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An Optimal Eighth-Order Derivative-Free Family of Potra-Pták's Method

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Academic Editor: Alicia Cordero

Received: 25 April 2015 / Accepted: 8 June 2015 / Published: 15 June 2015

Abstract: In this paper, we present a new three-step derivative-free family based on Potra-Pták's method for solving nonlinear equations numerically. In terms of computational cost, each member of the proposed family requires only four functional evaluations per full iteration to achieve optimal eighth-order convergence. Further, computational results demonstrate that the proposed methods are highly efficient as compared with many well-known methods.

Keywords: Steffensen-type methods; Kung-Traub conjecture; optimal order of convergence; efficiency index

1. Introduction

One of the most basic and earliest problem of numerical analysis concerns with finding efficiently and accurately the simple roots of a nonlinear equation of the form

$$f(x) = 0, \quad (1)$$

where $f : D \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is a nonlinear continuous function. Analytical methods for solving such equations are almost non-existent and therefore, it is only possible to obtain approximate solutions by relying on numerical methods based on iterative procedure (see e.g., [1–7]). Newton's method [5] is one of the most famous and basic method for solving such equations, which is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \dots \quad (2)$$

It converges quadratically for simple roots and linearly for multiple roots.

Multipoint iterative methods for solving nonlinear equation are of great practical importance since they overcome the limitations of one-point methods regarding the convergence order and computational efficiency. According to the Kung-Traub conjecture [2], the order of convergence of any multipoint method without memory requiring n function evaluations per iteration, cannot exceed the bound 2^{n-1} , called the optimal order. Thus, the optimal order for a method with three functional evaluations per step would be four.

As the order of an iterative method increases, so does the number of functional evaluations per step. Commonly, the efficiency of an iterative method is measured by the efficiency index defined by Ostrowski in [3] as $p^{1/d}$, where p is the order of convergence and d is the number of functional evaluations per step. To improve the order and efficiency of Newton’s method Equation (2), Potra and Pták [4] proposed the following third-order method:

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n - \frac{f(x_n) + f(y_n)}{f'(x_n)}, \end{cases} \quad n = 0, 1, 2, \dots \tag{3}$$

It satisfies the following error equation

$$e_{n+1} = 2c_2^2e_n^3 + (-9c_2^3 + 7c_2c_3)e_n^4 + O(e_n^5).$$

However, there are many practical situations in which the calculations of derivatives are expensive or it requires a great deal of time for them to be given or calculated. Therefore, the idea of removing derivatives from the iteration process is very significant.

In particular, when the first-order derivative $f'(x_n)$ in Newton’s method is replaced by forward-difference approximation $\frac{f(x_n+f(x_n))-f(x_n)}{f(x_n)}$, we get the well-known Steffensen method [6] as follows:

$$x_{n+1} = x_n - \frac{f(x_n)}{f[x_n, w_n]},$$

where $w_n = x_n + f(x_n)$ and $f[\cdot, \cdot]$ denotes the first order divided difference. As a matter of fact, both methods maintain quadratic convergence using only two functional evaluations per full step, but Steffensen method is derivative free, which is very useful in optimization problems. Recently, many higher-order derivative-free methods are built according to the Steffensen’s method, (cf. [7,8] and the references cited therein). Soleymani *et al.* in [9] presented the following fourth-order optimal Steffensen type methods given by

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f[x_n, w_n]}, \quad w_n = x_n + \beta f(x_n), \\ x_{n+1} = x_n - \frac{f(x_n) + f(y_n)}{f[x_n, w_n]} - \left(\frac{2f(x_n) + af(y_n)}{f[x_n, w_n]} \left(\frac{f(y_n)}{f(x_n)} \right)^2 \right) \left(1 - \frac{\beta f[x_n, w_n]}{2 + 2\beta f[x_n, w_n]} \right), \end{cases} \quad a \in \mathbb{R}, \tag{4}$$

where $\beta \in \mathbb{R} \setminus \{0\}$. The construction of this family is based on Potra-Pták’s method. However, we do not have any higher-order derivative-free modifications of Potra-Pták’s method till date.

With this aim, we intend to propose a new derivative-free modification of Potra-Pták’s method

having optimal eighth-order convergence. The construction of the proposed class is based on weight function approach. It is found by way of illustrations that the proposed methods are very useful in high precision computations.

2. Development of Derivative-Free Methods and Convergence Analysis

In this section, we intend to develop a new derivative-free class of three-point methods having optimal eighth-order convergence.

Thus, we consider the following iteration scheme

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - \frac{f(y_n)}{f'(x_n)}, \\ x_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)}, \end{cases} \tag{5}$$

where first two steps of the well-known Potra-Pták’s method are composed with the Newton step.

It satisfies the following error equation

$$e_{n+1} = 4c_2^5e_n^6 + (-36c_2^6 + 28c_2^4c_3)e_n^7 + O(e_n^8), \tag{6}$$

where $e_n = x_n - \alpha$ and $c_k = \frac{1}{k!} \frac{f^{(k)}(\alpha)}{f'(\alpha)}$, $k \geq 2$.

According to the Kung-Traub conjecture, the above scheme Equation (5) is not optimal because it has sixth-order convergence and requires five functional evaluations per full iteration. Following Cordero-Torregrosa conjecture [8], we replace derivatives in all three steps by suitable approximations that use available data. Therefore, we approximate

$$\begin{aligned} f'(x_n) &\approx f[x_n, w_n], \\ f'(z_n) &\approx f[x_n, w_n], \end{aligned} \tag{7}$$

where $w_n = x_n + \beta f(x_n)^3$, $\beta \in \mathbb{R} \setminus \{0\}$ and $f[x, y] = \frac{f(x) - f(y)}{x - y}$ denotes a divided difference (without index n).

Substituting these approximations in Equation (5), we get a derivative-free three-point iterative method given by

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f[x_n, w_n]}, \\ z_n = y_n - \frac{f(y_n)}{f[x_n, w_n]}, \\ x_{n+1} = z_n - \frac{f(z_n)}{f[x_n, w_n]}. \end{cases} \tag{8}$$

It satisfies the following error equation

$$e_{n+1} = 4c_2^3e_n^4 + (-26c_2^4 + 20c_2^2c_3)e_n^5 + O(e_n^6).$$

Again, the family of methods Equation (8) is not optimal according to the Kung-Traub conjecture. Therefore, to further improve its order of convergence, we shall now make use of weight function approach. Therefore, we consider

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f[x_n, w_n]}, & w_n = x_n + \beta f(x_n)^3, \\ z_n = x_n - \left(\frac{f(x_n) + f(y_n)}{f[x_n, w_n]} \right) G(\tau), & \tau = \frac{f(y_n)}{f(x_n)}, \\ x_{n+1} = z_n - \frac{f(z_n)}{f[x_n, w_n]} H(\tau, \phi), & \phi = \frac{f(z_n)}{f(y_n)}, \end{cases} \tag{9}$$

where $\beta \in \mathbb{R} \setminus \{0\}$ and G and H are parametric functions of one and two variables, respectively. Theorem (1) illustrates that under what conditions on weight functions, convergence order of family Equation (9) will arrive at the optimal level eight.

3. Convergence Analysis

Theorem 1. Assume that function $f : D \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is sufficiently differentiable and has a simple zero $\alpha \in D$. If an initial guess x_0 is sufficiently close to $\alpha \in D$, then the iterative scheme defined by Equation (9) has optimal convergence of order eight when

$$\begin{cases} G(0) = 1, G'(0) = 0, G''(0) = 4 \text{ and } |G^{(3)}(0)| \leq \infty, \beta \in \mathbb{R} \setminus \{0\}, \\ H_{00} = 1, H_{10} = 2, H_{01} = 1, H_{20} = \frac{G^{(3)}(0)}{3} + 6, H_{11} = 4, H_{30} = 3G^{(3)}(0) + \frac{G^{(4)}(0)}{4}, \end{cases}$$

where $H_{ij} = \frac{1}{i!j!} \frac{\partial H(u, v)}{\partial u^i \partial v^j} |_{(0,0)}$, $i = 0, 1, 2, 3$ and $j = 0, 1, 2, 3$.

It satisfies the following error equation

$$\begin{aligned} e_{n+1} = & \frac{1}{432} c_2 ((-18 + G^{(3)}(0))c_2^2 + 6c_3) \left[(H_{02}(-18 + G^{(3)}(0))^2 - 3(648 + 2H_{21}(-18 + G^{(3)}(0)) - 28G^{(3)}(0) \right. \\ & + 3G^{(4)}(0))c_2^4 + 36(-2 + G^{(2)}(0))c_3^2 - 12c_2^2(6\beta f'(\alpha)^3 + (-102 + 18H_{02} + 3H_{21} + G^{(3)}(0) - G^{(2)}(0)G^{(3)}(0))c_3 \\ & \left. - 72c_2c_4 \right] e_n^8 + O(e_n^9), \end{aligned} \tag{10}$$

where e_n and c_k are already defined in Equation (6).

Proof. Using Taylor’s series and symbolic computation, we can determine the asymptotic error constant of three-step derivative-free class of methods Equation (9). Furthermore, taking into account that $f(\alpha) = 0$, we can expand $f(x_n)$ about $x_n = \alpha$. Therefore, we get

$$f(x_n) = f'(\alpha)(e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + c_5e_n^5 + c_6e_n^6 + c_7e_n^7 + c_8e_n^8) + O(e_n^9). \tag{11}$$

Using that $w_n = x_n + \beta f(x_n)^3$, one gets

$$\begin{aligned} f[x_n, w_n] = & f'(\alpha) + 2f'(\alpha)c_2e_n + 3f'(\alpha)c_3e_n^2 + f'(\alpha) (\beta f'(\alpha)^3c_2 + 4c_4) e_n^3 \\ & + f'(\alpha) (3\beta f'(\alpha)^3c_2^2 + 3\beta f'(\alpha)^3c_3 + 5c_5) e_n^4 + O(e_n^5). \end{aligned} \tag{12}$$

From Equations (11) and (12), we have

$$y_n - \alpha = x_n - \alpha - \frac{f(x_n)}{f[x_n, w_n]} = c_2 e_n^2 + (-2c_2^2 + 2c_3) e_n^3 + (\beta f'(\alpha)^3 c_2 + 4c_2^3 - 7c_2 c_3 + 3c_4) e_n^4 + O(e_n^5). \tag{13}$$

Expanding $f\left(x_n - \frac{f(x_n)}{f[x_n, w_n]}\right)$ about $x_n = \alpha$, we have

$$f(y_n) = f\left(x_n - \frac{f(x_n)}{f[x_n, w_n]}\right) = f'(\alpha) c_2 e_n^2 + f'(\alpha) (-2c_2^2 + 2c_3) e_n^3 + f'(\alpha) (5c_2^3 + c_2 (\beta f'(\alpha)^3 - 7c_3) + 3c_4) e_n^4 + O[e_n]^5, \tag{14}$$

and

$$\tau = \frac{f(y_n)}{f(x_n)} = c_2 e_n + (-3c_2^2 + 2c_3) e_n^2 + (3c_4 - 10c_2 c_3 + 8c_2^3 + \beta f'(\alpha)^3 c_2) e_n^3 + O(e_n^4). \tag{15}$$

In the same vein, by considering $G(0) = 1, G'(0) = 0, G''(0) = 4$ and $|G^{(3)}(0)| \leq \infty$, we obtain

$$z_n - \alpha = x_n - \alpha - \left(\frac{f(x_n) + f(y_n)}{f[x_n, w_n]}\right) G(\tau) = \left(\left(3 - \frac{G^{(3)}(0)}{6}\right) c_2^3 - c_2 c_3\right) e_n^4 + O(e_n^5). \tag{16}$$

Moreover, we find

$$f(z_n) = f'(\alpha) \left[\left(3 - \frac{G^3(0)}{6}\right) c_2^3 - c_2 c_3\right] e_n^4 + f'(\alpha) \left[\left(-16 + \frac{3G^3(0)}{2} - \frac{G^{(4)}(0)}{24}\right) c_2^4 - 2c_2^3 - c_2^2 (\beta f'(\alpha)^3 + (-20 + G^3(0))c_3) - 2c_2 c_4\right] e_n^5 + O(e_n^6), \tag{17}$$

and

$$\phi = \frac{f(z_n)}{f(y_n)} = \frac{1}{6} (18c_2^2 - G^{(3)}(0)c_2^2 - 6c_3) e_n^2 + \frac{1}{24} (-24\beta f'(\alpha)^3 c_2 - 240c_2^3 + 28G^{(3)}(0)c_2^3 - G^{(4)}(0)c_2^3 + 288c_2 c_3 - 16G^{(3)}(0)c_2 c_3 - 48c_4) e_n^3 + O(e_n^4). \tag{18}$$

Since, it is clear from Equations (15) and (18) that τ and ϕ are of order e_n and e_n^2 , respectively. Therefore, we can expand weight function $H(\tau, \phi)$ in the neighborhood of origin by Taylor series expansion up to third order terms as follows:

$$H(\tau, \phi) = H_{00} + H_{10}\tau + H_{01}\phi + \frac{1}{2!} (H_{20}\tau^2 + 2H_{11}\tau\phi + H_{02}\phi^2) + \frac{1}{3!} (H_{30}\tau^3 + 3H_{21}\tau^2\phi + 3H_{12}\tau\phi^2 + H_{03}\phi^3). \tag{19}$$

Using Equations (17) and (19) in the last step of Equation (9), we get

$$e_{n+1} = \frac{1}{6} (-1 + H_{00}) c_2 \left((-18 + G^{(3)}(0)) c_2^2 + 6c_3 \right) e_n^4 + \left(\frac{1}{24} (-384 + 4H_{10}(-18 + G^{(3)}(0)) + 36G^{(3)}(0) - G^{(4)}(0) + H_{00}(528 - 44G^{(3)}(0) + G^{(4)}(0))) c_2^4 + 2(-1 + H_{00}) c_3^2 + c_2^2 (\beta f'(\alpha)^3 (-1 + H_{00}) + (20 + H_{10} + H_{00}(-22 + G^{(3)}(0)) - G^{(3)}(0)) c_3) + 2(-1 + H_{00}) c_2 c_4 \right) e_n^5 + \dots + O(e_n^9). \tag{20}$$

This implies that the derivative-free class of methods Equation (9) arrive at optimal eighth-order of convergence by choosing the weight functions as follow:

$$\begin{cases} G(0) = 1, G'(0) = 0, G''(0) = 4 \text{ and } |G^{(3)}(0)| \leq \infty, \beta \in \mathbb{R} \setminus 0, \\ H_{00} = 1, H_{10} = 2, H_{01} = 1, H_{20} = \frac{G^{(3)}(0)}{3} + 6, H_{11} = 4, H_{30} = 3G^{(3)}(0) + \frac{G^{(4)}(0)}{4}. \end{cases} \quad (21)$$

Finally, using Equations (21) in (20), we get the following error equation

$$\begin{aligned} e_{n+1} = & \frac{1}{432}c_2((-18 + G^{(3)}(0))c_2^2 + 6c_3) \left[(H_{02}(-18 + G^{(3)}(0))^2 - 3(648 + 2H_{21}(-18 + G^{(3)}(0)) - 28G^{(3)}(0) \right. \\ & + 3G^{(4)}(0))c_2^4 + 36(-2 + G^{(2)}(0))c_3^2 - 12c_2^2(6\beta f'(\alpha))^3 + (-102 + 18H_{02} + 3H_{21} + G^{(3)}(0) - G^{(2)}(0)G^{(3)}(0))c_3 \\ & \left. - 72c_2c_4 \right] e_n^8 + O(e_n^9). \end{aligned} \quad (22)$$

This concludes the proof. □

Remark 1. It is straightforward to see that all the proposed methods of family Equation (9) require four functional evaluations, viz., $f(x_n), f(y_n), f(w_n), f(z_n)$, per full iteration. Therefore, these methods are optimal in the sense of Kung-Traub conjecture and have the efficiency indices $E = 8^{\frac{1}{4}} \approx 1.682$. Furthermore, by choosing appropriate weight functions in family Equation (9), we can develop several new optimal derivative-free families having optimal eighth-order convergence.

Remark 2. From the application point of view, when the given problem is complicated, it becomes very difficult to evaluate derivatives. For example, the nonlinear function $h(x) = (\cot x)e^x \sqrt{1/(2x^2 \cosh x)}$ (see Figure 1), has a very complicated first derivative. Such shortcomings lead us to investigate new optimal iterative methods which are totally free from derivatives.

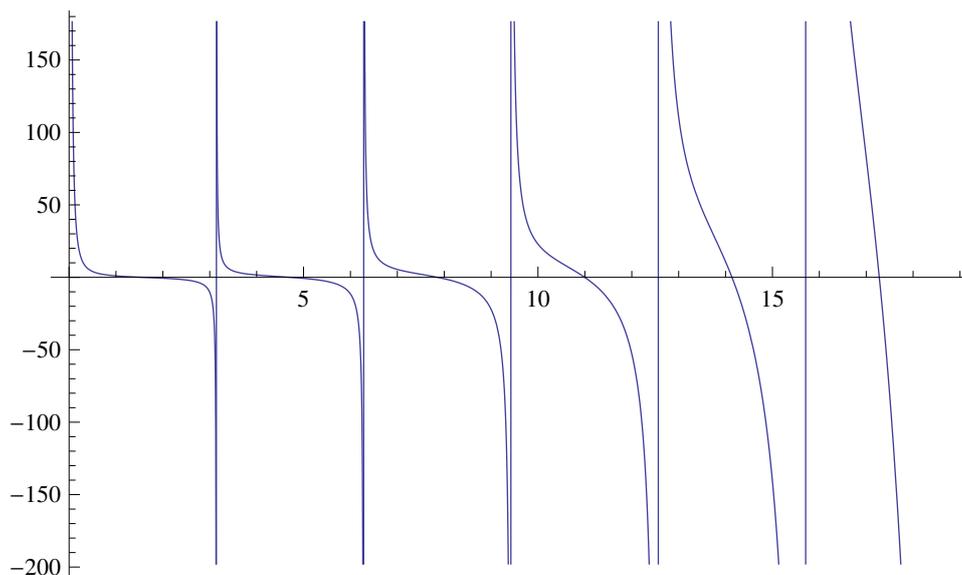


Figure 1. The graph of h(x) and its root.

4. Special Cases

In this section, we introduce some concrete methods based on the proposed class Equation (9).

Method 1 Let us consider the weight functions defined by

$$G(\tau) = \frac{\gamma}{6}\tau^3 + 2\tau^2 + 1 \text{ and } H(\tau, \phi) = \frac{\gamma}{2}\tau^3 + \left(\frac{\gamma}{6} + 3\right)\tau^2 + 4\tau\phi + 2\tau + \phi + 1,$$

where $\tau = \frac{f(y_n)}{f(x_n)}$, $\phi = \frac{f(z_n)}{f(y_n)}$ and γ is any free disposable parameter.

It can be easily seen that the above mentioned weight functions $G(\tau)$ and $H(\tau, \phi)$ satisfy all the conditions of Theorem (1). Therefore, we get a new derivative-free optimal family of eighth-order methods given by

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f[x_n, w_n]}, & w_n = x_n + \beta f(x_n)^3, \\ z_n = x_n - \left(\frac{f(x_n) + f(y_n)}{f[x_n, w_n]}\right) \left[\frac{\gamma}{6} \left(\frac{f(y_n)}{f(x_n)}\right)^3 + 2 \left(\frac{f(y_n)}{f(x_n)}\right)^2 + 1\right], \\ x_{n+1} = z_n - \frac{f(z_n)}{f[x_n, w_n]} \left[\frac{\gamma}{2} \left(\frac{f(y_n)}{f(x_n)}\right)^3 + \left(\frac{\gamma}{6} + 3\right) \left(\frac{f(y_n)}{f(x_n)}\right)^2 + 4 \frac{f(y_n) f(z_n)}{f(x_n) f(y_n)} + 2 \frac{f(y_n)}{f(x_n)} + \frac{f(z_n)}{f(y_n)} + 1\right]. \end{cases} \tag{23}$$

Method 2 Now, we consider the following weight functions

$$G(\tau) = \frac{\tau(1 - 12(\mu + 2)\tau) - 12}{\tau(1 - 12\mu\tau) - 12} \text{ and } H(\tau, \phi) = \frac{-24 + \left(\frac{299}{3} + 48\mu\right)\tau^3}{4(-6 + 6\phi + (12 - 5\tau)\tau)},$$

where $\tau = \frac{f(y_n)}{f(x_n)}$, $\phi = \frac{f(z_n)}{f(y_n)}$ and μ is any free disposable parameter.

These weight functions satisfy all the conditions of Theorem (1). Therefore, we obtain another new derivative-free optimal family of eighth-order methods given by

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f[x_n, w_n]}, & w_n = x_n + \beta f(x_n)^3, \\ z_n = x_n - \left(\frac{f(x_n) + f(y_n)}{f[x_n, w_n]}\right) \left[\frac{\frac{f(y_n)}{f(x_n)}(1 - 12(\mu + 2)\frac{f(y_n)}{f(x_n)}) - 12}{\frac{f(y_n)}{f(x_n)}(1 - 12\mu\frac{f(y_n)}{f(x_n)}) - 12}\right], \\ x_{n+1} = z_n - \frac{f(z_n)}{f[x_n, w_n]} \left[\frac{-24 + \left(\frac{299}{3} + 48\mu\right)\frac{f(y_n)^3}{f(x_n)}}{4(-6 + 6\frac{f(z_n)}{f(y_n)} + (12 - 5\frac{f(y_n)}{f(x_n)})\frac{f(y_n)}{f(x_n)})}\right]. \end{cases} \tag{24}$$

Method 3 Consider the weight functions defined by

$$G(\tau) = \frac{6\eta - \tau + 12\eta\tau^2 + (\eta - 2)\tau^3}{6\eta - \tau} \text{ and } H(\tau, \phi) = \frac{\tau^2 - 6\eta(12 + 25\tau^2)}{\tau^2 + 6\eta(-12 + 12\phi + (24 - 35\tau)\tau)},$$

where $\tau = \frac{f(y_n)}{f(x_n)}$, $\phi = \frac{f(z_n)}{f(y_n)}$ and η is any free disposable parameter.

These weight functions also satisfy all the conditions of Theorem (1). Therefore, we get another new optimal family of eighth-order methods given by

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f[x_n, w_n]}, & w_n = x_n + \beta f(x_n)^3, \\ z_n = x_n - \left(\frac{f(x_n) + f(y_n)}{f[x_n, w_n]} \right) \left[\frac{6\eta - \frac{f(y_n)}{f(x_n)} + 12\eta \frac{f(y_n)^2}{f(x_n)} + (\eta - 2) \frac{f(y_n)^3}{f(x_n)}}{6\eta - \frac{f(y_n)}{f(x_n)}} \right], \\ x_{n+1} = z_n - \frac{f(z_n)}{f[x_n, w_n]} \left[\frac{\frac{f(y_n)^2}{f(x_n)} - 6\eta \left(12 + 25 \frac{f(y_n)^2}{f(x_n)} \right)}{\frac{f(y_n)^2}{f(x_n)} + 6\eta(-12 + 12 \frac{f(z_n)}{f(y_n)} + (24 - 35 \frac{f(y_n)}{f(x_n)}) \frac{f(y_n)}{f(x_n)})} \right]. \end{cases} \tag{25}$$

Method 4 The method by Kung and Traub, see [2], denoted by KTM_8 , is

$$\begin{cases} y_n = x_n + \beta f(x_n), & \beta \in \mathbb{R} - \{0\}, \\ z_n = y_n - \beta \frac{f(x_n)f(y_n)}{f(y_n) - f(x_n)}, \\ w_n = z_n - \frac{f(x_n)f(y_n)}{f(z_n) - f(x_n)} \left[\frac{1}{f[y_n, x_n]} - \frac{1}{f[z_n, y_n]} \right], \\ x_{n+1} = z_n - \frac{f(x_n)f(y_n)f(z_n)}{f(w_n) - f(x_n)} \left[\frac{1}{f[y_n, x_n]} \left\{ \frac{1}{f[w_n, z_n]} - \frac{1}{f[z_n, y_n]} \right\} - \frac{1}{f[z_n, x_n]} \left\{ \frac{1}{f[z_n, y_n]} - \frac{1}{f[y_n, x_n]} \right\} \right]. \end{cases} \tag{26}$$

Method 5 The method by Soleymani, see [10], denoted by SM_8^1 , is

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f[x_n, w_n]}, & w_n = x_n + f(x_n), \\ z_n = y_n - \frac{f(y_n)}{f[x_n, y_n]} \left[1 + \frac{f(y_n)}{f(w_n)} + \left(\frac{f(y_n)}{f(w_n)} \right)^2 \right], \\ x_{n+1} = z_n - \frac{f(z_n)}{f[z_n, y_n]} \left[1 + \frac{1}{f[x_n, w_n] + 1} \left(\frac{f(y_n)}{f(x_n)} \right)^2 + (2 + f[x_n, w_n]) \frac{f(z_n)}{f(w_n)} \right]. \end{cases} \tag{27}$$

Method 6 The method by Zheng *et al.*, see [11], denoted by ZM_8 , is

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f[x_n, w_n]}, & w_n = x_n + \beta f(x_n), \\ z_n = y_n - \frac{f(y_n)}{f[x_n, y_n] + f[y_n, w_n] - f[x_n, w_n]}, \\ x_{n+1} = z_n - \frac{f(z_n)}{f[z_n, y_n] + f[z_n, x_n, x_n](z_n - y_n) + f[z_n, y_n, x_n, w_n](z_n - y_n)(z_n - x_n)}. \end{cases} \tag{28}$$

Method 7 The method by Soleymani, see [12], denoted by SM_8^2 , is

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f[x_n, w_n]}, & w_n = x_n + \beta f(x_n), \\ z_n = y_n - \frac{f(y_n)}{f[x_n, y_n] + f[y_n, w_n] - f[x_n, w_n]}, \\ x_{n+1} = z_n - \frac{f(z_n) \left\{ 1 + \left(\frac{f(y_n)}{f(x_n)} \right)^4 - (1 + \beta f[x_n, w_n]) \left(\frac{f(y_n)}{f(w_n)} \right)^3 - \left(\frac{f(z_n)}{f(y_n)} \right)^2 + \frac{f(z_n)}{f(w_n)} + \left(\frac{f(z_n)}{f(x_n)} \right)^2 \right\}}{f[x_n, z_n] + f[z_n, y_n] - f[x_n, y_n]} \end{cases} \quad (29)$$

5. Numerical Experiments

In this section, we shall check the effectiveness of the newly proposed methods. We employ the present methods Equation (23) (for $\beta = 1, \gamma = 12$), Equation (24) (for $\beta = 1, \mu = 12$) and Equation (25) (for $\beta = 1, \eta = 12$), denoted by MM_8^1, MM_8^2 and MM_8^3 , respectively to solve nonlinear equations. We compare them with Kung and Traub method Equation (26) ($KT M_8$), Soleymani method Equation (27) (SM_8^1), Zheng *et al.* method Equation (28) (ZM_8), Soleymani *et al.* method Equation (29) (SM_8^2), respectively. The test functions and their roots are displayed in Table 1. Comparison of different eighth-order derivative-free iterative methods with respect to the same number of functional evaluations (TNE = 12) are provided in Tables 2–4. All computations have been performed using the programming package *Mathematica* 9 with multiple precision arithmetic. We use $\epsilon = 10^{-35}$ as a tolerance error. The following stopping criteria are used for computer programs: (i) $|x_{n+1} - x_n| < \epsilon$, (ii) $|f(x_{n+1})| < \epsilon$. These methods are employed to solve some nonlinear equations of two classes: **smooth functions** and **non-smooth functions**:

$$g_1(x) = |x^2 - 2|, \quad \alpha \approx 1.4142135623730950488016887242096981, \quad x_0 = 1.3.$$

$$g_2(x) = \begin{cases} x(x - 1) & \text{if } x \leq 0 \\ -2x(x + 1) & \text{if } x \geq 0, \end{cases} \quad \alpha = 0, \quad x_0 = 0.5.$$

Table 1. Test functions and their roots.

Test functions	Roots	Initial guess
$f_1(x) = (\sin x)^2 + x$	$\alpha = 0$	$x_0 = 0.5$
$f_2(x) = x^2 - (1 - x)^{25}$	$\alpha \approx 0.14373925929975369826697493201066691 \dots$	$x_0 = 0.4$
$f_3(x) = \sin^{-1}(x^2 - 1) - \frac{x}{2} + 1$	$\alpha \approx 0.59481096839836917752265623515213618 \dots$	$x_0 = 0.3$
$f_4(x) = \tan(\log x) + \cos(x^3) \frac{1}{\sqrt{2x}}$	$\alpha \approx 0.44326078355676706795301995624689113 \dots$	$x_0 = 0.41$
$f_5(x) = 10xe^{-x^2} - 1$	$\alpha \approx 1.6796306104284499406749203388379704 \dots$	$x_0 = 1.5$

Table 2. Comparison of different methods for smooth functions.

f		KTM_8	SM_8^1	ZM_8	SM_8^2	MM_8^1	MM_8^2	MM_8^3
f_1	$ f(x_1) $	1.21e-3	1.22e-3	4.33e-3	1.67e-3	0.9e-3	5.86e-4	7.81e-4
	$ f(x_2) $	5.84e-22	3.06e-22	1.00e-13	4.85e-21	7.46e-24	1.44e-24	6.59e-25
	$ f(x_3) $	7.16e-168	4.67e-171	1.63e-77	2.60e-161	1.31e-184	1.92e-189	3.35e-193
f_2	$ f(x_1) $	4.37e-3	4.08e-3	1.18e-2	3.02e-3	2.08e-3	3.49e-5	3.49e-3
	$ f(x_2) $	3.21e-12	1.06e-11	1.16e-6	1.18e-12	2.69e-16	8.72e-20	8.09e-15
	$ f(x_3) $	1.01e-85	1.81e-80	5.09e-31	4.59e-88	1.06e-118	1.32e-144	1.26e-107
f_3	$ f(x_1) $	1.06e-7	6.23e-8	6.50e-6	3.92e-7	1.94e-8	4.81e-8	1.55e-8
	$ f(x_2) $	9.23e-59	1.07e-60	3.25e-33	2.65e-54	4.55e-66	1.73e-62	2.44e-66
	$ f(x_3) $	3.15e-467	8.39e-483	5.05e-197	1.15e-431	0.1e-490	4.93e-498	0.1e-492
f_4	$ f(x_1) $	9.31e-5	1.25e-4	1.46e-5	1.83e-5	1.00e-8	4.94e-7	1.21e-6
	$ f(x_2) $	8.46e-30	2.55e-29	5.66e-29	5.52e-36	1.08e-65	8.35e-49	9.70e-470
	$ f(x_3) $	3.95e-230	7.45e-227	1.92e-169	3.76e-280	0.1e-492	5.53e-383	1.68e-367
f_5	$ f(x_1) $	1.00e-3	3.79e-4	2.80e-3	1.78e-4	2.61e-5	1.79e-6	1.84e-6
	$ f(x_2) $	4.54e-26	9.35e-85	4.27e-49	7.58e-87	1.42e-39	1.06e-47	4.60e-48
	$ f(x_3) $	7.83e-205	4.28e-234	1.00e-101	5.17e-257	1.09e-313	1.58e-377	7.04e-381

Table 3. Comparison of different methods for non-smooth function $g_1(x)$.

Methods	$ g_1(x_1) $	$ g_1(x_2) $	$ g_1(x_3) $
Method KTM_8 Equation (26)	9.81e-3	4.14e-6	8.18e-13
Method SM_8^1 Equation (27)	1.03e-2	3.21e-6	3.21e-13
Method ZM_8 Equation (28)	3.28e-3	6.23e-11	8.36e-42
Method SM_8^2 Equation (29)	2.89e-2	6.28e-5	3.66e-10
Our Method MM_8^1 Equation (23)	2.97e-3	2.43e-22	4.69e-175
Our Method MM_8^2 Equation (24)	5.98e-4	3.56e-24	1.31e-188
Our Method MM_8^3 Equation (25)	1.33e-4	5.87e-33	6.67e-258

Table 4. Comparison of different methods for non-smooth function $g_2(x)$.

Methods	$ g_2(x_1) $	$ g_2(x_2) $	$ g_2(x_3) $
Method KTM_8 Equation (26)	D	D	D
Method SM_8^1 Equation (27)	D	D	D
Method ZM_8 Equation (28)	2.63e-1	6.16e-7	1.87e-40
Method SM_8^2 Equation (29)	0.786e+3	0.12e+1	1.03e-4
Our Method MM_8^1 Equation (23)	1.09e-1	7.44e-7	5.53e-13
Our Method MM_8^2 Equation (24)	2.61e-3	4.05e-25	1.40e-199
Our Method MM_8^3 Equation (25)	2.52e-3	3.94e-25	1.43e-199

D: stands for divergent.

6. Conclusions

In this study, we contribute further to the development of the theory of iteration processes and propose a new derivative-free optimal family of eighth-order methods for solving nonlinear equations numerically. It is noteworthy that the given scheme can produce several new derivative-free optimal eighth-order methods by choosing different types of weight functions. The asserted superiority of proposed methods is also corroborated by numerical results displayed in the Tables 2–4. The numerical experiments suggest that the new class would be valuable alternative for solving nonlinear equations.

Acknowledgments

We would like to express our gratitude to the anonymous referees for their insightful valuable comments and suggestions.

Author Contributions

The contributions of all of the authors have been similar. All of them have worked together to develop the present manuscript.

Conflicts of Interest

The authors declare no conflict of interest.

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