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The Uncapacitated Dynamic Single-Level Lot-Sizing Problem under a Time-Varying Environment and an Exact Solution Approach

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Abstract: The dynamic lot-sizing problem under a time-varying environment considers new features of the production system where factors such as production setup cost, unit inventory-holding cost, and unit price of manufacturing resources may vary in different periods over the whole planning horizon. Traditional lot-sizing theorems and algorithms are no longer fit for these situations as they had assumed constant environments. In our study, we investigated the dynamic lot-sizing problem with deteriorating production setup cost, a typical time-varying environment where the production setup is assumed to consume more preparing time and manufacturing resources as the production interval lasts longer. We proposed new lot-sizing models based on the traditional lot-sizing model considering the changing setup cost as a new constraint, called uncapacitated dynamic single-level lot-sizing under a time-varying environment (UDSLLS-TVE for short). The UDSLLS-TVE problem has a more realistic significance and higher research value as it is closer to reality and has higher computational complexity as well. We proposed two mathematical programming models to describe UDSLLS-TVE with or without nonlinear components, respectively. Properties of the UDSLLS-TVE models were extensively analyzed and an exact algorithm based on forward dynamic programming (FDP) was proposed to solve this problem with a complexity of $O(n^2)$. Comparative experiments with the commercial MIP solver CPLEX on synthesized problem instances showed that the FDP algorithm is a global optimization algorithm and has a high computational efficiency.

Keywords: lot-sizing problem; time-varying environment; deteriorating setup cost; dynamic programming; optimization

1. Introduction

The dynamic lot-sizing plan of a manufacturing system can be described as follows: In a given planning horizon T with multiple production periods and different demands from customers, the production setup is determined within aperiodic periods over T , and the production sizes/quantities cover all the requirements of products (internal or external) in all periods with or without backlogging, in order to minimize the sum of production costs, setup costs, and inventory-holding costs occurred in the whole planning horizon [1,2]. The earlier studies of lot-sizing problems were originated from the cost reduction requirement from inventory management [3]. Since Harris [4] published the famous economic order quantity (EOQ) formula, which can be viewed as the earliest form of single-point lot-sizing optimization, the trade-off optimization between the purchase/setup cost and inventory-hold cost has become a hot research topic both in industries and academia. In terms of the classification principles by Karimi et al. [5], lot-sizing problems can be classified into different categories such

as single or multi-level, capacitated or uncapacitated, dynamic or stationary, as shown in Table 1 as follows.

Table 1. General classification of lot-sizing problems.

Classification Principles	Categories
Product structure	Single-level; Multi-level
Product number	Single-item; Multi-item
Production capacity	No capacity constraint; Capacity constraint
Product demand	Dynamic; Stationary
Production schedule length	Finite horizon; Infinite horizon
Existence of cost adjustment	No setup cost; Setup cost
Existence of stock shortage	No backlog; Backlog allowed

The dynamic single-level lot-sizing (DSLLS) problem is a basic form of many variants of the lot-sizing problem, referring to the case of taking into account time-varying demands of single product with only one level of product structure. In this situation, all intermediate processes of a product, from raw materials (or purchased semi-finished products) to the end-product, are assumed to be processed within a merged phase. Wagner and Whitin [2] first proposed a basic formulation of the DSLLS problem and an exact algorithm (i.e., the well-known WW algorithm) based dynamic programming. Mathematically, the basic formulation can be formulated as an optimization model as follows.

Minimize:

$$TC = \sum_{t=1}^n (s_t \cdot y_t + h_t \cdot I_t) \quad (1)$$

Subject to:

$$I_t = I_{t-1} + X_t - D_t \quad \forall t > 1 \quad (2)$$

$$X_t - M \cdot y_t \leq 0 \quad \forall t \quad (3)$$

$$I_t \geq 0 \quad \forall t \quad (4)$$

$$X_t \geq 0 \quad \forall t \quad (5)$$

In the above formulation, the objective function in Equation (1) is to minimize the sum of total production setup costs, $s_t \cdot y_t$, and inventory-holding costs, $h_t \cdot I_t$, of all periods in the whole planning horizon, where n represents the number of periods, t is the index of period, h_t represents the unit inventory-holding cost at period t , I_t represents the inventory level at the end of period t , s_t represents the setup cost occurred in period t , and y_t is binary decision variables indicating a production is setup in period t (by $y_t = 1$) or not (by $y_t = 0$). Constraint (2) is the inventory flow representing the balance of production quantity, demand and inventory, X_t represents the production quantity in period t , and D_t represents the external requirements. Constraint (3) indicates that the production quantity of each period cannot exceed the maximum capacity M that can be a large number for uncapacitated capacity. Constraint (4) indicates that the stock shortage is not allowed. Constraint (5) indicates that the number of productions in each production period is nonnegative.

Wagner and Whitin [2] analyzed an important property of the model that in the n production plan scope, any period t ($0 < t < n$) should satisfy the following conditions.

$$f_t(I_{t-1}) = \min[h \cdot I_{t-1} + s \cdot y_t + f_{t+1}(I_t + x_{t+1} - D_{t+1})] \quad \forall t = 1, 2, \dots, n-1 \quad (6)$$

$$f_n(I_{n-1}) = \min(h \cdot I_{n-1} + s \cdot y_n) \quad \forall t = n \quad (7)$$

where f_t represents the optimal total costs in the period t and the Equations (6) and (7) represent the backward recursive equation based on dynamic programming. Based on these equations, Wagner and Whitin [2] put forward a famous WW algorithm for solving the dynamic single-level lot-sizing problem, based on the assumption that there are no capacity or resource limits in the production system. By using the backward dynamic programming, the single-level lot-sizing problem can be solved with exact solution efficiently. The WW algorithm can be represented by using the equation as follows:

$$F(t) = \min \left[\min_{1 \leq j < t} \left[s_j + \sum_{p=j}^{t-1} \sum_{k=p+1}^t h_p D_k + F(j-1) \right], s_t + F(t-1) \right] \quad (8)$$

The WW algorithm is an exact algorithm which reduces the problem complexity of $O(2^{n-1})$ to a computation complexity of $n(n+1)/2$. Several improved algorithms can be found in the literature [6–8]. Christophe et al. [6] proposed two polynomial time dynamic programming algorithms to solve the problem to optimality. Mathijn et al. [7] presented a Lagrangian heuristic to provide a feasible solution and lower bound for the problem. For costs and emissions such that the zero inventory property is satisfied, they gave a pseudo-polynomial algorithm, which can also be used to identify the complete set of Pareto optimal solutions of the bi-objective lot-sizing problem. Fanny [8] applied genetic programming (GP) to automatically generate specialized heuristics specific to the instance class. The contributions of these three papers are to reduce the computational complexity of the WW algorithm and save the calculation time. Lee et al. [9] studied the lot-sizing problem with time window and used the time window that can be adjusted by the user to ensure the demand in each period is being satisfied. For some early lot-sizing models, the demand for different periods is independent and can be considered as a constant. Macedo et al. [10] presents mixed-integer programming models to hybrid manufacturing and remanufacturing lot-sizing problems, in which remanufactured products are treated as new ones, so that both manufactured and remanufactured products compete to meet the demands. Sarakhsi et al. [11] studied the Joint Economic Lot-Sizing problem (JELS) for a single-vendor single-buyer system while demand is dependent on selling price. Rubasheuski et al. [12] considered the coordinated capacitated lot-sizing problem with stochastic non-stationary demand. Chowdhury et al. [13] presented a new $O(T)$ algorithm for the dynamic single-item lot-sizing problem. Similar studies are also available [14–16].

The multi-level lot-sizing (MLLS) problem is an extension of DSLLS where the interdependencies of multiple items within an assembly structure are considered in the lot-size optimization [17]. Sahebjamnia and Torabi [18] developed a fuzzy stochastic multi-objective linear programming model for a multi-level, capacitated lot-sizing problem in a mixed assembly shop. Furlan and Santos [19] proposed a hybrid heuristic based on the bees algorithm combined with the fix-and-optimize heuristic to solve the multi-level capacitated lot-sizing problem. For large-sized problems, meta heuristics algorithms were reported to obtain the near-optimal solutions with good computational efficiencies, such as the Genetic Algorithm [20], the particle swarm optimization (PSO) algorithm [21], and Variable Neighborhood Search [22–25].

However, traditional lot-sizing theorems and algorithms were assumed for a constant production environment where the unit costs of manufacturing resources were stable in involved multiple periods. This assumption does not fit for some industries where their production environments are suffering from constantly changing in different periods. A time-varying environment can be considered as some production factors of the manufacturing system, such as the availability of manufacturing resources, material/labor costs/prices, production capacities, or cost of inventory-holding, are constantly changing during the planning horizon. A number of studies can be found in literature on the lot-sizing problems involving time-varying production factors such as time-varying demand [26], time-varying pricing [27], and time-varying cost [28]. Haase and Kimms [28] studied the capacitated lot-sizing problem where production setup cost depends on the order of production steps. Federgruen and

Joern [27] designed a cost function that combined with the pricing model in time-varying demand and determined each lot-sizing quantity dynamically.

Other studies related to dynamic demand and inventory optimization in time-varying environments can be found in Dye et al. [29], Yang [3], Hong and Jern [30], Piperagkas et al. [31] and Tempelmeier and Hilger [32]. Dye et al. [29] proposed an approach for determining the optimal selling price with considering of the deterioration penalty of partial backlogging. Yang [3] and Balkhi [33] studied the lot-size inventory problem for deteriorating items considering partial backlogging. Hong and Jern [30] studied the dynamic lot-sizing and resource competition problem from the perspective of multiple firms and developed a capacity competition model. Piperagkas et al. [31] and Tempelmeier and Hilger [32] studied the dynamic lot-sizing problem with stochastic demands over the planning periods.

In this study, we investigated the uncapacitated dynamic lot-sizing problem in a time-varying environment (UDSLLS-TVE) and proposed an exact forward dynamic programming algorithm. In UDSLLS-TVE, we considered a deteriorating production setup in the production system, assuming that the cost for production setup/initialization will increase higher if the production interval lasts longer. This is reasonable in discrete manufacturing systems because the longer the production interval is, the more difficult the production setup/initialization for preparing, machine inspection, cleaning, and heating will be, and thus more preparing time and manufacturing resources will be consumed in the production initialization/setup stage. We assumed that there is a positive correlation between the setup cost and the production interval. That means, if the production interval is increased one by one, the cost of production initialization will increase by a fixed rate. Existing algorithms such as the WW algorithm and its variants, is no longer suitable for solving the UDSLLS-TVE problem. We conducted a deep analysis of the properties of the UDSLLS-TVE model and proposed an exact algorithm based on forward dynamic programming for UDSLLS-TVE. Computational experiments were conducted to verify the feasibility and optimality of proposed models and algorithm.

The under investigated UDSLLS-TVE problem in this paper is toward the minimization of the manufacturing costs. Thus, our study can help to improve the sustainability of a factory's production system and is also helpful for the economic sustainability of a country's industry. The current research has achieved an initial success by considering several practical factors in a TVE. However, the influences of in-plant transport organization studied by Taniguchi and Shimamoto [34] and Gola and Kłosowski [35], and the capacity exploitation studied by Nikiforos [36] Kłosowski et al. [37] on manufacturing costs, have not been included yet in this paper, but will be considered in our future work.

The following contents of the paper are organized as follows. In Section 2, we provide the formal formulation of the UDSLLS-TVE problem, then discuss the properties of the proposed model. In Section 3, we propose a dynamic programming algorithm for UDSLLS-TVE problem, including algorithm analysis and algorithm implementation. In Section 4, computational experiments are conducted to verify the performances of the proposed model and algorithm. Finally, in Section 5, we summarize the paper and give further research directions.

2. Problem Formulation

2.1. A Nonlinear Model of UDSLLS-TVE

To model the UDSLLS-TVE with deteriorating production setup, we first define two new factors that are to be used in the problem formulation: (1) The production interval L_t , and (2) the growth rate of setup cost $\alpha \in [0, \infty]$.

Definition 1. *Production interval L_t : It represents the number of unscheduled production periods between the period of t and the latest period t' to arrange production during the limited planning cycle n . The value of L_t ranges from $L_t \in [0, t - 2]$ ($t \geq 2$).*

$$L_t = \begin{cases} t - 1 - \max\{t' \cdot y_t | 1 \leq t' < t\}, & \forall t \neq 1 \\ L_0, & \forall t = 1 \end{cases} \quad (9)$$

In above definition, y_t is a binary decision variable indicating whether the production is arranged during period t . If the production is arranged, then $y_t = 1$, otherwise $y_t = 0$. In Equation (9) i is used to calculate the length of production interval L_t for period t , indicating the number of periods that have no production setup before period t . When $t = 1$, the production interval is initialized by the last period of previous planning cycle in accordance with the same rule. However, if only one planning cycle is considered, the production interval at the beginning period can be initialized as $L_1 = 0$. For other periods, we have $L_t \in [0, t - 2]$. Note that the production interval L_t will not be affected by whether a production setup is scheduled in period t or not.

Definition 2. Growth rate of setup cost α : A fixed rate representing the unit increase in setup costs for every additional period of production interval. This rate is determined in advance according to the actual time-varying production environment.

By the above definition, the production setup cost can be considered as a deteriorating variable calculated as $s'_t = s_t + \alpha \cdot L_t$. The new setup cost s'_t consists of two parts: (1) A fixed part of setup cost s_t and (2) time-varying part determined by variable L_t .

Notations of parameters and decision variables used to describe the UDSLSS-TVE problem are summarized in advance in Table 2 as follows.

Table 2. Symbol definition of the uncapacitated dynamic single-level lot-sizing under a time-varying environment (UDSLSS-TVE) model.

Symbol	Instructions
<i>Parameter:</i>	
n	Scope of the whole plan
t	Planning period
h_t	Holding costs of unit products at the end of period t
s_t	Setup cost during period t
D_t	External demand during period t
l	Leading time of production
I_0	Initial inventory at the beginning of the planning horizon
α	Setup cost growth rate
<i>Variables:</i>	
y_t	Binary decision variable indicating if a production is set up in period t or not
X_t	Non-negative decision variable indicating the production volume of lot-sizing in period t
I_t	Non-negative decision indicating the inventory level at the end of period t
L_t	Production interval

The UDSLSS-TVE problem is defined in a finite planning horizon with multiple production periods, in which time-varying demands from customers are known and must be met without backlogging. A nonlinear mathematical formulation of the UDSLSS-TVE problem is given as follows.

Minimize:

$$Total\ cost = \sum_{t=1}^n (h_t \cdot I_t + s_t \cdot y_t + \alpha L_t y_t) \quad (10)$$

Subject to:

$$I_t = \begin{cases} I_0 - D_t & \forall 1 \leq t < l \\ I_{t-1} + X_{t-l} - D_t & \forall l \leq t \leq n \end{cases} \quad (11)$$

$$L_t = t - 1 - \max\{t' \cdot y_{t'} | 1 \leq t' < t\} \quad \forall t \neq 1 \quad (12)$$

$$L_1 = 0 \quad (13)$$

$$I_t \geq 0 \quad \forall t \quad (14)$$

$$X_t \geq 0 \quad \forall t \quad (15)$$

$$y_t \in \{0, 1\} \quad \forall t \quad (16)$$

The objective function in Equation (10) is to minimize the sum of the production setup cost and inventory-holding cost in all periods, including the deteriorating effect of the production interval on the setup costs. Constraint (11) is the inventory flow along the periods in the horizon. It represents the balance between production batches, external demands and inventory. Constraints (12) and (13) calculate the production interval L_t for each period t according to Definition 1. The production interval of the first period is set to be 0. Constraint (14) indicates that backlogging is not allowed. Constraint (15) indicates that the production quantity in each period is non-negative. Constraint (16) defines that y_t is a binary decision variable, indicating that if a production setup is scheduled in period t , then, $y_t = 1$; otherwise $y_t = 0$.

The above UDSLSS-TVE model is nonlinear and has a higher level of complexity than traditional lot-sizing problems. In Section 3, we developed an exact solution approach based on forward dynamic programming with a computational complexity of $O(n)$. In Section 2.1, we provided a linear model for the UDSLSS-TVE problem which can be optimally solved by commercial MIP solvers such as CPLEX and Lingo. In the following, we analyzed the new features of the above UDSLSS-TVE model and brought about new properties regarding the feasibility and optimality of the solution.

Property 1. For an optimal solution of the UDSLSS-TVE model formulated in Equations (10)–(16), it satisfies $I_t \cdot X_{t-l} = 0$ and $I_t \cdot y_{t-l} = 0$ at any period, t and $t > l$, where l is the leading time.

Proof. This proof is by contradiction. Assuming there is an optimal solution with a period, t and $t > l$, that $I_t \cdot X_{t-l} = 0$ holds. That means a production setup has been arranged at period $t-l$ for the demand in period t (because the leading time is l) before the inventory being exhausted. We can always arrange the inventory quantity I_{t-l} to be produced in period $t-l$. This leads to save the inventory-holding cost resulting from inventory I_{t-l} and without changing the feasibility of the solution. Therefore, the newly arranged solution is better than the original optimal solution, which is contradictory to the assumption. Thus, the property $I_{t-l} \cdot I_{t-1} \geq 0$ holds. By a similar way, we can also prove that the property $I_t \cdot y_{t-l} = 0$ holds.

Property 2. For an optimal solution of the UDSLSS-TVE model formulated in Equation (10)–(16), it satisfies $X_t = 0$ or $\sum_{j=t+l}^k D_j$, where $k = \min\{t' | t' \geq t+l, y_{t'} = 1\}$ and l is the leading time.

Proof. This property is straightforward as it is a common property of traditional lot-sizing models, indicating that the production lot-sizing is either zero or a positive number that covers all the demands in periods from $t+l$ to the next period that has been scheduled with a production setup. The introduction of deteriorating production setup cost does not change this general lot-sizing rule.

Property 3. For an optimal solution of the UDSLLS-TVE model formulated in Equations (10)–(16), if the demand in the period t^* , i.e., D_{t^*} , is satisfied by the production quantity in the period t^{**} and $t^{**} < t^*$, i.e., $X_{t^{**}}$, the demand D_t for any period t between t^{**} and $t^* - l$ is also satisfied by $X_{t^{**}}$, where l is the leading time.

Proof. The property is in accordance with the general first-in-first-out (FIFO) rule of lot-sizing problems, indicating that the demand in each period must be satisfied by the latest produced products. The FIFO rule guarantees the lowest inventory-holding cost for produced product and violation of this rule will always lead to a sub-optimal solution.

Property 4. For an optimal solution of the UDSLLS-TVE model formulated in Equations (10)–(16), if $D_t = 0$ and $s_{t-l+1} - s_{t-l} - X_{t-l} \cdot h_{t-l} + \alpha < 0$, then $y_{t-l} = 0$ holds, where $t > l$ and l is the leading time.

Proof. This is proven by contradiction. Assuming there is an optimal solution with a period, t and $t > l$, that is contradictory to this property, i.e., $D_t = 0$, $s_{t-l} > \alpha$, and $y_{t-l} = 1$. We can always move the production setup in period $t - l$ to $t - l + 1$ by letting $y_{t-l} \leftarrow 0$ and $y_{t-l+1} \leftarrow 1$, such that the new solution remains feasible (because of $D_t = 0$) and the total cost change is $\Delta = s_{t-l+1} - s_{t-l} - X_{t-l} \cdot h_{t-l} + \alpha$. As $\Delta < 0$, thus the new solution is better than the original optimal solution, which is contradictory to the assumption.

2.2. A Linear Model of UDSLLS-TVE

To eliminate the nonlinear components introduced by the variable L_t in the model formulated in Equations (10)–(16), which are in the objective function Equation (10) and Constraint (12), we use the unit increment of setup u_t , instead of L_t , to convert the nonlinear model into a linear one. The linear model will be used to serve as the benchmark method in the computational section to provide optimal solutions for comparing with the proposed exact dynamic programming algorithm.

Definition 3. Increment of setup cost u_t : A binary decision variable indicating if one unit of setup cost occurred in period t (by $u_t = 1$) or not (by $u_t = 0$).

The variable u_t , as an alternative of L_t , plays the same role in representing the deteriorating effect of the production interval on the setup costs. Thus, we can reformulate the UDSLLS-TVE problem as an integer linear programming model for as follows.

Minimize:

$$\text{Total cost} = \sum_{t=1}^n (h_t \cdot I_t + s_t \cdot y_t + \alpha \cdot u_t) \quad (17)$$

Subject to:

$$I_t = \begin{cases} I_0 - D_t & \forall 1 \leq t < l \\ I_{t-1} + X_{t-l} - D_t & \forall l \leq t \leq n \end{cases} \quad (18)$$

$$u_t \geq u_{t+1} + y_{t+1} - 2 \cdot y_t \quad \forall t < n \quad (19)$$

$$u_n = 0 \quad (20)$$

$$I_t \geq 0 \quad \forall t \quad (21)$$

$$X_t \geq 0 \quad \forall t \quad (22)$$

$$y_t \in \{0, 1\} \quad \forall t \quad (23)$$

Note that the objective function in Equation (17) allows us to set an individual increment of setup cost α_t for each period t , which enhances the model to fit for a more complex production environment.

Constraints (19) and (20) are designed to guarantee that for each period t , an extra cost, α_t , always occurs once there is no production setup in period t (by $y_t = 0$), and there is always no extra cost for the last period. The rest of the other constraints are the same with those in Section 2.1. Note that since the above model formulated in Equations (17)–(23) is a linear model, it can be optimally solved by the MIP solver CPLEX used in the experiments section for small-sized instances.

3. Dynamic Programming Approach for UDSLLS-TVE Problem

3.1. Dynamic Programming Formulation

Before giving the dynamic programming formulation for the UDSLLS-TVE problem, several new definitions that are to be used in the formulation are introduced in advance as follows.

Definition 4. Sub-problem S_t . It represents the UDSLLS-TVE sub-problem that covers only the first t periods (i.e., from period 1 to t) of the original problem P .

Definition 5. Optimal cost of P_t at $y_t = 1$ with respect to different L_t : $S_t(L_t)$. It represents the optimal total cost of sub-problem P_t , including production setup cost and inventory-holding cost, at the conditions that $y_t = 1$ the production interval of period t takes the value of L_t .

Definition 6. Optimal cost of P_t at $y_t = 1$: $f(t)$. It represents the optimal total cost of the sub-problem P_t at the condition that a production setup is scheduled at period t , i.e., $y_t = 1$.

Definition 7. Optimum production interval before period t of P_t at $y_t = 1$: L_t^* . It indicates the optimal production interval before period t of problem P_t if a production setup is scheduled in period t (i.e., $y_t = 1$). Under this situation, the problem P_t has the cost of $f(t)$.

Without loss of generality and for the simplicity of the expression, we assumed a zero leading time in the formulation. According to the objective function (10) and Definition 5, the cost of sub-problem P_t at $y_t = 1$ can be formulated as a dynamic programming expressed as follows.

$$S_t(L_t) = \begin{cases} S_{t-1}(L_{t-1}^*) + s_t & \forall L_t = 0 \\ S_{t-1-L_t}(L_{t-1-L_t}^*) + s_t + \alpha L_t + \sum_{j=t-1-L_t}^{t-2} \sum_{k=j+1}^{t-1} h_j D_k & \forall 1 \leq L_t \leq t-2 \end{cases} \quad (24)$$

The above Equation (24) is a forward recursion dynamic formulation for calculating the values of $S_t(L_t)$ from the first period to the last period. The term $\sum_{j=t-1-L_t}^{t-2} \sum_{k=j+1}^{t-1} h_j D_k$ represents the inventory-holding cost occurred in the production intervals. According to Definition 6, $S_{t-1-L_t}(L_{t-1-L_t}^*)$ and $S_{t-1}(L_{t-1}^*)$ can be expressed as $f(t-1-L_t)$ and $f(t-1)$. Thus, the optimal cost of P_t at $y_t = 1$, i.e., $f(t)$, can be calculated with a forward dynamic programming expressed as follows.

$$f(t) = \min \left\{ \begin{array}{l} \min_{1 \leq L_t \leq t-2} \left[\alpha L_t + s_t + \sum_{j=t-1-L_t}^{t-2} \sum_{k=j+1}^{t-1} h_j D_k + f(t-1-L_t) \right] \\ \alpha + s_t + f(t-1) \end{array} \right\} \quad (25)$$

Definition 8. Optimal Cost of P_t at $y_t = 0$ with respect to different L_t : $H_t(L_t)$. It represents the optimal total cost of sub-problem P_t , at the condition that $y_t = 0$ and the production interval of period t takes the value of L_t .

Definition 9. Optimal cost of P_t at $y_t = 0$: $g(t)$. It represents the optimal total cost of the sub-problem P_t at the condition that a production setup is scheduled at period t , i.e., $y_t = 0$.

Definition 10. Optimum production interval before period t of P_t at $y_t = 0$: L_t' . It indicates the optimal production interval before period t of problem P_t if no production setup is scheduled in period t (i.e., $y_t = 0$). Under this situation, the problem P_t has the cost of $g(t)$.

According to Definitions 5 and 8, the cost of sub-problem P_t at $y_t = 0$ can be calculated by using the forwardly recursive formulation in Equation (26) as follows.

$$H_t(L_t) = \sum_{j=t-1-L_t}^{t-1} \sum_{k=j+1}^t h_j D_k + H_{t-1-L_t}(L_{t-1-L_t}) \quad \forall t > 1 \quad (26)$$

By above Equation (26), all values of $H_t(L_t)$ can be calculated with respect to each value of variable L_t in $[0, t-2]$. Thus, according to Definition 10, the function $g(t)$, i.e., the optimal cost of P_t at $y_t = 0$, takes the minimum value among them.

$$g(t) = \min_{0 \leq L_t \leq t-2} \left[\sum_{j=t-1-L_t}^{t-1} \sum_{k=j+1}^t h_j D_k + f(t-1-L_t) \right], \quad \forall t > 1 \quad (27)$$

Let $F(t)$ represent the optimal cost of the sub-problem P_t . Thus, $F(t)$ is determined by comparing the lower value between $f(t)$ and $g(t)$ for the case of period t being scheduled with a production setup (by $y_t = 1$) or not (by $y_t = 0$).

$$F(t) = \begin{cases} s_t & \forall t = 1 \\ \min[f(t), g(t)] & \forall 1 < t \leq n \end{cases} \quad (28)$$

Thus, the above Equations (25), (27) and (28) form a forward dynamic programming formulation, by which the optimal cost of sub-problem P_t can be determined forwardly from $t = 1$ to $t = n$, and finally the optimal cost of original problem P_n is determined.

3.2. Algorithm of Dynamic Programming

Based on the formulation in Section 3.1 for the UDSLTS-TVE model, we assumed that for a problem instance with n planning period, the dynamic programming algorithm for sub-problem P_{t^*} at any period t^* can be described by the steps as follows.

- (1) Suppose that a production setup is scheduled in period t^{**} ($t^{**} = 1, 2, \dots, t^* - 1$), the production quantity covers the demand of D_t for $t = t^{**}, t^{**} + 1, \dots, t^* - 1$, and the production interval corresponding to period t is $L_t = t^* - 1, t^* - 2, \dots, 1$.
- (2) Calculate $H_t(L_t)$ and $S_t(L_t)$ at the period t^* and for $L_t = t^* - 1, t^* - 2, \dots, 1$ according to Equations (24) and (26), respectively.
- (3) Determine $f(t)$, $g(t)$, and $F(t)$ at the period t^* according to Equations (25), (27) and (28), respectively.
- (4) If $t^* = n$, the algorithm ends; Otherwise, let $t^* \leftarrow t^* + 1$ and repeat steps (1), (2), (3) and (4).

According to the above formulation, we just need to make a calculation of $2(t-1)$ times for the sub-problem P_t at period t , and the optimal solution of the original problem can be obtained by no more than $n(n-1)$ times of calculation. Therefore, the algorithm has a polynomial computational complexity of $O(n(n-1))$. The pseudo code of the algorithm is provided in Figure 1.

- (1) Let $t = 1$, $y_1 = 1$, $L_1 = 0$, and $f(1) = S_1(0) = s_1$.
- (2) For $t = 2$ to n , find the optimal total cost $F(t)$ corresponding to period t , this includes two sub-steps of calculating the optimal production cost $f(t)$ and the optimal non-production cost $g(t)$ during the period t .
- (i) Calculate $f(t)$ for period t :
- ① When $L_t = 0$, $S_t(0) = \alpha + s_t + f(t-1)$;
 - ② When $1 \leq L_t \leq t-2$, $S_t(L_t) = \alpha L_t + s_t + \sum_{j=t-1-L_t}^{t-2} \sum_{k=j+1}^{t-1} h_j D_k + f(t-1-L_t)$;
 - ③ Compare the values for each $S_t(L_t)$, the minimum value is $f(t)$ and the correspond L_t is L'_t .
- (ii) Calculate $g(t)$ for period t :
- ① When $0 \leq L_t \leq t-2$, $H_t(L_t) = \sum_{j=t-1-L_t}^{t-1} \sum_{k=j+1}^t h_j D_k + H_{t-1-L_t}(L_{t-1-L_t})$
 - ② Compare the value for each $H_t(L_t)$, the minimum value is $g(t)$ and the L_t is L'_t .
- (ii) Compare $f(t)$ and $g(t)$, the smaller one is $F(t)$.
- (3) If $t = n$ then go to Step (4); otherwise, let $t \leftarrow t + 1$, repeat steps (2), (3), and (4).
- (4) When $t = n$, the $F(n)$ obtained is the optimal total cost for the entire planning period.

Figure 1. Pseudocode description of dynamic programming for the UDSLTS-TVE problem.

4. Computational Experiment

4.1. Example Verification of Dynamic Programming Algorithm

First, we verified the feasibility of the algorithm in Figure 1 by comparing the results with that obtained by the classic WW algorithm. The problem instance is given in Table 3, and includes the external demand, setup cost, and unit holding cost for a planning horizon with 12 periods/months. The production lot-size for each month is randomly determined. The unit increment of production setup cost is set by $\alpha = 50$. The leading time is zero.

Table 3. Algorithm description of the UDSLTS-TVE problem.

Month (t)	D_t	s_t	α	h_t
1	69	85	50	1
2	29	102	50	1
3	36	102	50	1
4	61	101	50	1
5	61	98	50	1
6	26	114	50	1
7	34	105	50	1
8	67	86	50	1
9	45	119	50	1
10	67	110	50	1
11	79	98	50	1
12	56	114	50	1
AVG	52.5	102.8	50	1

The forward dynamic programming algorithm proposed in Section 3.2 was coded by VC++ 6.0 language executed in a PC computer with a Windows 7 system. The RAM size is 2 GB and CPU is Core i3 dual-core M380 (4 CPUs) 2.53 GHz. The problem instances were solved with optimal solution $F(12) = 1091$. Detailed production schedules and the production lot-size in all planning periods are shown in Table 4.

Table 4. Solution results of the UDSLTS-TVE case.

Planning Period	1	2	3	4	5	6	7	8	9	10	11	12
y_t	1	0	1	1	1	0	1	1	0	1	1	0
X_t	98	0	36	61	87	0	34	112	0	67	135	0

In order to verify the optimality of the dynamic algorithm, we enumerated all combinations of feasible solutions and found that the optimal solution is the same with that found by the dynamic programming algorithm. In addition, the solution efficiency is greatly improved because the solution time is reduced from exponential time to polynomial time.

4.2. Comparison through Large-Scale Experiments

In order to better verify the dynamic programming algorithm proposed in this paper, large-scale calculation experiments were conducted. We randomly generated 100 problem instances with 12, 24, 48 planning periods, respectively. The forward dynamic programming algorithm was used to solve all these test problem instances. After that, we used commercial software AMPL/CPLEX (Version 12.0) to solve these problem instances with the linear UDSLTS-TVE model formulated in Equations (17)–(23). All tested instances were optimally solved by the AMPL/CPLEX software. In Tables 5–7, we show the first five instances of the tested 100 instances by two methods.

Table 5. Results of problem instances with 12 periods.

No.	Dynamic Programming		AMPL/CPLEX
	Cost	Time	Cost
1	769.93	<1 s	769.93
2	717.54	<1 s	717.54
3	510.17	<1 s	510.17
4	566.52	<1 s	566.52
5	720.04	<1 s	720.04
...	...	...	...
AVG of 100	712.37	<1 s	712.37

Table 6. Results of problem instances with 24 periods.

No.	Dynamic Programming		AMPL/CPLEX
	Cost	Time	Cost
1	1373.02	<1 s	1373.02
2	1588.75	<1 s	1588.75
3	1412.26	<1 s	1412.26
4	1103.34	<1 s	1103.34
5	1450.20	<1 s	1450.20
...	...	...	...
AVG of 100	1482.06	<1 s	1482.06

Table 7. Results of problem instances with 48 periods.

No.	Dynamic Programming		AMPL/CPLEX
	Cost	Time	Cost
1	2376.20	<1 s	2376.20
2	2869.74	<1 s	2869.74
3	3012.33	<1 s	3012.33
4	2879.96	<1 s	2879.96
5	2690.01	<1 s	2690.01
...	...	...	...
AVG of 100	2793.16	<1 s	2793.16

In order to present the results of computer experiments to the readers more conveniently and intuitively, we listed the first five lines of the calculation results under 12, 24, 48 planning periods respectively and omit the rest. For different kinds of planning periods, we provided the average value of 100 problem instances by two different methods. From Tables 5–7, it can be observed that the dynamic programming algorithm obtained the same optimal solution obtained by the AMPL/XPLEX MIP solver. All tested instances were in perfect agreement with the optimal solution by the AMPL/CPLEX software. In addition, the computational complexity of AMPL algorithm is 2^n while that of dynamic programming algorithm is n^2 . Computational time of AMPL/CPLEX increased more rapidly than that of the dynamic programming algorithm as the problem size increased. For all scales of problems tested in the experiments, the computational time spent by dynamic programming was always less than one second. That means the proposed model and approximation method are quite effective. The proposed models describe well the UDSLTS-TVE problem with mathematical formulations, and the proposed dynamic programming algorithm succeeds to yield optimal solutions of the UDSLTS-TVE model with high computational efficiencies.

5. Conclusions

The UDSLTS-TVE problem studied in this paper considered a continuous change of demands in different planning horizons and the changes of setup cost as well, and was formulated as mathematical programming models based the conventional lot-sizing model. We analyzed the influence of production interval on production setup cost in a time-varying environment. Two mathematical models were developed for the uncapacitated dynamic single-level lot-sizing problem under a time-varying environment (UDSLTS-TVE for short). Proposed properties of the UDSLTS-TVE problem were analyzed for developing efficient solution algorithms. An exact solution approach based on dynamic programming, which has a computational complexity of $O(n^2)$, was developed for the UDSLTS-TVE problem and significantly reduced the complexity of original problem, which is $O(2^n)$. Computational experiments were conducted to compare the obtained solution by our proposed algorithm with the optimal solution obtained by using AMPL/CPLEX software, which show the optimality and high efficiency of the dynamic programming algorithm. Though our study achieved a series of achievements mentioned above, the study was limited to machining process and did not take into account some other factors like in-plant transport organization and capacity exploitation. For future studies, multi-level product structures, in-plant transport organization, production capacity, and more time-varying variables can be taken into consideration to make the model more realistic.

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