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Optimal Replenishment for Perishable Products with Inventory-Dependent Demand and Backlogging under Continuous and Discrete Progressive Payments

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Received: 26 August 2018; Accepted: 26 September 2018; Published: 16 October 2018

Abstract: Managing material and cash flows attracts concerns of physical and financial departments in most companies. We, therefore, focus on optimizing the replenishment policy for a channel with stock-dependent demand considering item deterioration and order backlogging under two financial schemes of progressive trade credit periods. We take both the continuous payment regime (CPR) and discrete payment regime (DPR) into account in the progressive trading process, which generates ten distinct scenarios. We show that the profit functions may not necessarily be concave and accordingly give a corresponding computing algorithm, which relaxes the convexity assumptions of objective functions in the existing literature and consequently enrich the research. We address the formulation characterization and the logic of pursuing global optimization from models arising in all settings. Computational studies and simulations are conducted to illustrate the effect of various parameters on the optimal replenishing policy and profit. Numerical experiments show that the CPR scheme is dominantly prior to DPR for long replenishment time intervals, whereas it is exactly the opposite for short time intervals. We also examine the impact of the shortage cost and deteriorating rate on optimum ordering policy and channel performance. Finally, future research directions are addressed in the end.

Keywords: replenishing policy; stock-dependent demand; order backlogging; deterioration; progressive payments; operations optimization

1. Introduction

Managing material and cash flows is the common concern for both physical and financial departments both in most of business organizations [1]. In this sense, it is usually viewed as the interesting interface of financing and operations to seek trade-off between inventory control and accounting (including purchasing payment). For the transaction in supply chains, it is common in practice for the supplier to offer the downstream buyer different paying schemes after the transfer of the goods occurs, namely the *Progressive Payment Scheme* that has been extensively studied. This kind of strategy of payment delay is offered as an incentives of inspiring buyers to enlarge their purchasing amount, which is expected to incur similar effects resulting from a variety of incentive contracts [2] for enhancing supply chain performance. Aggarwal and Jaggi [3], Sarker et al. [4] and Soni and Shah [5] discuss the optimal ordering policy under the progressive payment scheme. In recent years, considering the progressive payment scheme, Pal [6] and Sen and Saha [7] also investigate the optimal

decisions of supply chain. From here we see that the progressive payment scheme is one of the most significant factors in ordering decisions, which is also the key factor for the research in this paper.

For the characteristics of market demand, practitioners in practice and academia have both observed the phenomenon that the demand rate is usually affected by the inventory on hand, which has been addressed by dozens of studies since Levin et al. [8]. Shukla et al. [9] developed two different mathematical models in terms of the different types of holding cost function with stock-level dependent demand for obtaining the optimal order quantity. Stock-dependent products is described as a key factor influencing the optimization problem in Fu and Chen [10], in which two functions of minimizing total cost with or without shortage are developed. Thus, it is important for a firm confronting the stock-dependent demand to find the optimum ordering policy associating its stock-on-hand evolving path in order to maximize (or minimize) the total profit (cost) in the long run.

On the other hand, some products may deteriorate depending on the storage conditions. The deteriorating items are classified into two categories in Li et al. [11], the first category are the decayed items, such as meat, fruits, vegetables, the other category are items that lose value, such as computer chips and mobile phone. The deteriorating products in this paper focus on the first category. As we all know, the deterioration of items shows a large impact on inventory decisions of firms. Therefore, how to control the items deterioration to mitigate its influence on inventory is a considerable question, which has gained much attention in academia such as Wang [12], Dye [13], Majumder et al. [14] and Ardak et al. [15].

We investigate the optimal reordering problem of deteriorating items with inventory-dependent demand in the scenario with a plausible delay payment scheme newly proposed. For this purpose, we clearly define two different delay payment regimes, *Continuous Payment Regime* (hereinafter referred as CPR) and *Discrete Payment Regime* (hereinafter referred as DPR) as follows: (1) *Continuous Payment Regime*: over some specific period, the buyer pays the supplier the money owed using the instantaneous revenue continuously until all purchasing cost has been paid back completely; (2) *Discrete Payment Regime*: the buyer accumulates revenue by depositing it in bank and pay money at some discrete specific time instants until all purchasing cost are cleared off.

The reminder of this paper is organized as follows. In Section 2, the relevant literatures are reviewed. In Section 3, we formulate the problem as well as define notations. In Section 4, we develop models in a variety of scenarios considering CPR and DPR in terms of the variation of permissible delay time, respectively. In Section 5, we firstly give a unified computational algorithm and then conduct computational experiments and sensitivity analyses to obtain some managerial insights. Future research and conclusions are offered in Section 6.

2. Literature Review

This study is typically featured in three aspects, namely, deterioration, stock-dependent demand and financially progressive payment credits. We therefore give a brief review over the literature from these three tracks. An increasing body of literature has focused on inventory control problems with deterioration in different settings. Wang [12], Dye [13] and Chang and [16] discussed the replenishment policy for deteriorating item considering backlogging. Yang et al. [17] formulate the inventory system for non-instantaneous deteriorating items with the price-dependent demand, aiming to find the optimal selling price and replenishment cycle time to maximize the total profit. Khanra et al. [18] assume the deterioration rate is constant, and developed the economic ordering quantity (EOQ) model for deteriorating item. Further, Singh and Pattnayak [19] explore the EOQ model by extending the aforesaid paper, suppose the deterioration rate is variable rather than constant. More studies related to the effect of deteriorating rate on inventory, and one can refer to Majumder et al. [14], Ardak et al. [15] and Pal and Samanta [20].

There is also a number of studies on modeling progressive payment schemes. Aggarwal and Jaggi [3], Sarker et al. [4] and Soni and Shah [5] are among early researches to develop some optimal ordering/quantity policies for system under progressive payment scheme. Chung [21] developed a

new solution procedure to overcome the shortcoming of that in Goyal et al. [22] in terms of the classical EOQ model considering payment delay. Mohan et al. [23] explored the optimal replenishment policy for multi-item in three different situations: the permissible delay is negative, is less than the replenishment interval and is greater than the replenishment interval. Recently, Chen and Teng [24] proposed a simple arithmetic-geometric method to solve the extended EOQ model under a conditionally permissible delay in payments. The same year, Li et al. [25] describe the inventory games where the retailer's delay in payments is permitted by the supplier, and proved that the core of inventory games is nonempty under a permissible delay. For more research on the replenishment considering the permissible delay in payment also see Li et al. [1], Vandana and Sharma [26], Sen and Saha [7] and Pal [6].

For the track of stock-dependent demand, Min and Zhou [27] addressed a deterministic inventory model with demand backlogging permitted and exploited a solution procedure for optimum replenishment policy in a similar way as Lee and Dye [28]. In a more complicated situation, Min et al. [29] considered sequential delayed payments between the retailer with her supplier and customers, respectively and focused on optimizing retailer's average profit. Although Sana [30] and Sana [31] also both considered stock dependent demand, they moved to study multiple products considering sales efforts' effect on demand as well. However, these two studies do not take delay payment into consideration while only showing the optimal replenishment policies. Extending the single-firm problem to the supply chain situation, Zhou et al. [32] analyzed the interactions of ordering and paying issues in a non-cooperative game theory framework. Their work opens the door to study stock-dependent demand featured ordering and trading policies with intertwined decision-makings in supply chain scenario. There are also a series of EOQ-based models with stock-dependent demand, such as Ray and Chaudhuri [33], Teng and Chang [34], Teng and Ouyang [35], Tyagi et al. [36], Shukla et al. [9], Ghosh et al. [37] and Fu and Chen [10].

Among the most relevant research to and our study, Teng et al. [38] provide a comprehensive extensions by allowing a non-zero ending inventory, bounded inventory capacity, and deterioration based on Soni and Shah [5]. Glock et al. [39] correct some formulation errors in Soni and Shah [5] and research a new scenario in which the retailer's credit interest rate is greater than the supplier's charging interest rate. However, they focus on variant problems using a similar framework of modeling and a solving method with that in Soni and Shah [5]. Different from the above studies, we not only simultaneously consider three factors of deterioration, stock-dependent demand and financially progressive payment credits, but also incorporate two delayed payment schemes and permit order backlogging, which is closer to reality yet seldom examined together.

Compared with the aforementioned existing literatures only adopting a DPR-like policy, our study considers CPR and DPR simultaneously in two main aspects: (1) under CPR, we initially formulate the reordering policy optimization problem, develop models and exploit associated solving processes; (2) under DPR, we reconstruct the formulation and modeling and moreover we permit the time point for clearing off purchasing cost (including charging interest) either in CPR or in DPR, which does not necessarily just hit the time instant T exactly as in Soni and Shah [5] and Teng et al. [38]. Namely, for $Z(T)$ the inventory level at time T equals to zero does not mean that the retailer needs to settle all unpaid money just at T . Relaxating the constraint of the clearing off time point at T is significant and will substantially affect the problem formulation, modeling and solving process. Second, when modeling and solving our problem, we remove the pre-conditions and assumptions that all objective functions are convex ensured by putting constraints upon the decision variable T which has been shown frequently in Soni and Shah [5], Teng et al. [38], Glock et al. [39] and so on. This relaxation is managerially significant again due to its impact on modeling and solving process and of course the optimal solution since the objective function should not be confined or necessarily convex (or concave) from the point of practice.

3. Preliminaries and Notations

Consider a retailer purchases finished goods from the upstream supplier in repeated cycles. The market demand facing the retailer is dependent on the timely stock level on hand, which is commonly assumed extensively in the literature and found in reality. Suppose the items are perishable goods that can deteriorate over time. Therefore, the inventory is depleted at three factors of stock level, deteriorating and constant consumption rates. Moreover, assume the retailer permits the order backlogging in each cycle and the unfilled orders will be served in the succeeding period. Hence, the schematic curve of inventory level over time can be plotted as shown in Figure 1, where the retailer receives $Q + V$ units goods from the supplier at the beginning of each replenishment period ($t = 0$) and immediately put V units of products to meet the backlogging orders inherited from the preceding period, and accordingly Q units are held as the initial inventory of the present period. In each replenishment cycle, the retailer can earn interest income from depositing sales revenue in the bank until the deadline of the permitted credit periods. The time instant T_c starts the shortage interval implying that the retailer needs to decide suitable V for backlogging all orders generated in this meantime.

Figure 1. The change of the inventory level over time.

In addition to the ordering and replenishment problem mentioned above, we also incorporate the issue of payment in instalment for goods into the study, which intertwines inventory ordering policy to substantially impact the system profit since the payment by instalment can also bring penalty and/or income. Therefore, considering the perishable goods with deterioration, we intend to investigate the system optimal ordering policy with shortage and stock-dependent demand under two trading regimes and credits. Specifically, several cases are discussed according to the variation of credit periods.

For the necessity of problem formulating, the following assumptions are adopted throughout the whole paper: (i) The demand rate, $R(t) = a + bZ(t) + \theta Z(t)$, is a function of the inventory level $Z(t)$, where a , b and θ are intrinsic demand, sensitivity parameter and deteriorating rate to inventory level, respectively, with $a > b > 0$, $0 \leq \theta < 1$ and $R(t) = a$ when $T_c \leq t \leq T$. (ii) Lead-time is neglected and replenishment usually occurs at the beginning of each period. Furthermore, assume the supplier's capacity is infinite without defective item. (iii) as for the payment schedule, the retailer will charge additional fees for the unpaid balances as follows: nothing in the case of payment clearance before time M , at a rate of I_{c1} supposing payment clearance during time interval $[M, N]$ and a rate of I_{c2} given payment clearance after N with $I_{c2} > I_{c1} > I_e$ and I_e denoting interest the rate of banks.

4. Mathematical Models

According to the illustrations and assumptions in Section 2, the depletion of inventory is due to three aspects: the constant demand a , the stock-depended demand and the deterioration in the

products. Thereby, we use the differential equations below to govern the instantaneous state of inventory at time t

$$dZ(t)/dt = -a - bZ(t) - \theta Z(t), \quad 0 \leq t \leq T_c \quad (1)$$

$$dZ(t)/dt = -a, \quad T_c < t \leq T \quad (2)$$

Then the stock level can be obtained below in view of the boundary conditions $Z(T_c) = 0$ and $Z(0) = Q$.

$$Z(t) = \frac{a}{b+\theta} [e^{(b+\theta)(T_c-t)} - 1], \quad 0 \leq t \leq T_c \quad (3)$$

$$Z(t) = -a(t - T_c), \quad T_c < t \leq T \quad (4)$$

Combining the condition $Z(T) = -V$ and Function (4) yields

$$T_c = T - V/a \quad (5)$$

Thus, the inventory level in period $[0, T]$ can be summarized as below

$$Z(t) = \begin{cases} \frac{a}{b+\theta} [e^{(b+\theta)(T-V/a-t)} - 1], & 0 \leq t \leq T - V/a \\ a(T - V/a - t), & T - V/a < t \leq T \end{cases} \quad (6)$$

Additionally, the retailer's other costs per unit time after value assignment include ordering cost A/T , shortage cost c_0V/T , and holding cost $Z(t) = \begin{cases} \frac{a}{b+\theta} [e^{(b+\theta)(T-V/a-t)} - 1], & 0 \leq t \leq T - V/a \\ a(T - V/a - t), & T - V/a < t \leq T \end{cases}$.

In view of diverse credit periods with different penalty interests charged by supplier as mentioned previously, four major cases with subcases will be discussed as follows, after which we will compare all cases under different progressive payment schemes to recognize the optimum replenishing policy.

4.1. Case 1 $T_c < T \leq M < N$

We first consider an extreme but simplest situation where the replenishment cycle is less than the first credit time M (shown in Figure 2). In this case, assume the retailer intends to exploit the advantage of no penalty incurred in the case of payment clearance before the first credit time M . Thus, the rational retailer will definitely clear all outstanding payment at M since she can deposit the sales revenue in the bank to earn interest at the rate of I_e . Considering the costs incurred by ordering, wholesaling, shortage and inventory holding, the retailer's problem is to determine suitable replenishing cycle time and shortage to maximize her annual profit, which can be expressed as:

$$\begin{aligned} \underset{(T,V)}{\text{Max}} \quad AP_1(T, V) = & \frac{1}{T} \left\{ p \left[\int_0^{T_c} R(t) dt + V \right] + pI_e VM + pI_e \int_0^{T_c} R(t)(M-t) dt \right. \\ & \left. - c(Q+V) - A - c_0V - h \int_0^{T_c} Z(t) dt \right\} \\ \text{s.t.} \quad & \begin{cases} T_c = T - V/a \\ T_c < T \leq M < N \\ V \leq V_0 \end{cases} \end{aligned} \quad (7)$$

The first term, the sum of second and third terms and the rest in the curly braces of above Equation (7) represent sales revenue, interest earned and manifold aforesaid costs, respectively.

4.2. Case 2 $M < T_c < N$ ($M < T_c \leq N < T$ or $M < T_c < T \leq N$)

We now consider the situation in which the replenishment time interval can be divided into two parts: $[0, M]$ and $[M, T]$ (shown in Figure 3). The profit of retailer includes both sales revenue and interest earned during the credit period $[0, M]$. While penalty interest is charged at rate of I_{c1} during in

the period $[M, T]$, the wise retailer will repay the wholesale cost at the time M to avoid the penalty interest charged by the supplier. The interest earned during the credit period $[0, M]$ takes the form:

$$IE_{[0,M]} = pI_e VM + pI_e \int_0^M R(t)(M-t)dt \quad (8)$$

Figure 2. Inventory level over time in case 1.

Figure 3. Inventory level over time in case 2.

The remaining deposit of the retailer after repaying the wholesale cost to the supplier at time M can be expressed as follows:

$$U_{2-1} = pV + p \int_0^M R(t)dt + pI_e VM + pI_e \int_0^M R(t)(M-t)dt - c(Q+V) \quad (9)$$

The first two terms of the Equation (9) represents the sales revenue, the sum of the third and fourth terms denotes the earned interest of the retailer by depositing the sales revenue in the bank, and the last term of the function represents the wholesale cost.

It should be noted that whether the revenue including sales revenue and earned interest is sufficient to repay all the wholesale cost. Considering the problem, two subcases are discussed below.

4.2.1. Subcase 2-1 The Revenue Being Sufficient to Repay the Wholesale Cost at Time M

The retailer repays all the wholesale cost at time M , according to the Equation (9), immediately, the annual profit of retailer in the whole replenishment time interval takes the form:

$$\begin{aligned} \text{Max}_{(T,V)} AP_{2-1}(T, V) = & \frac{1}{T} \left\{ p \int_M^{T_c} R(t) dt + p I_e \int_M^{T_c} R(t) (T-t) dt + U_{2-1} [1 + I_e (T-M)] - A - c_0 V - h \int_0^{T_c} Z(t) dt \right\} \\ \text{s.t.} \left\{ \begin{array}{l} U_{2-1} = pV + p \int_0^M R(t) dt + p I_e VM + p I_e \int_0^M R(t) (M-t) dt - c(Q+V) \\ M < T_c < N \\ p \left[\int_0^M R(t) dt + V \right] + p I_e VM + p I_e \int_0^M R(t) (M-t) dt \geq c(Q+V) \\ V \leq V_0 \end{array} \right. \end{aligned} \quad (10)$$

4.2.2. Subcase 2-2 The Revenue Is Insufficient to Repay the Wholesale Cost at Time M

In this subcase, we consider that the revenue is insufficient to repay the wholesale cost in the permissible credit period. For the wise retailer, all revenue including sales revenue and earned interest will be spent on repaying the wholesale cost at time M to avoid paying higher penalty interest. The unpaid money accordingly at time M is shown as:

$$U(M) = c(Q+V) - pV - p \int_0^M R(t) dt - p I_e VM - p I_e \int_0^M R(t) (M-t) dt \quad (11)$$

Obviously, the unpaid money must be paid by retailer and the penalty interest will be incurred beyond the credit period M . Meanwhile, the supplier provides two ways for the retailer regarding how to pay the remaining balance: continuous and discrete payment regimes.

Subcase 2-2a Continuous Payment Regime (CPR)

With the continuous payment regime, the retailer continuously uses the instantaneous revenue to repay the debt until all wholesale costs are completely paid off. Denote t_a the time at which the unpaid balance is cleared off under the continuous payment regime through the paper, $M < t_a \leq T_c$. Especially, the profit of the retailer in the period ($T_c < t_a$) is negative, to facilitate the research, the situation is excluded in this model.

The unpaid money during the $[M, T_c]$ can be calculated from

$$dU(t)/dt = -pR(t) + I_{c1}U(t), \quad M < t < T_c \quad (12)$$

Combining the Equations (12) and (13), the unpaid money takes the form:

$$U(t) = \frac{pa\theta e^{-I_{c1}t}}{I_{c1}(b+\theta)} + \frac{abpe^{-I_{c1}t}e^{(b+\theta)(T-V/a-t)}}{(b+\theta)(b+\theta+I_{c1})} + U(M) - \frac{pa\theta e^{-I_{c1}M}}{I_{c1}(b+\theta)} + \frac{abpe^{-I_{c1}t}e^{(b+\theta)(T-V/a-M)}}{(b+\theta)(b+\theta+I_{c1})} \quad (13)$$

The clearance point t_a can be obtained by letting $U(t) = 0$. Furthermore, the earned interest in the interval $[t_a, T_c]$ is

$$IE_{2-2a} = p \int_{t_a}^{T_c} R(t) dt + p I_e \int_{t_a}^{T_c} R(t) (T-t) dt \quad (14)$$

Immediately, the annual profit of the retailer during the whole cycle period can be expressed as:

$$\begin{aligned} \text{Max}_{(T,V)} AP_{2-2a}(T, V) = & \frac{1}{T} [p \int_{t_a}^{T_c} R(t) dt + p I_e \int_{t_a}^{T_c} R(t) (T-t) dt - A - c_0 V - h \int_0^{T_c} Z(t) dt] \\ \text{s.t.} \left\{ \begin{array}{l} U(M) > 0, U(T_c) < 0 \\ t_a = \arg\{U(t_a) = 0\} \\ M < T_c < N, M < t_a \leq T_c \\ V \leq V_0 \end{array} \right. \end{aligned} \quad (15)$$

Subcase 2-2b Discrete Payment Regime (DPR)

Different from the continuous payment regime, the retailer will pay all the remaining debt at the specific time t_b with the discrete payment regime. Similarly, the case with $T < t_b$ is not discussed in the paper.

The charged interest during the period $[M, t_b]$ can be written as:

$$IC_{2-2b} = I_{c1} U(M)(t_b - M) \quad (16)$$

The earned interest during the period $[M, t_b]$ can be written as:

$$IE_{2-2b} = pI_e \int_M^{t_b} R(t)(t_b - t)dt \quad (17)$$

The revenue during the period $[t_b, T_c]$ can be written as:

$$R_{2-2b} = p \int_{t_b}^{T_c} R(t)dt + pI_e \int_{t_b}^{T_c} R(t)(T - t)dt \quad (18)$$

Considering the retailer settles all unpaid balance at time t_b , the following equation holds

$$p \int_M^{t_b} R(t)dt + pI_e \int_M^{t_b} R(t)(t_b - t)dt = U(M) + I_{c1} U(M)(t_b - M) \quad (19)$$

The left-hand side (LHS) of above equation represents the sales revenue and interest earned in the interval $[M, t_b]$, and the right-hand side (RHS) denotes the unpaid balance and interest charged. Solving the equation below can obtain t_b

$$\phi(x) = p \int_M^x R(t)dt + pI_e \int_M^x R(t)(x - t)dt - U(M) - I_{c1} U(M)(x - M), M < x \leq T_c \quad (20)$$

where $\phi(M) < 0$, $\phi(t_b) = 0$, $\phi(T_c) > 0$. Subsequently, the annual profit of the retailer during the whole cycle period is shown as:

$$\begin{aligned} \text{Max}_{(T,V)} AP_{2-2b}(T, V) &= \frac{1}{T} [p \int_{t_b}^{T_c} R(t)dt + pI_e \int_{t_b}^{T_c} R(t)(T - t)dt - A - c_0 V - h \int_0^{T_c} Z(t)dt] \\ \text{s.t.} \quad &\begin{cases} \phi(M) < 0, \phi(T_c) \geq 0 \\ t_b = \arg\{\phi(t_b) = 0\} \\ M < T_c < N, M < t_b \leq T_c \\ V \leq V_0 \end{cases} \end{aligned} \quad (21)$$

4.3. Case 3 $M < N < T_c < T$

Consider the different penalty interest rates over progressive delays in payments with the relation $M < N < T_c < T$ (shown in Figure 4). In this case, for necessity of analysis the whole cycle period is divided to three intervals $[0, M]$, $[M, N]$ and $[N, T]$, each of which corresponds to a subcase as discussed below. Similarly, whether the retailer can settle all unpaid balance at time M generates the following two subcases.

4.3.1. Subcase 3-1 The Revenue Is Sufficient to Repay the Wholesale Cost at Time M

This case is no different from the Case 2-1 except for the credit period, which will not be reiterated here.

4.3.2. Subcase 3-2 The Revenue Is Insufficient to Repay the Wholesale Cost at Time M

Consider the subcase in which total purchasing cost cannot be settled at time M . As mentioned previously, assume the supplier offers the retailer two progressive payment regimes, i.e. continuous and discrete payment regimes, whose essential difference reflects the way the retailer behaves continuously or discretely for doing the payment. Then we explore the subcases below.

Subcase 3-2a Continuous Payment Regime (CPR)

With the continuous payment regime, the retailer continuously transfers the instantaneous income to defray the debt until all balance is cleared off at a certain time t_a . However, there are different penalty interests incurred when the clearance point t_a belongs to the intervals $[M, N]$ or $[N, T_c]$ due to the principles described in supplier's installment payment schemes. Hence, the Subcase 3-2a can be further decomposed into two separate parts below.

Subcase 3-2a' $M < t_a \leq N$

This subcase is similar to the Subcase 2-2a as described previously, so it will not be addressed here to avoid the repetition.

Subcase 3-2a'' $N < t_a \leq T_c$

In this subcase, the retailer is not able to settle all unpaid balance in the two progressive periods. Under continuous payment regime, the retailer will pay the penalty interest at the rate I_{c1} during the time interval $[M, N]$ and I_{c2} during the period $[N, T_c]$. Then the differential equations below hold

$$dU(t)/dt = \begin{cases} -pR(t) + I_{c1}U(t), & M < t \leq N \\ -pR(t) + I_{c2}U(t), & N < t \leq T_c \end{cases} \quad (22)$$

According to the Equation (12) and conditions $U(N) > 0$ and $U(T_c) \leq 0$, the unpaid balance is governed by the equations below

$$U(t) = \begin{cases} \frac{pa\theta(e^{-I_{c1}t} - e^{-I_{c1}M})}{I_{c1}(b+\theta)} + \frac{abpe^{-I_{c1}t}[e^{(b+\theta)(T-V/a-t)} - e^{(b+\theta)(T-V/a-M)}]}{(b+\theta)(b+\theta+I_{c1})} + U(M), & M < t \leq N \\ \frac{pa\theta(e^{-I_{c2}t} - e^{-I_{c2}N})}{I_{c2}(b+\theta)} + \frac{abpe^{-I_{c2}t}[e^{(b+\theta)(T-V/a-t)} - e^{(b+\theta)(T-V/a-N)}]}{(b+\theta)(b+\theta+I_{c2})} + U(N), & N < t \leq T_c \end{cases} \quad (23)$$

Hence, retailer's annual profit in whole period in Subcase 3-2a'' can be expressed as:

$$\begin{aligned} \underset{(T,V)}{\text{Max}} AP_{3-a''}(T, V) &= \frac{1}{T} [p \int_{t_a}^{T_c} R(t) dt + p I_e \int_{t_a}^{T_c} R(t)(T-t) dt - A - c_0 V - h \int_0^{T_c} Z(t) dt] \\ \text{s.t.} \quad &\begin{cases} U(M) > 0, U(N) > 0, U(T_c) \leq 0 \\ t_a = \arg\{U(t_a) = 0\} \\ M < N < t_a < T_c \\ V \leq V_0, T \leq T_0 \end{cases} \end{aligned} \quad (24)$$

Subcase 3-2b Discrete Payment Regime (DPR)

Now we pause here to discuss the subcase under the other progressive payment regime, i.e. discrete payment regime. As the name suggests, DPR is just on the opposite of CPR, namely, the retailer deposits money in some discrete times until all unpaid balance is cleared off. at time t_b . Similarly, depending on intervals $[M, N]$ or $[N, T_c]$, to which the point t_b belongs, we further discuss the problem in two separate situations, respectively.

Subcase 3-2b' $M < t_b \leq N$

Similarly, this subcase is the same as the Subcase 2-2b, so we again omit the deriving process.

Subcase 3-2b'' $N < t_b \leq T_c$

Considering the penalty interest charged by the supplier, the retailer tries to settle the unpaid balance by time N and then he pays off all the debt at the time t_b under the discrete payment regime. Hence, the unpaid balance at time N can be written as:

$$U(N) = I_{c1}U(M)(N - M) + U(M) - p \int_M^N R(t)dt - pI_e \int_M^N R(t)(N - t)dt \quad (25)$$

Accordingly, the interest charged and the interest earned during the credit period $[N, t_b]$ are shown respectively as:

$$IC_{3-2b''} = I_{c2}U(N)(t_b - N) \quad (26)$$

$$IE_{3-2b''} = pI_e \int_N^{t_b} R(t)(t_b - t)dt \quad (27)$$

Note that the retailer can square up all unpaid balance at the time t_b only when her accumulated income equals the sum of unpaid money and charged interest, that is the equation below holds

$$p \int_N^{t_b} R(t)dt + pI_e \int_N^{t_b} R(t)(t_b - t)dt = U(N) + I_{c2}U(N)(t_b - N) \quad (28)$$

where t_b is determined by the following function as well as its boundary conditions

$$\phi(x) = p \int_N^x R(t)dt + pI_e \int_N^x R(t)(x - t)dt - U(N) - I_{c2}U(N)(t_b - N), \quad N < x \leq T_c \quad (29)$$

with $\phi(N) < 0$, $\phi(t_b) = 0$, $\phi(T_c) > 0$.

Given the above, the retailer's average annual profit can be expressed as follows

$$\begin{aligned} \underset{(T,V)}{\text{Max}} \quad AP_{3-2b''}(T, V) &= \frac{1}{T} [p \int_{t_b}^{T_c} R(t)dt + pI_e \int_{t_b}^T R(t)(T - t)dt - A - h \int_0^{T_c} Z(t)dt] \\ \text{s.t.} \quad &\begin{cases} \phi(M) > 0, \phi(N) > 0, \phi(T_c) \leq 0 \\ t_b = \arg\{\phi(t_b) = 0\} \\ M < N < t_b \leq T_c \\ V \leq V_0, T \leq T_0 \end{cases} \end{aligned} \quad (30)$$

Figure 4. Inventory level over time in case 3.

4.4. Case 4 $T_c < M < T$ ($T_c < M < N < T$ or $T_c < M < T < N$)

Now we shed the light on the case in which the relationship $T_c < M < T$ holds, which can be decomposed into sub-cases $T_c < M < N < T$ or $T_c < M < T < N$ (shown in Figure 5). In view of the

retailer's rationality, she will return a large amount money at time M while she earns income from sales and deposit interest. Then retailer's remaining bank savings after payment at time M can be expressed as follows:

$$U(M) = p[\int_0^{T_c} R(t)dt + V] + pI_e VM + pI_e \int_0^{T_c} R(t)(M-t)dt - c(Q+V) \quad (31)$$

where the first term in RHS of Equation (32) represents the sales revenue, the last term the total purchasing cost and the second term the earned interest of the retailer by depositing the sales revenue in the bank. Furthermore, the remaining savings of retailer is continuously deposited in bank to earn the interest during the credit period $[M, T]$, and it takes the form

$$IE_4 = I_e U(M)(T-M) \quad (32)$$

Hence, the average annual profit of retailer takes can be formulated as follows

$$\begin{aligned} \text{Max}_{(T, V)} AP_4(T, V) &= \frac{1}{T} \{U(M)[1 + I_e(T-M)] - A - c_0 V - h \int_0^{T_c} Z(t)dt\} \\ \text{s.t.} \quad &\begin{cases} U(M) = p[\int_0^{T_c} R(t)dt + V] + pI_e VM + pI_e \int_0^{T_c} R(t)(M-t)dt - c(Q+V) \geq 0 \\ T_c < M < N \\ V \leq V_0, T \leq T_0 \end{cases} \end{aligned} \quad (33)$$

Figure 5. Inventory level over time in case 4.

4.5. The Characterization and Logic of Global Optimization

Given all the formulation above, there exist, in total, ten subcases explored in terms of two progressive payment approaches and a comparison of trade credits and the replenishing cycle, which is shown below:

$$T_c = T - \frac{V}{a} \Rightarrow \begin{cases} T_c \leq M \begin{cases} T \leq M \Rightarrow \text{case 1} \\ T > M \Rightarrow \text{case 4} \end{cases} \\ T_c > M \begin{cases} T_c \leq N \Rightarrow \text{case 2} \begin{cases} \text{paid off before } M \Rightarrow \text{subcase 2-1} \\ \text{paid off after } M \Rightarrow \begin{cases} \text{CPR : subcase 2-2a} \\ \text{DPR : subcase 2-2b} \end{cases} \end{cases} \\ T_c > N \Rightarrow \text{case 3} \begin{cases} \text{paid off before } M \Rightarrow \text{subcase 3-1} \\ \text{paid off in } [M, N] \Rightarrow \begin{cases} \text{CPR : subcase 3-2a'} \\ \text{DPR : subcase 3-2b'} \end{cases} \\ \text{paid off after } N \Rightarrow \begin{cases} \text{CPR : subcase 3-2a''} \\ \text{DPR : subcase 3-2b''} \end{cases} \end{cases} \end{cases} \end{cases}$$

Essentially, it is a process to pursue global optimization based on multiple piecewise nonlinear optimizations. Its logic is to first solve each subcase by getting the local optimum solution, and then compare all of these ten local optimum values to determine the global optimum solution. In view of the complication and nonlinearity of the whole problem, we are not able to give the elegantly analytical solution, so an effective way is to conduct numerical experiments to investigate the problem's characterization as well as managerial insights. Note that the detailed exposition of formulation and representations for all ten cases are addressed in the Appendix.

5. Numerical Examples

To illustrate the previously developed models, the paper utilizes Particle Swarm Optimization (PSO), an extensively used intelligent optimization algorithm, to solve the problem. For more detailed information about the PSO, one can refer to Shi and Eberhart [40] and Eberhart and Shi [41]. Here some computational experiments are proposed under given parameters below, $A = 200$, $c = 20$, $c_0 = 3$, $p = 30$, $h = 6$, $a = 1000$, $b = 0.25$, $M = 17/365$, $N = 35/365$, $\theta = 0.1$, $I_e = 12\%$, $I_{c_1} = 13\%$ and $I_{c_2} = 20\%$. In addition, assume the upper bounds of shortage quantity and replenishment time interval are $V_0 = 500$ and $T_0 = 2$, respectively. The scale of particle swarm is 100, the iteration times 300 and the calculation time is 10.

Example 1. The annual profit (maximum between CPR and DPR) of the retailer is increasing in the replenishment time interval originally, however, with the growth of the replenishment time interval, the annual profit presents a downtrend. This is the same as our intuition that the annual profit is decreasing in shortage quantity. In addition, the annual profit is zero in some situations as shown in Figure 6. The reason for rendering the phenomenon is that the constraint condition $T_c = T - V/a \geq 0$ is not satisfied. In view of the situation, we define the annual profit equals to zero when $T_c = T - V/a < 0$. Observing Figure 7, when there is a short replenishment time interval, the profit of the retailer in the discrete payment regime (DPR) is greater than that in the continuous payment regime (CPR), while CPR is superior to DPR with the long replenishment time interval. The result reveals that it is beneficial for the retailer to choose DPR with a short replenishment time interval, otherwise, the CPR will be the best choice.

Figure 6. The annual profit of retailer.

Figure 7. The difference of the annual profit between CPR and DPR.

Example 2. Figures 8 and 9 depict how the unit shortage cost and deteriorating rate influence on the retailer's profit in the two different payment regimes, and some results can be obtained from the figures. First, the profit is decreasing in unit shortage cost (the same result described aforesaid) and deteriorating rate. Second, the time hitting the profit peak shifts right with the increase of shortage quantity, that is, the retailer obtains the maximum profit in a short replenishment time interval with a small shortage quantity. Third, originally, the profit of retailer in DPR is higher than that in CPR, while, as the growth of replenishment time interval, the CPR is superior to the DPR. The result illustrates when there is a short replenishment time interval, and it is beneficial to the retailer with DPR, otherwise, the CPR will be chosen by the retailer (the same result as obtained in Example 1).

Figure 8. The annual profit versus c_0 variation.

Figure 9. The annual profit versus θ variation.

Example 3. Replenishment time interval, order quantity and profit of the retailer are decreasing in unit shortage cost, as shown in Table 1, and some information is reflected in this phenomenon. With the increases of the unit shortage cost, the retailer decreases the replenishment time interval and order quantity by increasing the replenishment times for reducing the shortage cost. Nevertheless, the profit of the retailer is still decreasing in the shortage cost. In addition, the shortage quantity closes to the maximum shortage quantity with a low shortage cost. However, there is no shortage with the high shortage cost.

Table 1. Effect of c_0 on decision parameters.

c_0	T	V	$\bar{Q}$	AP
2	0.8167	497.7667	835.1714	9057.6313
3	0.8832	491.0036	911.3325	8467.8100
4	0.2180	0	226.4905	8410.6412
5	0.2178	0	226.3488	8410.6407
6	0.2176	0	226.1124	8410.6383

Example 4. Observing Tables 2 and 3, some conclusions can be found. As the increase of stock-dependent demand parameter b and the credit periods M and N , the annual profit of retailer is increasing, while it is decreasing in penalty interest rates I_{c_1} and I_{c_2} . The phenomenon indicates that when the demand is more dependent on the stock-level or there is a long credit period, the retailer should increase the ordering quantity and extend the replenishment time interval for getting high profits. However, considering the penalty interest, the retailer must shorten the replenishment time interval and reduce the ordering quantity to mitigate the negative effect of penalty interest. In reality, firms should focus on two issues for getting high profit. First, investigate the sensitivity of the consumer on inventory to evaluate the value of b , then decide to increase or reduce the order quantity and replenishment cycle interval for profiting more in term of the value of b . Second, retailers should strengthen the bargaining power with upstream companies to get the long credit periods and low penalty interest rate.

Table 2. Effect of b , M and I_{c1} on decision parameters.

b	M				
	20/365 ($I_{c1}=15\%$)	30/365 ($I_{c1}=15\%$)	40/365 ($I_{c1}=15\%$)	40/365 ($I_{c1}=25\%$)	40/365 ($I_{c1}=35\%$)
0.15					
T	0.8441	0.8605	0.8927	0.8837	0.8716
V	488.4832	489.1813	471.2593	486.4183	472.1411
$\bar{Q}$	860.3828	878.2310	891.2670	914.0644	892.2445
AP	8404.4491	8477.7351	8537.7501	8532.4892	8526.0081
0.2					
T	0.8646	0.8697	0.8988	0.8976	0.8846
V	493.0853	491.1428	490.7041	489.9936	481.6495
$\bar{Q}$	886.0368	892.0257	924.8738	923.5452	909.9451
AP	8448.1874	8521.0173	8589.4862	8588.9267	8581.8973
0.25					
T	0.8866	0.8950	0.9082	0.9083	0.8938
V	500	500	498.2631	498.0957	488.6078
$\bar{Q}$	913.9737	923.5793	939.1119	939.1504	923.8838
AP	8497.2087	8573.3328	8645.2952	8645.1166	8636.5492
0.3					
T	0.8981	0.9134	0.9198	0.9193	0.9011
V	496.6770	500	500	499.8087	489.0922
$\bar{Q}$	932.1163	949.5101	957.0938	956.4898	936.9380
AP	8540.1315	8621.0579	8696.7935	8696.4078	8686.3345
0.35					
T	0.9102	0.9313	0.9386	0.9346	0.9091
V	490.3020	497.0977	498.5971	498.0310	489.3622
$\bar{Q}$	952.4840	976.6036	985.1538	980.3983	951.3962
AP	8584.4509	8670.4279	8749.4627	8747.2465	8738.3301

Table 3. Effect of b , N and I_{c1} on decision parameters.

b	N				
	50/365 ($I_{c2}=30\%$)	70/365 ($I_{c2}=30\%$)	90/365 ($I_{c2}=30\%$)	90/365 ($I_{c2}=40\%$)	90/365 ($I_{c2}=50\%$)
0.15					
T	0.8595	0.8701	0.8701	0.8701	0.8633
V	491.9821	500	500	500	494.4807
$\bar{Q}$	876.8582	887.7219	887.7933	887.7933	880.7802
AP	8480.8280	8488.3131	8488.3071	8488.3071	8483.1111
0.2					
T	0.8729	0.8795	0.8749	0.8700	0.8617
V	495.9561	500	500	491.2119	488.7023
$\bar{Q}$	895.0049	901.9195	896.7598	892.3055	883.3820
AP	8525.3818	8529.1272	8529.2760	8520.6942	8518.6131
0.25					
T	0.8836	0.8852	0.8868	0.8867	0.8868
V	498.4398	500	500	500	500
$\bar{Q}$	910.7060	912.3960	914.2018	914.1471	914.0690
AP	8570.8756	8572.3571	8572.3734	8572.3734	8572.3734
0.3					
T	0.8971	0.8985	0.8995	0.9054	0.8962
V	498.5952	499.7340	500	500	495.6017
$\bar{Q}$	930.6382	932.0134	933.1913	940.1691	930.0751
AP	8616.7980	8617.8926	8618.1521	8617.9222	8613.8861
0.35					
T	0.9001	0.9100	0.9134	0.9120	0.9101
V	494.9595	495.6477	500	498.8823	497.4994
$\bar{Q}$	939.3413	951.1595	954.3575	950.6287	953.2017
AP	8661.7442	8662.6566	8666.9194	8665.8131	8664.4647

6. Concluding Remarks

In this research, we investigated the optimizing ordering policy for a perishable product supply chain with stock-dependent demand facing items deterioration and order backlogging under progressive payment schemes. We considered the continuous payment regime (CPR) and discrete payment regime (DPR) both in the progressive trading process to generate piecewise nonlinear optimization problem of ten distinct scenarios. We demonstrated the characterization of problem formulation as well as the logic of exploring global optimization from all piecewise local optimizations. We further conducted computational studies and numerical simulation to illustrate the impact of parameters on the optimal ordering policy and channel performance.

Some key findings of this study are summarized as follows: (i) we decompose the problem into an equivalent set of multiple piecewise nonlinear optimizations and elaborate its formulation characterization and logic, which provides an integral approach for exploring the optimal replenishing policy in this specific problem. (ii) comparing the profits incurred by two progressive payment schemes implies that the relatively long replenishment cycle will let the retailer better off under CPR, whereas the relatively short cycle works better under DPR. Therefore, for firms, it is best choice to repay the money by CPR when there is long replenishment cycle, otherwise, the DPR will be the best way for getting more profits. (iii) reducing order backlogging quantity may benefit retailer's pursuing greater profit in short time range. Thus, the one who wants to maximize the profit in a short time would better reduce order backlogging quantity. (iv) the increasing of stock-dependent demand parameter and the length of the progressive credit period will push up the ordering quantity and replenishment cycle for getting more profit. The result implies two directions that the firms should pay much attention: first, increasing the sensitivity of the consumer on inventory, and second, enhancing bargaining power to obtain a long progressive credit period.

Our research contributes to the track of managing material and cash flows. The proposed model can be extended in several ways. For example, the model may extend the constant deterioration rate to time-varying deterioration. Furthermore, retailers are also likely to provide the permissible delay in payment to consumers in day-to-day deal; then the model will become more complex, and more significant results may be explored.

Author Contributions: L.H. contributed to design, conduct this research and write and revise the whole paper. H.G. and X.Z. contributed to jointly conduct the research and solve some of models and are responsible for doing the numerical studies. Q.W. and C.H. contributed to join the research, discuss models, give valuable suggestions and revise the paper.

Funding: This work is partially supported by NSFC Grants (No. 91646118, 71701144, 71602142, 71528002). It is also supported by Natural Science Foundation of Hebei Province of China under Grant No. G2017207015 and Humanity and Social Science Youth Foundation of Ministry of Education of China under Grant No. 18YJC630183.

Conflicts of Interest: The authors declare no conflict of interest.

Notation/Description

The notations used in this paper are summarized as below:

h	Inventory holding cost per unit per year excluding interest charges
p	Unit selling price
c	Unit wholesale cost ($c < p$)
A	Fixed ordering cost/ order
M	The first permissible delay in settling account (i.e., the trade credit period)
N	The second permissible delay in settling account and $N > M$
Q	The on-hand inventory at the beginning of each replenishment period
c_0	Unit shortage cost
V	The shortage quantity (the decision variable)
V_0	The maximum permissible shortage quantity
T	The replenishment cycle time
T_0	The maximum replenishment time interval
T_c	The time point that on-hand products are depleted to zero
$\bar{Q}$	The order quantity at the beginning of the replenishment period
t_a	the time at which the unpaid balance is cleared off under continuous payment regime
t_b	the time at which the unpaid balance is cleared off under discrete payment regime
$AP(T, V)$	The annual profit of the retailer

Appendix A

This appendix addresses the detailed exposition of formulation and representations for all ten cases (subcases).

Case 1

$$\begin{aligned}
 \max_{(T,V)} AP_1(T, V) &= \frac{1}{T} \left\{ p \left[\int_0^{T_c} R(t) dt + V \right] + p I_e V M + p I_e \int_0^{T_c} R(t) (M - t) dt - c(Q + V) - A - c_0 V - h \int_0^{T_c} Z(t) dt \right\} \\
 &= \frac{1}{T} \left\{ (p + p I_e M) \left[V + \frac{a \theta T_c}{b + \theta} + \frac{ab(e^{(b+\theta)T_c} - 1)}{(b + \theta)^2} \right] - p I_e \left[\frac{a \theta (T_c)^2}{2(b + \theta)} - \frac{ab T_c}{(b + \theta)^2} + \frac{ab(e^{(b+\theta)T_c} - 1)}{(b + \theta)^3} \right] \right. \\
 &\quad \left. - c(Q + V) - A - c_0 V - h \left[\frac{a(e^{(b+\theta)T_c} - 1)}{(b + \theta)^2} - \frac{a T_c}{b + \theta} \right] \right\} \\
 \text{s.t. } &\begin{cases} T_c = T - V/a, T_c < T \leq M < N \\ V \leq V_0 \end{cases}
 \end{aligned}$$

Case 2

Subcase 2-1

$$\begin{aligned}
 \max_{(T,V)} TP_{2-1}(T, V) &= \frac{1}{T} \left\{ p \int_M^{T_c} R(t) dt + p I_e \int_M^{T_c} R(t) (T - t) dt + U_{2-1} [1 + I_e (T - M)] - A - c_0 V - h \int_0^{T_c} Z(t) dt \right\} \\
 &= \frac{1}{T} \left\{ (p + p I_e T) \left[\frac{a \theta}{b + \theta} (T_c - M) + \frac{ab(e^{(b+\theta)(T_c-M)} - 1)}{(b + \theta)^2} \right] + U_{2-1} [1 + I_e (T - M)] - A - c_0 V - h \left[\frac{a(e^{(b+\theta)T_c} - 1)}{(b + \theta)^2} - \frac{a T_c}{b + \theta} \right] \right. \\
 &\quad \left. - p I_e \left[\frac{a \theta [(T_c)^2 - M^2]}{2(b + \theta)} - \frac{ab [M e^{(b+\theta)(T_c-M)} - T_c]}{(b + \theta)^2} + \frac{ab [e^{(b+\theta)(T_c-M)} - 1]}{(b + \theta)^3} \right] \right\} \\
 \text{s.t. } &\begin{cases} T_c = T - V/a \\ U_{2-1} = \left\{ pV + p I_e V M + (p + p I_e M) \left[\frac{a \theta}{b + \theta} M + \frac{ab(e^{(b+\theta)T_c} - e^{(b+\theta)(T_c-M)})}{(b + \theta)^2} \right] \right. \\ \quad \left. - p I_e \left[\frac{a \theta M^2}{2(b + \theta)} - \frac{ab M e^{(b+\theta)(T_c-M)}}{(b + \theta)^2} + \frac{ab(e^{(b+\theta)T_c} - e^{(b+\theta)(T_c-M)})}{(b + \theta)^3} \right] - c(Q + V) \right\} \geq 0 \\ M < T_c < N, V \leq V_0 \end{cases}
 \end{aligned}$$

Subcase 2-2

Subcase 2-2a

$$\begin{aligned}
\text{Max}_{(T,V)} AP_{2-2a}(T,V) &= \frac{1}{T} [p \int_{t_a}^{T_c} R(t) dt + p I_e \int_{t_a}^{T_c} R(t)(T-t) dt - A - c_0 V - h \int_0^{T_c} Z(t) dt] \\
&= \frac{1}{T} \left\{ (p + p I_e T) \left[\frac{a\theta}{b+\theta} (T_c - t_a) + \frac{ab(e^{(b+\theta)(T_c-t_a)} - 1)}{(b+\theta)^2} \right] - p I_e \left[\frac{a\theta[(T_c)^2 - t_a^2]}{2(b+\theta)} - \frac{ab[t_a e^{(b+\theta)(T_c-t_a)} - T_c]}{(b+\theta)^2} + \frac{ab[e^{(b+\theta)(T_c-t_a)} - 1]}{(b+\theta)^3} \right] \right. \\
&\quad \left. - A - c_0 V - h \left[\frac{a}{(b+\theta)^2} (e^{(b+\theta)(T-V/a)} - 1) - \frac{a(T-V/a)}{b+\theta} \right] \right\} \\
\text{s.t.} \quad &\begin{cases} U(M) = - \left\{ pV + p I_e VM + (p + p I_e M) \left[\frac{a\theta M}{b+\theta} + \frac{ab(e^{(b+\theta)T_c} - e^{(b+\theta)(T_c-M)})}{(b+\theta)^2} \right] \right. \\ \quad \left. - p I_e \left[\frac{a\theta M^2}{2(b+\theta)} - \frac{abMe^{(b+\theta)(T_c-M)}}{(b+\theta)^2} + \frac{ab(e^{(b+\theta)T_c} - e^{(b+\theta)(T_c-M)})}{(b+\theta)^3} \right] - c(Q+V) \right\} > 0 \\ U(T_c) = \left\{ \frac{pa\theta e^{-I_{c1}T_c}}{I_{c1}(b+\theta)} + \frac{abpe^{-I_{c1}T_c}}{(b+\theta)(b+\theta+I_{c1})} + U(M) - \frac{pa\theta e^{-I_{c1}M}}{I_{c1}(b+\theta)} + \frac{abpe^{-I_{c1}M}e^{(b+\theta)(T-V/a-M)}}{(b+\theta)(b+\theta+I_{c1})} \right\} < 0 \\ t_a = \arg \left\{ \frac{pa\theta e^{-I_{c1}t_a}}{I_{c1}(b+\theta)} + \frac{abpe^{-I_{c1}t_a}}{(b+\theta)(b+\theta+I_{c1})} + U(M) - \frac{pa\theta e^{-I_{c1}M}}{I_{c1}(b+\theta)} + \frac{abpe^{-I_{c1}M}e^{(b+\theta)(T-V/a-M)}}{(b+\theta)(b+\theta+I_{c1})} \right\} = 0 \\ M < T_c < N, M < t_a \leq T_c \\ V \leq V_0 \end{cases}
\end{aligned}$$

Subcase 2-2b

$$\begin{aligned}
\text{Max}_{(T,V)} AP_{2-2b}(T,V) &= \frac{1}{T} [p \int_{t_b}^{T_c} R(t) dt + p\gamma \int_{t_b}^{T_c} R(t)(T-t) dt - A - h \int_0^{T_c} Z(t) dt] \\
&= \frac{1}{T} \left\{ (p + p I_e T) \left[\frac{a\theta}{b+\theta} (T_c - t_b) + \frac{ab(e^{(b+\theta)(T_c-t_b)} - 1)}{(b+\theta)^2} \right] - A - c_0 V - h \left[\frac{a(e^{(b+\theta)(T-V/a)} - 1)}{(b+\theta)^2} - \frac{a(T-V/a)}{b+\theta} \right] \right. \\
&\quad \left. - p I_e \left[\frac{a\theta[(T_c)^2 - t_b^2]}{2(b+\theta)} - \frac{ab[t_b e^{(b+\theta)(T_c-t_b)} - T_c]}{(b+\theta)^2} + \frac{ab[e^{(b+\theta)(T_c-t_b)} - 1]}{(b+\theta)^3} \right] \right\} \\
\text{s.t.} \quad &\begin{cases} T_c = T - V/a \\ \phi(M) = \left\{ pV + p I_e VM + (p + p I_e M) \left[\frac{a\theta}{b+\theta} M + \frac{ab(e^{(b+\theta)T_c} - e^{(b+\theta)(T_c-M)})}{(b+\theta)^2} \right] \right. \\ \quad \left. - p I_e \left[\frac{a\theta M^2}{2(b+\theta)} - \frac{abMe^{(b+\theta)(T_c-M)}}{(b+\theta)^2} + \frac{ab(e^{(b+\theta)T_c} - e^{(b+\theta)(T_c-M)})}{(b+\theta)^3} \right] - c(Q+V) \right\} < 0 \\ \phi(T_c) = \left\{ p(1 + I_e T_c) \left[\frac{a\theta}{b+\theta} (T_c - M) + \frac{ab(e^{(b+\theta)(T_c-M)} - 1)}{(b+\theta)^2} \right] + \phi(M) [1 + I_{c1}(T_c - M)] \right. \\ \quad \left. - p I_e \left[\frac{a\theta(T_c^2 - M^2)}{2(b+\theta)} - \frac{ab(Me^{(b+\theta)(T_c-M)} - T_c)}{(b+\theta)^2} + \frac{ab(e^{(b+\theta)(T_c-M)} - 1)}{(b+\theta)^3} \right] \right\} \geq 0 \\ t_b = \arg \left\{ p(1 + I_e t_b) \left[\frac{a\theta}{b+\theta} (t_b - M) + \frac{ab(e^{(b+\theta)(T_c-M)} - e^{(b+\theta)(T_c-t_b)})}{(b+\theta)^2} \right] + \phi(M) [1 + I_{c1}(t_b - M)] \right. \\ \quad \left. - p I_e \left[\frac{a\theta(t_b^2 - M^2)}{2(b+\theta)} - \frac{ab(Me^{(b+\theta)(T_c-M)} - xe^{(b+\theta)(T_c-t_b)})}{(b+\theta)^2} + \frac{ab(e^{(b+\theta)(T_c-M)} - e^{(b+\theta)(T_c-t_b)})}{(b+\theta)^3} \right] \right\} = 0 \\ M < t_b < T_c < N \\ V \leq V_0, T \leq T_0 \end{cases}
\end{aligned}$$

$$\begin{aligned}
\phi(x) &= p \int_M^x R(t) dt + p I_e \int_M^x R(t)(x-t) dt - U(M) - I_{c1} U(M)(x-M) \\
&= p(1 + I_e x) \left\{ \frac{a\theta}{b+\theta} (x-M) + \frac{ab}{(b+\theta)^2} [e^{(b+\theta)(T-V/a-M)} - e^{(b+\theta)(T-V/a-x)}] \right\} \\
&\quad - p I_e \left\{ \frac{a\theta(x^2 - M^2)}{2(b+\theta)} - \frac{ab}{(b+\theta)^2} [Me^{(b+\theta)(T-V/a-M)} - xe^{(b+\theta)(T-V/a-x)}] + \frac{ab}{(b+\theta)^3} [e^{(b+\theta)(T-V/a-M)} - e^{(b+\theta)(T-V/a-x)}] \right\} \\
&\quad - [1 + I_{c1}(x-M)] \left\{ \frac{ac}{(b+\theta)} [e^{(b+\theta)(T-V/a)} - 1] + Vc - pV(1 + I_e M) - \frac{pa\theta M}{b+\theta} (1 + I_e M/2) \right. \\
&\quad \left. - \frac{pab}{(b+\theta)^2} [e^{(b+\theta)(T-V/a)} - e^{(b+\theta)(T-V/a-M)} + I_e Me^{(b+\theta)(T-V/a)}] - \frac{pI_e ab}{(b+\theta)^3} [e^{(b+\theta)(T-V/a-M)} - e^{(b+\theta)(T-V/a)}] \right\} \\
&= p(1 + I_e x) \left[\frac{a\theta}{b+\theta} (x-M) + \frac{ab(e^{(b+\theta)(T_c-M)} - e^{(b+\theta)(T_c-x)})}{(b+\theta)^2} \right] + \phi(M) [1 + I_{c1}(x-M)] \\
&\quad - p I_e \left[\frac{a\theta(x^2 - M^2)}{2(b+\theta)} - \frac{ab(Me^{(b+\theta)(T_c-M)} - xe^{(b+\theta)(T_c-x)})}{(b+\theta)^2} + \frac{ab(e^{(b+\theta)(T_c-M)} - e^{(b+\theta)(T_c-x)})}{(b+\theta)^3} \right]
\end{aligned}$$

Case 3

Subcase 3-1

$$\begin{aligned}
 \text{Max}_{(T,V)} AP_{3-1}(T,V) &= \frac{1}{T} \left\{ p \int_M^{T_c} R(t) dt + p I_e \int_M^{T_c} R(t)(T-t) dt + U_{3-1} [1 + I_e(T-M)] - A - c_0 V - h \int_0^{T_c} Z(t) dt \right\} \\
 &= \frac{1}{T} \left\{ a(p + p I_e T) \left[\frac{\theta(T_c - M)}{b + \theta} + \frac{b(e^{(b+\theta)(T_c - M)} - 1)}{(b + \theta)^2} \right] + U_{3-1} [1 + I_e(T - M)] - A - c_0 V - ah \left[\frac{e^{(b+\theta)T_c} - 1}{(b + \theta)^2} - \frac{T_c}{b + \theta} \right] \right. \\
 &\quad \left. - ap I_e \left[\frac{\theta[(T_c)^2 - M^2]}{2(b + \theta)} - \frac{b M e^{(b+\theta)(T_c - M)} - T_c}{(b + \theta)^2} + \frac{b[e^{(b+\theta)(T_c - M)} - 1]}{(b + \theta)^3} \right] \right\} \\
 \text{s.t.} \quad &\begin{cases} T_c = T - V/a, M < N < T_c \\ U_{3-1} = \left\{ pV + p I_e VM + a(p + p I_e M) \left[\frac{\theta}{b + \theta} M + \frac{b(e^{(b+\theta)T_c} - e^{(b+\theta)(T_c - M)})}{(b + \theta)^2} \right] \right. \\ \quad \left. - ap I_e \left[\frac{\theta M^2}{2(b + \theta)} - \frac{b M e^{(b+\theta)(T_c - M)}}{(b + \theta)^2} + \frac{b[e^{(b+\theta)T_c} - e^{(b+\theta)(T_c - M)})]}{(b + \theta)^3} \right] - c(Q + V) \right\} \geq 0 \\ V \leq V_0 \end{cases}
 \end{aligned}$$

Subcase 3-2a'

$$\begin{aligned}
 \text{Max}_{(T,V)} AP_{3-2a'}(T,V) &= \frac{1}{T} \left[p \int_{t_a}^{T_c} R(t) dt + p I_e \int_{t_a}^{T_c} R(t)(T-t) dt - A - c_0 V - h \int_0^{T_c} Z(t) dt \right] \\
 &= \frac{1}{T} \left\{ (p + p I_e T) \left[\frac{a\theta}{b + \theta} (T_c - t_a) + \frac{ab(e^{(b+\theta)(T_c - t_a)} - 1)}{(b + \theta)^2} \right] - A - c_0 V - h \left[\frac{a(e^{(b+\theta)(T - V/a)} - 1)}{(b + \theta)^2} - \frac{a(T - V/a)}{b + \theta} \right] \right. \\
 &\quad \left. - p I_e \left[\frac{a\theta[(T_c)^2 - t_a^2]}{2(b + \theta)} - \frac{ab[t_a e^{(b+\theta)(T_c - t_a)} - T_c]}{(b + \theta)^2} + \frac{ab[e^{(b+\theta)(T_c - t_a)} - 1]}{(b + \theta)^3} \right] \right\} \\
 \text{s.t.} \quad &\begin{cases} U(M) = - \left\{ pV + p I_e VM + (p + p I_e M) \left[\frac{a\theta M}{b + \theta} + \frac{ab(e^{(b+\theta)T_c} - e^{(b+\theta)(T_c - M)})}{(b + \theta)^2} \right] \right. \\ \quad \left. - p I_e \left[\frac{a\theta M^2}{2(b + \theta)} - \frac{ab M e^{(b+\theta)(T_c - M)}}{(b + \theta)^2} + \frac{ab(e^{(b+\theta)T_c} - e^{(b+\theta)(T_c - M)})}{(b + \theta)^3} \right] - c(Q + V) \right\} > 0 \\ U(N) = \left\{ \frac{pa\theta e^{-I_{c1}N}}{I_{c1}(b + \theta)} + \frac{abpe^{-I_{c1}N}}{(b + \theta)(b + \theta + I_{c1})} + U(M) - \frac{pa\theta e^{-I_{c1}M}}{I_{c1}(b + \theta)} + \frac{abpe^{-I_{c1}M} e^{(b+\theta)(T - V/a - M)}}{(b + \theta)(b + \theta + I_{c1})} \right\} < 0 \\ t_a = \arg \left\{ \frac{pa\theta e^{-I_{c1}t_a}}{I_{c1}(b + \theta)} + \frac{abpe^{-I_{c1}t_a}}{(b + \theta)(b + \theta + I_{c1})} + U(M) - \frac{pa\theta e^{-I_{c1}M}}{I_{c1}(b + \theta)} + \frac{abpe^{-I_{c1}M} e^{(b+\theta)(T - V/a - M)}}{(b + \theta)(b + \theta + I_{c1})} \right\} = 0 \\ M < t_a < N < T_c \\ V \leq V_0, T \leq T_0 \end{cases}
 \end{aligned}$$

Subcase 3-2a''

$$\begin{aligned}
 \text{Max}_{(T,V)} AP_{3-2a''}(T,V) &= \frac{1}{T} \left[p \int_{t_a}^{T_c} R(t) dt + p I_e \int_{t_a}^{T_c} R(t)(T-t) dt - A - c_0 V - h \int_0^{T_c} Z(t) dt \right] \\
 &= \frac{1}{T} \left\{ (p + p I_e T) \left[\frac{a\theta(T_c - t_a)}{b + \theta} + \frac{ab(e^{(b+\theta)(T_c - t_a)} - 1)}{(b + \theta)^2} \right] - A - c_0 V - h \left[\frac{a(e^{(b+\theta)(T - V/a)} - 1)}{(b + \theta)^2} - \frac{a(T - V/a)}{b + \theta} \right] \right. \\
 &\quad \left. - p I_e \left[\frac{a\theta[(T_c)^2 - t_a^2]}{2(b + \theta)} - \frac{ab[t_a e^{(b+\theta)(T_c - t_a)} - T_c]}{(b + \theta)^2} + \frac{ab[e^{(b+\theta)(T_c - t_a)} - 1]}{(b + \theta)^3} \right] \right\} \\
 \text{s.t.} \quad &\begin{cases} T_c = T - V/a \\ U(M) = - \left\{ pV + p I_e VM + (p + p I_e M) \left[\frac{a\theta}{b + \theta} M + \frac{ab(e^{(b+\theta)T_c} - e^{(b+\theta)(T_c - M)})}{(b + \theta)^2} \right] \right. \\ \quad \left. - p I_e \left[\frac{a\theta M^2}{2(b + \theta)} - \frac{ab M e^{(b+\theta)(T_c - M)}}{(b + \theta)^2} + \frac{ab(e^{(b+\theta)T_c} - e^{(b+\theta)(T_c - M)})}{(b + \theta)^3} \right] - c(Q + V) \right\} > 0 \\ U(N) = \left\{ \frac{pa\theta e^{-I_{c1}N}}{I_{c1}(b + \theta)} + \frac{abpe^{-I_{c1}N}}{(b + \theta)(b + \theta + I_{c1})} + U(M) - \frac{pa\theta e^{-I_{c1}M}}{I_{c1}(b + \theta)} + \frac{abpe^{-I_{c1}M} e^{(b+\theta)(T - V/a - M)}}{(b + \theta)(b + \theta + I_{c1})} \right\} > 0 \\ U(T_c) = \frac{pa\theta(e^{-I_{c2}T_c} - e^{-I_{c2}N})}{I_{c2}(b + \theta)} + \frac{abpe^{-I_{c2}t} [e^{(b+\theta)(T - V/a - T_c)} - e^{(b+\theta)(T - V/a - N)}]}{(b + \theta)(b + \theta + I_{c2})} + U(N) \leq 0 \\ t_a = \arg \left\{ \frac{pa\theta(e^{-I_{c2}t_a} - e^{-I_{c2}N})}{I_{c2}(b + \theta)} + \frac{abpe^{-I_{c2}t} [e^{(b+\theta)(T - V/a - t_a)} - e^{(b+\theta)(T - V/a - N)}]}{(b + \theta)(b + \theta + I_{c2})} + U(N) = 0 \right\} \\ M < N < t_a < T_c \\ V \leq V_0, T \leq T_0 \end{cases}
 \end{aligned}$$

Subcase 3-2b'

$$\begin{aligned}
\text{Max}_{(T,V)} AP_{3-2b'}(T,V) &= \frac{1}{T} [p \int_{t_b}^{T_c} R(t) dt + p I_e \int_{t_b}^{T_c} R(t)(T-t) dt - A - c_0 V - h \int_0^{T_c} Z(t) dt] \\
&= \frac{1}{T} \left\{ (p + p I_e T) \left[\frac{a\theta(T_c - t_b)}{b+\theta} + \frac{ab(e^{(b+\theta)(T_c - t_b)} - 1)}{(b+\theta)^2} \right] - A - c_0 V - h \left[\frac{a(e^{(b+\theta)T_c} - 1)}{(b+\theta)^2} - \frac{aT_c}{b+\theta} \right] \right. \\
&\quad \left. - p I_e \left[\frac{a\theta[(T_c)^2 - t_b^2]}{2(b+\theta)} - \frac{ab[t_b e^{(b+\theta)(T_c - t_b)} - T_c]}{(b+\theta)^2} + \frac{ab[e^{(b+\theta)(T_c - t_b)} - 1]}{(b+\theta)^3} \right] \right\} \\
\text{s.t.} \quad &\begin{cases} T_c = T - V/a \\ \phi(M) = \left\{ pV + p I_e VM + (p + p I_e M) \left[\frac{a\theta}{b+\theta} M + \frac{ab(e^{(b+\theta)T_c} - e^{(b+\theta)(T_c - M)})}{(b+\theta)^2} \right] \right. \\ \quad \left. - p I_e \left[\frac{a\theta M^2}{2(b+\theta)} - \frac{abMe^{(b+\theta)(T_c - M)}}{(b+\theta)^2} + \frac{ab(e^{(b+\theta)T_c} - e^{(b+\theta)(T_c - M)})}{(b+\theta)^3} \right] - c(Q + V) \right\} < 0 \\ \phi(N) = \left\{ p(1 + I_e N) \left[\frac{a\theta(N - M)}{b+\theta} + \frac{ab(e^{(b+\theta)(N - M)} - 1)}{(b+\theta)^2} \right] + \phi(M) [1 + I_{c1}(N - M)] \right. \\ \quad \left. - p I_e \left[\frac{a\theta(N^2 - M^2)}{2(b+\theta)} - \frac{ab(Me^{(b+\theta)(N - M)} - T_c)}{(b+\theta)^2} + \frac{ab(e^{(b+\theta)(N - M)} - 1)}{(b+\theta)^3} \right] \right\} \geq 0 \\ t_b = \arg \left\{ p(1 + I_e t_b) \left[\frac{a\theta(t_b - M)}{b+\theta} + \frac{ab(e^{(b+\theta)(T_c - M)} - e^{(b+\theta)(T_c - t_b)})}{(b+\theta)^2} \right] + \phi(M) [1 + I_{c1}(t_b - M)] \right. \\ \quad \left. - p I_e \left[\frac{a\theta(t_b^2 - M^2)}{2(b+\theta)} - \frac{ab(Me^{(b+\theta)(T_c - M)} - x e^{(b+\theta)(T_c - t_b)})}{(b+\theta)^2} \right] + \frac{ab(e^{(b+\theta)(T_c - M)} - e^{(b+\theta)(T_c - t_b)})}{(b+\theta)^3} \right\} = 0 \\ M < t_b \leq N \leq T_c \\ V \leq V_0, T \leq T_0 \end{cases}
\end{aligned}$$

Subcase 3-2b''

$$\begin{aligned}
\text{Max}_{(T,V)} AP_{3-2b''}(T,V) &= \frac{1}{T} [p \int_{t_b}^{T_c} R(t) dt + p I_e \int_{t_b}^{T_c} R(t)(T-t) dt - A - c_0 V - h \int_0^{T_c} Z(t) dt] \\
&= \frac{1}{T} \left\{ (p + p I_e T) \left[\frac{a\theta(T_c - t_b)}{b+\theta} + \frac{ab(e^{(b+\theta)(T_c - t_b)} - 1)}{(b+\theta)^2} \right] - A - c_0 V - h \left[\frac{a(e^{(b+\theta)T_c} - 1)}{(b+\theta)^2} - \frac{aT_c}{b+\theta} \right] \right. \\
&\quad \left. - p I_e \left[\frac{a\theta[(T_c)^2 - t_b^2]}{2(b+\theta)} - \frac{ab[t_b e^{(b+\theta)(T_c - t_b)} - T_c]}{(b+\theta)^2} + \frac{ab[e^{(b+\theta)(T_c - t_b)} - 1]}{(b+\theta)^3} \right] \right\} \\
\text{s.t.} \quad &\begin{cases} M < N < t_b \leq T_c = T - V/a \\ \phi(M) = \left\{ pV + p I_e VM + (p + p I_e M) \left[\frac{a\theta}{b+\theta} M + \frac{ab(e^{(b+\theta)T_c} - e^{(b+\theta)(T_c - M)})}{(b+\theta)^2} \right] \right. \\ \quad \left. - p I_e \left[\frac{a\theta M^2}{2(b+\theta)} - \frac{abMe^{(b+\theta)(T_c - M)}}{(b+\theta)^2} + \frac{ab(e^{(b+\theta)T_c} - e^{(b+\theta)(T_c - M)})}{(b+\theta)^3} \right] - c(Q + V) \right\} < 0 \\ \phi(N) = \left\{ p(1 + I_e N) \left[\frac{a\theta(N - M)}{b+\theta} + \frac{ab(e^{(b+\theta)(N - M)} - 1)}{(b+\theta)^2} \right] + \phi(M) [1 + I_{c1}(N - M)] \right. \\ \quad \left. - p I_e \left[\frac{a\theta(N^2 - M^2)}{2(b+\theta)} - \frac{ab(Me^{(b+\theta)(N - M)} - T_c)}{(b+\theta)^2} + \frac{ab(e^{(b+\theta)(N - M)} - 1)}{(b+\theta)^3} \right] \right\} < 0 \\ \phi(T_c) = \left\{ p(1 + I_e T_c) \left[\frac{a\theta}{b+\theta} (T_c - N) + \frac{ab(e^{(b+\theta)(T_c - N)} - 1)}{(b+\theta)^2} \right] + \phi(N) [1 + I_{c2}(T_c - N)] \right. \\ \quad \left. - p I_e \left[\frac{a\theta(T_c^2 - N^2)}{2(b+\theta)} - \frac{ab(Ne^{(b+\theta)(T_c - N)} - T_c)}{(b+\theta)^2} + \frac{ab(e^{(b+\theta)(T_c - N)} - 1)}{(b+\theta)^3} \right] \right\} \leq 0 \\ t_b = \arg \left\{ p(1 + I_e t_b) \left[\frac{a\theta(t_b - N)}{b+\theta} + \frac{ab(e^{(b+\theta)(T_c - N)} - e^{(b+\theta)(T_c - t_b)})}{(b+\theta)^2} \right] + \phi(N) [1 + I_{c2}(t_b - N)] \right. \\ \quad \left. - p I_e \left[\frac{a\theta(t_b^2 - N^2)}{2(b+\theta)} - \frac{ab(Ne^{(b+\theta)(T_c - N)} - T_c e^{(b+\theta)(T_c - t_b)})}{(b+\theta)^2} + \frac{ab(e^{(b+\theta)(T_c - N)} - e^{(b+\theta)(T_c - t_b)})}{(b+\theta)^3} \right] \right\} = 0 \\ V \leq V_0, T \leq T_0 \end{cases}
\end{aligned}$$

Case 4

$$\begin{aligned}
\text{Max}_{(T,V)} AP_4(T,V) &= \frac{1}{T} [U(M) I_e (T - M) - A - c_0 V - h \int_0^{T_c} Z(t) dt] \\
&= \frac{1}{T} \left\{ U(M) I_e (T - M) - A - c_0 V - h \left[\frac{a(e^{(b+\theta)T_c} - 1)}{(b+\theta)^2} - \frac{aT_c}{b+\theta} \right] \right\} \\
\text{s.t.} \quad &\begin{cases} T_c = T - V/a \\ U(M) = \left\{ (p + p I_e M) \left[V + \frac{a\theta T_c}{b+\theta} + \frac{ab(e^{(b+\theta)T_c} - 1)}{(b+\theta)^2} \right] \right. \\ \quad \left. - p I_e \left[\frac{a\theta(T_c)^2}{2(b+\theta)} - \frac{abT_c}{(b+\theta)^2} + \frac{ab(e^{(b+\theta)T_c} - 1)}{(b+\theta)^3} \right] - c(Q + V) \right\} \geq 0 \\ T_c < M < N \\ V \leq V_0, T \leq T_0 \end{cases}
\end{aligned}$$

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