

Article

Linear Approximation of Deformation in Soft Robotics and Kinematic Links

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Abstract

Modeling large shape changes efficiently is an important challenge in deformable kinematics, soft robotics, compliant mechanisms, geometric modeling, and deformation-based path planning. This paper presents Projective Estimation with Per-Point Scales (PEPS), a family of linear estimation methodologies for obtaining a compact global approximation of the deformation between two configurations of a body from corresponding points. The deformation is represented by a single projective deformation matrix, 3 by 3 in two dimensions and 4 by 4 in three dimensions, which provides a unified representation of translation, rotation, scaling, shearing, and projective effects. Three related formulations are developed and compared. PEPS-1 explicitly introduces an independent homogeneous scale for each point correspondence. PEPS-2 eliminates these additional variables by enforcing projective collinearity and has an algebraic structure closely related to the classical Direct Linear Transform. PEPS-3 augments the collinearity formulation with equivalent constraints derived in a translated coordinate frame to investigate improvements in numerical conditioning. Isotropic coordinate normalization is applied to all three formulations to reduce sensitivity to coordinate magnitude, reference-frame placement, and point distribution. The resulting systems are estimated efficiently using singular value decomposition. The methodologies are evaluated on representative nonlinear shape transformations, including square-to-circle, cube-to-sphere, and cube-to-ellipsoid mappings. Reconstruction accuracy is assessed both at the correspondences used for estimation and at additional points, allowing fitting performance to be distinguished from generalization over the complete shape. The results show that a single projective deformation matrix can effectively capture the dominant global characteristics of nonlinear shape changes. PEPS-2 and PEPS-3 generally provide smaller linear systems, improved numerical conditioning, and lower computational cost than PEPS-1, while PEPS-1 can offer greater fitting flexibility for certain redundant or geometrically dependent correspondence sets. Comparisons with iterative nonlinear optimization demonstrate that the proposed linear formulations achieve comparable global approximations at substantially lower computational cost. These characteristics make the PEPS framework particularly suitable for fast deformation representation, deformation-aware kinematics, soft and continuum robotics, compliant mechanisms, and geometry-based path planning and control.



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conditions of the [Creative Commons](#)[Attribution \(CC BY\) license](#).**Keywords:** projective transformation; deformable kinematics; homogeneous coordinates; deformation matrix

1. Introduction

Recent advances in soft robotics, continuum robotics, and compliant mechanisms have increased the need for efficient representations of complex shape changes and deformation behaviors [1–3]. Data-driven and geometry-based approaches have been explored for shape reconstruction, kinematic modeling, and the control of highly deformable systems [2,4]. However, many existing approaches rely on dense sensing, high-dimensional deformation representations, or application-specific models. This motivates the investigation of compact geometric mappings capable of approximating global shape deformations from sparse correspondence information.

The estimation of geometric transformations from corresponding points is a fundamental problem in kinematics, robotics, computer vision, and geometric modeling. Such transformations arise in a wide range of applications, including camera calibration [5], panoramic image stitching [6], visual odometry and simultaneous localization and mapping (SLAM) [7], and three-dimensional reconstruction from multiple views [8]. In robotics and kinematic modeling, transformation estimation also plays an important role in pose determination, coordinate registration, motion analysis, and geometric mapping between different configurations of a system. Homogeneous coordinates provide a convenient mathematical framework for representing these relationships, enabling translation, rotation, scaling, shear, and projective effects to be expressed within a unified matrix formulation.

Beyond rigid-body motion, many engineering systems involve substantial geometric deformation between configurations. Examples include soft-link and continuum robots, compliant mechanisms, deformable structures, biological systems, shape morphing, and geometric reconstruction problems. In such applications, the objective is often to characterize the overall deformation of a body rather than merely determine its rigid-body displacement. Efficient mathematical representations of deformation are essential for analysis, simulation, path planning, control, and design optimization.

A common requirement across these domains is the recovery of a transformation directly from sparse point correspondences. While highly accurate solutions can often be obtained through nonlinear optimization procedures, iterative refinement techniques, or physics-based models, such approaches are typically computationally demanding and may require careful initialization, dense measurements, or repeated numerical optimization. In contrast, linear formulations remain attractive because they provide deterministic solutions, low computational cost, and straightforward implementation. However, the numerical behavior of linear methods can be strongly influenced by point distribution, coordinate magnitude, and reference-frame placement. Correspondences clustered within a small region of the workspace or located far from the coordinate origin often produce poorly conditioned systems, leading to reduced numerical accuracy and increased sensitivity to noise [9,10].

Projective geometry offers an attractive framework for addressing these challenges. Although traditionally associated with computer vision and image geometry [8], projective transformations possess a rich representational capability that extends beyond rigid motions and affine mappings. When expressed in homogeneous coordinates, a single projective transformation can capture translation, rotation, scaling, shear, and perspective effects simultaneously. Consequently, projective mappings provide a compact means of approximating complex geometric deformations and shape changes using a unified matrix representation.

Despite the extensive literature on projective transformation estimation, existing linear formulations have been developed primarily for camera geometry, image registration, and related computer vision problems. Their potential for approximating global shape deformations and supporting deformation-aware kinematic analysis has received compar-

atively little attention. Moreover, the influence of alternative projective formulations on numerical conditioning, computational efficiency, and reconstruction accuracy has not been systematically investigated.

To address these gaps, this paper presents the Projective Estimation with Per-Point Scales (PEPS) framework, a family of linear methodologies for estimating projective deformation mappings directly from corresponding points. The central idea is to determine a homogeneous deformation matrix that provides a compact global approximation of the deformation between two configurations. Such representations are particularly attractive for deformation-aware kinematics, soft robotics, compliant mechanisms, geometric modeling, shape reconstruction, and deformation-based path planning.

Elements of the PEPS framework were developed in the authors' earlier work [11,12]. In particular, the normalization strategy adopted throughout this paper was selected based on a previous investigation of its influence on numerical conditioning and estimation accuracy and is applied consistently to all formulations considered here. Building on these foundations, the present paper provides a unified treatment of the PEPS family and a comprehensive comparative analysis of its principal formulations. Three related methodologies are considered. PEPS-1 employs an independent scale parameter for each correspondence and recovers the deformation through a homogeneous SVD-based formulation. PEPS-2 eliminates the correspondence-dependent scale variables through homogeneous collinearity constraints, reducing the number of unknowns while preserving projective consistency. PEPS-3 extends this formulation through coordinate-shift symmetrization to investigate the influence of coordinate translation on the numerical behavior of the estimation process.

Although a projective deformation matrix is defined up to an arbitrary scale factor, the proposed PEPS formulations retain the scale factor as an unknown. This leads to a homogeneous linear system, which is estimated using SVD. In practice, additional point correspondences are used to obtain an overdetermined system, improving numerical stability.

To further reduce sensitivity to coordinate-frame placement and point distribution, isotropic coordinate normalization based on Hartley's method [9] is incorporated into the framework. The proposed methodologies are evaluated using planar and spatial point correspondences generated from synthetic nonlinear shape deformations, including square-to-circle, cube-to-sphere, and cube-to-ellipsoid mappings. For each case, affine and projective transformation models are computed as the closest linear approximations to the underlying nonlinear deformation. The results demonstrate that complex nonlinear shape changes can be effectively approximated using projective transformation models while maintaining high computational efficiency and numerical robustness.

2. Materials and Methods

2.1. PEPS-1: Direct Correspondence Formulation

This section introduces the Projective Estimation with Per-Point Scales (PEPS), a linear framework for estimating projective mappings in homogeneous coordinates. The method assigns an independent scale parameter to each correspondence, allowing perspective effects to be modeled while preserving linearity.

To model general deformations, including non-rigid and perspective effects, we introduce a deformation matrix $\Delta \in \mathbb{R}^{4 \times 4}$ acting on homogeneous coordinates:

$$\mathbf{P}' = \Delta \mathbf{P} = \begin{bmatrix} u_j \\ v_j \\ q_j \\ \omega_j \end{bmatrix}, \quad (1)$$

where $\mathbf{P}$ denotes the original point in homogeneous coordinates and $\mathbf{P}'$ denotes the intermediate homogeneous point obtained by applying the deformation matrix Δ :

$$\Delta = \begin{bmatrix} \mathbf{M} & \mathbf{t} \\ \mathbf{v}^T & \sigma \end{bmatrix}, \quad (2)$$

where $\mathbf{M} \in \mathbb{R}^{3 \times 3}$ represents linear deformation effects (rotation, scaling, and shear), $\mathbf{t} \in \mathbb{R}^3$ represents translation, and $\mathbf{v}$ together with σ capture projective effects. After transformation, Cartesian coordinates are recovered through dehomogenization:

$$[x' \ y' \ z']^T = \frac{\mathbf{M}[x \ y \ z]^T + \mathbf{t}}{\mathbf{v}^T[x \ y \ z]^T + \sigma} \quad (3)$$

The dehomogenization step assumes that the homogeneous denominator remains nonzero. If the denominator approaches zero, the corresponding point is mapped close to the projective plane at infinity, which can result in unstable Cartesian coordinates. Therefore, practical implementations should monitor the dehomogenization denominator and handle near-singular cases appropriately.

Consider two sets of corresponding points,

$$\mathbf{P} = \{\mathbf{P}_1, \mathbf{P}_2, \dots, \mathbf{P}_n\}, \quad \mathbf{P}'' = \{\mathbf{P}_1'', \mathbf{P}_2'', \dots, \mathbf{P}_n''\}, \quad (4)$$

where $\mathbf{P}$ denotes the original point configuration and $\mathbf{P}''$ denotes the corresponding target point configuration. The intermediate homogeneous points are given by $\mathbf{P}' = \Delta\mathbf{P}$, while the target points $\mathbf{P}''$ are obtained by dehomogenizing $\mathbf{P}'$. Each pair $(\mathbf{P}_j, \mathbf{P}_j'')$ represents a known correspondence. Points are represented in homogeneous coordinates as

$$\mathbf{P}_j = \begin{bmatrix} x_j \\ y_j \\ z_j \\ 1 \end{bmatrix}, \quad \mathbf{P}_j'' = \begin{bmatrix} x_j'' \\ y_j'' \\ z_j'' \\ 1 \end{bmatrix}, \quad j = 1, \dots, n \quad (5)$$

and the deformation between configurations is modeled by the deformation matrix $\Delta \in \mathbb{R}^{4 \times 4}$.

The PEPS formulation is written as

$$\Delta\mathbf{P}_j = s(j)\mathbf{P}_j'', \quad j = 1, \dots, n, \quad (6)$$

where $s(j) = \omega_j \in \mathbb{R}$ is an explicit scale parameter associated with the j -th correspondence. Expanding all correspondence equations yields the homogeneous linear system

$$\mathbf{A}_{PEPS1}\mathbf{x} = 0, \quad (7)$$

where $\mathbf{A}_{PEPS1} \in \mathbb{R}^{4n \times (16+n)}$ is the coefficient matrix and

$$\mathbf{x} = \begin{bmatrix} \text{vec}(\Delta) \\ s(1) \\ \vdots \\ s(n) \end{bmatrix} \in \mathbb{R}^{16+n} \quad (8)$$

contains the unknown deformation parameters and scale factors.

It should be emphasized that the linearity of the proposed PEPS methodologies refers to the parameter estimation procedure and the homogeneous matrix representation. Once

the deformation matrix Δ has been estimated, the corresponding Cartesian coordinates are obtained through dehomogenization, resulting in the standard fractional-linear mapping associated with projective transformations.

For a non-degenerate 3D configuration, each point correspondence contributes four linearly independent homogeneous constraints. Therefore, the number of independent equations is $4n$, while the PEPS-1 formulation contains $16 + n$ unknowns, including the 16 deformation parameters and the n correspondence-dependent scale factors. The minimum number of correspondences is obtained from

$$4n \geq 16 + n \quad (9)$$

which gives $n \geq 5.33$. Since the number of correspondences must be an integer, at least six correspondences are required. This condition assumes a non-degenerate point configuration; if the selected points contain collinear or coplanar arrangements, some constraints become linearly dependent, causing the rank of the coefficient matrix $\mathbf{A}_{PEPS1}$ to decrease and preventing reliable estimation of the deformation parameters [8]. Therefore, such configurations should be avoided when selecting correspondence subsets.

The solution is obtained using the singular value decomposition SVD) [13]

$$\mathbf{A}_{PEPS1} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T, \quad (10)$$

where the desired solution corresponds to the right singular vector associated with the smallest singular value. The resulting deformation matrix Δ defines the projective mapping that approximates the underlying deformation within the PEPS framework.

The planar homography case follows as a special case by restricting points to $z = 0$, reducing the formulation to a 3×3 homogeneous transformation in 2D.

PEPS-1 represents a full projective transformation by introducing an independent scale factor for each point correspondence. While this formulation preserves linearity and directly enforces point-to-point consistency in homogeneous coordinates, it requires n additional unknowns beyond the 16 elements of the deformation matrix. As a result, the estimation problem contains $16 + n$ unknown parameters for n correspondences, leading to larger linear systems and increased computational cost. This observation motivates the development of an alternative formulation that eliminates the explicit scale variables while retaining the ability to estimate the same class of projective transformations.

2.2. PEPS-2: Collinearity-Based Geometric Formulation

Rather than introducing an explicit scale parameter for each correspondence, PEPS-2 exploits a fundamental property of homogeneous coordinates: two vectors represent the same geometric point if they differ only by a nonzero scale factor. Consequently, the projective consistency condition can be enforced implicitly through a collinearity constraint, eliminating the need for per-point scale variables and yielding a more compact estimation problem.

Given a correspondence $\mathbf{P}_j \leftrightarrow \mathbf{P}_j''$ in homogeneous coordinates (both with homogeneous component equal to 1), the goal is to find a 4×4 matrix Δ (defined up to scale) such that the transformed point is

$$\mathbf{P}_j' = \Delta \mathbf{P}_j = \begin{bmatrix} u_j \\ v_j \\ q_j \\ \omega_j \end{bmatrix} \quad (11)$$

and is collinear with the target point $\mathbf{P}_j'' = [x_j'', y_j'', z_j'', 1]^T$ in projective space. Geometrically, this means that the vectors $\overrightarrow{OP_j''}$ and $\overrightarrow{OP_j'}$ lie on the same line through the origin, i.e., there exists a nonzero scalar ω_j such that

$$\mathbf{P}_j' = \omega_j \mathbf{P}_j''. \quad (12)$$

After dehomogenization, this collinearity condition ensures

$$\frac{u_j}{\omega_j} = x_j'', \quad \frac{v_j}{\omega_j} = y_j'', \quad \frac{q_j}{\omega_j} = z_j'', \quad (13)$$

which is precisely the desired mapping $\mathbf{P}_j'' \simeq \Delta \mathbf{P}_j$.

Unlike PEPS-1, which introduces an explicit scale factor for each point correspondence, PEPS-2 eliminates the explicit scale variables by enforcing projective collinearity through the cross-product formulation. This formulation avoids explicit division by the homogeneous coordinate ω_j during the computation phase.

In 2D, the cross product in $\mathbb{R}^3$ (including the homogeneous coordinate) gives three equations:

$$\mathbf{P}_j'' \times (\Delta \mathbf{P}_j) = \begin{bmatrix} y_j'' \omega_j - v_j \\ u_j - x_j'' \omega_j \\ x_j'' v_j - y_j'' u_j \end{bmatrix} = \mathbf{0}. \quad (14)$$

The homogeneous cross product produces three scalar equations; however, only two are linearly independent because the collinearity constraint in the projective plane has two independent degrees of freedom. Consequently, it is sufficient to retain only the first two equations:

$$\begin{cases} y_j'' \omega_j - v_j = 0 \\ u_j - x_j'' \omega_j = 0 \end{cases} \quad (15)$$

Each correspondence therefore contributes exactly two linear constraints on the nine elements of Δ .

Let $\Delta = [d_{kl}]_{k,l=1}^3$ and let $\mathbf{d} = \text{vec}(\Delta^\top) = [d_{11}, d_{12}, d_{13}, d_{21}, \dots, d_{33}]^\top \in \mathbb{R}^9$ be the column-major vectorization.

Substituting $\Delta \mathbf{P}_j = [u_j, v_j, \omega_j]^\top$ into (15) yields the following row vectors:

$$\text{Row 1: } [0, 0, 0, -x_j, -y_j, -1, y_j'' x_j, y_j'' y_j, y_j''] \mathbf{d} = 0, \quad (16)$$

$$\text{Row 2: } [x_j, y_j, 1, 0, 0, 0, -x_j'' x_j, -x_j'' y_j, -x_j''] \mathbf{d} = 0. \quad (17)$$

Stacking these rows for all N correspondences produces the homogeneous system

$$\mathbf{A}_{PEPS2} \mathbf{d} = \mathbf{0}, \quad \mathbf{A}_{PEPS2} \in \mathbb{R}^{2N \times 9}. \quad (18)$$

The unknown vector contains nine elements of the homogeneous transformation matrix. Under non-degenerate configurations, each correspondence contributes two linearly independent equations, yielding $2N$ equations in 9 unknowns. Consequently, the minimum configuration consists of five non-collinear point correspondences (with no three points collinear), for which the homogeneous system contains 10 equations and 9 unknowns.

In 3D, points are represented in homogeneous coordinates in $\mathbb{P}^3$, and the transformation matrix $\Delta \in \mathbb{R}^{4 \times 4}$. Unlike the 2D case, the cross product is not defined in $\mathbb{R}^4$, so collinearity between $\mathbf{P}_j'$ and $\mathbf{P}_j''$ is enforced through scalar proportionality.

Let

$$\mathbf{P}'_j = \Delta \mathbf{P}_j = \begin{bmatrix} u_j \\ v_j \\ q_j \\ \omega_j \end{bmatrix} \sim \mathbf{P}''_j = \begin{bmatrix} x''_j \\ y''_j \\ z''_j \\ 1 \end{bmatrix}. \quad (19)$$

Collinearity implies the existence of a nonzero scalar ω_j such that

$$\mathbf{P}'_j = \omega_j \mathbf{P}''_j. \quad (20)$$

This yields the three independent linear constraints:

$$\begin{cases} \omega_j x''_j - u_j = 0 \\ \omega_j y''_j - v_j = 0 \\ \omega_j z''_j - q_j = 0 \end{cases} \quad (21)$$

Each correspondence therefore contributes exactly three linear constraints on the sixteen elements of Δ . Let $\Delta = [d_{kl}]_{k,l=1}^4$ and let $\mathbf{d} = \text{vec}(\Delta^\top) \in \mathbb{R}^{16}$ be the column-major vectorization.

Substituting $\Delta \mathbf{P}_j = [u_j, v_j, q_j, \omega_j]^\top$ into (21) yields the row vectors:

$$\text{Row 1: } [-x_j, -y_j, -z_j, -1, 0, 0, 0, 0, 0, 0, 0, 0, x''_j x_j, x''_j y_j, x''_j z_j, x''_j] \mathbf{d} = 0, \quad (22)$$

$$\text{Row 2: } [0, 0, 0, 0, -x_j, -y_j, -z_j, -1, 0, 0, 0, 0, y''_j x_j, y''_j y_j, y''_j z_j, y''_j] \mathbf{d} = 0, \quad (23)$$

$$\text{Row 3: } [0, 0, 0, 0, 0, 0, 0, 0, -x_j, -y_j, -z_j, -1, z''_j x_j, z''_j y_j, z''_j z_j, z''_j] \mathbf{d} = 0. \quad (24)$$

Stacking these rows for all N correspondences produces the homogeneous system

$$\mathbf{A}_{PEPS2} \mathbf{d} = \mathbf{0}, \quad \mathbf{A}_{PEPS2} \in \mathbb{R}^{3N \times 16}. \quad (25)$$

Under non-degenerate configurations (with no three points collinear and no four points coplanar), each correspondence contributes three linearly independent equations, yielding $3N$ equations in 16 unknowns. Consequently, the minimum configuration consists of six non-collinear and non-coplanar point correspondences, for which the homogeneous system contains 18 equations and 16 unknowns. The corresponding coefficient matrix $\mathbf{A}_{PEPS2} \in \mathbb{R}^{18 \times 16}$ has full column rank in generic configurations.

The matrix $\mathbf{A}_{PEPS2}$ is sparse and block-structured, with each group of rows corresponding to a single point correspondence (two rows per point in 2D and three rows per point in 3D). In the general position, the coefficient matrix has full column rank (9 in 2D and 16 in 3D). The solution vector is obtained as the right singular vector corresponding to the smallest singular value of $\mathbf{A}_{PEPS2}$, thereby minimizing the residual $\|\mathbf{A}_{PEPS2} \mathbf{d}\|$ subject to $\|\mathbf{d}\| = 1$:

$$\mathbf{A}_{PEPS2} = \mathbf{U} \Sigma \mathbf{V}^\top \Rightarrow \mathbf{d} = \mathbf{V}(:, \text{end}). \quad (26)$$

When the point configuration is degenerate (e.g., collinear points in 2D and collinear or coplanar configurations in 3D), the rank of $\mathbf{A}_{PEPS2}$ decreases, resulting in an ill-conditioned estimation problem. Such degeneracies are known to produce ambiguous or unstable projective estimates because the correspondence constraints no longer provide sufficient independent information [8]. In such cases, the SVD-based solution becomes sensitive to numerical perturbations due to the poor conditioning of the coefficient matrix, which may lead to unreliable deformation estimates [10,13]. Therefore, the experiments in this work

focus on non-degenerate correspondence configurations to evaluate the performance of the proposed formulations.

PEPS-2 offers several advantages over the PEPS-1 formulation. By eliminating the correspondence-dependent scale variables through homogeneous collinearity constraints, it provides a more compact projective estimation problem while retaining the ability to model perspective effects. The resulting algebraic residual is directly related to the collinearity condition between homogeneous vectors, providing a scale-invariant formulation. A remaining limitation of this formulation is that the numerical conditioning of the associated linear system may depend on the coordinate distribution of the input points. This motivates the development of PEPS-3, which introduces additional constraints from an equivalent translated coordinate frame to investigate the influence of coordinate-shift symmetrization on the numerical behavior of the estimation while preserving the projective consistency of PEPS-2.

2.3. PEPS-3: Coordinate-Shift Symmetrization for Numerical Stability

While PEPS-2 improves upon PEPS-1 by eliminating the correspondence-dependent scale variables through homogeneous collinearity constraints, the numerical conditioning of the associated linear system may still depend on the distribution of the point coordinates. Although the underlying projective formulation is invariant under coordinate transformations, different coordinate representations may lead to coefficient matrices with different numerical properties. To address this, PEPS-3 introduces an additional set of constraints derived from an equivalent translated coordinate frame and combines them with the original constraints into a single estimation problem. The estimated deformation is subsequently mapped back to the original coordinate frame, preserving the underlying projective mapping while aiming to improve the numerical properties of the estimation.

To construct the additional constraint system, PEPS-3 translates the entire point set along the x -axis so that all points lie in the negative x -half-space (the x -axis is chosen arbitrarily for simplicity). The translated coordinates provide an equivalent representation of the same projective mapping, from which additional collinearity constraints are constructed. The translation is defined by

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 & -T_x \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (3D \text{ case}), \quad (27)$$

with the obvious 3×3 counterpart in 2D. The parameter $T_x > 0$ is chosen such that the maximum x -coordinate of both source and target points becomes negative after translation:

$$T_x = \max_j (\max(\mathbf{P}_j(1), \mathbf{P}_j''(1))) + \Delta_t, \quad (28)$$

where $\Delta_t > 0$ is a safety margin (e.g., $\Delta_t \approx 5$ – 10 units, depending on the coordinate scale).

The transformed points are then given by

$$\tilde{\mathbf{P}}_j = \mathbf{T}\mathbf{P}_j, \quad \tilde{\mathbf{P}}_j'' = \mathbf{T}\mathbf{P}_j''. \quad (29)$$

The translated coordinate frame provides an alternative representation of the same projective mapping while preserving projective equivalence. By incorporating constraints derived from both the original and translated coordinate frames, PEPS-3 constructs a more symmetric linear system, which can improve the numerical conditioning of the estimation

problem. This approach is conceptually related to normalization strategies commonly employed in projective estimation to improve numerical stability (see, e.g., [8,10]).

In the shifted coordinate frame, we apply the same collinearity constraint as in PEPS-2, but now it is applied to the transformed points:

$$\tilde{\mathbf{P}}_j'' \times (\tilde{\mathbf{A}}\tilde{\mathbf{P}}_j) = \mathbf{0}, \quad (30)$$

where $\tilde{\mathbf{A}}$ is the homography expressed in the shifted frame. This yields a second constraint matrix $\tilde{\mathbf{A}}$, constructed exactly as in (18) but using $\tilde{\mathbf{P}}_j$ and $\tilde{\mathbf{P}}_j''$ instead of the original points.

The relationship between the original and shifted homographies is given by the conjugation

$$\mathbf{\Delta} = \mathbf{T}^{-1}\tilde{\mathbf{A}}\mathbf{T}, \quad (31)$$

Because both $\mathbf{\Delta}$ and $\tilde{\mathbf{A}}$ are defined only up to scale, this conjugation preserves the projective equivalence [14].

To solve for a single consistent $\mathbf{d} = \text{vec}(\mathbf{\Delta}^\top)$ in the original frame, we transform the shifted constraints back using the Kronecker product identity [15]. Let $\tilde{\mathbf{d}} = \text{vec}(\tilde{\mathbf{A}}^\top)$. The conjugation implies

$$\tilde{\mathbf{d}} = (\mathbf{T} \otimes \mathbf{T}^{-\top})\mathbf{d}, \quad (32)$$

where $\otimes$ denotes the Kronecker product and $\mathbf{T}^{-\top}$ is the transpose of the inverse. After accounting for the row-major vectorization convention via the commutation matrix $\mathbf{C}$ (a permutation matrix), the transformed constraint matrix for the shifted system can be written as [15]

$$\mathbf{A}_{\text{shifted}} = \tilde{\mathbf{A}}\mathbf{K}, \quad \mathbf{K} = \mathbf{C}(\mathbf{T} \otimes \mathbf{T}^{-\top})\mathbf{C}^\top. \quad (33)$$

The full PEPS-3 system is then obtained by stacking the original and transformed constraints

$$\mathbf{A}_{\text{PEPS3}} = \begin{bmatrix} \mathbf{A} \\ \mathbf{A}_{\text{shifted}} \end{bmatrix}, \quad (34)$$

and the solution is given by the right singular vector corresponding to the smallest singular value of $\mathbf{A}_{\text{PEPS3}}$.

By incorporating constraints derived from both the original and translated coordinate frames, PEPS-3 combines both sets of constraints into a single linear estimation problem while preserving the underlying projective mapping through conjugation. This results in a more symmetric constraint system that can improve numerical conditioning for certain point distributions.

2.4. Relationship to Existing Projective Estimation Methods

The PEPS-2 formulation was derived from the homogeneous collinearity condition between transformed and target points. Subsequent comparison with the literature revealed a close relationship to the classical Direct Linear Transform (DLT) formulation commonly used in projective geometry and computer vision [8]. Both approaches enforce point correspondence consistency through homogeneous constraints and recover the transformation from the null space of a linear system using singular value decomposition.

Despite these similarities, the present formulation was developed from a projective deformation perspective and is expressed using a general homogeneous transformation matrix. Rather than treating the projective matrix primarily as a camera or image-mapping operator, the PEPS framework interprets it as a compact global approximation of the deformation between two configurations of a deformable body. In the three-dimensional case, all sixteen elements of the 4×4 deformation matrix are intentionally retained as unknowns, with the solution determined only up to an arbitrary scale factor. This differs from con-

ventional DLT formulations, where the projective scale ambiguity is typically removed by fixing one parameter or by otherwise normalizing the transformation, resulting in fifteen independent unknowns. Furthermore, the PEPS framework comprises three complementary formulations: PEPS-1 explicitly retains one scale parameter for each correspondence, PEPS-2 eliminates these variables through homogeneous collinearity constraints, and PEPS-3 augments the estimation using an equivalent translated coordinate frame to improve numerical behavior. Together, these formulations provide distinct trade-offs in system size, numerical conditioning, computational cost, and reconstruction accuracy while forming a unified framework for approximating translation, rotation, scaling, shear, and projective effects in applications such as soft robotics, compliant mechanisms, deformable kinematics, and deformation-based path planning.

2.5. Source of Instability

While the proposed formulations are computationally efficient and easily interpretable, numerical experiments revealed sensitivity to the conditioning of the linear system. Since the entries of $\mathbf{A}$ depend directly on point coordinates, translating the same geometric configuration can alter matrix scales and lead to poor numerical conditioning.

To analyze this effect, consider translating both point sets by the same translation vector $\mathbf{t} \in \mathbb{R}^3$. The corresponding homogeneous translation vector is defined as

$$\bar{\mathbf{t}} = \begin{bmatrix} \mathbf{t} \\ 0 \end{bmatrix} = \begin{bmatrix} t_x \\ t_y \\ t_z \\ 0 \end{bmatrix}. \quad (35)$$

The translated source and target points are then given by

$$\mathbf{P}_j^{(\mathbf{t})} = \mathbf{P}_j + \bar{\mathbf{t}}, \quad \mathbf{P}_j''^{(\mathbf{t})} = \mathbf{P}_j'' + \bar{\mathbf{t}}. \quad (36)$$

Using the same deformation matrix Δ , the residual for the j -th correspondence becomes

$$\mathbf{r}_j^{(\mathbf{t})} = \Delta(\mathbf{P}_j^{(\mathbf{t})}) - \mathbf{P}_j''^{(\mathbf{t})}. \quad (37)$$

Expanding the expression gives

$$\begin{aligned} \mathbf{r}_j^{(\mathbf{t})} &= \Delta(\mathbf{P}_j) + \Delta(\bar{\mathbf{t}}) - \mathbf{P}_j'' - \bar{\mathbf{t}} \\ &= \underbrace{\Delta(\mathbf{P}_j) - \mathbf{P}_j''}_{\mathbf{r}_j} + \underbrace{\Delta(\bar{\mathbf{t}}) - \bar{\mathbf{t}}}_{\Delta\bar{\mathbf{t}}}. \end{aligned} \quad (38)$$

Thus,

$$\mathbf{r}_j^{(\mathbf{t})} = \mathbf{r}_j + \Delta\bar{\mathbf{t}}. \quad (39)$$

In general, $\Delta(\mathbf{t}) \neq \bar{\mathbf{t}}$ because the transformation may contain scaling, shearing, or projective components. Consequently, the residual norm and total error are not translation invariant:

$$E^{(\mathbf{t})} = \sum_{j=1}^n \|\mathbf{r}_j^{(\mathbf{t})}\|_2 \neq \sum_{j=1}^n \|\mathbf{r}_j\|_2 = E. \quad (40)$$

This indicates that the numerical behavior of PEPS methodologies may depend on the absolute placement of the object within the coordinate system rather than solely on the underlying geometry. Similar conditioning issues are well known in classical projective estimation methods such as the Direct Linear Transform (DLT) [8,9].

These observations motivate introducing a normalization procedure prior to constructing the PEPS formulations. By centering point sets at the origin and scaling them to a consistent range, translation invariance is preserved while improving numerical conditioning. The adopted normalization strategy is presented in the next subsection.

2.6. Isotropic Normalization of Point Correspondences

To improve conditioning, a normalization procedure is applied prior to constructing the linear systems associated with the PEPS formulations. The goal is to remove the influence of absolute position and scale while preserving the geometric relationships between corresponding points. Let $\mathbf{P}_j$ and $\mathbf{P}_j''$ denote corresponding points in the original and target configurations, expressed in homogeneous coordinates as shown in (5). Following classical normalization approaches [8,16], each point set is independently translated and uniformly scaled.

The centroid of the original point set is computed as

$$\mathbf{c}_o = \frac{1}{n} \sum_{j=1}^n \begin{bmatrix} x_j \\ y_j \\ z_j \end{bmatrix}, \quad (41)$$

with an analogous centroid $\mathbf{c}_t$ computed for the target points. Each point set is then translated so its centroid coincides with the origin.

After centering, an uniform scaling factor is introduced to normalize the spatial extent of the coordinates. Let $\mathbf{P}_j^*$ denote centered points. Their average Euclidean distance from the origin is

$$r_o = \frac{1}{n} \sum_{j=1}^n \|\mathbf{P}_j^*\|_2, \quad (42)$$

and the scaling factor is chosen so the average distance becomes $\sqrt{3}$:

$$\alpha_o = \frac{\sqrt{3}}{r_o}. \quad (43)$$

A similar factor α_t is computed for the target point set.

The combined translation and scaling operations can be expressed through homogeneous similarity transformations

$$\mathbf{T}_s = \begin{bmatrix} \alpha_o \mathbf{I}_3 & -\alpha_o \mathbf{c}_o \\ \mathbf{0}^T & 1 \end{bmatrix}, \quad \mathbf{T}_t = \begin{bmatrix} \alpha_t \mathbf{I}_3 & -\alpha_t \mathbf{c}_t \\ \mathbf{0}^T & 1 \end{bmatrix}, \quad (44)$$

which are applied to the original and transformed points, respectively. The normalized coordinates are therefore

$$\tilde{\mathbf{P}}_j = \mathbf{T}_s \mathbf{P}_j, \quad \tilde{\mathbf{P}}_j'' = \mathbf{T}_t \mathbf{P}_j''. \quad (45)$$

The family of PEPS estimation procedure is then applied to the normalized point sets, producing a deformation matrix $\tilde{\Delta}$ that relates the normalized coordinates:

$$\tilde{\mathbf{P}}_j'' \approx \tilde{\Delta} \tilde{\mathbf{P}}_j. \quad (46)$$

Finally, the deformation matrix corresponding to the original coordinate frame is recovered by reversing the normalization transformations:

$$\Delta = \mathbf{T}_t^{-1} \tilde{\Delta} \mathbf{T}_s. \quad (47)$$

By centering and scaling the coordinates, the entries of the system matrix of each PEPS formulation become more balanced, improving least-squares conditioning and reducing sensitivity to global translations and coordinate scale.

2.7. General PEPS Estimation Procedure

Algorithm 1 summarizes the implementation procedure of the proposed PEPS framework. The three formulations differ only in the construction of the coefficient matrix, while the remaining estimation steps are identical.

Algorithm 1 General PEPS Estimation Framework

Require: Source point set $\mathbf{P}$, target point set $\mathbf{P}''$, selected PEPS formulation

Ensure: Deformation matrix Δ and transformed point set

- 1: Normalize the source and target point sets using isotropic normalization.
 - 2: Construct the coefficient matrix corresponding to the selected PEPS formulation.
 - 3: Compute the economy-size singular value decomposition of the coefficient matrix.
 - 4: Extract the right singular vector associated with the smallest singular value.
 - 5: Recover the normalized deformation matrix $\tilde{\Delta}$.
 - 6: Denormalize the deformation matrix using the inverse normalization transformations.
 - 7: Transform the source points using Δ .
 - 8: Dehomogenize the transformed homogeneous points to obtain Cartesian coordinates.
 - 9: Compute the reconstruction error if required.
- return** Δ and the transformed point set.
-

The coefficient matrix in Step 2 depends on the selected formulation. PEPS-1 constructs the system by introducing one explicit scale variable for each correspondence. PEPS-2 eliminates the explicit scale variables by enforcing projective collinearity through the cross-product formulation. PEPS-3 extends the PEPS-2 formulation by incorporating constraints from both the original and translated coordinate frames before recovering the deformation matrix through conjugation.

3. Results

A comparative analysis of the three PEPS formulations (PEPS-1, PEPS-2, and PEPS-3) is conducted through two- and three-dimensional experiments assessing numerical stability and geometric accuracy. Isotropic coordinate normalization is applied as a preprocessing step in all experiments prior to constructing the PEPS systems. The goal is to examine the quality and consistency of the estimated deformations across different geometric configurations and point selections.

The experiments use synthetic point correspondences generated from nonlinear transformations, including square-to-circle mappings in 2D and cube-to-sphere and cube-to-ellipsoid mappings in 3D. In each case, the deformation matrix is estimated from a subset of correspondences and evaluated both on the selected points and on additional points not used during estimation, allowing separate evaluation of fitting accuracy and generalization performance.

Two error measures are used throughout the experiments. The partial error is computed over the estimation subset and measures how accurately the constraints are satisfied, while the total error is computed over all correspondences and evaluates generalization. This distinction is important because improved fitting accuracy does not necessarily imply improved generalization.

Although this work focuses on the linear PEPS formulations, nonlinear optimization can further reduce geometric reprojection error [8,17,18]. To provide a benchmark, a Levenberg–Marquardt (LM) refinement is implemented following standard computer vision practice. To evaluate the nonlinear optimizer as a standalone estimation method,

rather than as a refinement initialized from a projective estimate, the optimization is initialized with the identity deformation matrix. Step and function tolerances of 10^{-12} are used, with a maximum of 500 iterations.

The experiments compare the performance of PEPS-1, PEPS-2, and PEPS-3 and include a nonlinear optimization benchmark for reference.

3.1. Two-Dimensional Example: Square-to-Circle Transformation

The performance of the three PEPS formulations is evaluated through a comparative analysis of numerical stability and geometric reconstruction accuracy. In all experiments, isotropic coordinate normalization is applied as a preprocessing step prior to constructing the corresponding PEPS systems. The objective is to compare PEPS-1, PEPS-2, and PEPS-3 with respect to reconstruction accuracy, numerical conditioning, and robustness to point selection.

Figure 1 presents a representative square-to-circle deformation obtained using the three PEPS formulations and a nonlinear optimization benchmark. Figure 1a–c correspond to PEPS-1, PEPS-2, and PEPS-3, respectively, while Figure 1d shows the result obtained using Levenberg–Marquardt optimization. All three PEPS formulations produce visually similar reconstructions and closely approximate the target geometry.

The transformed points $\hat{\mathbf{P}}'_j$ obtained from the estimated deformation matrix are compared with the corresponding target points $\mathbf{P}''_j$ in Cartesian coordinates. The reconstruction accuracy is evaluated using the root mean square error (RMSE), defined as

$$E_{\text{RMSE}} = \sqrt{\frac{1}{n} \sum_{j=1}^n \|\mathbf{P}''_j - \hat{\mathbf{P}}'_j\|_2^2}. \quad (48)$$

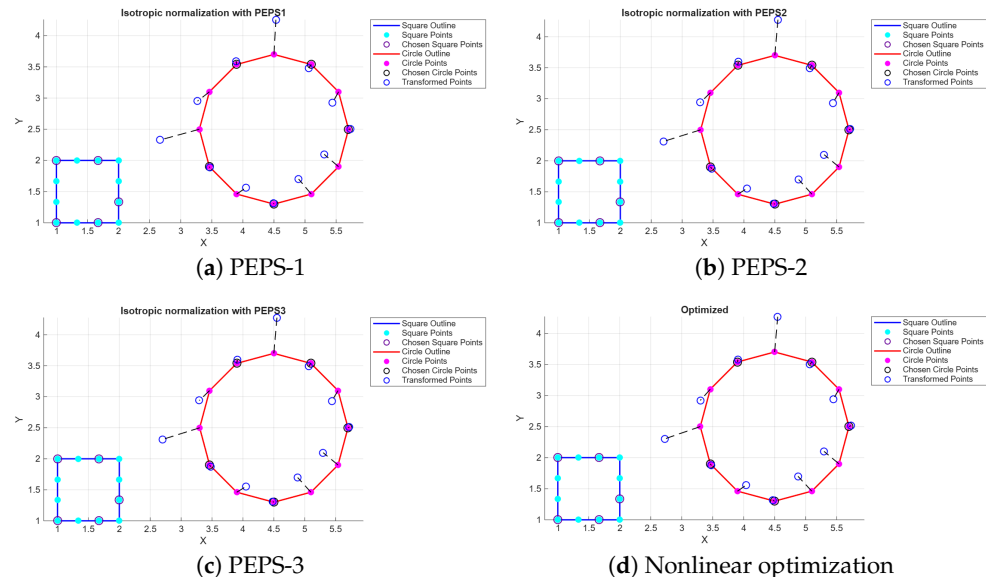


Figure 1. Comparison of the estimated square-to-circle deformation.

In each experiment, the deformation matrix is estimated using five correspondences selected from a set of twelve points located along the boundary of the square. Ten correspondence subsets were generated through random sampling without replacement, while rejecting subsets containing three collinear points. These subsets were then fixed and reused throughout the reported trials to ensure a consistent comparison between the different PEPS formulations. The remaining correspondences are reserved for evaluation. Two error measures are considered throughout the analysis. The partial error is computed over the five estimation points and measures how accurately the available constraints are satisfied,

whereas the total error is computed over all twelve correspondences and evaluates the ability of the estimated deformation to generalize beyond the estimation subset.

For the representative example shown in Figure 1, PEPS-1 yields a total RMSE of 0.2988, while PEPS-2 and PEPS-3 both achieve a slightly lower total RMSE of 0.2966. The nonlinear optimization benchmark produces the lowest total RMSE of 0.2927.

The same ordering is observed for the partial error computed over the five estimation correspondences. PEPS-1 produces a partial RMSE of 0.0428, whereas PEPS-2 and PEPS-3 reduce the error to 0.0401. The nonlinear optimization benchmark again achieves the lowest error, with a partial RMSE of 0.0383.

The coefficient matrices associated with the three PEPS formulations exhibit a similar pattern. PEPS-1 produces a condition number of 32.699, whereas PEPS-2 and PEPS-3 yield condition numbers of 13.997. Thus, while all three formulations provide comparable reconstruction accuracy, PEPS-2 and PEPS-3 result in more favorably conditioned linear systems. For reference, the nonlinear optimization benchmark converged in 13 iterations.

To assess robustness with respect to point selection, ten independent trials are performed using different randomly selected subsets of five correspondences. An additional reference case (Trial 11) uses all twelve correspondences for estimation. Since all points participate in the estimation process for Trial 11, only the total error is reported for this case.

Figure 2 summarizes the reconstruction errors obtained across all trials. The behavior of the three PEPS formulations differs when comparing partial and total errors. For the partial error, computed only over the five correspondences used during estimation, a highly consistent trend is observed across all trials. PEPS-1 consistently produces the largest error, while PEPS-2 and PEPS-3 yield identical and lower errors. The nonlinear optimization benchmark achieves the lowest partial error in every case. This behavior indicates that the cross-product-based formulations satisfy the available constraints more accurately than PEPS-1.

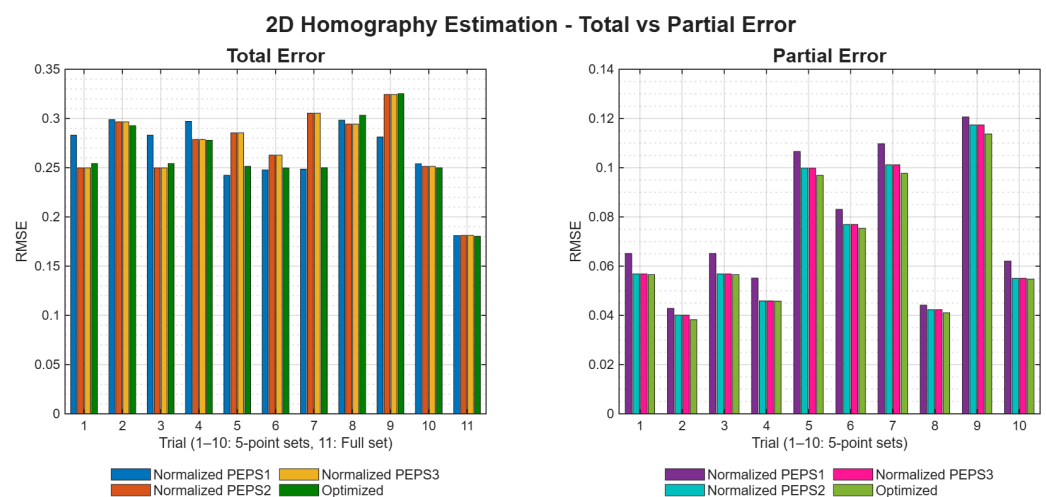


Figure 2. RMSE comparison across 11 experimental cases for square-to-circle deformation.

The total error exhibits a less uniform pattern. Although PEPS-2 and PEPS-3 generally produce lower total errors than PEPS-1, several trials show the opposite behavior, with PEPS-1 achieving a lower total error than the other formulations. This apparent inconsistency arises because the total error evaluates the quality of the deformation over all correspondences, including points that were not used during estimation.

The distribution of the selected correspondences plays an important role in this behavior. When the five estimation points are well distributed along the boundary of the square and their corresponding locations on the circle, both partial and total errors tend to be low. However, when the selected points are concentrated within a limited region, the estimated

deformation may fit those points particularly well, resulting in a lower partial error, while providing a less accurate mapping of the remaining correspondences. Consequently, a reduction in partial error does not necessarily imply a reduction in total error.

An interesting observation can be made from Trial 11, where all twelve correspondences are used during estimation. As expected, the reconstruction error is lower than in the preceding trials, since the deformation matrix is estimated from the complete set of available information rather than from only five correspondences. However, unlike the trend observed in most of the preceding trials, PEPS-2 and PEPS-3 do not produce the lower reconstruction error in this case. A possible explanation is related to the geometric structure of the dataset. Since all points lie on the boundary of the square, many correspondences are located on common edges and are therefore collinear. In PEPS-1, the homogeneous scale associated with each correspondence is treated as an independent unknown and estimated together with the deformation matrix. In contrast, PEPS-2 and PEPS-3 eliminate these scale factors through cross-product constraints, resulting in fewer unknowns and a more compact formulation. Consequently, the additional degrees of freedom available in PEPS-1 may provide greater flexibility when fitting a large set of geometrically dependent correspondences. Although PEPS-2 and PEPS-3 yield more favorably conditioned coefficient matrices, improved numerical conditioning does not necessarily imply lower reconstruction error, particularly when all available correspondences are used during estimation.

Overall, all three PEPS formulations provide accurate deformation estimates and produce comparable geometric reconstructions. PEPS-2 and PEPS-3 consistently achieve lower partial errors and improved numerical conditioning relative to PEPS-1, indicating more accurate satisfaction of the estimation constraints. However, the total error is influenced not only by the formulation itself but also by the geometric distribution and dependency of the selected correspondences. As demonstrated by Trial 11, improved numerical conditioning does not necessarily translate into lower reconstruction error. In particular, when all boundary correspondences are used during estimation, the additional scale parameters introduced in PEPS-1 may provide greater flexibility in accommodating geometrically dependent observations, resulting in slightly lower reconstruction error despite a less favorably conditioned coefficient matrix.

3.2. Three-Dimensional Example: Cube-to-Sphere Transformation

To further assess the performance of the three PEPS formulations, the experiments are extended to a three-dimensional cube-to-sphere deformation. Compared with the two-dimensional square-to-circle example, this problem involves a larger number of deformation parameters and a more restrictive geometric configuration, providing a more demanding test of reconstruction accuracy and numerical stability.

As in the previous example, isotropic coordinate normalization is applied as a preprocessing step prior to constructing the PEPS systems. The deformation matrix is estimated from a subset of point correspondences and evaluated using both partial and total RMSE.

In the three-dimensional case, at least six point correspondences are required to estimate the deformation matrix. To generate estimation subsets, points are randomly selected while enforcing non-degeneracy constraints, namely that no three points are collinear and no four points are coplanar. Ten correspondence configurations are generated and stored prior to the experiments, and the same subsets are reused for all evaluated methodologies to ensure a consistent comparison. These conditions guarantee that the selected subsets provide sufficient geometric information to estimate the deformation while avoiding degenerate configurations.

Figure 3 presents a representative cube-to-sphere deformation obtained using PEPS-1, PEPS-2, PEPS-3, and the nonlinear optimization benchmark. As in the two-dimensional

example, all three formulations successfully reconstruct the target geometry and produce visually similar results. For the representative cube-to-sphere deformation shown in Figure 3, PEPS-1 yields a total RMSE of 2.6300, whereas PEPS-2 and PEPS-3 reduce the total error to 2.1960. The nonlinear optimization benchmark achieves the lowest total RMSE of 2.0527.

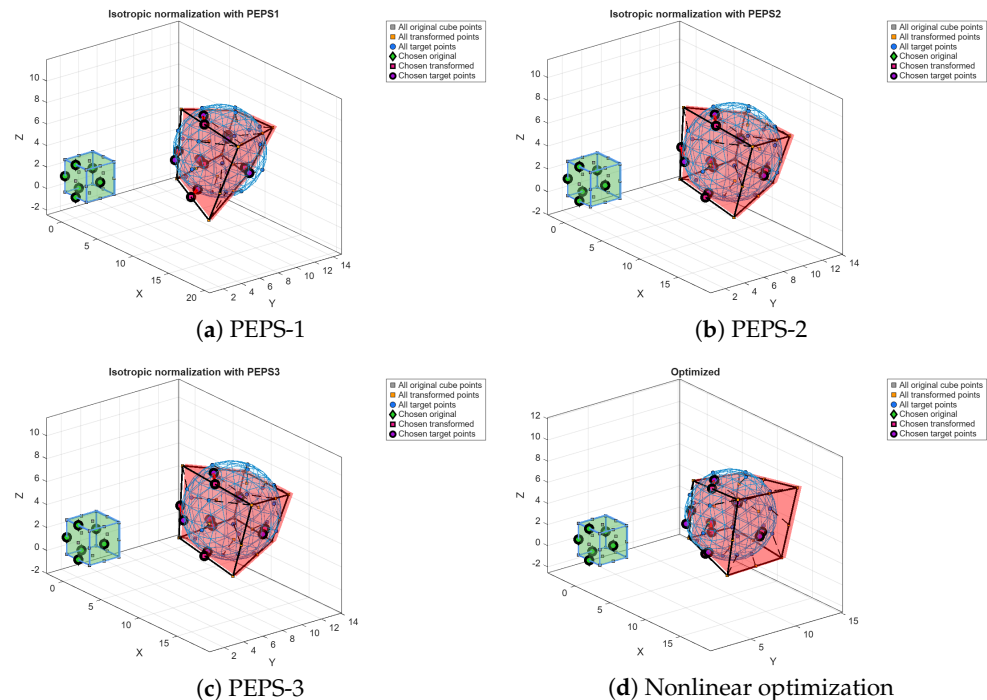


Figure 3. Comparison of the estimated cube-to-sphere deformation.

A similar trend is observed for the partial error computed over the estimation correspondences. PEPS-1 produces a partial RMSE of 1.1504, while PEPS-2 and PEPS-3 reduce the error to 1.0647. The nonlinear optimization benchmark again yields the lowest error, with a partial RMSE of 0.9114.

The condition numbers of the corresponding coefficient matrices are 12.4292 for PEPS-1 and 11.9804 for both PEPS-2 and PEPS-3. Unlike the two-dimensional example, the differences in numerical conditioning are relatively small. Nevertheless, the cross-product-based formulations provide a noticeable reduction in both total and partial reconstruction errors. The nonlinear optimization benchmark converged in 27 iterations.

To evaluate robustness with respect to point selection, an additional set of experiments is performed using different subsets of six correspondences satisfying the non-collinearity and non-coplanarity requirements. As in the two-dimensional example, ten independent trials are conducted using randomly selected estimation subsets, while an additional reference case uses all available correspondences during estimation.

Figure 4 summarizes the reconstruction errors obtained across all trials. The overall behavior is consistent with the representative example. PEPS-2 and PEPS-3 produce identical results and generally achieve lower total and partial errors than PEPS-1. The nonlinear optimization benchmark yields the lowest errors in most cases, although the differences are less pronounced for certain point selections.

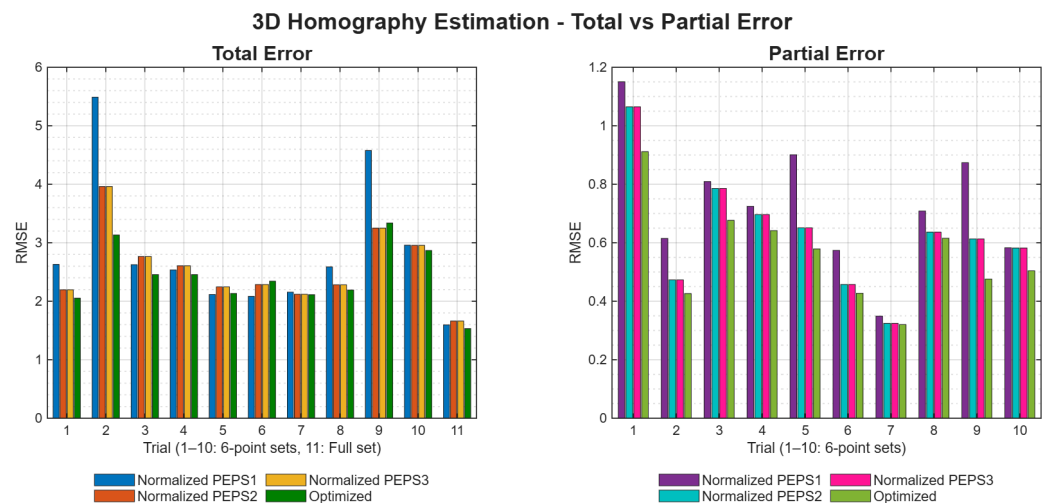


Figure 4. RMSE comparison across 11 experimental cases for cube-to-sphere deformation.

As in the two-dimensional experiments, the partial error exhibits a more consistent ordering among the methods than the total error. The total error remains strongly influenced by the spatial distribution of the selected correspondences and therefore provides a measure of generalization beyond the estimation subset. Nevertheless, the results indicate that PEPS-2 and PEPS-3 maintain their advantage over PEPS-1 across a variety of admissible point configurations.

Overall, the three-dimensional experiments exhibit behavior similar to that observed in the two-dimensional case. PEPS-2 and PEPS-3 consistently outperform PEPS-1 in terms of reconstruction accuracy while producing identical results to one another. Although the improvement in numerical conditioning is modest, the reduction in both total and partial RMSE demonstrates that the alternative constraint formulations are better suited to the cube-to-sphere deformation considered here. As expected, the nonlinear optimization benchmark provides the most accurate reconstruction at the expense of iterative computation.

The trial using all 26 correspondences exhibits behavior similar to that observed in the two-dimensional example. Although PEPS-2 and PEPS-3 provide better numerical conditioning, PEPS-1 achieves a slightly lower reconstruction error, suggesting that the additional scale parameters may offer greater flexibility when a large number of geometrically dependent correspondences are included in the estimation process.

3.3. Three-Dimensional Example: Cube-to-Ellipsoid Transformation

As a final validation, the experiments are performed on a cube-to-ellipsoid deformation. Compared with the cube-to-sphere example, the ellipsoidal target introduces anisotropic geometric distortion, providing an additional test of the robustness of the three PEPS formulations.

Unlike the previous examples, the representative case shown in Figure 5 uses all 26 available correspondences during estimation. This allows the behavior of the three formulations to be examined when the complete set of geometric information is available.

Figure 5 presents the estimated cube-to-ellipsoid deformation obtained using PEPS-1, PEPS-2, PEPS-3, and the nonlinear optimization benchmark. All methods successfully reconstruct the target geometry and produce visually similar deformations.

For this example, PEPS-1 yields a total RMSE of 4.6738, whereas PEPS-2 and PEPS-3 both produce total RMSEs of 5.0536. The nonlinear optimization benchmark achieves the lowest total RMSE of 4.5251.

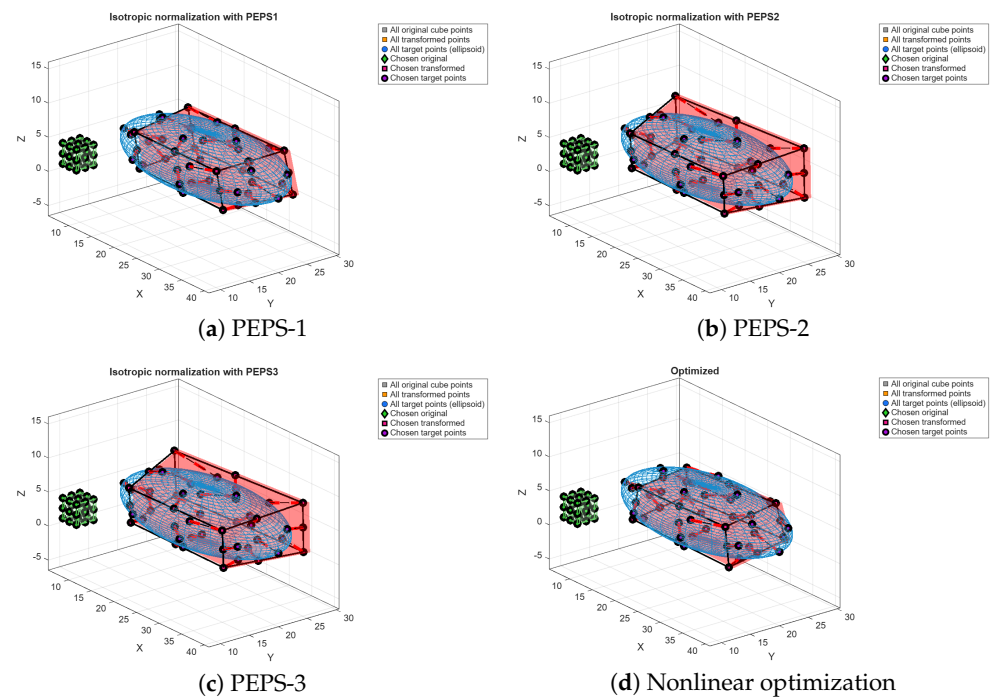


Figure 5. Comparison of the estimated cube-to-ellipsoid deformation.

The condition numbers of the corresponding coefficient matrices are 10.9951 for PEPS-1 and 5.6843 for both PEPS-2 and PEPS-3, indicating improved numerical conditioning for the cross-product-based formulations. Nevertheless, despite their lower condition numbers, PEPS-2 and PEPS-3 produce slightly larger reconstruction errors than PEPS-1. Similar behavior was observed in Trial 11 of the square-to-circle and cube-to-sphere examples and may be attributed to the different treatment of homogeneous scale factors in the three formulations. The nonlinear optimization benchmark converged in 20 iterations.

To evaluate robustness with respect to point selection, ten independent trials are performed using different subsets of six correspondences satisfying the non-collinearity and non-coplanarity requirements. Figure 6 summarizes the reconstruction errors obtained across all trials.

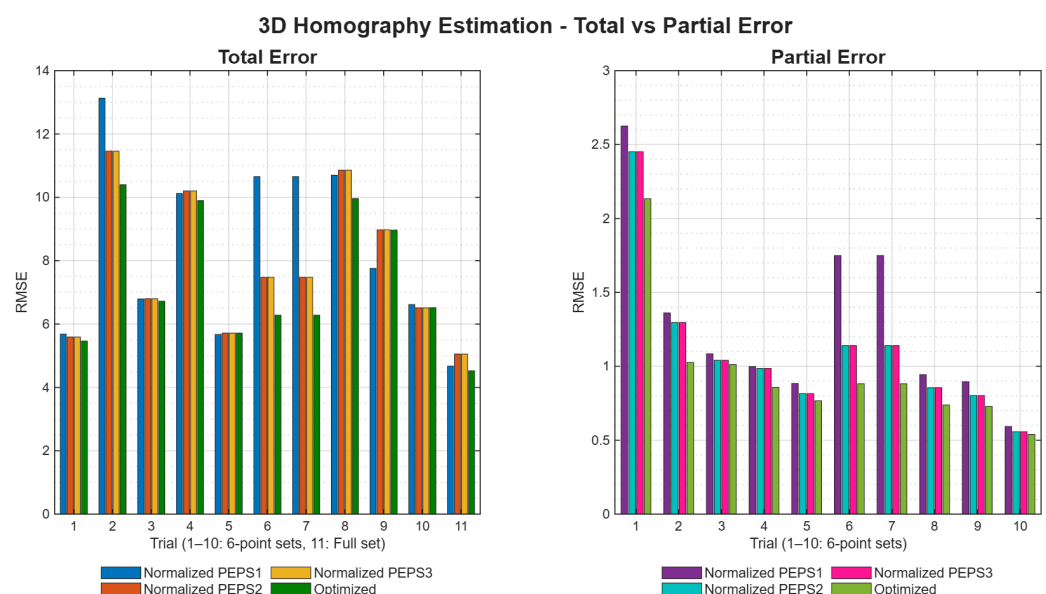


Figure 6. RMSE comparison across 11 experimental cases for cube-to-ellipsoid deformation.

The overall behavior is consistent with the previous experiments. PEPS-2 and PEPS-3 produce identical results and generally achieve lower reconstruction errors than PEPS-1, while the nonlinear optimization benchmark yields the lowest errors in most cases. As before, the partial error exhibits a more consistent ordering among the methods than the total error, indicating that generalization remains strongly influenced by the spatial distribution of the selected correspondences.

Overall, the cube-to-ellipsoid experiments confirm the trends observed throughout the study. PEPS-2 and PEPS-3 consistently produce identical solutions and generally outperform PEPS-1 while yielding more favorably conditioned coefficient matrices. At the same time, the results further demonstrate that numerical conditioning and reconstruction accuracy should be regarded as complementary rather than equivalent performance measures.

3.4. Computational Efficiency

The computational efficiency of the proposed PEPS formulations was evaluated through both theoretical complexity analysis and experimental runtime measurements. The derived computational complexity was verified by measuring execution times for increasing numbers of point correspondences, while representative runtimes for the experimental cases are reported later in this section. All methods construct a homogeneous linear system and solve it via SVD. In all cases, isotropic coordinate normalization is applied as a preprocessing step. Since normalization consists only of centering and scaling operations, its computational cost grows linearly with the number of correspondences and is negligible compared with the subsequent singular value decomposition (SVD).

The computational efficiency analysis assumes the use of an economy-size dense SVD, as implemented in MATLAB, to compute the solution. Under this formulation, the computational cost is dominated by the dimensions of the coefficient matrix, and the scaling behavior of each methodology follows from the number of rows and unknown parameters. For PEPS-1, the homogeneous scale associated with each correspondence is treated as an independent unknown. Consequently, the coefficient matrix contains $(16+N)$ unknowns and has dimensions approximately $\mathbf{A}_{\text{PEPS1}} \in \mathbb{R}^{4N \times (16+N)}$, which yields the following computational complexity:

$$\mathcal{O}_{\text{PEPS1}} = \mathcal{O}\left(4N(16+N)^2\right) \approx \mathcal{O}(N^3). \quad (49)$$

In contrast, PEPS-2 and PEPS-3 eliminate the scale factors through cross-product constraints, leaving only the 16 deformation parameters as unknowns. The corresponding coefficient matrices therefore have fixed width of $\mathbf{A}_{\text{PEPS2}} \in \mathbb{R}^{3N \times 16}$ and $\mathbf{A}_{\text{PEPS3}} \in \mathbb{R}^{6N \times 16}$, yielding

$$\mathcal{O}_{\text{PEPS2}} = \mathcal{O}(3N \cdot 16^2) \approx \mathcal{O}(N) \quad (50)$$

$$\mathcal{O}_{\text{PEPS3}} = \mathcal{O}(6N \cdot 16^2) \approx \mathcal{O}(N), \quad (51)$$

where the number of rows grows linearly with the number of correspondences while the number of columns remains constant.

Thus, PEPS-2 and PEPS-3 exhibit linear computational complexity with respect to the number of correspondences, whereas PEPS-1 exhibits cubic complexity due to the introduction of correspondence-dependent scale variables. Although the normalization preprocessing improves numerical stability, it does not alter the asymptotic computational complexity of any of the formulations. The same complexity behavior is observed in the two-dimensional case, where PEPS-1 similarly introduces one scale variable per correspondence while PEPS-2 and PEPS-3 retain a fixed number of unknown deformation parameters.

For comparison, the nonlinear Levenberg-Marquardt benchmark is iterative. Each iteration requires evaluating the residual vector and Jacobian over all correspondences,

resulting in a per-iteration complexity of approximately $\mathcal{O}(N)$. The overall complexity is therefore

$$\mathcal{O}_{\text{LM}} = \mathcal{O}(IN), \quad (52)$$

where (I) denotes the number of iterations required for convergence. Consequently, the computational efficiency of the nonlinear method depends not only on the number of correspondences but also on the convergence behavior and initialization of the optimization process.

To further assess computational efficiency, 100 trials were conducted for each experimental case. All runtime experiments were executed on a Dell Latitude 7450 laptop equipped with an Intel® Core™ Ultra 7 165U processor (12 cores, 14 logical processors) and 32 GB RAM. The algorithms were implemented in MATLAB R2026a and executed under Microsoft Windows 11 Enterprise. No specialized hardware or GPU acceleration was used. The reported runtimes include matrix construction, coordinate normalization, and singular value decomposition (SVD) computations. Mean, minimum, and maximum runtimes are reported in Table 1.

Table 1. Computational efficiency results over 100 trials for the 2D and 3D experimental cases.

Case	Methodology	Computational Time [ms]		
		Mean	Min	Max
2D Example	PEPS-1	0.109	0.098	0.170
	PEPS-2	0.053	0.047	0.083
	PEPS-3	0.061	0.054	0.106
	Nonlinear Optimization (LM)	2.387	1.719	5.313
3D Example I	PEPS-1	0.248	0.186	0.325
	PEPS-2	0.129	0.100	0.144
	PEPS-3	0.177	0.131	0.424
	Nonlinear Optimization (LM)	2.938	2.535	6.019
3D Example II	PEPS-1	0.248	0.216	0.297
	PEPS-2	0.133	0.124	0.181
	PEPS-3	0.180	0.168	0.211
	Nonlinear Optimization (LM)	3.967	3.455	5.596

Across all experiments, the three linear PEPS formulations require only a fraction of a millisecond to estimate the deformation matrix, substantially outperforming the nonlinear optimization benchmark. PEPS-2 consistently exhibits the lowest computational cost, while PEPS-3 produces comparable runtimes. Both cross-product-based formulations are approximately two times faster than PEPS-1 due to the elimination of correspondence-dependent scale variables and the resulting reduction in the number of unknowns. Despite this difference, all three linear formulations remain significantly more efficient than nonlinear optimization. Depending on the experimental case, PEPS-1 is approximately 15–25 times faster than the Levenberg–Marquardt benchmark, while PEPS-2 and PEPS-3 provide an additional twofold reduction in computational time.

In terms of numerical conditioning, consistent trends are observed across all experimental cases. PEPS-2 and PEPS-3 generally produce more favorably conditioned coefficient matrices than PEPS-1, reflecting the benefits of the cross-product-based formulations. The improvement is particularly noticeable in the two-dimensional experiments, whereas the

differences become less pronounced in the three-dimensional examples. Nevertheless, PEPS-2 and PEPS-3 exhibit identical conditioning behavior throughout all experiments.

The nonlinear optimization benchmark requires between 12 and 24 iterations for the 2D example, between 14 and 68 iterations for the cube-to-sphere deformation, and between 17 and 148 iterations for the cube-to-ellipsoid deformation. These results demonstrate that the proposed PEPS formulations provide a substantial computational advantage while maintaining reconstruction accuracy comparable to that of iterative nonlinear optimization.

4. Conclusions

This paper presented the Projective Estimation with Per-Point Scales (PEPS) framework, a family of linear methodologies for estimating projective deformation mappings from sparse point correspondences. The framework provides a unified approach for approximating planar and spatial shape transformations using a single homogeneous deformation matrix estimated from the prescribed correspondences.

Evaluation on a range of representative deformation scenarios, including square-to-circle, cube-to-sphere, and cube-to-ellipsoid mappings, demonstrated that a single projective deformation matrix can provide accurate global approximations of nonlinear shape transformations while maintaining very low computational cost. The results further showed that the reduced formulations, PEPS-2 and PEPS-3, consistently improve numerical conditioning and computational efficiency relative to PEPS-1 while yielding comparable reconstruction accuracy. Overall, the three methodologies offer complementary trade-offs: PEPS-1 may provide lower reconstruction errors for certain deformation scenarios, whereas PEPS-2 and PEPS-3 offer improved numerical properties and reduced computational complexity through their scale-free formulations. In practice, PEPS-2 and PEPS-3 are recommended when the minimum number of non-degenerate point correspondences is used for estimation. When additional correspondences are available, PEPS-1 may be advantageous if the correspondence set contains collinear (2D) or coplanar (3D) points, as its scale-based formulation was observed to be less sensitive to such configurations. For reliable performance, the selected correspondences should form a non-degenerate configuration and be distributed as uniformly as possible over the shape boundary.

More broadly, the significance of the PEPS framework extends beyond projective transformation estimation. By combining the simplicity of linear algebraic solutions with the expressive power of projective geometry, PEPS provides a new tool for deformation-aware kinematics and shape mapping. Potential applications include soft-link and continuum robotics, compliant mechanisms, geometric modeling, shape reconstruction, deformation-based path planning, and the analysis of deformable mechanical systems.

The results suggest that compact projective mappings can provide effective global approximations of the nonlinear shape transformations considered in this study. This capability enables efficient deformation representation, with potential applications in fast modeling, real-time implementation, and geometry-based control and design methodologies for robotic and mechanical systems. While the proposed framework provides an efficient representation of correspondence-based deformation mappings, additional validation involving physical constraints, orientation preservation, and continuum deformation behavior is required for application-specific kinematic modeling. For the deformation scenarios considered in this study, PEPS-2 and PEPS-3 produced identical numerical performance. While no additional benefit of the coordinate-shift formulation was observed in these experiments, it remains of theoretical interest, and its potential advantages for more challenging deformation scenarios will be investigated in future work. Furthermore, since PEPS estimates a single global projective transformation, it provides an approximation of the deformation behavior and cannot represent arbitrary local nonlinear deformation

fields. Accordingly, comparisons with local deformation and non-rigid registration methods, such as spline-, mesh-, or physics-based approaches, remain an important direction for future research. The accuracy of the estimated mapping also depends on the quality and spatial distribution of the available point correspondences, motivating future work on dense deformation validation, application-specific extensions, and further evaluation of the coordinate-shift formulation introduced in PEPS-3 for more challenging deformation scenarios and real-world applications.

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