Algorithms **2009**, *2*, 1221-1231; doi:10.3390/a2031221

OPEN ACCESS

algorithms

ISSN 1999-4893

www.mdpi.com/journal/algorithms

Article

Multiplication Symmetric Convolution Property for Discrete Trigonometric Transforms

Do Nyeon Kim * and K. R. Rao

Department of Electrical Engineering, University of Texas at Arlington, Box 19016 Arlington, TX 76019, USA; E-Mail: rao@uta.edu

* Author to whom correspondence should be addressed; E-Mail: dnkim@uta.edu; Tel: +1 (817) 272-2097, Fax: +1 (817) 272-2253.

Received: 25 June 2009; in revised form: 31 August 2009 / Accepted: 15 September 2009 / Published: 22 September 2009

Abstract: The symmetric-convolution multiplication (SCM) property of discrete trigonometric transforms (DTTs) based on unitary transform matrices is developed. Then as the reciprocity of this property, the novel multiplication symmetric-convolution (MSC) property of discrete trigonometric transforms, is developed.

Keywords: symmetric/asymmetric convolution; trigonometric transforms

1. Introduction

Shen *et al*. [1] developed fast DCT-domain convolution for time/spatial-domain multiplication using DCT type-2. They exploited symmetry and orthogonality for the fast algorithm. Logo-keying operation in the spatial domain can be done in the DCT domain for compressed image/video editing. However, they have not derived the convolution from the symmetric-convolution multiplication (SCM) property of discrete trigonometric transforms (DTTs), which will be the focus of this paper.

Time-domain symmetric convolution for a linear phase filtering application has been developed by Martucci [2] and is called the symmetric-convolution multiplication property of DTTs. Based on the SCM property, Zou *et al*. [3] have developed a symmetric convolution for linear phase FIR filtering. Filter coefficients consist of symmetric and asymmetric parts.

Based on those earlier works, Reju *et al*. [4] have finally developed fast circular convolution using DTTs. The input sequences to be convolved need not be symmetric or asymmetric. Thus fast DCTs and DSTs can be used instead of the FFTs for FIR filtering. Generalized fast convolution using numerous transforms as well as the DFT and DTTs has been studied by Korohoda *et al*. in [5].

In this paper we show that swapping the forward and inverse transforms in the SCM property [2] yields a multiplication-convolution (MSC) property. That is, the convolution of transformed sequences gives the same results as the forward transform after element-by-element multiplication of the data sequences. The necessary scaling factor M for these new properties has been described in the third line below Eq. (20) in [2], saying "it is possible to swap the usage of the forward and inverse transform; in that case, an *extra scaling factor* may be required". Here M is the size of the generalized DFT (GDFT) when a DTT is derived from an M-point GDFT. We can get 40 types of MSCs corresponding to 40 types of SCMs in Tables VI, VII of [2].

In (1) of [6], the formulation of the SCM for the unitary matrices is presented, with only the exception of some of the 40 options mentioned by Martucci [2], because they cannot be expressed with the assumed tools. Here, we fill that gap. Consider the following matrix relationship between convolutional/unnormalized DTTs (denoted with lower-case subscript like C_{1e}) and unitary DTTs (denoted with capital-letter subscript like C_{IE}).

$$C_{1e} = \sqrt{2N}\, A_1^{-1}\, C_{IE}\, A_1 \qquad C_{2e} = \sqrt{2N}\, A_2^{-1} C_{IIE} \tag{1}$$
$$C_{3e} = \sqrt{2N}\, C_{IIIE}\, A_2 \qquad C_{4e} = \sqrt{2N}\, C_{IVE} \tag{2}$$
$$S_{1e} = \sqrt{2N}\, S_{IE} \qquad S_{2e} = \sqrt{2N}\, A_3^{-1} S_{IIE} \tag{3}$$
$$S_{3e} = \sqrt{2N}\, S_{IIIE}\, A_3 \qquad S_{4e} = \sqrt{2N}\, S_{IVE} \tag{4}$$
$$C_{1o} = \sqrt{2N-1}\, A_2^{-1} C_{IO}\, A_2 \qquad C_{2o} = \sqrt{2N-1}\, A_2^{-1} C_{IIO}\, A_4 \tag{5}$$
$$C_{3o} = \sqrt{2N-1}\, A_4^{-1}\, C_{IIIO}\, A_2 \qquad C_{4o} = \sqrt{2N-1}\, C_{IVO} \tag{6}$$
$$S_{1o} = \sqrt{2N-1}\, S_{IO} \qquad S_{2o} = \sqrt{2N-1}\, S_{IIO} \tag{7}$$
$$S_{3o} = \sqrt{2N-1}\, S_{IIIO} \qquad S_{4o} = \sqrt{2N-1}\, A_4^{-1}\, S_{IVO}\, A_4 \tag{8}$$

Here the fact that scalars commute with matrices is used for C_{3e} and others.

$$A_1 = [\sqrt{k_n}\,]_{N+1} = [\sqrt{k_m}\,]_{N+1} = \mathrm{diag}(\tfrac{1}{\sqrt{2}}, 1, 1, \ldots, 1, \tfrac{1}{\sqrt{2}}), \qquad \text{where } m, n = 0, 1, \ldots, N \tag{9}$$
$$A_2 = [\sqrt{k_n}\,]_{N} = [\sqrt{k_m}\,]_{N} = \mathrm{diag}(\tfrac{1}{\sqrt{2}}, 1, 1, \ldots, 1), \qquad \text{where } m, n = 0, 1, \ldots, N-1 \tag{10}$$
$$A_3 = [\sqrt{k_n}\,]_{N} = [\sqrt{k_m}\,]_{N} = \mathrm{diag}(1, 1, \ldots, 1, \tfrac{1}{\sqrt{2}}), \qquad \text{where } m, n = 1, 2, \ldots, N \tag{11}$$
$$A_4 = [\sqrt{l_n}\,]_{N} = [\sqrt{l_m}\,]_{N} = \mathrm{diag}(1, 1, \ldots, 1, \tfrac{1}{\sqrt{2}}), \qquad \text{where } m, n = 0, 1, \ldots, N-1 \tag{12}$$

$$k_p = \begin{cases} 1/2 & p = 0, N \\ 1 & p = 1, 2, \ldots, N-1 \end{cases} \tag{13}$$
$$l_p = \begin{cases} 1 & p = 0, 1, \ldots, N-2 \\ 1/2 & p = N-1 \end{cases} \tag{14}$$

where $\mathrm{diag}(a_{11}, a_{11}, \ldots, a_{NN})$ implies a diagonal matrix with the diagonal elements as $(a_{11}, a_{11}, \ldots, a_{NN})$. I_N is the identity matrix of size $N \times N$.

Let matrices J, K, Q, and R be defined as:

$$\mathrm{J} = \begin{pmatrix} I_N \\ O_{1\times N} \end{pmatrix} \qquad \mathrm{K} = \begin{pmatrix} O_{1\times N} \\ I_N \end{pmatrix} \qquad \mathrm{Q} = \begin{pmatrix} I_{N-1} \\ O_{1\times(N-1)} \end{pmatrix} \qquad \mathrm{R} = \begin{pmatrix} O_{1\times(N-1)} \\ I_{N-1} \end{pmatrix} \tag{15}$$

$$(N+1)\times N \qquad (N+1)\times N\,N \qquad \times(N-1) \qquad N\times(N-1)$$

where $O_{1\times N}$ is the zero row-vector, with N elements, whose entries are all zero. Multiplying one of those matrices with an input vector appends a single zero on the top of the first or under the last element of the vector. Multiplying the transpose of one of those matrices with an input vector discards the first or last element of the vector.

2. Symmetric-Convolution Multiplication Property

One of the forty cases of symmetric convolution is proven as an example.

Property. $\mathrm{y} = \mathrm{A}_1^{-1}\mathrm{C}_{IE}^{-1}\ \mathrm{J}\ \mathcal{H}_{\mathrm{C}2e}\ \mathrm{C}_{\mathrm{II}E}\ \mathrm{x}$ [(A8) in Table 3]

Proof. Equation (A.1) in the appendix of [2] can be rewritten as:

$$\mathrm{C}_{1e} = 2\ \hat{C}_{1e}\ [k_n]_{N+1} \tag{16}$$

where $\hat{C}_{1e}$ is the kernel.

$$\mathrm{C}_{IE} = \sqrt{2/N}\ [\sqrt{k_m}\ ]_{N+1}\ \hat{C}_{1e}\ [\sqrt{k_n}\ ]_{N+1} \tag{17}$$

$$([\sqrt{k_n}\ ]_{N+1})^{-1} = [\sqrt{1/k_n}\ ]_{N+1} \tag{18}$$

From (16), (17) and (18):

$$\mathrm{C}_{1e} = [\sqrt{2N/k_m}\ ]_{N+1}\ \mathrm{C}_{IE}\ [\sqrt{k_n}\ ]_{N+1} \tag{19}$$

$$\mathrm{C}_{1e}^{-1} = [\sqrt{1/k_n}\ ]_{N+1}\ \mathrm{C}_{IE}^{-1}\ [\sqrt{k_m/(2N)}\ ]_{N+1} \tag{20}$$

From (1) and (10):

$$\mathrm{C}_{2e}^{-1} = \mathrm{C}_{\mathrm{II}E}^{-1}\ [\sqrt{k_m/(2N)}\ ]_N \tag{21}$$

Let x and w be input column vectors in time domain. Let y be an output column vector in time domain. Let $\mathcal{H}_{\mathrm{C}2e}$ be a diagonal matrix defined as $\mathcal{H}_{\mathrm{C}2e} = \mathrm{diag}([\mathrm{C}_{2e}\ \mathrm{w}]^T)$, where superscript T denotes the transpose operator. We rewrite the 8th property in Table VI of [2] as:

$$\begin{aligned} \mathrm{y} &= \mathrm{C}_{1\mathrm{e}}^{-1}\ \{\ \mathcal{H}_{\mathrm{C}2e}\ \mathrm{C}_{2\mathrm{e}}\ \mathrm{x}\} \\ &= \mathrm{C}_{1e}^{-1}\ \{\ \mathcal{H}_{\mathrm{C}2e}\ [\sqrt{2N/k_m}\ ]_N\ \mathrm{C}_{\mathrm{II}E}\ \mathrm{x}\} \\ &= [\sqrt{1/k_n}\ ]_{N+1}\ \mathrm{C}_{IE}^{-1}\ [\sqrt{k_m}\ ]_{N+1}\ \mathrm{J}\ \{\ \mathcal{H}_{\mathrm{C}2e}\ [\sqrt{1/k_m}\ ]_N\ \mathrm{C}_{\mathrm{II}E}\ \mathrm{x}\ \} \end{aligned} \tag{22}$$

The matrix J shows up in the last line of (22) for zero padding. Since $\mathcal{H}_{\mathrm{C}2e}$ and $[\sqrt{1/k_m}\ ]_N$ are diagonal matrices, they can commute. Thus:

$$\begin{aligned} \mathrm{y} &= [\sqrt{1/k_n}\ ]_{N+1}\ \mathrm{C}_{IE}^{-1}\ \mathrm{J}\ \{\ \mathcal{H}_{\mathrm{C}2e}\ \mathrm{C}_{\mathrm{II}E}\ \mathrm{x}\ \} \\ &= [\sqrt{1/k_n}\ ]_{N+1}\ \mathrm{C}_{IE}^{-1}\ \mathrm{J}\ \mathcal{H}_{\mathrm{C}2e}\ \mathrm{C}_{\mathrm{II}E}\ \mathrm{x} \qquad \blacksquare \end{aligned} \tag{23}$$

Another example is shown in (11) of [6]. Equation (12) of [6] is expanded to full for our derivation and only results are listed in the first column of Tables 3 and 4 in the Appendix.

3. Multiplication Symmetric-Convolution Property

Let X and Y be transformed input and output data vectors. Since there are one-to-one correspondences between unitary discrete trigonometric transforms (DTTs), we can exchange the forward transform for inverse one and vice versa as follows. In other words, a pair has the same matrix but has different names. Define h_{C3e} as:

$$h_{C3e} = \mathrm{diag}(\ [(C^{3e})^{-1}\ W]^T\ ) \tag{24}$$

where C^{3e} will be defined in (27). That is, $\mathcal{H}_{C2e}$ and h_{C3e} are the same matrix with different names (thus names of DTTs need to change).

Then from (23):

$$Y = [\ \sqrt{1/k_m}\ ]_{N+1} C_{IE}\ J\ h_{C3e}\ C_{IIIE}^{-1}\ X \tag{25}$$

Notice the forward transform matrix C_{IIE} is replaced by the inverse transform matrix C_{IIIE}^{-1} since they are the same matrix, and vice versa. Now this equation represents a SCM of DTTs. A key point of this new property is that we need to redefine convolution forms of DCTs and DSTs. The factor of M is divided for the inverse DCT of the convolution form in [2] whereas it is divided for the forward DCT of the new convolution form. M is $2N$ for even and $2N - 1$ for odd. Now new convolution form for DCT 2 is denoted as C^{2e}:

	Forward		Inverse	
(old)	C_{2e}	$\Leftrightarrow$	$C_{2e}^{-1} = \frac{1}{M}\ C_{3e}$	(26)
(new)	$C^{2e} = \frac{1}{M}\ C_{2e}$	$\Leftrightarrow$	$(C^{2e})^{-1} = C_{3e}$ or $(C^{3e})^{-1} = C_{2e}$	(27)

The rest of DTTs can be readily obtained from the appendix of [2].

MSC properties can be described in terms of convolutional / unnormalized DTTs to obtain similar results:

(11th SCM in Table IV of [2]) $\quad C_{3e}^{-1}\ \ (C_{3e} \times C_{3e})$ (28)

(ours, MSC) $\quad M\ \ C^{2e}\ ((C^{2e})^{-1} \times (C^{2e})^{-1})$ (29)

Only results are listed. Matrices J, K, Q, and R are required for different index ranges between operands. Link between the index range-based maneuvers described by Martucci [2] and the introduction of the J, K, Q and R matrices is presented in Tables 1 and 2 for $M = 2N$ and $M = 2N - 1$, respectively.

4. Applications

For an image resizing (filter) application, one of 40 MSC properties is used. The definition of a normalizing parameter $F(k)$ in [7] needs a minor change as:

$$F(\mathrm{k}) = \begin{cases} \sqrt{2} & k = 0 \\ 1 & 1 \le k \le N - 1 \end{cases}$$

Then we can derive one of MSC properties, which is (4) in [7], from our equation as follows:

$$C_{II}\{x(n) \times w(n)\}\ = C_{II}\{x(n)\} \otimes\ C^{2e}\ \{w(n)\}$$

$$= C_{II}\{x(n)\} \otimes \tfrac{1}{2N}\, C_{2e}\,\{w(n)\}$$

where the symbol $\otimes$ denotes symmetric convolution. Since $C_{2e} = \sqrt{2N}\ A_2^{-1}C_{IIE}$ in (1),

$$C_{II}\{x(n) \times w(n)\} = C_{II}\{x(n)\} \otimes \tfrac{1}{\sqrt{2N}}\, A_2^{-1}\, C_{II}\{w(n)\}$$

By the associativity of (continuous and discrete) convolution:

$$\begin{aligned} C_{II}\{x(n) \times w(n)\} &= \tfrac{1}{\sqrt{2N}}\, C_{II}\{x(n)\} \otimes A_2^{-1}C_{II}\{w(n)\} \\ &= \tfrac{1}{\sqrt{2N}}\, A_2\,[A_2^{-1}C_{II}\{x(n)\} \otimes A_2^{-1}C_{II}\{w(n)\}] \end{aligned} \tag{30}$$

This is shown in block diagram format in Fig. 1(b). Since $G(k)$ and $F(k)$ are A_2^{-1} and $\tfrac{1}{\sqrt{2N}}\, A_2$ in matrix form:

$$C_{II}\{x(n) \times w(n)\} = G(k)\ [\ F(k)\ C_{II}\{x(n)\} \otimes F(k)\ C_{II}\{w(n)\}\ ]$$

Convolution has the property of associativity with scalar multiplication. Let F and G be any real sequences. Then:

$$a(F \otimes G) = (aF) \otimes G = F \otimes (aG)$$

for any real (or complex) number a.

For a numerical example, let:

$w = (1, 2, 3, 4)^T$, $x = (1, 0, 3, 2)^T$ and $N = 4$

Then the time domain element-by-element multiplication of the two vectors is:

$$w \times x = (1, 0, 9, 8)^T$$

$$\begin{aligned} W = A_2^{-1}\, C_{II}\, w &= (a_0, a_1, a_2, a_3)^T \\ &= (7.071, -2.230, 0, -0.159)^T \end{aligned}$$

$$\begin{aligned} X &= A_2^{-1}\, C_{II}\, x \\ &= (4.243, -1.465, 0, 1.689)^T \end{aligned}$$

$$\begin{aligned} Y &= W \otimes X = (W_t + W_h)\, X \\ &= (36, -19.823, 0, 11.272)^T \end{aligned} \tag{31}$$

where:

$$W = (W_t + W_h) = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ a_1 & a_0 & a_1 & a_2 \\ a_2 & a_1 & a_0 & a_1 \\ a_3 & a_2 & a_1 & a_0 \end{pmatrix} + \begin{pmatrix} 0 & a_1 & a_2 & a_3 \\ 0 & a_2 & a_3 & 0 \\ 0 & a_3 & 0 & -a_3 \\ 0 & 0 & -a_3 & -a_2 \end{pmatrix} \tag{32}$$

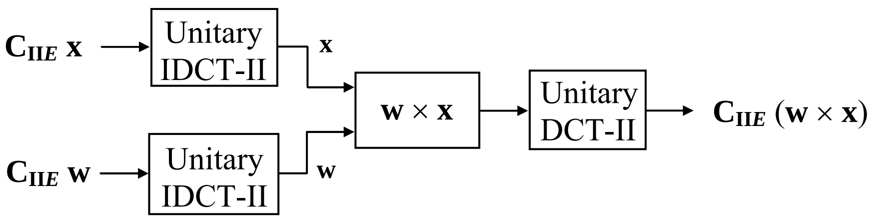
Equation (32) is $\left[y^{3e}_{(N),t}\right] + \left[y^{3e}_{(N),h}\right]$ in [8, p. 2635], and W_t and W_h are a symmetric Toeplitz matrix and a Hankel matrix [9]. Symmetric convolution is represented in matrix multiplication form of (31) using (32). Since the expression inside the square brackets of (30) is the matrix Y defined in (31):

$$Y = \sqrt{2N}\, A_2^{-1}C_{II}\,(w \times x) \tag{33}$$

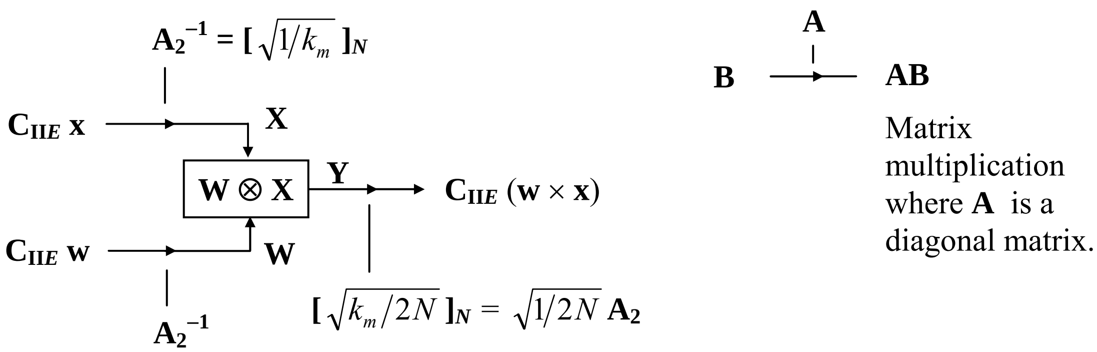
$$C_{II}\,(w \times x) = \tfrac{1}{\sqrt{2N}}\ A_2 Y \tag{34}$$

Equation (34) corresponds to (30). Thus Y can be computed by using either (33) or (31). In other words, the symmetric convolution of DCT coefficients, Figure 1(b) is an alternative method to computing the DCT of multiplication of two time sequences, Figure 1(a).

Figure 1. For compressed image / video editing, logo-keying operation (alpha blending) can be done in (a) the spatial and (b) transform domains [1]. The symbol × denotes the element-by-element multiplication of the two vectors and ⊗ denotes the symmetric convolution of the two vectors.



(a)



(b)

5. Conclusions

For logo-keying operation or alpha blending, one image is scaled by α and the other is scaled by $(1 - \alpha)$, where α is a value between zero and one, and then two images are added up. For compressed image / video editing, multiplication operation in logo-keying operation in the spatial can be done in the DTT domain by convolving the unitary DTT coefficients of the two images. For those applications, we have developed 40 types of the MSC properties for the unitary DTTs (DCTs and DSTs [11]–[13]).

Acknowledgements

The authors wish to thank the reviewers.

Appendix

Table 1. Link between the index range-based maneuvers described by Martucci [2] and the introduction of the J, K, Q and R matrices, $M = 2N$.

SCM (Symmetric-Conv. Mult.)	Forward transform			$\mathcal{H}$			Inverse transform		
	Input index range	Output index range		Input index range	Output index range		Input index range	Output index range	
$y = S_{IE}^{-1} (R^T J^T \mathcal{H}_{C1e}) S_{IE}\, x$				$0 \to N$	$1 \to N-1$	$R^T J^T$			
$y = S_{IE}^{-1} \mathcal{H}_{S1e} R^T J^T C_{IE} (A_1 x)$	$0 \to N$	$1 \to N-1$	$R^T J^T$						
$y = -A_1^{-1} C_{IE}^{-1} J R \mathcal{H}_{S1e} S_{IE}\, x$	$1 \to N-1$	$0 \to N$	J R						
$y = C_{IIE}^{-1} \mathcal{H}_{C2e} J^T C_{IE} (A_1 x)$	$0 \to N$	$0 \to N-1$	J^T						
$y = S_{IIE}^{-1} \mathcal{H}_{S2e} K^T C_{IE} (A_1 x)$	$0 \to N$	$1 \to N$	K^T						
$y = -C_{IIE}^{-1} R \mathcal{H}_{S1e} Q^T S_{IIE}\, x$	$1 \to N$	$1 \to N-1$	Q^T				$1 \to N-1$	$0 \to N-1$	R
$y = -C_{IIE}^{-1} R (Q^T \mathcal{H}_{S2e}) S_{IE}\, x$				$1 \to N$	$1 \to N-1$	Q^T	$1 \to N-1$	$0 \to N-1$	R
$y = A_1^{-1} C_{IE}^{-1} J \mathcal{H}_{C2e} C_{IIE}\, x$							$0 \to N-1$	$0 \to N$	J
$y = S_{IE}^{-1} (Q^T \mathcal{H}_{S2e}) R^T C_{IIE}\, x$	$0 \to N-1$	$1 \to N-1$	R^T	$1 \to N$	$1 \to N-1$	Q^T			
$y = S_{IE}^{-1} (R^T \mathcal{H}_{C2e}) Q^T S_{IIE}\, x$	$1 \to N$	$1 \to N-1$	Q^T	$0 \to N-1$	$1 \to N-1$	R^T			
$y = -A_1^{-1} C_{IE}^{-1} K \mathcal{H}_{S2e} S_{IIE}\, x$							$1 \to N$	$0 \to N$	K

Table 2. Link between the index range-based maneuvers described by Martucci [2] and the introduction of the J, K, Q and R matrices, $M = 2N - 1$.

SCM (Symmetric-Conv. Mult.)	Forward transform			$\mathcal{H}$			Inverse transform		
	Input index range	Output index range		Input index range	Output index range		Input index range	Output index range	
$y = S_{IO}^{-1} (R^T \mathcal{H}_{C1o}) S_{IO}\, x$				$0 \to N-1$	$1 \to N-1$	R^T			
$y = S_{IO}^{-1} \mathcal{H}_{S1o} R^T C_{IO} (A_2 x)$	$0 \to N-1$	$1 \to N-1$	R^T						
$y = -A_2^{-1} C_{IO}^{-1} R \mathcal{H}_{S1o} S_{IO}\, x$							$1 \to N-1$	$0 \to N-1$	R
$y = S_{IIO}^{-1} \mathcal{H}_{S1o} R^T C_{IIO} (A_4 x)$	$0 \to N-1$	$1 \to N-1$	R^T						
$y = S_{IIO}^{-1} (R^T \mathcal{H}_{C2o}) S_{IO}\, x$				$0 \to N-1$	$1 \to N-1$	R^T			
$y = S_{IIO}^{-1} (R^T \mathcal{H}_{C1o}) S_{IIo}\, x$				$0 \to N-1$	$1 \to N-1$	R^T			
$y = S_{IIO}^{-1} \mathcal{H}_{S2o} R^T C_{IO} (A_2 x)$	$0 \to N-1$	$1 \to N-1$	R^T						
$y = -A_4^{-1} C_{IIO}^{-1} R \mathcal{H}_{S1o} S_{IIO}\, x$							$1 \to N-1$	$0 \to N-1$	R
$y = -A_4^{-1} C_{IIO}^{-1} R \mathcal{H}_{S2o} S_{IO}\, x$							$1 \to N-1$	$0 \to N-1$	R

Table 2. *Cont.*

SCM (Symmetric-Conv. Mult.)	Forward transform			$\mathcal{H}$			Inverse transform		
	Input index range	Output index range		Input index range	Output index range		Input index range	Output index range	
$y = S_{IO}^{-1}\, \mathcal{H}_{S2o}\, R^T\, C_{IIO}\, (A_4\, x)$	$0 \to N-1$	$1 \to N-1$	R^T						
$y = S_{IO}^{-1}\, (R^T\, \mathcal{H}_{C2o})\, S_{IIO}\, x$				$0 \to N-1$	$1 \to N-1$	R^T			
$y = -\, A_2^{-1}\, C_{IO}^{-1}\, R\, \mathcal{H}_{S2o}\, S_{IIO}\, x$							$1 \to N-1$	$0 \to N-1$	R
$y = S_{IIIO}^{-1}\, \mathcal{H}_{S3o}\, Q^T\, C_{IIIO}\, (A_2\, x)$	$0 \to N-1$	$0 \to N-2$	Q^T						
$y = S_{IIIO}^{-1}\, Q^T\, \mathcal{H}_{C3o}\, Q\, S_{IIIO}\, x$	$0 \to N-2$	$0 \to N-1$	Q				$0 \to N-1$	$0 \to N-2$	Q^T
$y = -\, A_2^{-1}\, C_{IIIO}^{-1}\, Q\, \mathcal{H}_{S3o}\, S_{IIIO}\, x$							$0 \to N-2$	$0 \to N-1$	Q
$y = C_{IVO}^{-1}\, (Q^T\, \mathcal{H}_{C3o})\, C_{IVO}\, x$				$0 \to N-1$	$0 \to N-2$	Q^T			
$y = C_{IVO}^{-1} \mathcal{H}_{C4o}\, Q^T\, C_{IIIO}\, (A_2\, x)$	$0 \to N-1$	$0 \to N-2$	Q^T						
$y = A_4^{-1}\, S_{IVO}^{-1}\, Q\, \mathcal{H}_{S3o}\, C_{IVO}\, x$							$0 \to N-2$	$0 \to N-1$	Q
$y = A_4^{-1}\, S_{IVO}^{-1}\, Q\, \mathcal{H}_{C4o}\, S_{IIIO}\, x$							$0 \to N-2$	$0 \to N-1$	Q
$y = -C_{IVO}^{-1} \mathcal{H}_{S3o}\, Q^T\, S_{IVO}\, (A_4\, x)$	$0 \to N-1$	$0 \to N-2$	Q^T						
$y = -C_{IVO}^{-1}\, (Q^T\, \mathcal{H}_{S4o})\, S_{IIIO}\, x$				$0 \to N-1$	$0 \to N-2$	Q^T			
$y = A_2^{-1}\, C_{IIIO}^{-1}\, Q\, \mathcal{H}_{C4o}\, C_{IVO}\, x$							$0 \to N-2$	$0 \to N-1$	Q
$y = S_{IIIO}^{-1}\, (Q^T\, \mathcal{H}_{S4o})\, C_{IVO}\, x$				$0 \to N-1$	$0 \to N-2$	Q^T			
$y = S_{IIIO}^{-1} \mathcal{H}_{C4o}\, Q^T\, S_{IVO}\, (A_4\, x)$	$0 \to N-1$	$0 \to N-2$	Q^T						

Table 3. 20 of 40 types of SCM and MSC properties for the DTTs, $M = 2N$.

SCM (Symmetric-Conv. Mult.)	MSC (Mult. Symmetric-Conv.)	
$y = A_1^{-1} C_{IE}^{-1} \mathcal{H}_{C1e}\, C_{IE}\, (A_1\, x)$	$Y = A_1^{-1} C_{IE}\, h_{C1e}\, C_{IE}^{-1} (A_1\, X)$	(A1)
$y = S_{IE}^{-1}\, (R^T\, J^T\, \mathcal{H}_{C1e})\, S_{IE}\, x$	$Y = S_{IE} (R^T\, J^T\, \mathcal{H}_{C1e})\, S_{IE}^{-1}\, X$	(A2)
$y = S_{IE}^{-1}\, \mathcal{H}_{S1e}\, R^T\, J^T\, C_{IE}\, (A_1\, x)$	$Y = S_{IE} \mathcal{H}_{S1e}\, R^T\, J^T\, C_{IE}^{-1}\, (A_1\, X)$	
$y = -\, A_1^{-1} C_{IE}^{-1}\, J\, R\, \mathcal{H}_{S1e}\, S_{IE}\, x$	$Y = -\, A_1^{-1} C_{IE}\, J\, R\, h_{S1e}\, S_{IE}^{-1}\, X$	(A3)
$y = C_{IIE}^{-1} \mathcal{H}_{C1e}\, C_{IIE}\, x$	$Y = C_{IIIE}\, h_{C1e}\, C_{IIIE}^{-1}\, X$	(A4)
$y = C_{IIE}^{-1} \mathcal{H}_{C2e}\, J^T\, C_{IE} (A_1\, x)$	$Y = C_{IIIE}\, h_{C3e}\, J^T\, C_{IE}^{-1} (A_1\, X)$	
$y = S_{IIE}^{-1} \mathcal{H}_{S1e}\, C_{IIE}\, x$	$Y = S_{IIIE}\, h_{S1e}\, C_{IIIE}^{-1}\, X$	(A5)
$y = S_{IIE}^{-1} \mathcal{H}_{C2e}\, S_{IE}\, x$	$Y = S_{IIIE}\, h_{C3e}\, S_{IE}^{-1}\, X$	
$y = S_{IIE}^{-1} \mathcal{H}_{C1e}\, S_{IIE}\, x$	$Y = S_{IIIE}\, h_{C1e}\, S_{IIIE}^{-1}\, X$	(A6)
$y = S_{IIE}^{-1} \mathcal{H}_{S2e}\, K^T\, C_{IE}\, (A_1\, x)$	$Y = S_{IIE}\, h_{S3e}\, K^T\, C_{IE}^{-1} (A_1\, X)$	

Table 3. *Cont.*

SCM (Symmetric-Conv. Mult.)	MSC (Mult. Symmetric-Conv.)	
$y = -C_{\mathrm{II}E}^{-1} R \mathcal{H}_{\mathrm{S}1e} Q^T S_{\mathrm{II}E} x$ $y = -C_{\mathrm{II}E}^{-1} R (Q^T \mathcal{H}_{\mathrm{S}2e}) S_{\mathrm{I}E} x$	$Y = -C_{\mathrm{III}E} R h_{\mathrm{S}1e} Q^T S_{\mathrm{III}E}^{-1} X$ $Y = -C_{\mathrm{III}E} R (Q^T h_{\mathrm{S}3e}) S_{\mathrm{I}E}^{-1} X$	(A7)
$y = A_1^{-1} C_{\mathrm{I}E}^{-1} J \mathcal{H}_{\mathrm{C}2e} C_{\mathrm{II}E} x$	$Y = A_1^{-1} C_{\mathrm{I}E} J h_{\mathrm{C}3e} C_{\mathrm{III}E}^{-1} X$	(A8)
$y = S_{\mathrm{I}E}^{-1} (Q^T \mathcal{H}_{\mathrm{S}2e}) R^T C_{\mathrm{II}E} x$ $y = S_{\mathrm{I}E}^{-1} (R^T \mathcal{H}_{\mathrm{C}2e}) Q^T S_{\mathrm{II}E} x$	$Y = S_{\mathrm{I}E} (Q^T h_{\mathrm{S}3e}) R^T C_{\mathrm{III}E}^{-1} X$ $Y = S_{\mathrm{I}E} (R^T h_{\mathrm{C}3e}) Q^T S_{\mathrm{III}E}^{-1} X$	(A9)
$y = -A_1^{-1} C_{\mathrm{I}E}^{-1} K \mathcal{H}_{\mathrm{S}2e} S_{\mathrm{II}E} x$	$Y = -A_1^{-1} C_{\mathrm{I}E} K h_{\mathrm{S}3e} S_{\mathrm{III}E}^{-1} X$	(A10)
$y = A_2^{-1} C_{\mathrm{III}E}^{-1} \mathcal{H}_{\mathrm{C}3e} C_{\mathrm{III}E} (A_2 x)$	$Y = A_2^{-1} C_{\mathrm{II}E} h_{\mathrm{C}2e} C_{\mathrm{II}E}^{-1} (A_2 X)$	(A11)
$y = A_3^{-1} S_{\mathrm{III}E}^{-1} \mathcal{H}_{\mathrm{S}3e} C_{\mathrm{III}E} (A_2 x)$ $y = A_3^{-1} S_{\mathrm{III}E}^{-1} \mathcal{H}_{\mathrm{C}3e} S_{\mathrm{III}E} (A_3 x)$	$Y = A_3^{-1} S_{\mathrm{II}E} h_{\mathrm{S}2e} C_{\mathrm{II}E}^{-1} (A_2 X)$ $Y = A_3^{-1} S_{\mathrm{II}E} h_{\mathrm{S}2e} S_{\mathrm{II}E}^{-1} (A_3 X)$	(A12)
$y = A_2^{-1} C_{\mathrm{III}E}^{-1} \mathcal{H}_{\mathrm{S}3e} S_{\mathrm{III}E} (A_3 x)$	$Y = A_2^{-1} C_{\mathrm{II}E} h_{\mathrm{S}2e} S_{\mathrm{II}E}^{-1} (A_3 X)$	(A13)
$y = C_{\mathrm{IV}E}^{-1} \mathcal{H}_{\mathrm{C}3e} C_{\mathrm{IV}E} x$ $y = C_{\mathrm{IV}E}^{-1} \mathcal{H}_{\mathrm{C}4e} C_{\mathrm{III}E} (A_2 x)$	$Y = C_{\mathrm{IV}E} h_{\mathrm{C}2e} C_{\mathrm{IV}E}^{-1} X$ $Y = C_{\mathrm{IV}E} h_{\mathrm{C}4e} C_{\mathrm{II}E}^{-1} (A_2 X)$	(A14)
$y = S_{\mathrm{IV}E}^{-1} \mathcal{H}_{\mathrm{S}3e} C_{\mathrm{IV}E} x$ $y = S_{\mathrm{IV}E}^{-1} \mathcal{H}_{\mathrm{C}4e} S_{\mathrm{III}E} (A_3 x)$	$Y = S_{\mathrm{IV}E} h_{\mathrm{S}2e} C_{\mathrm{IV}E}^{-1} X$ $Y = S_{\mathrm{IV}E} h_{\mathrm{C}4e} S_{\mathrm{II}E}^{-1} (A_3 X)$	(A15)
$y = S_{\mathrm{IV}E}^{-1} \mathcal{H}_{\mathrm{C}3e} S_{\mathrm{IV}E} x$ $y = S_{\mathrm{IV}E}^{-1} \mathcal{H}_{\mathrm{S}4e} C_{\mathrm{III}E} (A_2 x)$	$Y = S_{\mathrm{IV}E} h_{\mathrm{C}2e} S_{\mathrm{IV}E}^{-1} X$ $Y = S_{\mathrm{IV}E} h_{\mathrm{S}4e} C_{\mathrm{II}E}^{-1} (A_2 X)$	(A16)
$y = -C_{\mathrm{IV}E}^{-1} \mathcal{H}_{\mathrm{S}3e} S_{\mathrm{IV}E} x$ $y = -C_{\mathrm{IV}E}^{-1} \mathcal{H}_{\mathrm{S}4e} S_{\mathrm{III}E} (A_3 x)$	$Y = -C_{\mathrm{IV}E} h_{\mathrm{S}2e} S_{\mathrm{IV}E}^{-1} X$ $Y = -C_{\mathrm{IV}E} h_{\mathrm{S}4e} S_{\mathrm{II}E}^{-1} (A_3 X)$	(A17)
$y = A_2^{-1} C_{\mathrm{III}E}^{-1} \mathcal{H}_{\mathrm{C}4e} C_{\mathrm{IV}E} x$	$Y = A_2^{-1} C_{\mathrm{II}E} h \mathcal{H}_{\mathrm{C}4e} C_{\mathrm{IV}E}^{-1} X$	(A18)
$y = A_3^{-1} S_{\mathrm{III}E}^{-1} \mathcal{H}_{\mathrm{S}4e} C_{\mathrm{IV}E} x$ $y = A_3^{-1} S_{\mathrm{III}E}^{-1} \mathcal{H}_{\mathrm{C}4e} S_{\mathrm{IV}E} x$	$Y = A_3^{-1} S_{\mathrm{II}E} h_{\mathrm{S}4e} C_{\mathrm{IV}E}^{-1} X$ $Y = A_3^{-1} S_{\mathrm{II}E} h_{\mathrm{C}4e} S_{\mathrm{IV}E}^{-1} X$	(A19)
$y = -A_2^{-1} C_{\mathrm{III}E}^{-1} \mathcal{H}_{\mathrm{S}4e} S_{\mathrm{IV}E} x$	$Y = -A_2^{-1} C_{\mathrm{II}E} h_{\mathrm{S}4e} S_{\mathrm{IV}E}^{-1} X$	(A20)

Table 4. 20 of 40 types of SCM and MSC properties for the DTTs, $M = 2N - 1$.

SCM (Symmetric-Conv. Mult.)	MSC (Mult. Symmetric-Conv.)	
$y = A_2^{-1} C_{\mathrm{I}O}^{-1} \mathcal{H}_{\mathrm{C}1o} C_{\mathrm{I}O} (A_2 x)$	$Y = A_2^{-1} C_{\mathrm{I}O} h_{\mathrm{C}1o} C_{\mathrm{I}O}^{-1} (A_2 X)$	(A21)
$y = S_{\mathrm{I}O}^{-1} (R^T \mathcal{H}_{\mathrm{C}1o}) S_{\mathrm{I}O} x$ $y = S_{\mathrm{I}O}^{-1} \mathcal{H}_{\mathrm{S}1o} R^T C_{\mathrm{I}O} (A_2 x)$	$Y = S_{\mathrm{I}O} (R^T h_{\mathrm{C}1o}) S_{\mathrm{I}O}^{-1} X$ $Y = S_{\mathrm{I}O} h_{\mathrm{S}1o} R^T C_{\mathrm{I}O}^{-1} (A_2 X)$	(A22)
$y = -A_2^{-1} C_{\mathrm{I}O}^{-1} R \mathcal{H}_{\mathrm{S}1o} S_{\mathrm{I}O} x$	$Y = -A_2^{-1} C_{\mathrm{I}O} R h_{\mathrm{S}1o} S_{\mathrm{I}O}^{-1} X$	(A23)
$y = A_4^{-1} C_{\mathrm{II}O}^{-1} \mathcal{H}_{\mathrm{C}1o} C_{\mathrm{II}O} (A_4 x)$ $y = A_4^{-1} C_{\mathrm{II}O}^{-1} \mathcal{H}_{\mathrm{C}2o} C_{\mathrm{I}O} (A_2 x)$	$Y = A_4^{-1} C_{\mathrm{III}O} h_{\mathrm{C}1o} C_{\mathrm{III}O}^{-1} (A_4 X)$ $Y = A_4^{-1} C_{\mathrm{III}O} h_{\mathrm{C}3o} C_{\mathrm{I}O}^{-1} (A_2 X)$	(A24)
$y = S_{\mathrm{II}O}^{-1} \mathcal{H}_{\mathrm{S}1o} R^T C_{\mathrm{II}O} (A_4 x)$ $y = S_{\mathrm{II}O}^{-1} (R^T \mathcal{H}_{\mathrm{C}2o}) S_{\mathrm{I}O} x$	$Y = S_{\mathrm{III}O} h_{\mathrm{S}1o} R^T C_{\mathrm{III}O}^{-1} (A_4 X)$ $Y = S_{\mathrm{III}O} (R^T h_{\mathrm{C}3o}) S_{\mathrm{I}O}^{-1} X$	(A25)

Table 4. *Cont.*

SCM (Symmetric-Conv. Mult.)	MSC (Mult. Symmetric-Conv.)	
$y = S_{IIO}^{-1}\,(R^T\,\mathcal{H}_{C1o})\,S_{IIo}\,x$ $y = S_{IIO}^{-1}\mathcal{H}_{S2o}\,R^T\,C_{IO}\,(A_2\,x)$	$Y = S_{IIIO}\,(R^T\,h_{C1o})\,S_{IIIO}^{-1}\,X$ $Y = S_{IIIO}\;\;h_{S3o}\,R^T\,C_{IO}^{-1}\,(A_2\,X)$	(A26)
$y = -\,A_4^{-1}\,C_{IIO}^{-1}\,R\,\mathcal{H}_{S1o}\,S_{IIO}\,x$ $y = -\,A_4^{-1}\,C_{IIO}^{-1}\,R\,\mathcal{H}_{S2o}\,S_{IO}\,x$	$Y = -\,A_4^{-1}\,C_{IIIO}\,R\,h_{S3o}\,S_{IO}^{-1}\,X$ $Y = -\,A_4^{-1}\,C_{IIIO}\,R\,h_{S1o}\,S_{IIIO}^{-1}\,X$	(A27)
$y = A_2^{-1}\,C_{IO}^{-1}\,\mathcal{H}_{C2o}\,C_{IIO}\,(A_4\,x)$	$Y = A_2^{-1}\,C_{IO}\,h_{C3o}\,C_{IIIO}^{-1}\,(A_4\,X)$	(A28)
$y = S_{IO}^{-1}\,\mathcal{H}_{S2o}\,R^T\,C_{IIO}\,(A_4\,x)$ $y = S_{IO}^{-1}\,(R^T\,\mathcal{H}_{C2o})\,S_{IIO}\,x$	$Y = S_{IO}\;\;h_{S3o}\,R^T\,C_{IIIO}^{-1}\,(A_4\,X)$ $Y = S_{IO}^{-1}\,(R^T\,h_{C3o})\,S_{IIIO}\,X$	(A29)
$y = -\,A_2^{-1}\,C_{IO}^{-1}\,R\,\mathcal{H}_{S2o}\,S_{IIO}\,x$	$Y = -\,A_2^{-1}\,C_{IO}\;\;R\;\;h_{S3o}\,S_{IIIO}^{-1}\,X$	(A30)
$y = A_2^{-1}\,C_{IIIO}^{-1}\mathcal{H}_{C3o}\,C_{IIIO}\,(A_2\,x)$	$Y = A_2^{-1}\,C_{IIO}\;\;h_{C2o}\,C_{IIO}^{-1}(A_2\,X)$	(A31)
$y = S_{IIIO}^{-1}\,\mathcal{H}_{S3o}\,Q^T\,C_{IIIO}\,(A_2\,x)$ $y = S_{IIIO}^{-1}\,Q^T\,\mathcal{H}_{C3o}\,Q\,S_{IIIO}\,x$	$Y = S_{IIO}\;\;h_{S2o}\,Q^T\,C_{IIO}^{-1}(A_2\,X)$ $Y = S_{IIO}\;\;Q^T\,h_{S2o}\,Q\,S_{IIO}^{-1}\,X$	(A32)
$y = -\,A_2^{-1}\,C_{IIIO}^{-1}\,Q\,\mathcal{H}_{S3o}\,S_{IIIO}\,x$	$Y = -\,A_2^{-1}\,C_{IIO}\;\;Q\,h_{S2o}\,S_{IIO}^{-1}\,X$	(A33)
$y = C_{IVO}^{-1}\,(Q^T\,\mathcal{H}_{C3o})\,C_{IVO}\,x$ $y = C_{IVO}^{-1}\mathcal{H}_{C4o}\,Q^T\,C_{IIIO}\,(A_2\,x)$	$Y = C_{IVO}\,(Q^T\,h_{C2o})\,C_{IVO}^{-1}\,X$ $Y = C_{IVO}\;\;h_{C4o}\,Q^T\,C_{IIO}^{-1}(A_2\,X)$	(A34)
$y = A_4^{-1}\,S_{IVO}^{-1}\,Q\,\mathcal{H}_{S3o}\,C_{IVO}\,x$ $y = A_4^{-1}\,S_{IVO}^{-1}\,Q\,\mathcal{H}_{C4o}\,S_{IIIO}\,x$	$Y = A_4^{-1}\,S_{IVO}\,Q\;\;h_{S2o}\,C_{IVO}^{-1}\,X$ $Y = A_4^{-1}\,S_{IVO}\,Q\;\;h_{C4o}\,S_{IIO}^{-1}\,X$	(A35)
$y = A_4^{-1}\,S_{IVO}^{-1}\mathcal{H}_{C3o}\,S_{IVO}\,(A_4\,x)$ $y = A_4^{-1}\,S_{IVOE}^{-1}\mathcal{H}_{S4o}\,C_{IIIO}\,(A_2\,x)$	$Y = A_4^{-1}\,S_{IVO}\;\;h_{C2o}\,S_{IVO}^{-1}\;\;(A_4\,X)$ $Y = A_4^{-1}\,S_{IVO}\,h_{S4o}\,C_{IIO}^{-1}\,(A_2\,X)$	(A36)
$y = -\,C_{IVO}^{-1}\mathcal{H}_{S3o}\,Q^T\,S_{IVO}\,(A_4\,x)$ $y = -\,C_{IVO}^{-1}\,(Q^T\,\mathcal{H}_{S4o})\,S_{IIIO}\,x$	$Y = -\,C_{IVO}\,h_{S2o}\,Q^T\,S_{IVO}^{-1}\,(A_4\,X)$ $Y = -\,C_{IVO}\,(Q^T\,h_{S4o})\,S_{IIIO}^{-1}\,X$	(A37)
$y = \;A_2^{-1}\,C_{IIIO}^{-1}\,Q\,\mathcal{H}_{C4o}\,C_{IVO}\,x$	$Y = A_2^{-1}\,C_{IIO}\,Q\;\;h_{C4o}\,C_{IVO}^{-1}\,X$	(A38)
$y = S_{IIIO}^{-1}\,(Q^T\,\mathcal{H}_{S4o})\,C_{IVO}\,x$ $y = S_{IIIO}^{-1}\mathcal{H}_{C4o}\,Q^T\,S_{IVO}\,(A_4\,x)$	$Y = S_{IIO}\,(Q^T\,h_{S4o})\,C_{IVO}^{-1}\,x$ $Y = S_{IIO}^{-1}\,h_{C4o}\,Q^T\,S_{IVO}^{-1}\,(A_4\,X)$	(A39)
$y = -\,A_2^{-1}\,C_{IIIO}^{-1}\mathcal{H}_{S4o}\,S_{IVO}\,(A_4\,x)$	$Y = -\,A_2^{-1}\,C_{IIO}\,h_{S4o}\,S_{IVO}^{-1}\,(A_4\,x)$	(A40)

References and Notes

1. Shen, Bo; Sethi, I.K.; Bhaskaran, V. DCT convolution and its application in compressed domain. *IEEE T. Circ. Syst. Vid.* **1998**, *8*, 947–952.
2. Martucci, S.A. Symmetric convolution and the discrete sine and cosine transforms. *IEEE Trans. SP* **1994**, *42*, pp. 1038–1051.
3. Zou, X.; Muramatsu, S.; Kiya, H. The generalized overlap-add and overlap-save methods using discrete sine and cosine transforms for FIR filtering. *Proceedings of the 3rd IEEE Int'l Conf. on Signal Processing, ICSP'96*, Beijing, China, October 1996; pp. 91–94.

4. Reju, V.G.; Koh, S.N.; Soon, I.Y. Convolution using discrete sine and cosine transforms. *IEEE Signal Proc. Let.* **2007**, *14*, 445–448.
5. Korohoda, P.; Dabrowski, A. Generalized convolution as a tool for the multi-dimensional filtering tasks. *Multidim. Syst. Sign. P.* **2008**, *19*, 361–377.
6. Foltz, T.M.; Welsh, B.M.; Holmberg, C.D. Symmetric convolution using unitary transform matrices. *IEEE T. Signal Proces.* **2000**, *48*, 2691–2692.
7. Park, Y.S.; Park H.W. Design and analysis of an image resizing filter in the block-DCT domain. *IEEE T. Circ. Syst. Vid.* **2004**, *14*, 274–279.
8. Sánchez, V.; García, P.; Peinado, A. M.; Segura, J.C. ; Rubio, A. J. Diagonalizing properties of the discrete cosine transforms. *IEEE T. Signal Proces.* **1995**, *43*, 2631–2641.
9. Sánchez, V.; García, P.; Peinado, A. M.; Segura, J.C. ; Rubio, A. J. Generating matrices for the discrete sine transforms. *IEEE T. Signal Proces.* **1996**, *44*, 2644–2646.
10. Martucci, S.A. Image resizing in the discrete sine and cosine domain. *Proceedings of the IEEE International Conference on Image Processing 1995*, Washington, DC, USA, October 1995; Volume 2, pp. 244–247.
11. Foltz, T.M.; Welsh, B.M. Symmetric convolution of asymmetric multidimensional sequences using discrete trigonometric transforms. *IEEE T. Image Proces.* **1999**, *8*, 640–651.
12. Rao, K.R.; Yip, P. *Discrete Cosine Transform: Algorithms, Advantages, And Applications*, Academic Press: London, England, 1990.
13. Britanak, V.; Yip, P.; Rao, K.R. *Discrete Cosine and Sine Transforms*, Academic Press (Elsevier): Orlando, FL, USA, 2007; pp. 32–38.