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Shipment Policy for an Economic Production Quantity Model Considering Imperfection and Transportation Cost

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Abstract: Determining replenishment lot size and number of shipments in a traditional production setup has been of great interest among researchers during the last decades. In order to survive modern competition, the manufacturer has to make good decisions about the lot size that is to be shipped to the retailer. Recently, several researchers have developed mathematical models for modelling different real-world situations; however, these models are lacking due to a combination of imperfection in process and shipment lot sizing. Therefore, in the proposed research, shipment policy for an imperfect production setup has been developed with transportation costs taken into consideration. The model analyzed lot sizing for manufacturers and retailers with imperfections in terms of equally sized shipments. Furthermore, an all-unit-discount policy for shipment is considered in the proposed research, and at the end, numerical computation and sensitivity analyses are carried out to gain more insight into the specifications of the model.

Keywords: economic production quantity; imperfection in production process; backorders; shipment policy; transportation cost

1. Introduction

Recently, determining the replenishment lot size and number of shipments in a production setup has been a prime focus for scholars. Therefore, researchers have developed different mathematical models considering various conditions in order to reduce overall cost. Taft [1] developed economic production quantity, which is considered to be the first inventory type model in a production environment. Since then, that model has been studied and extended in many ways by several researchers. Economic production quantity (EPQ) determines the quantity a company has to order to minimize its total inventory cost by balancing between inventory holding and fixed ordering costs. It is a very robust model and has no significant variation when input variables are slightly changed. It was taken as an assumption in the development of EPQ that there would be no defects in production processes, but in practical life situations there are reasons that defects occur in processes. These reasons may come down to a difference in operator experience, raw materials supplied and the capabilities of machines. However, different scholars have taken these reasons into consideration. Jamal et al. [2] studied the special environment of a production system for a single stage and determined the optimum batch size for two different cases where poor-quality items were reprocessed. In the first

case, reprocessing of defective or poor quality items occurred in the same cycle, and in the second case, reprocessing was performed after certain production cycles. Cárdenas-Barrón et al. [3] carried out further work on the first case, i.e., reprocessing in the same cycle case as Jamal et al. [2]. This research allows fixed and linear backorders in order to imitate problems as they occur in real situations. Both these models assumed a constant and known defective rate, but in reality, the defective rate may be random since it follows different distribution functions. Therefore, Cárdenas-Barrón et al.'s [3] work was extended by Sarkar et al. [4], who considered the defective rate to be random, following uniform, triangular or beta distribution.

Similar mathematical research has caught the attention of different scholars. Chang et al. [5] formulated a case where two stages were taken into consideration. In the first case, the production assembly was considered as automatic, and in the second case, it was considered as a manual system. In both these cases, the production was considered to be in an imperfect setup in which there were possibilities of producing poor quality items. In the proposed model, run time and production rate were taken as decision variables. Widyadana and Wee [6] also developed a mathematical model with the intention of reworking defective items. In this model, different production lines were considered for actual production and for the reprocessing of poor-quality products. Several production lines were considered for actual production and only one line for the reprocessing of defective products. Sensitivity analysis was performed in terms of production rate, production time, total cost, setup cost, serviceable cost and demand rate. Using the influence of inflation, Sarkar [7] developed a model that focused common phenomena, concluding that machines or processes may become out of control if they last for long periods of time. The model also reprocessed poor-quality items into high-quality items, while also considering some of the expenses. The model considered the system to be more reliable when reducing the presence of an increasing number of poor-quality items in production. Cárdenas-Barrón and Taleizadeh [8] analyzed a case where the number of shipments and replenishment batches were obtained for the inventory models obtained previously.

Along with these models, several researchers developed transportation policies for different setups. For example, Goyal and Nebebe [9] developed a shipment policy by considering the shipment of the first lot to be relatively small and the lots that follow to be relatively large. The larger lots were assumed to be of an equal size. Swenseth and Godfrey [10] formulated mathematical models for the influence of the straight freight discount on decisions by considering over-declaration as feasible. Ertogral, Darwish et al. [11] determined the shipment lot size in a manufacturer–buyer setup for three different cases. In first case, the model was developed without considering transportation cost. In second case, transportation cost was taken into account while developing the model, whereas in the third case the over-declaring of batch size was considered. Sajadieh and Jokar [12] developed a marketing-based inventory-related model to find the appropriate values of decision variables and increase overall profit for organizations. Chiu, Liu et al. [13] found an appropriate replenishment for EPQ, considering multiple shipments and rework processes. The model considered the failure of some of the reprocessed items during the reworking phase and turned them into scrap. Chiu, Lin et al. [14] examined a manufacturer–retailer combined setup with reprocessing and better items shipment to reduce inventory carrying cost. The model incorporated the $n + 1$ shipment policy and determined shipments and replenished items. Kropf et al. [15] developed a model by considering a fixed cost per shipment. To conclude the size of the fixed cost, the model used data from the Swiss border to determine performance level. Ekici, Altan et al. [16] developed a model by considering order size restriction and the benefits of order size consolidations. The model also observed the non-trivial pricing behavior of the manufacturer, as well as that of suppliers, by including different settings. Giri, Chakraborty et al. [17] worked with consignment stock policy, considering a case in which a single type of item was delivered by a single producer to only one buyer in an un-equal size shipment. Sağlam and Banerjee et al. [18] formulated a mathematical model for the integration of manufactured items and scheduling issue regarding transportation decisions. The model considered

both full- and less-than-truck-load shipments. The basic goal of the work was to minimize overall cost on the manufacturer's side.

Eduardo et al. [19] presented a simple derivation of the two inventory policies suggested by Jamal et al. [2] with better results and easier manual computation. Sana et al. [20] developed a production policy for finding out production lot size, optimal production rate and optimal safety stock. The work presented by Sarkar [21] was concerned with the joint determination of reliability parameter, safety stock and optimal production lot size. Sarkar et al. [22] worked on an economic manufacturing quantity model for stock-dependent demand in an imperfect production process. A solution procedure was presented by Chung [23] to find an optimal solution of $TC(Q,B)$. Keeping the production of imperfect quality items in mind, Sarkar et al. [24] considered an EPQ model with a stochastic demand pattern. A time-dependent quadratic demand model was considered by Sarkar et al. [25] under the effect of time-value of money and inflation with variable reliability parameter. Cárdenas-Barrón et al. [26] determined both optimal number of shipments and optimal replenishment lot size jointly for an economic manufacturing quantity (EMQ) model with multiple shipments and rework. A production inventory model was developed by Sarkar [27] for a deteriorating product in a two-echelon supply chain management. Yaghin et al. [28] developed an integrated marketing inventory model in a two-echelon supply chain model involving customer behavior more realistically and discount promotion to determine pricing quantities, shipping and optimal ordering simultaneously. Taleizadeh et al. [29] studied the production and inventory problem in a three-layer supply chain under two scenarios. In the first scenario, all defective products were considered as scrape while in the second scenario, the defective products were reworked and sold to customers as perfect products. Hong-Fwu Yu et al. [30] investigated a single retailer single-vender production inventory model under an unequal-sized policy with immediate return for imperfect products. The contributions of various researchers in the similar research area are given in Table 1.

Table 1. Comparison of proposed research with existing literature.

Reference	IPS *	B **	SP †	SIP ††
Jamal et al. [2]	✓	×	×	×
Cárdenas-Barrón et al. [3]	✓	✓	×	×
Sarkar et al. [4]	✓	✓	×	×
Chang, Su et al. [5]	✓	✓	×	×
Widyadana and Wee [6]	✓	×	×	×
Sarkar [7]	✓	×	×	×
Cárdenas-Barrón et al. [8]	✓	×	×	×
Goyal and Nebebe [9]	×	×	✓	×
Swenseth and Godfrey [10]	×	×	✓	×
Ertogral and Darwish [11]	×	×	✓	×
Ekici, Altan et al. [16]	×	×	✓	×
Giri, Chakraborty et al. [17]	×	×	✓	×
Sağlam and Banerjee [18]	×	×	✓	×
Cárdenas-Barrón et al. [19]	✓	×	×	×
Sana and Chaudhuri [20]	✓	×	×	×
Sarkar, Sana et al. [21]	✓	×	×	×
Sarkar, Chaudhuri et al. [22]	✓	×	×	×
Chung et al. [23]	✓	✓	×	×
Sarkar, Sana et al. [24]	✓	×	×	×
Sarkar, Sana et al. [25]	✓	×	×	×
Cárdenas-Barrón, Sarkar et al. [26]	✓	×	×	×
Sarkar [27]	✓	×	×	×
Yaghin, Fatemi Ghomi [28]	×	×	✓	×
Taleizadeh et al. [29]	×	×	✓	×
Yu and Hsu et al. [30]	×	×	✓	×
Our proposed model	✓	✓	✓	✓

* Imperfect Production System, ** Backorder, † Shipment Policy, †† Shipment in an Imperfect Production.

A sustainable economic order quantity problem was examined by Lee et al. [31] with multi-modal transportation options and a stochastic lead-time. Using numerical experiments, a mathematical model of the concomitant sustainable economic order quantity was presented and various scenarios were explored to determine the effects of incorporating sustainability considerations into the traditional inventory model on operational decisions which include sourcing decisions and the choice of transportation model combinations. Nardo et al. [32] considered smart factory topics and logistics 4.0 and focused on vertical integration for the implementation of reconfigurable and flexible smart production systems by the use of information system integration for the purpose of optimizing the flow of material in a 4.0 full service approach. Sarkar et al. [33] presented sustainable inventory management with the objective of total profit maximization by the development of a synergic economic order quantity model and considered rework, multi-trade-credit policy and shortages simultaneously. The model helped in making decisions for the enhancement of sustainable inventory management performance by controlling the cycle time and a fraction of time for a global supply chain. Shen et al. [34] investigated a production inventory model for product deterioration under a carbon tax policy and collaborative preservation technology investment from the perspective of supply chain integration with the main purpose to determine the optimal delivery, ordering, production and investment policies for a vendor and buyer that maximizes the profit in a unit time.

The current research considered the transportation cost as part of the ordering cost and independent of the shipment size. As transportation has significant impact on economic activities of any organization, special attention should be given to the shipment policies. In this direction, the proposed research extends the Sarkar et al. [3] inventory model by developing a shipment policy with transportation cost consideration.

The rest of this paper is organized as follows. Section 2 lists the notations and assumptions which are used in the models. Section 3 describes the model in detail. Section 4 shows a numerical example and sensitivity analysis to illustrate the managerial insights of the models. A brief conclusion is given in Section 5.

2. Model Specification

In this model, a system with a single-manufacturer and single-retailer is designed in such a way that the manufacturer makes finished products by processing raw material in a single stage production setup as shown in Figure 1. The production rate of the finished products by the manufacturer is “P”. During the production run, defective parts are also produced which are reprocessed according to a backward rework policy in the same production cycle to make good quality products. These products are then transported to the retailer according to equal sized shipment policy. Hence, transportation cost is considered for both manufacturer and retailer for the transportation of finished products to the retailer. In this case, the manufacturer also allows backorders. Therefore, in addition to the basic costs, there will also be linear and fixed backordering costs associated with the inventory.

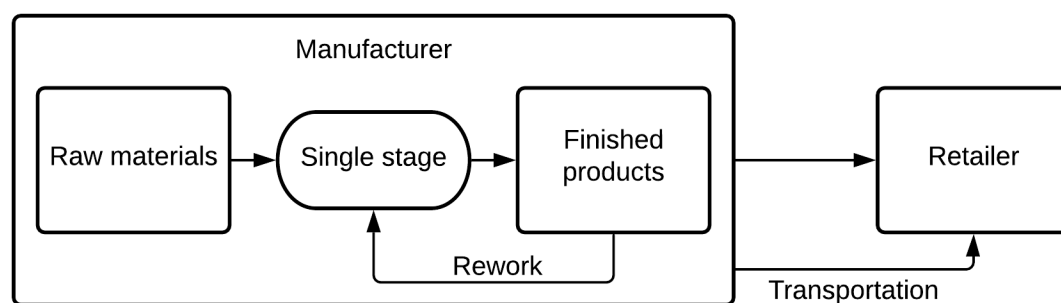


Figure 1. Inventory flow in an imperfect production setup with transportation to retailer.

The existing models of Goyal and Nebebe [9], Swenseth and Godfrey [10], Ertogral and Darwish [11], Ekici, Altan et al. [16], Giri, Chakraborty et al. [17], Sağlam and Banerjee [18], Yaghin, Fatemi Ghomi [28], Taleizadeh et al. [29] and Yu and Hsu et al. [30] have developed shipment policies for perfect production systems and are lacking in the consideration of imperfection in the processes. Hence, in this direction, the proposed work extends the existing research through the development of a shipment policy for an imperfect production system and considering transportation costs. The model also describes a way to calculate the total cost of the system in the next section.

The mathematical model is characterized by the following parameters and decision variables.

Parameters

P	production rate
K_m	manufacturer production setup cost
K_r	retailer ordering cost
H_m	manufacturer inventory carrying cost per product per unit time
H_r	retailer inventory carrying cost per product per unit time
C	manufacturing cost of a product
D	demand rate
s	unit transportation cost
Q	batch size
B	size of backorders
G_s	system average inventory
W	backorder cost per product per unit of time
F	backordering cost per product
J	Average backorders

Decision variables

q	shipment lot size
n	number of shipments

The following assumptions are considered during the mathematical modelling

- Demand and production rates are constant and known over the planning horizon. Production rate is greater than demand rate ($P > D$), hence no shortages.
- After every production cycle, the products are 100% screened at the manufacturing side of setup and screening cost is ignored.
- Two types of backordering costs are considered. Linear i.e., the backordering cost is applied to average backorder, and fixed i.e., the backordering cost is applied to maximum level of backorders.
- There is no scrap within the cycle and all defective products are reworked to make perfect quality products.
- The beginning inventory is enough to fulfil the demand of retailer during the first production cycle.
- Only one kind of item is considered in the development of the model.
- Carrying or holding costs are considered for average inventory.
- Production and rework are done in the same manufacturing system at the same production rate.

3. Model Formulation

As the model considers the shipment of finished products from manufacturer to retailer, the model is composed of the following types of costs i.e., manufacturer's setup cost, retailer's ordering cost, inventory carrying cost of manufacturer and retailer, fixed and linear backordering costs, production cost and unit transportation cost. In this setup, imperfect items are also produced which are reprocessed in the same production cycle to perfect items. It is assumed that this defect rate follows uniform distribution and has the same production cost whenever defective products are reprocessed.

Setup cost of manufacturer

The setup cost of manufacturer is calculated using Equation (1). The annual setup cost for manufacturer is D over Q times the cost per cycle.

$$\text{Setup cost} = \frac{KD}{Q} \quad (1)$$

Ordering cost of retailer

As a retailer can place n number of orders, total ordering cost of the retailer is equal to ordering cost times the number of orders. Equation (2) is used to evaluate ordering cost of the retailer.

$$\text{Ordering cost of the retailer} = \frac{nK_r D}{Q} \quad (2)$$

Inventory carrying cost of manufacturer

At the start of the cycle, minimum inventory exists which is enough to fulfil the need of a retailer for the first cycle and the total inventory increases at a rate of $(P - D)$. Therefore,

$$\text{Maximum inventory} = \left[\frac{Dq}{p} + \frac{(P - D)Q}{2P} \right] \quad (3)$$

The inventory carrying cost of the manufacturer is maximum inventory times the manufacturer inventory carrying cost per product per unit time as formulated in Equation (4)

$$\text{Inventory carrying cost of manufacturer} = H_m \left[\frac{Dq}{p} + \frac{(P - D)Q}{2P} \right] \quad (4)$$

Inventory carrying cost of retailer

The average inventory carried by the retailer is half q . Therefore, the inventory carrying cost of the retailer is equal to the average retailer's inventory times the difference between inventory carrying costs per product of manufacturer to retailer and is enumerated using Equation (5). Where ΔH is the difference between H_m and H_r .

$$\text{Inventory carrying cost of the retailer} = \frac{\Delta H q}{2} \quad (5)$$

Fixed and linear backordering costs

Fixed and linear backordering costs for the system are calculated using Equations (6) and (7), respectively.

$$\text{Fixed backordering cost} = \frac{BD}{Q} \times F \quad (6)$$

$$\text{Linear backordering cost} = WJ \quad (7)$$

Production cost

In each production cycle, the batch quantity produced is calculated using Equation (8).

$$\text{Batch quantity} = Q[1 + E(R)] \quad (8)$$

As the production cost per product is C . Therefore, the production cost per batch in a cycle is C times batch quantity and is calculated using Equation (9). The total production cost per year is enumerated using Equation (10), which on simplification results in Equation (11).

$$\text{Production cost per batch in a cycle} = CQ[1 + E(R)] \quad (9)$$

$$\text{Production cost} = CQ[1 + E(R)] \times \left(\frac{D}{Q}\right) \quad (10)$$

$$\text{Production cost} = CD[1 + E(R)] \quad (11)$$

Unit transportation cost

Unit transportation cost is the cost of transporting a single unit from manufacturer to retailer. For a given lot size $q \in [X_i, X_{i+1}]$, the transportation cost per unit time is equal to transportation cost per production lot cycle divided by the duration of production lot cycle as given in Equation (12), which upon simplification results in Equation (13).

$$\text{Unit transportation cost} = s_i n q \times \frac{D}{nq} \quad (12)$$

$$\text{Unit transportation cost} = s_i D \quad (13)$$

Hence, using all equations explained above, the total cost function is given as follows:

Total cost = manufacturer setup cost + retailer ordering cost + manufacturer inventory carrying cost + retailer inventory carrying cost + fixed backordering cost + linear backordering cost + production cost + unit transportation cost.

Incorporating the values for these cost functions, the total cost is then equal to:

$$TC(q, n) = \frac{K_m D}{Q} + \frac{n K_r D}{Q} + H_m \left[\frac{Dq}{P} + \frac{(P-D)Q}{2P} \right] + \frac{\Delta H q}{2} + \frac{FBD}{Q} + \frac{WB^2 A}{2QE} + CD(2-A) + S_i D \quad (14)$$

The simplification of Equation (14) results into Equation (15).

$$TC(q, n) = \frac{(K_m + n K_r) D}{nq} + H_m \left[\frac{Dq}{P} + \frac{(P-D)nq}{2P} \right] + \frac{\Delta H q}{2} + \frac{FBD}{nq} + \frac{WB^2 A}{2nqE} + CD(2-A) + S_i D \quad (15)$$

As the total system cost equation consists of two decision variables i.e., shipment lot size (q) and number of shipments (n), for minimization of total cost, the second order Hessian Matrix is positive definite which implies all principle minors are positive. Furthermore, for this problem the sufficient conditions are as follows:

$$\frac{\partial^2 TC(q, n)}{\partial q^2} > 0 \text{ and } \frac{\partial^2 TC(q, n) \partial^2 TC(q, n)}{\partial q^2 \partial n^2} > 0$$

Taking partial derivative of Equation (15) with respect to q and n , the simplification results into Equations (16) and (17), respectively.

$$\frac{\partial TC}{\partial q} = -\frac{K_m D}{nq^2} - \frac{K_r D}{q^2} + \frac{H_m D}{P} + \frac{H_m (P-D)n}{2P} + \frac{\Delta H}{2} - \frac{FBD}{nq^2} - \frac{WB^2 A}{2nq^2 E} \quad (16)$$

And

$$\frac{\partial TC}{\partial n} = -\frac{K_m D}{n^2 q} + \frac{H_m (P-D)q}{2P} - \frac{FBD}{n^2 q} - \frac{WB^2 A}{2n^2 q E} \quad (17)$$

From sufficient conditions,

$$\frac{\partial^2 TC}{\partial q^2} = \left(\frac{2}{nq^3} \right) \left[(K_m + FB)D + \frac{WB^2A}{2E} \right] + \frac{2K_r D}{q^3} \quad (18)$$

$$\frac{\partial^2 TC}{\partial n^2} = \left(\frac{2}{n^3 q} \right) \left[(K_m + FB)D + \frac{WB^2A}{2E} \right] \quad (19)$$

$$\frac{\partial^2 TC}{\partial q \partial n} = \left(\frac{1}{n^2 q^2} \right) \left[(K_m + FB)D + \frac{WB^2A}{2E} \right] + \frac{H_m(P-D)}{2P} \quad (20)$$

$$\left(\frac{\partial^2 TC}{\partial q \partial n} \right)^2 = \left(\frac{1}{n^2 q^2} \right)^2 \left[(K_m + FB)D + \frac{WB^2A}{2E} \right]^2 + \left[\frac{H_m(P-D)}{2P} \right]^2 + 2 \left(\frac{1}{n^2 q^2} \right) \left[(K_m + FB)D + \frac{WB^2A}{2E} \right] \left[\frac{H_m(P-D)}{2P} \right] \quad (21)$$

$$\frac{\partial^2 TC \partial^2 TC}{\partial q^2 \partial n^2} = \left(\frac{4}{n^4 q^4} \right) \left[(K_m + FB)D + \frac{WB^2A}{2E} \right]^2 + \left(\frac{4K_r D}{n^3 q^4} \right) \left[(K_m + FB)D + \frac{WB^2A}{2E} \right] \quad (22)$$

Subtracting Equation (21) from Equation (22), the difference is given in Equation (23).

$$\begin{aligned} \frac{\partial^2 TC \partial^2 TC}{\partial q^2 \partial n^2} - \left(\frac{\partial^2 TC}{\partial q \partial n} \right)^2 &= \left(\frac{4}{n^4 q^4} \right) \left[(K_m + FB)D + \frac{WB^2A}{2E} \right]^2 + \left(\frac{4K_r D}{n^3 q^4} \right) \left[(K_m + FB)D + \frac{WB^2A}{2E} \right] \\ &- \left(\frac{1}{n^2 q^2} \right)^2 \left[(K_m + FB)D + \frac{WB^2A}{2E} \right]^2 \\ &- \left[\frac{H_m(P-D)}{2P} \right]^2 - 2 \left(\frac{1}{n^2 q^2} \right) \left[(K_m + FB)D + \frac{WB^2A}{2E} \right] \left[\frac{H_m(P-D)}{2P} \right] \end{aligned} \quad (23)$$

Let

$$M = \left(\frac{1}{n^2 q^2} \right) \left[(K_m + FB)D + \frac{WB^2A}{2E} \right] \quad (24)$$

$$N = \left(\frac{4K_r D}{n^3 q^4} \right) \left[(K_m + FB)D + \frac{WB^2A}{2E} \right] \quad (25)$$

$$O = \left[\frac{H_m(P-D)}{2P} \right] \quad (26)$$

Therefore,

$$\frac{\partial^2 TC \partial^2 TC}{\partial q^2 \partial n^2} - \left(\frac{\partial^2 TC}{\partial q \partial n} \right)^2 = 4M^2 + N - M^2 - O^2 - 2MO \quad (27)$$

$$\frac{\partial^2 TC \partial^2 TC}{\partial q^2 \partial n^2} - \left(\frac{\partial^2 TC}{\partial q \partial n} \right)^2 = 3M^2 + N - O^2 - 2MO \quad (28)$$

For cost equation minimization, the condition is $3M^2 + N - O^2 - 2MO > 0$ i.e., if the expression $3M^2 + N - O^2 - 2MO$ is greater than 0, it means that sufficient condition of optimality criteria is satisfied. Therefore, it can be concluded that the cost equation is convex when the expression $3M^2 + N - O^2 - 2MO > 0$ is satisfied.

To obtain the optimum points for this case, the first order partial derivatives with respect to the variables are separately equal to 0. Therefore, for this case the necessary conditions are given as follows:

$$\frac{\partial TC(q, n)}{\partial q} = 0 \text{ and } \frac{\partial TC(q, n)}{\partial n} = 0$$

Putting Equation (16) equal to zero,

$$0 = -\frac{(K_m + nK_r)D}{nq^2} + \frac{H_mD}{P} + \frac{H_m(P-D)n}{2P} + \frac{\Delta H}{2} - \frac{FBD}{nq^2} - \frac{WB^2A}{2nq^2E} \quad (29)$$

After simplifications,

$$q^* = \sqrt{\frac{[2ED(K_m + nK_r + FB) + WB^2A]}{[2nE\left(H_m\left(\frac{P}{D} + \frac{(P-D)n}{2P}\right) + \frac{\Delta H}{2}\right)]}} \quad (30)$$

Now putting Equation (17) equal to zero,

$$0 = -\frac{K_mD}{n^2q} + \frac{H_m(P-D)q}{2P} - \frac{FBD}{n^2q} - \frac{WB^2A}{2n^2qE} \quad (31)$$

After simplification,

$$n^* = \frac{\sqrt{[8E^2PDH_m(K_m + FB)(P-D) + 4EPDWB^2AH_m(P-D)]}}{2qEH_m(P-D)} \quad (32)$$

Solving Equations (30) and (32) simultaneously,

$$q^* = \frac{\sqrt{[8E^2PDH_m(K_m + FB)(P-D) + 4EPDWB^2AH_m(P-D)]}}{2nEH_m(P-D)} \quad (33)$$

Subtracting Equation (33) from Equation (30)

$$\sqrt{\frac{[2ED(K_m + nK_r + FB) + WB^2A]}{[2nE\left(H_m\left(\frac{P}{D} + \frac{(P-D)n}{2P}\right) + \frac{\Delta H}{2}\right)]}} - \frac{\sqrt{[8E^2PDH_m(K_m + FB)(P-D) + 4EPDWB^2AH_m(P-D)]}}{2nEH_m(P-D)} = 0 \quad (34)$$

After simplification

$$n^* = \sqrt{\frac{[2ED(K_m + FB) + WB^2A][P\Delta H + 2H_mD]}{2EDH_mK_r(P-D)}} \quad (35)$$

Hence, incorporating the values of optimum number of shipments n^* and lot size q^* in Equation (15) we get the optimum total cost equation as:

$$TC^*(q^*, n^*) = \frac{(K_m + n^*K_r)D}{n^*q^*} + H_m\left[\frac{Dq^*}{P} + \frac{(P-D)n^*q^*}{2P}\right] + \frac{\Delta Hq^*}{2} + \frac{FBD}{n^*q^*} + \frac{WB^2A}{2n^*q^*E} + CD(2-A) + S_iD \quad (36)$$

The proposed model analyzes the case when transportation cost is considered as a function of shipment lot size and in all-unit-discount cost format rather than considering it to be part of setup or order cost or to be insignificant. As the model considers shipment of finished products from manufacturer to retailer with consideration of transportation cost, all unit transportation cost structure has been proposed for this case.

In unit transportation cost structure, for ranges $0 \leq q < X_1$, $X_1 \leq q < X_2$, $\dots$, $X_{m-1} \leq q < X_m$ and $X_m \leq q$ units, the corresponding unit transportation costs are USD $s_0, s_1, \dots, s_{m-1}$ and s_m , respectively. Where, $s_0 > s_1 > \dots, s_m$.

The transportation cost structure indicates that when the lot size is in the range 0 to X_1 , then the unit transportation cost from manufacturer to retailer is s_0 while for lot size greater than or equal to X_1 and less than X_2 , the unit transportation cost from manufacturer to retailer is s_1 and so on. Similarly, if the lot size is equal to or greater than X_m , then the unit transportation cost is s_m ,

which is the minimum unit transportation cost. Therefore, in the equations for transportation cost, the corresponding expressions for unit transportation costs $s_0D, s_1D, \dots, s_mD$ are lot size ranges $q \in (0, X_1), q \in (X_1, X_2), \dots, q \in (X_m, \text{infinity})$, respectively.

4. Numerical Computation and Sensitivity Analysis

To illustrate the specifications and provide an additional insight into the model, this section performs numerical computations and sensitivity analysis. The example considers a model with transportation cost. Furthermore, the example uses the data from the Sarkar, Cárdenas-Barrón et al. [4] and Ertogral and Darwish et al. [11] models. In order to check the effect of key parameters on overall cost, sensitivity analysis is performed.

Numerical Example

A manufacturer produces 550 items per year. The yearly demand placed by customers is 300 units. From the previous year record, it is known that the manufacturer inventory carrying cost per product per unit of time is USD 50, retailer inventory carrying cost per product per unit of time is USD 10, linear backorder cost per product per unit of time is USD 10, fixed backorder cost per product is USD 10, manufacturing cost of a product is USD 7, manufacturer production setup cost is USD 50, retailer ordering cost is USD 10 and backorder size is 33 items. If the values of other parameters are assumed as $a = 0.03$ and $b = 0.07$, then the optimum number of shipments, optimum quantity, and the total optimum cost can be calculated as follows:

Solution Procedure

$$\text{Step1: } q_T^* = 9.61 \\ n^* = 6$$

$$\text{Step2: As } q_T^* < 15 \quad \text{therefore, go to Step 3}$$

$$\text{Step3: } n_{up} = 6 \quad \text{and} \quad n_{lw} = \max \left\{ 1, \sqrt{\frac{[8E2PDHm(Km + FB)(P - D)] + \left[\frac{4EPWB2AHm(P - D)}{2EHm(P - D)Xm} \right]}{}} \right\} \\ n_{lw} = 3.83$$

n_{lw} is when rounded up, it results into 4

$$\text{Step4: For } n = 4, 5, 6$$

$$(a) \quad n = 4, q = 12.64, l = 2$$

$$(b) \quad TC(12.64, 4) = 5130.22 \\ TC(15, 4) = 5099.04$$

$$(a) \quad n = 5, q = 10.87, l = 2$$

$$(b) \quad TC(10.87, 5) = 5102.83 \\ TC(15, 5) = 5143.87$$

$$(a) \quad n = 6, q = 9.61, l = 1$$

$$(b) \quad TC(9.61, 6) = 5186.16 \\ TC(10, 6) = 5097.67 \\ TC(10, 6) = 5230.56$$

$$\text{Step5: By inspecting the optimal solution obtained for all values of } n, \\ \text{it can be concluded that the overall optimal solution is:} \\ n = 6, q = 10 \text{ and } TC(10, 6) = \$5097.67$$

In unit transportation cost structure, for ranges $0 \leq q < 5$, $5 \leq q < 10$, $10 \leq q < 15$ and $15 \leq q$, the corresponding unit transportation values are 4, 3.5, 3.2 and 3, respectively, as can be seen in Figure 2.

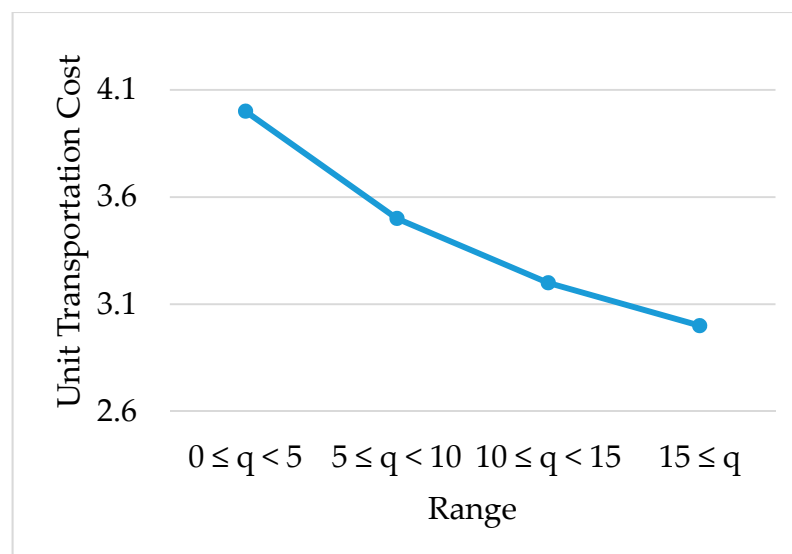


Figure 2. Structure for the unit transportation cost.

Sensitivity Analysis

Sensitivity analysis is performed to point out the detailed specifications of the proposed model. Sensitivity analysis determines how different values of an independent variable affect a particular dependent variable under a given set of assumptions. In other words, sensitivity analyses show how various sources of uncertainty in a mathematical model contribute to overall uncertainty of the model. Sensitivity analysis is usually used in the production and business world commonly used by engineers and financial analysts. Sensitivity analyses for key parameters of the example are given in Figure 3.

As shown in Figure 3a, there is a direct relation between production rate and the total cost of the system. It can also be observed that the production rate is very sensitive for decreasing its value by 50% of its original value, which reduces the total cost of the system by 26.88% as compared to the increase by the same rate i.e., 50% which increases the total cost of the system by just 2.27%. It means that the production rate will not be allowed to increase that much as it will not have much of an effect on the total cost but rather it will increase inventory holding cost. However, if the production rate is decreased by 50%, then it must be done as it will reduce the total cost of the system by a suitable amount. Furthermore, -25% to 25% change is not that sensitive in the case of the production rate; therefore, if it falls in the mentioned range, it will not have that much effect on the total cost of the system.

As can be seen in Figure 3b, there is a direct relationship between manufacturing production setup cost and the total cost of the system. It can be seen that the manufacturing production setup cost is not that much sensitive and increasing its value by 50% of its original value will increase the total cost of the system by 2.45%. Similarly decreasing its value by 50% of its original value reduces the total cost of the system by just 2.45%. In the same way, an increase or decrease in its value by 25% increases or decreases the total cost of the system by 1.23%, respectively. Hence, if the values are changed in these mentioned ranges, the total cost of the system is not that much affected.

As depicted in Figure 3c, there is a direct relationship between ordering cost of retailers and the total cost of the system. It can be noticed that ordering cost of retailers is not that sensitive and increasing its value by 50% of its original value increases the total cost of system by 2.94%. Similarly decreasing its value by 50% of its original value reduces the total cost of the system by just 2.94%. Likewise, the increase or decrease in its value by 25% increases or decreases the total cost of the system

by 1.47%, respectively. Hence, the change in values in these ranges have no significant effect on the total cost of the system.

As shown in Figure 3d, there is a direct relationship between manufacturer inventory carrying cost and the total cost of the system. It can be observed that the manufacturer inventory carrying cost is more sensitive as compared to the manufacturer production setup cost. Increasing its value by 50% of its original value increases the total system cost by 6.91%. Similarly decreasing its value by 50% of its original value reduces the total cost of the system by just 6.91%. In the same manner, increasing or decreasing its value by 25%, increases or decreases the total cost of the system by 3.45% and 3.46%, respectively.

As can be seen in Figure 3e, there is a direct relationship between retailers' inventory carrying cost and the total cost of the system. It can be seen that the retailers' inventory carrying cost is not that much sensitive. Increasing its value by 50% of its original value increases the total cost of the system by 2.94%. Similarly decreasing its value by 50% of its original value reduces the total cost of the system by 2.94%. Likewise, increasing or decreasing its value by 25%, increases or decreases the total cost of the system by 1.47%, respectively. Hence, any change in values in these mentioned ranges will not affect the total cost of the system significantly.

A direct relationship can be seen between manufacturing cost of the products and the total cost of the system as depicted in Figure 3f. It can be noticed that manufacturing cost of the products is the second most sensitive parameter among all other parameters. Increasing the value of manufacturing cost of the products by 50% of its original value increases the total cost of the system by 21.63%. Similarly decreasing its value by 50% of its original value reduces the total cost of the system by just 21.63%. Likewise, increasing or decreasing its value by 25% increases or decreases the total cost of the system by 10.81%, respectively. Therefore, a change in manufacturing cost of the products in the given ranges will affect the total cost of the system significantly.

As shown in Figure 3g, there is a direct relationship between demand rate and the total cost of the system. It can be observed that demand rate is the most sensitive parameter among all other parameters. An increase in its value by 50% of its original value increases the total cost of the system by 41.36%. Furthermore, in a similar manner, decreasing its value by 50% of its original value reduces the total cost of the system by 34.39%. In the same way, increasing or decreasing the value of demand rate by 25% increases or decreases the total cost of the system by 18.48% and 17.41%, respectively. Hence, changing the values of demand rate in these stated ranges will affect the total cost of the system remarkably.

As can be seen in Figure 3h, there is a direct relationship between unit transportation cost and the total cost of the system. It can be noticed that unit transportation cost is also a sensitive parameter up to some extent. Increasing the value of unit transportation cost by 50% of its original value increases the total cost of the system by 9.42% and decreasing its value by 50% of its original value reduces the total cost of the system by 9.42%. Likewise, increasing or decreasing the unit transportation value by 25% increases or decreases the total cost of the system by 4.71%, respectively. Hence, by changing the unit transportation cost values, the total cost of the system is affected slightly.

As depicted in Figure 3i, there is a direct relationship between number of shipments and the total cost of the system. An interesting phenomenon can be seen in case of changing the values of the number of shipments. The increase or decrease in the number of shipments increases the total cost of the system. Increasing the value of number of shipments by 50% of its original value increases the total cost of the system by 5.63%, and decreasing its value by 50% of its original value increases the total cost of the system by 2.58%. In the same manner, increasing or decreasing its value by 25% increases the total cost of the system by 0.76% and 0.81%, respectively. Hence, by changing values of the number of shipments, the total cost of the system is increased.

As shown in Figure 3j, there is a direct relationship between number of shipments lot size and the total cost of the system. An interesting phenomenon can also be seen in case of changing the values of shipment lot size. By increasing or decreasing the shipments lot size, the total cost of the system is

increased. Increasing its value by 50% of its original value increases the total cost of the system by 8.35%, and decreasing its value by 50% of its original value increases the total cost of the system cost by 3.78%. Similarly, increasing or decreasing its value by 25% increases the total cost of the system by 1.14% and 1.28%, respectively. Hence, for any change in the values of shipment lot size, the total cost of the system is increased.

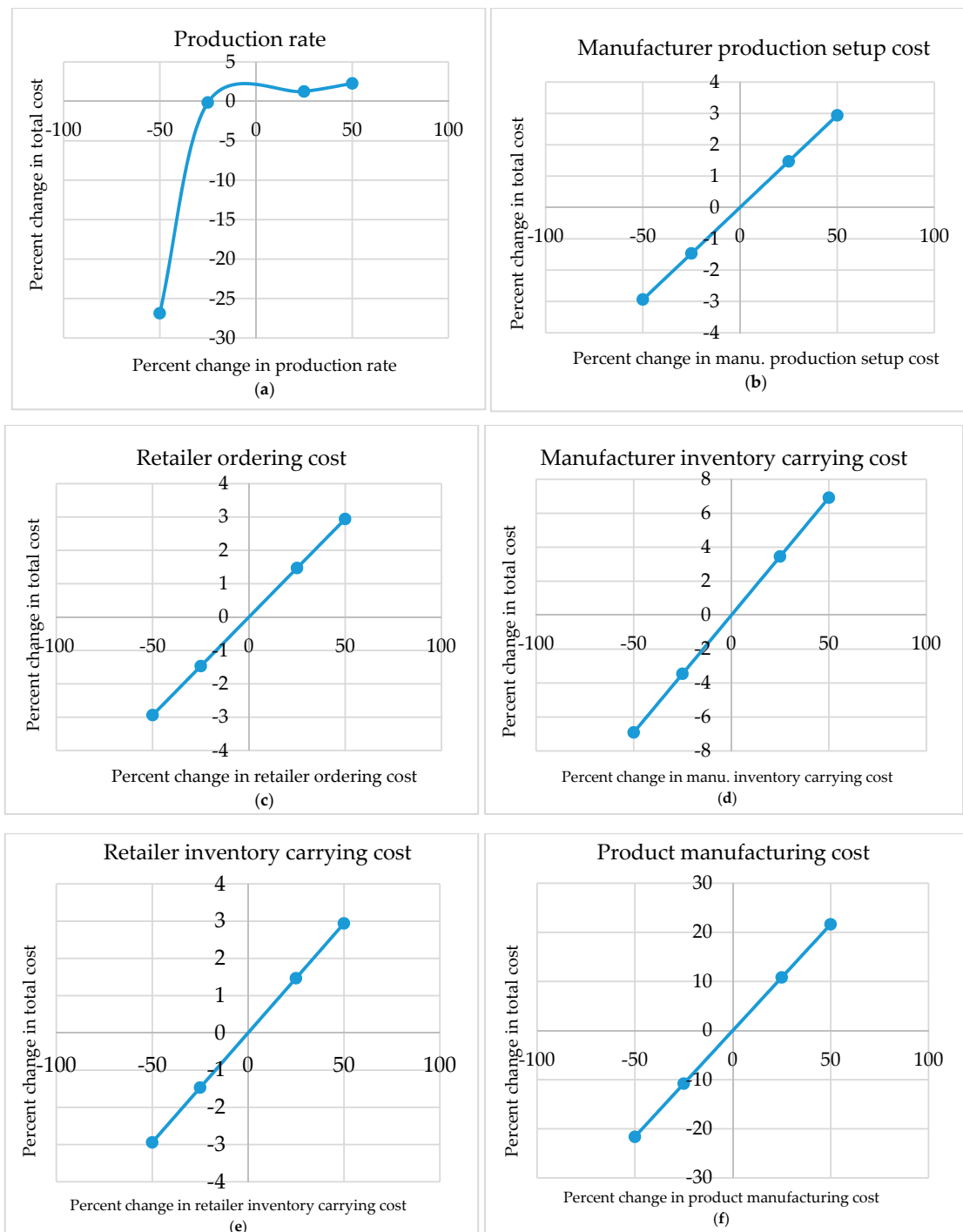


Figure 3. Cont.

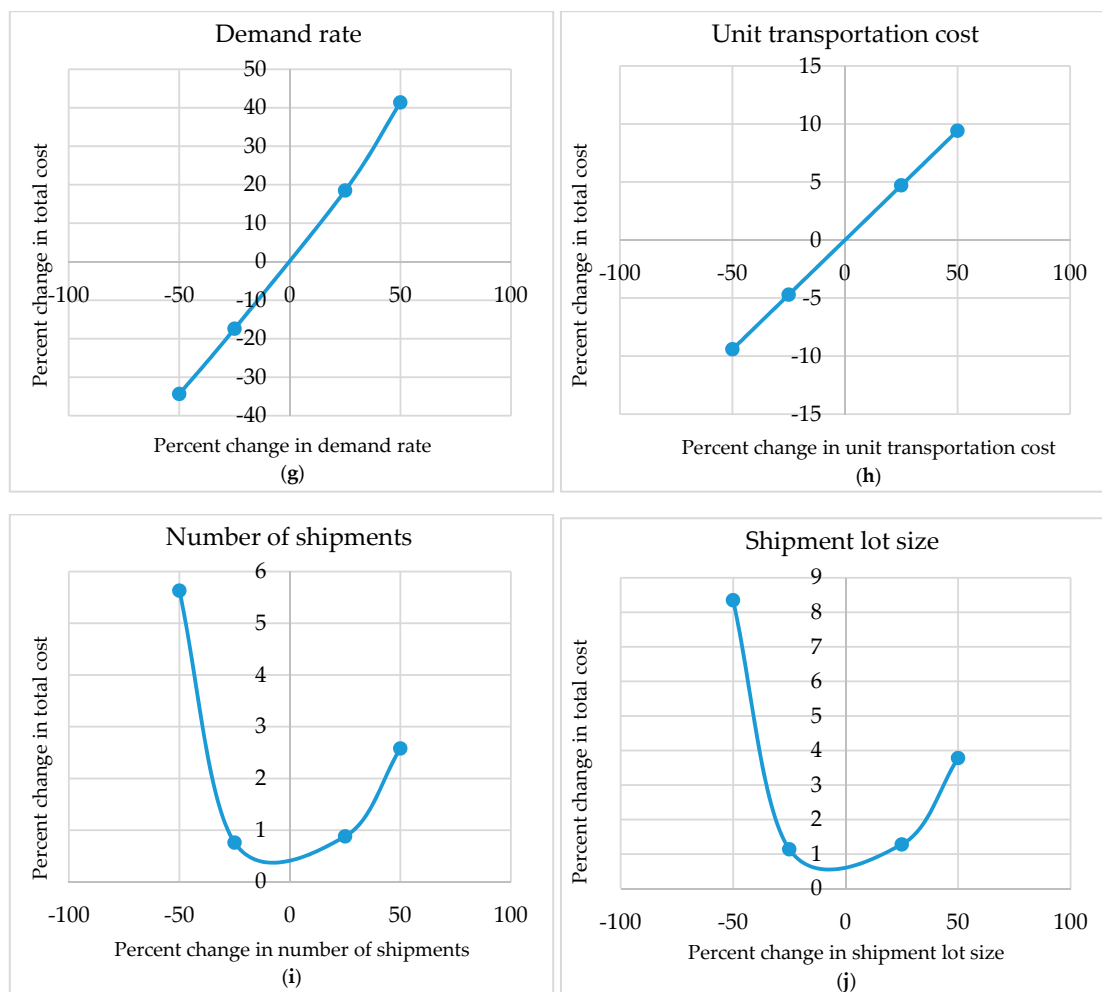


Figure 3. Sensitivity analysis for the key parameters of example.

It can be concluded from the sensitivity analysis that by increasing the values of parameters such as fixed setup cost (K_m), fixed ordering cost (K_r), unit inventory carrying cost (H_m and H_r), fixed cost per backorder (F), linear backorder cost (W), unit production cost (C) and demand (D) results in the increased value of total cost of the system, and decreasing values of these parameters decreases the value of the total cost of the system. However, interesting results can be found for (q) and (n) as by changing the values of these parameters, the total cost of the system is increased. Furthermore, it is observed that demand (D) and unit production cost (C) are the most sensitive parameters compared to others.

5. Conclusions and Future Recommendations

As already discussed, many researchers worked on an imperfect production system, backorder, and shipment policy, while our work considered all these factors as well as shipment in an imperfect production. The proposed research considered the vendor–buyer shipment size problem for an equal-size shipment policy in an imperfect production system. All unit discount transportation costs have been analyzed and optimal solution procedures for the proposed model have been developed. It can be observed from the solution procedure that the shipment decision varies as transportation cost is incorporated into the system. Further, numerical computations and sensitivity analysis are performed to point out specifications of this work. It can be observed from sensitivity analysis that by increasing values of the parameters like fixed setup cost (K_m), fixed ordering cost (K_r), unit inventory carrying cost (H_m and H_r), fixed cost per backorder (F), linear backorder cost (W), unit production cost (C) and demand (D) resulted in increased values of the total cost of the system and decreasing

values of these parameters resulted in decreased values of the total cost of the system. However, different results can be seen for (q) and (n), which is that -50% to $+50\%$ changes in shipment lot size (q) and number of shipments (n) show an increase in the overall cost of the system. Furthermore, it is observed that demand (D) and unit production cost (C) are the most sensitive parameters compared to all other parameters.

The model considers only one type of items in a single stage production system. However, in reality there may be situations where different types of products are produced in a multi-stage production system. Therefore, the present research can be extended for production of multi-product in multi-stage production system. Only the manufacturer and retailer stages are considered in the system. Therefore, future work can consider the supplier to manufacturer stage as well. Furthermore, shipment policy has been developed by considering one retailer, whereas there can be more than one retailer in a system. Therefore, future research can also consider many retailers in the system. Furthermore, the demand is considered as constant in the planning horizon in our work, and hence, this research can be extended by incorporating variable demand rate into the model.

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