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An Optimal Order Method for Multiple Roots in Case of Unknown Multiplicity

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Abstract: In the literature, recently, some three-step schemes involving four function evaluations for the solution of multiple roots of nonlinear equations, whose multiplicity is not known in advance, are considered, but they do not agree with Kung–Traub's conjecture. The present article is devoted to the study of an iterative scheme for approximating multiple roots with a convergence rate of eight, when the multiplicity is hidden, which agrees with Kung–Traub's conjecture. The theoretical study of the convergence rate is investigated and demonstrated. A few nonlinear problems are presented to justify the theoretical study.

Keywords: nonlinear equations; iterative method; multiple root; unknown multiplicity; optimal convergence rate

1. Introduction

In the present report, we discuss an iterative scheme for calculating the approximate value of a multiple root ξ of a nonlinear equation $\psi(x) = 0$, in $\mathcal{R}$. The modified Newton's scheme [1] is probably the first method for calculating the estimated value of the multiple root when multiplicity m is available in advance, with a convergence rate twice that in some open set around ξ, under suitable regularity assumptions. For the same case, various schemes are presented in the literature by using numerous modes (one can see [2–13] for a few of them).

For the second case, *i.e.*, when the multiplicity m is not available in advance, for calculating the multiple root of $\psi(x) = 0$, Traub [14] used the relation $f(x) = \psi(x)/\psi'(x)$. In this stage, the problem of finding multiple roots of $\psi(x) = 0$ is equivalent to calculating the simple root of $f(x) = 0$. However, for this substitution, Newton's method involved the first and second derivatives also. To fend off these evaluations, King [15] used finite difference approximation.

Working with a parallel idea, Iyengar and Jain [16] developed two iterative methods of order three and four for approximating the multiple roots. The cubic convergence order scheme is as:

$$x_{n+1} = x_n - \frac{f(x_n)}{g(x_n)} - \frac{f\left(x_n - \frac{f(x_n)}{g(x_n)}\right)}{g(x_n)} \tag{1}$$

and the fourth-order method is expressed as:

$$x_{n+1} = x_n - \frac{f(x_n)}{g(x_n)} - \frac{f\left(x_n - \frac{f(x_n)}{g(x_n)}\right)}{g(x_n)} - \frac{f\left(x_n - \frac{f(x_n)}{g(x_n)} - \frac{f\left(x_n - \frac{f(x_n)}{g(x_n)}\right)}{g(x_n)}\right)}{g(x_n)} \tag{2}$$

where $g(x_n) = \frac{f(x_n+\beta f(x_n))-f(x_n)}{\beta f(x_n)}$.

Wu and Fu [17] also used a similar transformation and presented a uni-parametric scheme of second order for handling similar type problems. In addition, Wu *et al.* [18] prescribed another transformation given by:

$$f(x) = \frac{sign(\psi(x))\rho}{sign(\psi(x + sign(\psi(x))|\psi(x)|^{1/m}) - \psi(x))\rho + \psi(x + sign(\psi(x))|\psi(x)|^{1/m}) - \psi(x)} \tag{3}$$

where $\rho = \psi(x)|\psi(x)|^{1/m}$ and m represents the multiplicity, and engaged the modified Steffensen's method (see [19,20]):

$$x_{n+1} = x_n - h_n \frac{f^2(x_n)}{tf^2(x_n) + f(x_n + f(x_n)) - f(x_n)} \tag{4}$$

for evaluating the estimated root of $f(x) = 0$, where $h_n(> 0)$ is the iteration step size and $|t| < \infty$. Parida and Gupta [21] implied a different conversion:

$$\begin{aligned} f(x) &= \frac{\psi^2(x)}{\delta + \psi(x + \psi(x)) - \psi(x)}, \quad if\ \psi(x) \neq 0 \\ &= 0, \quad\quad\quad\quad\quad\quad\quad\quad\quad if\ \psi(x) = 0 \end{aligned} \tag{5}$$

where $\delta = sign(\psi(x + \psi(x)) - \psi(x))\psi^2(x)$, for converting the assignment of calculating multiple roots of ψ to find the simple root of f.

Yun [22] gave a new conversion of $\psi(x)$, which is expressed as follows:

$$H_\epsilon(x) = x_n - \frac{\epsilon\psi^2(x)}{\psi(x + \epsilon\psi(x)) - \psi(x)} \tag{6}$$

and considered ϵ in such a way that $max_{a\leq x\leq b}|\epsilon\psi(x)| = \delta$, as well as presented the iterative scheme of the following form:

$$x_{n+1} = x_n - \frac{2(x_{n+1} - x_n)H_\epsilon(x_n)}{H_\epsilon(2x_n - x_{n-1}) - H_\epsilon(x_{n-1})} \tag{7}$$

To get a higher order method, Li *et al.* [23] proposed an iterative scheme for approximating multiple roots of $f(x) = 0$ with fifth-order convergence, by using the transformation $f(x) = \psi(x_n)/\psi'(x_n)$, which is given as:

$$\begin{aligned} y_n &= x_n - \frac{f^2(x_n)}{f(x_n + f(x_n)) - f(x_n)} \\ u_n &= y_n - \frac{f(y_n)f(x_n)}{f(x_n + f(x_n)) - f(x_n)} \\ x_{n+1} &= u_n - \frac{f(u_n)}{f[u_n, y_n] + f[z_n, x_n, x_n](z_n - y_n)} \end{aligned} \tag{8}$$

where $f[.,.]$ and $f[.,.,.]$ represent the first- and second-order divided differences of f, respectively.

More recently, by using the same transformation, Sharma and Bahl [24] proposed an iterative scheme with a convergence rate of six and expressed as:

$$\begin{aligned} y_n &= x_n - \frac{f^2(x_n)}{f(x_n + f(x_n)) - f(x_n)} \\ u_n &= y_n - \frac{f(y_n)f(x_n)}{f(x_n + f(x_n)) - f(x_n)} \\ x_{n+1} &= u_n - \frac{f(u_n)f[x_n, y_n]}{f[x_n, u_n]f[y_n, u_n]} \end{aligned} \tag{9}$$

The above mentioned fifth- and sixth-order methods involved four function evaluations $[f(x_n), f(x_n + f(x_n)), f(y_n), f(u_n)]$, and by Kung and Traub's [25] conjecture, the optimal order should be eight. Motivated by this theory, we are going to present an optimal eighth-order method for multiple roots in case of unknown multiplicity: to our best knowledge, this is the first method of the optimal eighth order. Theoretically, its local rate of convergence is proven and finally supported by numerical testing.

2. Scheme and Analysis of the Local Convergence Rate

In this paper, we deal with the below mentioned conversion [14,26]:

$$\begin{aligned} f(x) &= \frac{\psi(x)}{\psi'(x)}, \ if\ \psi(x) \neq 0 \\ &= 0, \qquad if\ \psi(x) = 0 \end{aligned} \tag{10}$$

and employ a three-step Newton's scheme given by:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)} \\ u_n &= y_n - \frac{f(y_n)}{f'(y_n)} \\ x_{n+1} &= u_n - \frac{f(u_n)}{f'(u_n)} \end{aligned} \tag{11}$$

To escape the computations of primary differentiations of $f(x)$, $f(y(x))$ and $f(u(x))$, we estimate them as mentioned below:

$$f'(x_n) \approx \frac{f(z_n) - f(x_n)}{f(x_n)} = g_1(x_n, z_n) \tag{12}$$

where $z_n = x_n + f(x_n)$.

$$f'(y_n) \approx \frac{f[x_n, y_n]f[y_n, z_n]}{f[x_n, z_n]} = g_2(x_n, y_n, z_n) \tag{13}$$

and:

$$f'(u_n) \approx b_2 - b_1 b_4 = g_3(x_n, y_n, z_n, u_n) \tag{14}$$

where $b_1 = f(u_n)$, $b_4 = \frac{f[y_n,u_n,x_n] - f[y_n,u_n,z_n]}{f[y_n,z_n] - f[y_n,x_n]}$, $b_3 = f[y_n, u_n, z_n] + b_4 f[y_n, z_n]$, $b_2 = f[y_n, u_n] - b_3(y_n - u_n) + f(y_n)b_4$ (for more details, one can follow Equations (6) and (19), respectively, of [27]). Using these estimates of $f'(x_n)$, $f'(y_n)$ and $f'(u_n)$, given by Equations (12)–(14), respectively, in the scheme Equation (11), we have the scheme given by:

$$
\begin{aligned}
y_n &= x_n - \frac{f(x_n)}{g_1(x_n, z_n)} \\
u_n &= y_n - \frac{f(y_n)}{g_2(x_n, z_n, y_n)} \\
x_{n+1} &= u_n - \frac{f(u_n)}{g_3(x_n, z_n, y_n, u_n)}
\end{aligned} \tag{15}
$$

In order to discuss the convergence analysis of the scheme defined by Equation (15), we are moving to validate the following result:

Theorem 1. *Assume that $f \in C^8(D)$ $(D \subseteq R \to R)$, and contains only one root $\xi \in D$, where D is an open subset of R and s is sufficiently large. If the starting guess x_0 is near enough to ξ, the iterative scheme defined by Equation (15) has an optimal convergence rate of eight.*

Proof of Theorem 1. Let us suppose that $\psi(x)$ can be expressed as:

$$\psi(x) = (x - \xi)^m h(x) \tag{16}$$

where ξ is a multiple zero of $\psi(x) = 0$, with multiplicity m and $h(\xi) \neq 0$. According to Equation (16), we can write:

$$\psi'(x) = m(x - \xi)^{m-1} h(x) + (x - \xi)^m h'(x) \tag{17}$$

After taking the ratio of Equation (16) and (17), this can be written as:

$$f(x) = \frac{\psi(x)}{\psi'(x)} = \frac{(x - \xi)h(x)}{mh(x) + (x - \xi)h'(x)} \tag{18}$$

It is clear from the above Equation (18) that the task of approximating the multiple zero of $\psi(x) = 0$ is identical to the effort of estimating the simple root ξ of $f(x) = 0$. By virtue of Taylor's series expansion, we can write:

$$h(x_n) = h(\xi)[1 + b_1 e_n + b_2 e_n^2 + b_3 e_n^3 + b_4 e_n^4 + b_5 e_n^5 + b_6 e_n^6 + b_7 e_n^7 + b_8 e_n^8 + b_9 e_n^9 + o(e_n^{10})] \tag{19}$$

where $b_k = \frac{h^{(k)}(\xi)}{k! h(\xi)}$, $k = 1, 2, \ldots$ and $e_n = x_n - \xi$. By Equation (19), we attain:

$$h'(x_n) = h(\xi)[b_1 + 2b_2 e_n + 3b_3 e_n^2 + 4b_4 e_n^3 + 5b_5 e_n^4 + 6b_6 e_n^5 + 7b_7 e_n^6 + 8b_8 e_n^7 + 9b_9 e_n^8 + o(e_n^9)] \tag{20}$$

Using the expression Equations (19) and (20) in Equation (18) and then using symbolic computation software, Mathematica [28], we get:

$$
\begin{aligned}
f(x_n) &= \frac{e_n h(x_n)}{mh(x_n) + e_n h'(x_n)} \\
&= D_1 e_n + D_2 e_n^2 + D_3 e_n^3 + D_4 e_n^4 + D_5 e_n^5 + D_6 e_n^6 + D_7 e_n^7 + D_8 e_n^8 + o(e_n^9)
\end{aligned} \tag{21}
$$

where:

$$D_1 = \frac{1}{m},\ D_2 = \frac{-b_1}{m^2},\ D_3 = \frac{b_1^2 + mb_1^2 - 2mb_2}{m^2}$$

$$D_4 = \frac{-b_1^3 - 2mb_1^3 - m^2b_1^3 + 4mb_1b_2 + 3m^2b_1b_2 - 3m^2b_3}{m^4}$$

$$\begin{aligned} D_5 &= \frac{1}{m^5}\{(1+m)^3b_1^4 - 2m(1+m(3+2m)b_1^2b_2 + 2m^2(3+2m)b_1b_3 \\ &+2m^2((2+m)b_2^2 - 2mb_4)\} \end{aligned}$$

$$\begin{aligned} D_6 &= \frac{1}{m^6}\{-(1+m)^4b_1^5 + m(1+m)^2(8+5m)b_1^3b_2 - m^2(1+m)(9+5m)b_1^2b_3 \\ &+m^2b_1(-(2+m)(6+5m)b_2^2 + m(8+5m)b_4) + m^3((12+5m)b_2b_3 - 5mb_5)\} \end{aligned}$$

$$\begin{aligned} D_7 &= \frac{1}{m^7}\{(1+m)^5b_1^6 - 2m(1+m)^3(5+3m)b_1^4b_2 + 6m^2(1+m)^2(2+m)b_1^3b_3 \\ &+3m^2(1+m)(2+m)b_1^2((4+3m)b_2^2 - 2mb_4) \\ &+2m^3b_1(-2(9+m(11+3m))b_2b_3 + m(5+3m)b_5) \\ &+m^3(-2(2+m)^2b_2^3 + 2m(8+3m)b_2b_4 + 3m((3+m)b_3^2 - 2mb_6))\} \end{aligned}$$

$$\begin{aligned} &D_8 = \frac{1}{m^8}\{-(1+m)^6b_1^7 + m(1+m)^4(12+7m)b_1^5b_2 - m^2(1+m)^3(15+7m)b_1^4b_3 \\ &+m^2(1+m)^2b_1^3(-2(2+m)(10+7m)b_2^2 + m(16+7m)b_4) \\ &+m^3(1+m)b_1^2(3(24+m(27+7m))b_2b_3 - m(15+7m)b_5) \\ &+m^3b_1((2+m)^2(8+7m)b_2^3 - 2m(24+7m(4+m))b_2b_4 \\ &+m(-(3+m)(9+7m)b_3^2 + m(12+7m)b_6)) \\ &+m^4(-(2+m)(18+7m)b_2^2b_3 + m(20+7m)b_2b_5 + m((24+7m)b_3b_4 - 7mb_7))\} \end{aligned}$$

By virtue of the relation Equation (21), we can obtain the Taylor's series expansion for $f(z_n)$, and then, using its expressions together with $f(x_n)$, in the first sub-step of scheme Equation (15) and after simplifying, we have:

$$y_n = \xi + F_2e_n^2 + F_3e_n^3 + F_4e_n^4 + F_5e_n^5 + F_6e_n^6 + F_7e_n^7 + F_8e_n^8 + o(e_n^9) \tag{22}$$

where:

$$F_2 = -\frac{(1+m)b_1}{m^2}, F_3 = \frac{((2+3m+2m^2)b_1^2 - 2(1+3m+2m^2)b_2)}{m^3}$$

$$F_4 = \frac{1}{m^4}\{-(3+6m+5m^2+3m^3)b_1 + (5+16m+16m^2+9m^3)b_1b_2 - 3(1+4m+6m^2+3m^3)b_3\}$$

$$\begin{aligned} &F_5 = \frac{1}{m^5}\{(2+5m+5m^2+3m^3+2m^4)b_1^4 - (4+15m+19m^2+13m^3+8m^4)b_1^2b_2 \\ &+(-1+m+6m^2+6m^3+4m^4)b_2^2 + (5+16m+23m^2+17m^3+8m^4)b_1b_3 \\ &-2(1+5m+10m^2+10m^3+4m^4)b_4\} \end{aligned}$$

$$\begin{aligned}
F_6 = \ & \frac{1}{m^6}\{-(5+15m+18m^2+11m^3+5m^4+5m^5)b_1^5 \\
& +(10+46m+71m^2+52m^3+28m^4+25m^5)b_1^3b_2 \\
& -(20+58m+85m^2+72m^3+42m^4+25m^5)b_1^2b_3 \\
& +(-7-18m-3m^2+28m^3+33m^4+25m^5)b_2b_3 \\
& +b_1(-(-9-2m+29m^2+36m^3+27m^4+25m^5)b_2^2 \\
& +(17+70m+115m^2+108m^3+63m^4+25m^5)b_4) \\
& -5(1+6m+15m^2+20m^3+15m^4+5m^5)b_5\}
\end{aligned}$$

$$\begin{aligned}
F_7 = \ & \frac{1}{m^7}\{(6+21m+30m^2+21m^3+6m^4+m^5+6m^6)b_1^6 \\
& -2(5+30m+57m^2+47m^3+15m^4+5m^5+18m^6)b_1^4b_2 \\
& -2(1-3m-3m^2+m^3-m^4+m^5+6m^6)b_2^3 \\
& +2(15+42m+61m^2+53m^3+27m^4+15m^5+18m^6)b_1^3b_3 \\
& +2(-5-15m-17m^2-3m^3+17m^4+23m^5+18m^6)b_2b_4 \\
& +b_1^2((-24-30m+30m^2+58m^3+18m^4+15m^5+54m^6)b_2^2 \\
& -2(21+78m+118m^2+107m^3+66m^4+33m^5+18m^6)b_4) \\
& +2b_1((19+54m+56m^2+19m^3-4m^4-22m^5-36m^6)b_2b_3 \\
& +(13+66m+138m^2+155m^3+110m^4+53m^5+18m^6)b_5) \\
& +3((-2-14m-30m^2-28m^3-10m^4+3m^5+6m^6)b_3^2 \\
& -2(1+7m+21m^2+35m^3+35m^4+21m^5+6m^6)b_6)\}
\end{aligned}$$

$$\begin{aligned}
F_8 = \ & \frac{1}{m^8}\{(7+28m+47m^2+40m^3+13m^4-7m^5-7m^6+7m^7)b_1^7 \\
& -(7+66m+161m^2+163m^3+42m^4-61m^5-44m^6+49m^7)b_1^5b_2 \\
& +(35+94m+125m^2+95m^3+10m^4-49m^5-18m^6+49m^7)b_1^4b_3 \\
& +(11+4m-74m^2-128m^3-136m^4-128m^5-36m^6+49m^7)b_2^2b_3 \\
& +(17+136m+416m^2+640m^3+542m^4+232m^5+8m^6-49m^7)b_3b_4 \\
& -b_1^3((49+107m+32m^2-53m^3+44m^4+164m^5+86m^6-98m^7)b_2^2 \\
& +(77+274m+395m^2+320m^3+160m^4+50m^5+33m^6+49m^7)b_4) \\
& -(-13-44m-44m^2-8m^3+20m^4+52m^5+66m^6+49m^7)b_2b_5 \\
& -b_1^2((-119-384m-573m^2-514m^3-402m^4-306m^5-75m^6+147m^7)b_2b_3 \\
& -(77+354m+679m^2+712m^3+480m^4+234m^5+101m^6+49m^7)b_5) \\
& -((13-20m-44m^2-20m^3-50m^4-100m^5-46m^6+49m^7)b_2^3 \\
& +(38+241m+515m^2+583m^3+410m^4+190m^5+23m^6-49m^7)b_3^2 \\
& +2(32+102m+161m^2+178m^3+132m^4+60m^5-14m^6-49m^7)b_2b_4 \\
& +(37+224m+574m^2+812m^3+700m^4+404m^5+166m^6+49m^7)b_6) \\
& +(7+56m+196m^2+392m^3+490m^4+392m^5+196m^6+49m^7)b_7\}
\end{aligned}$$

In a similar way to the first sub-step, the Taylor's series for $f(y_n)$ can be derived using Equation (22), and then, using the involved expressions from the above relations, we can write:

$$u_n = \xi + H_4e_n^4 + H_5e_n^5 + H_6e_n^6 + H_7e_n^7 + H_8e_n^8 + o(e_n^9) \tag{23}$$

where:

$$H_4 = \frac{1}{m^5}\{(1+m)^2 b_1(-1+m)b_1^2 - 2mb_2)e^4\}$$

$$H_5 = -\frac{1}{m^6}\{(1+m)((-5-5m+4m^3)b_1^4 + (6+7m-9m^2-14m^3)b_1^2 b_2$$
$$+4m(1+3m+2m^2)b_2^2 + 3m(1+3m+2m^2)b_1 b_3\}$$

$$H_6 = \frac{1}{m^7}\{(-18-49m-50m^2-18m^3+9m^4+10m^5)b_1^5$$
$$+(39+129m+140m^2+30m^3-65m^4-47m^5)b_1^3 b_2$$
$$+(-9-23m-4m^2+52m^3+75m^4+33m^5)b_1^2 b_3$$
$$-6m(1+m)^2(2+7m+7m^2)b_2 b_3$$
$$+2(1+m)b_1((-6-19m-7m^2+22m^3+24m^4)b_2^2 - 2m(1+4m+6m^2+3m^3)b_4)\}$$

$$H_7 = \frac{1}{m^9}\{(-1+51m+179m^2+254m^3+182m^4+56m^5-14m^6-20m^7)b_1^6$$
$$+(2-158m-659m^2-1044m^3-782m^4-206m^5+126m^6+116m^7)b_1^4 b_2$$
$$+m(66+243m+360m^2+206m^3-72m^4-188m^5-96m^6)b_1^3 b_3$$
$$+mb_1^2((102+544m+1015m^2+813m^3+149m^4-253m^5-174m^6)b_2^2$$
$$+(-12-39m-18m^2+91m^3+192m^4+170m^5+60m^6)b_4)$$
$$+m(1+m)(4(-2-14m-27m^2-16m^3+5m^4+10m^5)b_2^3$$
$$-9m(1+6m+14m^2+15m^3+6m^4)b_3^2 - 8m(2+11m+24m^2+25m^3+10m^4)b_2 b_4)$$
$$+m(1+m)b_1((-36-129m-115m^2+122m^3+304m^4+208m^5)b_2 b_3$$
$$-5m(1+5m+10m^2+10m^3+4m^4)b_5)\}$$

$$H_8 = \frac{1}{m^{10}}\{(9-113m-530m^2-960m^3-947m^4-541m^5-153m^6+14m^7+35m^8)b_1^7$$
$$+(-27+446m+2426m^2+4908m^3+5221m^4+3091m^5+828m^6-173m^7-240m^8)b_1^5 b_2$$
$$+(3-294m-1304m^2-2526m^3-2633m^4-1417m^5-150m^6+333m^7+214m^8)b_1^4 b_3$$
$$+b_1^3((16-428m-2914m^2-6922m^3-8196m^4-5129m^5-1298m^6+479m^7+478m^8)b_2^2$$
$$+m(99+431m+744m^2+599m^3+43m^4-393m^5-408m^6-163m^7)b_4)$$
$$+mb_1^2((333+1927m+4528m^2+5351m^3+2921m^4-127m^5-1258m^6-663m^7)b_2 b_3$$
$$+(-15-59m-48m^2+141m^3+405m^4+485m^5+320m^6+95m^7)b_5)$$
$$+2m(1+m)((-18-140m-379m^2-446m^3-179m^4+90m^5+116m^6)b_2^2 b_3$$
$$-6m(2+14m+41m^2+63m^3+51m^4+17m^5)b_3 b_4$$
$$-5m(2+13m+35m^2+50m^3+39m^4+13m^5)b_4 b_5$$
$$-mb_1(2(-50-421m-1244m^2-1728m^3-1206m^4-326m^5+134m^6+121m^7)b_2^3$$
$$-2(-24-127m-224m^2-55m^3+386m^4+682m^5+544m^6+182m^7)b_2 b_4$$
$$-3(1+m)((-9-36m-46m^2+23m^3+125m^4+144m^5+73m^6)b_3^2$$
$$-2m(1+6m+15m^2+20m^3+15m^4+5m^5)b_6))\}$$

Finally, we obtain the Taylor's series expansion of $f(u_n)$, with the help of just the previous equation, and using the required relations obtained above, in the last sub-step of scheme Equation (15), we get the final error expression as follows:

$$\begin{aligned} e_{n+1} &= \frac{(1+m)^4 b_1((-1+m)b_1^2 - 2mb_2)(-(-1+m)b_1^4 + m(2+m)b_1^2 b_2 - 4m^2 b_2^2 + 3m^2 b_1 b_3}{m^{11}} e_n^8 \\ &\quad + o(e_n^9) \end{aligned} \tag{24}$$

Thus, the proof is completed. □

We further discuss the procedure for approximating the multiplicity of the zero ξ calculated by the discussed iterative scheme. If x_n is the n-th estimate of the multiple root computed by using scheme Equation (15), then in view of Equation (21), we get:

$$f(x_n) \approx \frac{(x_n - \xi)h(x_n)}{mh(x_n) + (x_n - \xi)h'(x_n)} = \frac{e_n h(x_n)}{mh(x_n) + e_n h'(x_n)}$$

By the fact that e_n is small, we have $f(x_n) \approx \frac{e_n}{m}$. Similarly, we can find that $f(x_{n+1}) \approx \frac{e_{n+1}}{m}$. Moreover, $e_{n+1} - e_n = x_{n+1} - x_n$. Thus, we can write:

$$m \approx \frac{x_{n+1} - x_n}{f(x_{n+1}) - f(x_n)}$$

Hence, m may be approximated by the reverse of the first-order divided difference of f for consecutive estimates x_n and x_{n+1} (see [15,21]).

For comparing different iterative schemes abstractly, the concept of the efficiency index is presented by Owtrowski [29] and given by $d^{1/n}$, where d is the convergence rate and n is the total evaluations involved in each iteration. That means an iterative scheme with a greater efficiency index is more adequate. By using this formula, the efficiency index of the considered scheme is $8^{1/4} = 1.8618$, which is more than $6^{1/4} = 1.5651$ of $MM6$ and $5^{1/4} = 1.4953$ of $MM5$.

3. Numerical Testing with the Conclusion

In this portion, we employ the proposed three-step method Equation (15) (MM8) for solving five nonlinear equations and scrutinize them by the methods given by Sharma and Bahl Equation (9) (MM6), in [24], and the Li *et al.* Equation (8) (MM5), in [23]. Displayed in Tables 1–3 are the absolute error in the root, the absolute value of the function and the absolute error in the approximation of unknown multiplicity, for three iterations. All of the computations were done by using Mathematica 8. We mention below five test equations along with their exact roots ξ (the functions, as well as initial guesses are taken from [23]):

1. $f_1 = \dfrac{(x - 5^{1/2})^4}{(x-1)^2 + 1}, \; m = 4, \; \xi = 2.2360...,$
2. $f_2 = (8xe^{-x^2} - 2x - 3)^8, \; m = 8, \; \xi = 1.7903...,$
3. $f_3 = (ln(x^2 + 3x + 5) - 2x + 7)^8, \; m = 8, \; \xi = 5.4690...,$
4. $f_4 = \dfrac{(x-2)^4}{(x-1)^2 + 1}, \; m = 4, \; \xi = 2.000...,$
5. $f_5 = \left(x^{1/2} - \dfrac{1}{x} - 1\right)^7, \; m = 7, \; \xi = 2.1478....$

Table 1. Numerical results for $MM8$.

f	n	$\|x_n - \xi\|$	$\|f(x_n)\|$	$\|m - m_n\|$
f_1	1	3.0654e−4	7.6640e−4	1.1458e+0
	2	4.4333e−32	1.1083e−32	2.9976e−4
	3	8.4937e−255	2.1234e−255	4.3356e−32
f_2	1	2.1643e−3	2.7085e−4	3.5724e−1
	2	3.9285e−23	4.9107e−24	9.1312e−3
	3	4.6110e−181	5.7638e−182	1.6577e−22
f_3	1	5.7113e−9	7.1391e−10	1.4907e−1
	2	2.4732e−77	3.0916e−78	4.5992e−10
	3	3.0587e−624	3.8234e−625	1.9917e−78
f_4	1	4.4515e−5	1.1129e−5	1.0345e+0
	2	1.6081e−38	4.0203e−39	4.45154e−5
	3	4.6651e−306	1.1663e−306	1.6081e−38
f_5	1	4.7605e−4	6.8015e−5	1.7797e+0
	2	2.7073e−30	3.8675e−31	8.3970e−4
	3	2.9694e−240	4.2420e−241	4.7766e−30

Table 2. Numerical results for $MM6$.

f	n	$\|x_n - \xi\|$	$\|f(x_n)\|$	$\|m - m_n\|$
f_1	1	7.6409e−3	1.9138e−3	1.1487e+0
	2	1.7779e−15	4.4447e−16	7.4627e−3
	3	2.8524e−92	7.1310e−92	1.7387e−15
f_2	1	4.8261e−3	6.0480e−4	3.5708e−1
	2	3.1700e−15	3.9625e−16	2.0356e−2
	3	2.5528e−88	3.1910e−89	1.3377e−14
f_3	1	1.2583e−7	1.5729e−8	1.4907e−1
	2	1.6225e−50	2.0282e−51	1.0133e−8
	3	7.4567e−308	9.3209e−309	1.3066e−51
f_4	1	4.1460e−3	1.0376e−3	1.0366e+0
	2	3.9308e−17	9.8270e−18	4.1460e−3
	3	2.8495e−101	7.1237e−102	3.9308e−17
f_5	1	4.4860e−3	6.4158e−4	1.7819e+0
	2	9.8136e−17	1.4019e−17	7.8955e−3
	3	1.0975e−98	1.5679e−99	1.7315e−16

Table 3. Numerical results for $MM5$.

f	n	$\|x_n - \xi\|$	$\|f(x_n)\|$	$\|m - m_n\|$
f_1	1	7.1979e−3	1.8026e−3	1.1485e+0
	2	5.9419e−14	1.4855e−14	70306e−3
	3	2.3260e−69	5.8149e−70	5.8109e−14
f_2	1	4.7520e−3	5.9550e−4	3.5708e−1
	2	6.2369e−14	7.7961e−15	2.0043e−2
	3	2.4558e−68	3.0697e−69	2.6318e−13
f_3	1	2.2101e−7	2.7627e−8	1.4907e−1
	2	1.8202e−42	2.2753e−43	1.7798e−8
	3	6.8964e−218	8.6205e−219	1.4658e−42
f_4	1	3.5022e−3	8.7632e−4	1.0363e+0
	2	1.8093e−15	4.5231e−16	3.5022e−3
	3	6.6558e−77	1.6639e−77	1.8093e−15
f_5	1	4.7515e−3	6.7960e−4	1.7821e+0
	2	3.6795e−15	5.2564e−16	8.3616e−3
	3	1.0884e−75	1.5548e−76	6.4920e−15

The numerical values in Tables 1–3 validate that the presented scheme MM8 performs better, not only for the absolute error in the root and the absolute value of the function as compared to MM6 and MM5, but also in the approximation of unknown multiplicity. The techniques explained in this note could also be adapted to new high order methods designed recently (see [30] and the references therein): we will explore this matter in future research.

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