also supported students in representing their own ways of understanding using concrete representation and visuals, and to make connections between them. Thus, students represented two versions of the algebraic forms of $\sum_{k=1}^{n} k^2$ as (a) $\frac{n(n+1)(2n+1)}{6}$ and (b) $\mathrm{n}(\mathrm{n}+1)\frac{2n+1}{2} \div 3$. She also highlighted the core structure of the models from the other students' presentations, and *repeated* what the student had presented to the whole group. During this time, Ms. K did not *rephrase* but rather *repeated*, as the student's presentation was very clear, so that other students could easily interpret the model and make connections among the concrete, visual, and abstract representations.

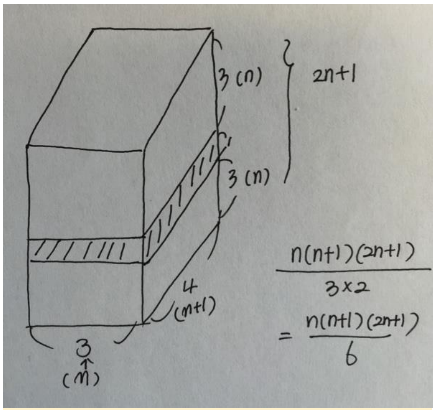


Figure 12. The visual model for $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$ presented by a student representing her group.

5. Conclusions and Discussion

The ways in which Ms. K supported her students' conceptual development can be described within the concreteness fading strategy framework. The critical features of her instructional support strategies were helping students to not only merely interact with manipulatives (i.e., concrete representations), but also to use different representations and make connections between concrete, visual, and abstract representations. Her supporting ways were to fade out the superficial features that are not helpful in developing concepts with abstract representations, and to make connections among the different representations that are the key for conceptual understanding [27]. This conclusion is also supported by a recent experimental study which revealed that children with explicit guides for making connections between different representations showed better outcomes than those who were provided with comparisons between representations [28,29].

The concept of sequences extends to early algebra for exploring patterns using different representations and strategies. Furthermore, it is critical to connect between representations and ultimately use symbols to develop algebraic thinking [30] or conceptual understanding [27]. High school mathematics teachers sometimes fail in letting students explore mathematical concepts using various representations and in making connections between representations; thus, an ultimate understanding of the core concepts is often lacking. The case described in this study allows for the development of alternative perspectives and strategies for mathematics instruction that will ultimately contribute to the arsenal of diverse methods of mathematical learning among the students. The instructional strategies that Ms. K used in each phase described in Table 1 provide insights into how other teachers can support their students to do conceptual understanding in their mathematics classrooms. The specifically described strategies may be useful to apply them in their classrooms.

I believe that concreteness fading strategies in school mathematics will help students develop a better conceptual understanding of mathematics. Some articles have pointed out that using concrete manipulatives alone, or merely comparing two different representations does not guarantee successful conceptual understanding [28,31], even though concrete manipulatives offer opportunities to do that. This current study also points out that the concreteness fading instructional strategy emphasizes the connection stages between the phases. One of the challenges teachers face while using concrete materials in their mathematics classrooms are the lack of explicit guides for linking phases [31]. This case study contributes not only to the possibility of expanding the concreteness fading instructional strategy framework in mathematics classrooms, but also has implications for explicit instructional strategies of linking phases, especially in high school mathematics.

Concreteness fading instructional strategy is a promising and sustainable instructional strategy for fostering students' conceptual understanding, especially for students with different learning strategies in mathematics. Although this is a high school case, so we may need more cases at different school levels, there are more empirical studies to support this promising strategy (e.g., [29,31,32]). I do not argue that this strategy is a panacea. However, it supports students' conceptual understanding by using multiple representations and explicit guides for making connections between them. Thus, this study of a concreteness fading instructional strategy also has implications on pre-service teacher education that has been put in place since the educational reform movement and curriculum change (e.g., [33]) in Korea. The specified instructional strategies from this case and the proposed theoretical framework can be used in teacher education and professional development resources, which will produce sustainable educational reforms.

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Appendix A. The Activity Sheet that Ms. K Created and Provided to Students

1. $\sum_{k=1}^{n} k$

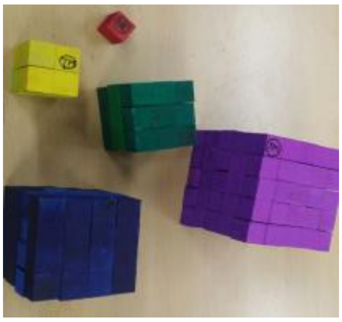
(a) Using blocks, let's represent 1 + 2 + 3 in a simpler expression.
(b) How can we figure out 1 + 2 + 3 + 4?
(c) Let's figure out 1 + 2 + 3 +…+ 10,000.
(d) Let's represent $\sum_{k=1}^{n} k$ using another expression without Σ, and represent it as a simpler expression.

2. $\sum_{k=1}^{n} k^3$

(a) Using blocks, let's represent $1^3 + 2^3 + 3^3$ in a simpler expression.
(b) How can we figure out $1^3 + 2^3 + 3^3 + 4^3$?
(c) Let's figure out

$1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + 7^3 + \cdots + 100^3$.

(d) Let's represent $\sum_{k=1}^{n} k^3$ using another expression without Σ, and represent it as a simpler expression.



3. $\sum_{k=1}^{n} k^2$

(a) Using blocks, let's represent $1^2 + 2^2 + 3^2$ in a simpler expression.
(b) How can we figure out $1^2 + 2^2 + 3^2 + 4^2$?
(c) Let's figure out $1^2 + 2^2 + 3^2 + \cdots + 100^2$
(d) Let's represent $\sum_{k=1}^{n} k^2$ using another expression without Σ, and represent it as a simpler expression.

Appendix B. The Visual Hint Sheet that Ms. K Provided to Students Who Struggled

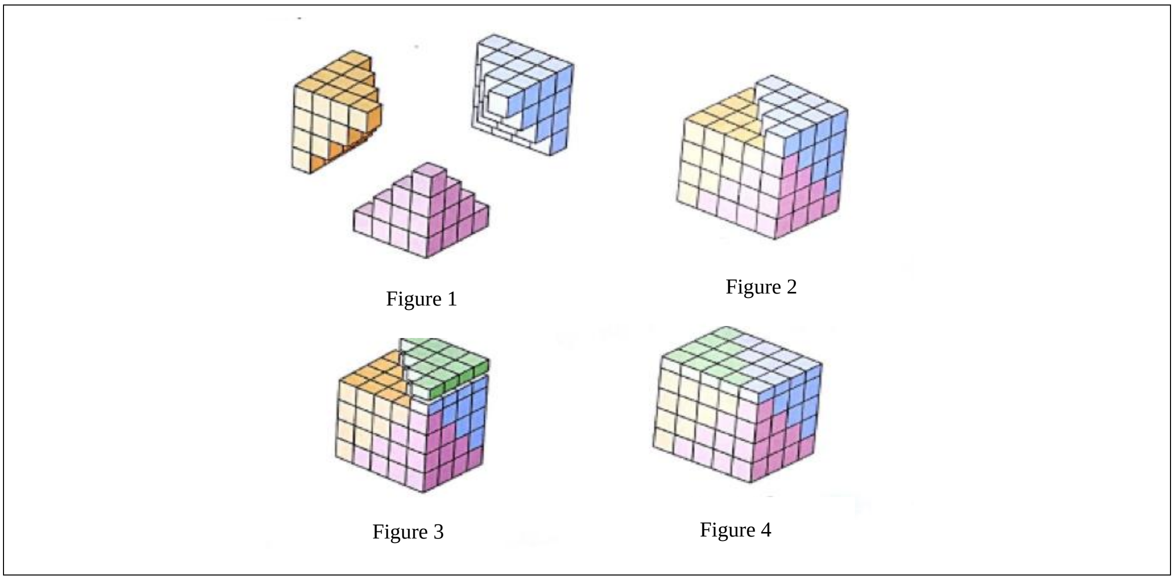


Figure A1. Visual hint sheet 1.

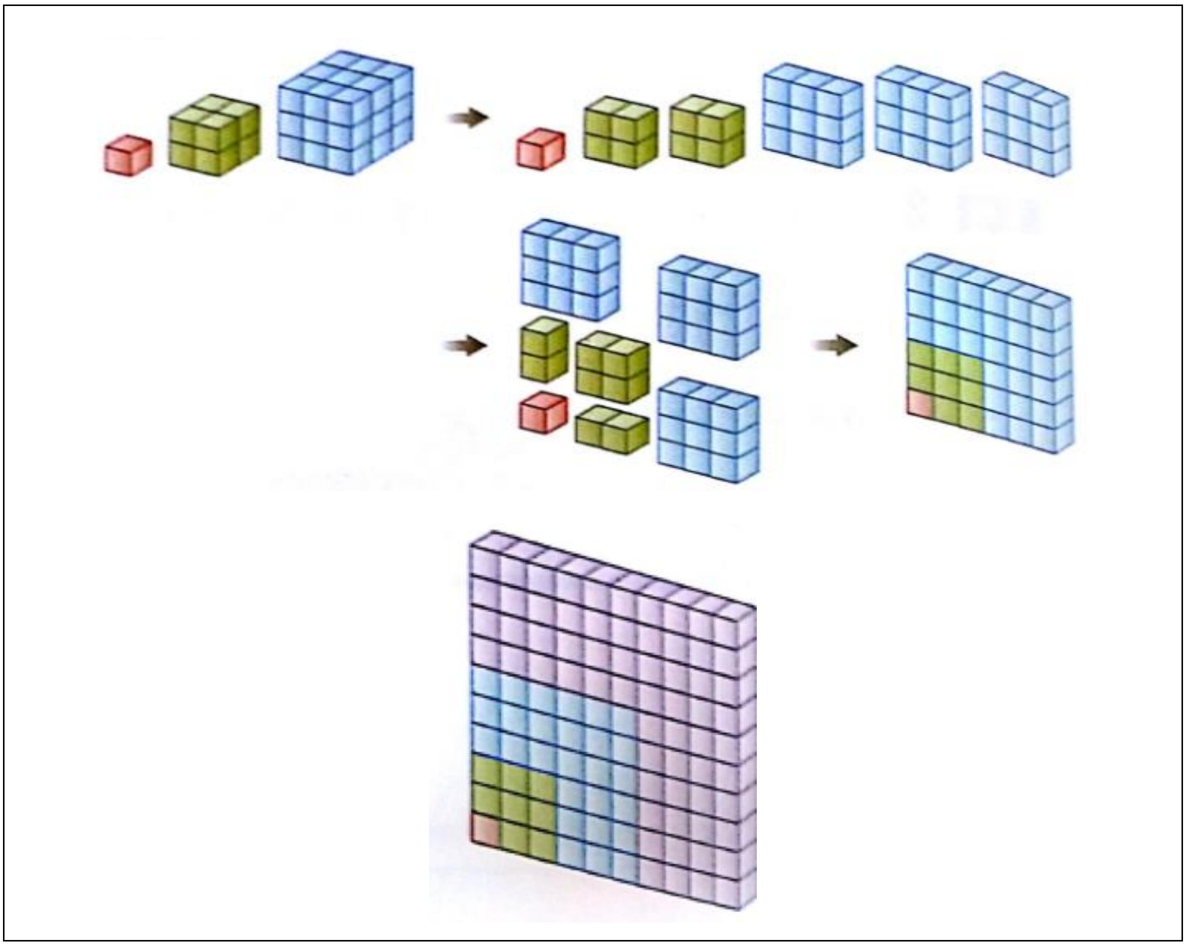


Figure A2. Visual hint sheet 2.

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